Baryons in the light-front approach: The three-quark picture

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In this work, a three-quark picture is constructed using a bottom-up approach for baryons in a light-front quark model. The shape parameters, which characterize the momentum distribution inside a baryon, are determined with the help of the pole residue of the baryon. The relation between the three-quark picture and the diquark picture is clarified. When building the model, we find that Lorentz boost plays a crucial role, and the bottom-up modeling approach can be generalized to multiquark states. Based on this, a unified theoretical framework for describing multiquark states may be established. As a by-product of model construction, we can easily obtain a newly improved definition of baryon interpolating current. The hadron interpolating currents are the starting point of lattice QCD and QCD sum rules, and therefore are of great importance.

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I. INTRODUCTION

Recently, there have been some big events in the field of heavy flavor physics, including the discovery of CP violation (CPV) in neutral charm mesons [1] and the discovery of doubly charmed baryons [2]. On the one hand, CPV has been confirmed in many meson systems [3–5]; however, CPV has never been observed in the baryon sector, and to search for baryonic CPV is becoming particularly urgent [6]. On the other hand, the discovery of doubly charmed baryon has attracted great interest both theoretically and experimentally. The study of the properties of heavy baryons plays an important role in accurately testing the standard model, searching for the origin of CP violation and new physics, and understanding strong interactions.

From a theoretical perspective, baryons are generally more complicated than mesons. In spite of this, there have been many methods to study the decay properties of heavy flavor baryons, including the light-front quark model, SU(3) flavor symmetry, effective field theory, QCD and light-cone sum rules, perturbative QCD, and lattice QCD. Some recent progress can be found in Refs. [7–27].

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In Ref. [7], the light-front quark model (LFQM) is used to investigate the weak decays of doubly heavy baryons under the diquark picture. In this picture, the two quarks that do not participate in weak interactions are considered to form a loosely bound system—a diquark. In this way, the relatively complicated three-body problem inside a baryon is reduced to a relatively simple two-body problem. However, this diquark picture is criticized by some people. Take the weak decays of $\Xi_{bc}(bcq)$ as an example. When the b quark decays, (cq) is considered as a diquark, while when the c quark decays, (bq) is considered as a diquark. From this, one can see that the diquark picture is actually a matter of expediency. In addition, the diquark picture inevitably contains more parameters, such as the diquark masses. Even for the 0^+ diquark $[ud]$ and the 1^+ diquark $\{ud\}$ containing the same quark components, their masses are considered to be different [28].

Under the diquark picture, in Ref. [29], the baryon-quark-diquark vertex is given by

$$-\frac{1}{\sqrt{3}}\gamma_5 \not{\epsilon}^*(p_2,\lambda_2), \tag{1}$$

while in Ref. [30], it is

$$\frac{1}{\sqrt{3}}\gamma_5 \not{\epsilon}^*(\bar{P},\lambda_2) = \frac{1}{\sqrt{3}}\gamma_5\left(\not{\epsilon}^*(p_2,\lambda_2) - \frac{M_0+m_1+m_2}{p_2\cdot\bar{P}+m_2M_0}\epsilon^*(p_2,\lambda_2)\cdot\bar{P}\right), \tag{2}$$

where $p_{1,2}$ $(m_{1,2})$ are, respectively, the momenta (masses) of quark 1 and the diquark, λ_2 (ϵ) is the helicity (polarization vector) of the diquark, $\bar{P} \equiv p_1 + p_2$, and $M_0^2 \equiv \bar{P}^2$. In fact, except for an unimportant negative sign, there is a Lorentz boost between the two expressions, and the latter have correctly considered this effect. Later, one can see that this Lorentz boost effect plays a crucial role in our model construction.

Early in 1998, the authors of Ref. [31] developed the three-quark picture of heavy baryons. Recently, the three-quark picture has been used to study the weak decays of heavy flavor baryons in Refs. [32–37], and this work aims to highlight the following points:

- (i) Spin wave function will be constructed in a bottom-up approach. In this work, we are limited to considering ground state baryons. Several typical baryons include Λ_Q, Σ_Q, and Σ_Q^*. This is due to the following coupling in spin space: $(\frac{1}{2}\otimes\frac{1}{2})\otimes\frac{1}{2} = (0\oplus 1)\otimes\frac{1}{2} = \frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$. One of our potentially important discoveries is that the Lorentz boost may become more important when constructing the quark model of multiquark states, which have a certain size, and are multibody QCD bound states. Since we may still lack a quark model that can describe all the multiquark states, this discovery may be a key to unlock the door. It is worth noting that LFQM has been applied to some relatively simple multiquark states in Refs. [38,39]. In addition, as a by-product of model construction, we can easily obtain improved definitions of interpolating currents of baryons.
- (ii) A method for determining the shape parameters will be proposed. The shape parameters [see Eq. (15)] characterize the momentum distribution inside a hadron. For the meson case, the shape parameter is determined by the decay constant of the meson [40]. The "decay constant" of a baryon is the pole residue, which is used to determine the shape parameters of the baryon (see below).
- (iii) The relation between the diquark picture and the three-quark picture will be clarified. In this work, we will consider three weak decays: $\Lambda_b \to \Lambda_c$, $\Sigma_b \to \Sigma_c$, and $\Xi_{cc} \to \Lambda_c$. For the first two processes, the spectator quarks are, respectively, a scalar and an axial-vector diquark, while for the last process, a diquark is broken up in the initial state and a new diquark emerges by rearranging quarks in the final state. Here, for $\Xi_{cc} \to \Lambda_c$, the two charmed quarks in Ξ_{cc} are usually considered as an axial-vector diquark and the u, d quarks in Λ_c form a scalar one. In the diquark picture, overlap factors are important quantities for obtaining the physical form factors [7]. We will derive the overlap factors for $\Xi_{cc} \to \Lambda_c$, through which the relation between the diquark picture and the three-quark picture can be illustrated.

The rest of this article is arranged as follows. In Sec. II, theoretical framework and some applications are introduced, including the following: the definitions of baryon states; the determination of the shape parameters; the form factors of $\Lambda_b \to \Lambda_c$, $\Sigma_b \to \Sigma_c$, and $\Xi_{cc} \to \Lambda_c$; the relation between the two pictures; and improved definitions of interpolating currents of baryons. In Sec. III, numerical results of shape parameters, form factors, and semileptonic decay widths will be shown and will be compared with others in the literature. We conclude this article in the last section.

II. THEORETICAL FRAMEWORK AND SOME APPLICATIONS

A. The baryon states

In this section, we will consider three baryon states: Λ_Q, Σ_Q, and Σ_Q^*. They all have the same quark components udQ and are all S-wave baryons, and their spins are, respectively, $1/2$, $1/2$, and $3/2$.

Under the three-quark picture, the baryon state in LFQM is expressed as

$$
\begin{aligned}
|\mathcal{B}(P,S,S_z)\rangle = \int &\{d^3\tilde{p}_1\}\{d^3\tilde{p}_2\}\{d^3\tilde{p}_3\}2(2\pi)^3 \\
&\times \delta^3(\tilde{P}-\tilde{p}_1-\tilde{p}_2-\tilde{p}_3)\frac{1}{\sqrt{P^+}} \\
&\times \sum_{\lambda_1,\lambda_2,\lambda_3} \Psi^{SS_z}(\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\lambda_1,\lambda_2,\lambda_3) \\
&\times C^{ijk}|q_1^i(p_1,\lambda_1)q_2^j(p_2,\lambda_2)q_3^k(p_3,\lambda_3)\rangle, \qquad (3)
\end{aligned}
$$

where p_i (λ_i) is the light-front momentum (helicity) of the ith quark, the color wave function $C^{ijk} = \epsilon^{ijk}/\sqrt{6}$, and the spin and momentum wave functions are contained in Ψ^{SS_z}. The light-front momentum is decomposed into $p_i = (p_i^-, p_i^+, p_{i\perp})$ and

$$
\tilde{p}_i = (p_i^+, p_{i\perp}), \qquad p_{i\perp} = (p_i^1, p_i^2),
$$
$$
p_i^- = \frac{m_i^2 + p_{i\perp}^2}{p_i^+}, \qquad \{d^3\tilde{p}_i\} = \frac{dp_i^+ d^2 p_{i\perp}}{2(2\pi)^3}. \qquad (4)
$$

The intrinsic variables $(x_i, k_{i\perp})$ are introduced through

$$
p_i^+ = x_i P^+, \qquad p_{i\perp} = x_i P_\perp + k_{i\perp},
$$
$$
\sum_{i=1}^{3} x_i = 1, \qquad \sum_{i=1}^{3} k_{i\perp} = 0, \qquad (5)
$$

where x_i is the light-front momentum fraction constrained by $0 < x_i < 1$. The invariant mass M_0 is defined by $M_0^2 \equiv \bar{P}^2$ with $\bar{P} = p_1 + p_2 + p_3$, and it can be shown that

$$
M_0^2 = \frac{k_{1\perp}^2 + m_1^2}{x_1} + \frac{k_{2\perp}^2 + m_2^2}{x_2} + \frac{k_{3\perp}^2 + m_3^2}{x_3}. \qquad (6)
$$

M is in general different from the baryon mass M which obeys the condition $M^2 = P^2$. This is due to the fact that the baryon and the constituent quarks cannot be on their mass shell simultaneously. However, $\gamma^+ u(\bar{P}) = \gamma^+ u(P)$ holds [30]. The internal momenta are defined as

$$k_i = (k_i^-, k_i^+, k_{i\perp}) = (e_i - k_{iz}, e_i + k_{iz}, k_{i\perp}) = \left(\frac{m_i^2 + k_{i\perp}^2}{x_i M_0}, x_i M_0, k_{i\perp}\right), \quad (7)$$

and then it is easy to obtain

$$e_i = \frac{x_i M_0}{2} + \frac{m_i^2 + k_{i\perp}^2}{2 x_i M_0}, \qquad k_{iz} = \frac{x_i M_0}{2} - \frac{m_i^2 + k_{i\perp}^2}{2 x_i M_0}, \quad (8)$$

where e_i denotes the energy of the ith quark in the rest frame of $\bar{P}$. The momenta $k_{i\perp}$ and k_{iz} constitute a momentum three-vector $\vec{k}_i = (k_{i\perp}, k_{iz})$.

For Λ_Q, in which the u, d quarks are considered to form a 0^+ diquark, Ψ in Eq. (3) can be shown as

$$\Psi_0^{S=\frac{1}{2},S_z}(\tilde{p}_i, \lambda_i) = A_0 \bar{u}(p_3, \lambda_3)(\bar{\not{P}} + M_0)(-\gamma_5) C \bar{u}^T(p_2, \lambda_2) \times \bar{u}(p_1, \lambda_1) u(\bar{P}, S_z) \Phi(x_i, k_{i\perp}), \quad (9)$$

for Σ_Q, in which the u, d quarks are considered to form a 1^+ diquark,

$$\Psi_1^{S=\frac{1}{2},S_z}(\tilde{p}_i, \lambda_i) = A_1 \bar{u}(p_3, \lambda_3)(\bar{\not{P}} + M_0)(\gamma^\mu - v^\mu) C \bar{u}^T(p_2, \lambda_2) \times \bar{u}(p_1, \lambda_1)\left(\frac{1}{\sqrt{3}}\gamma_\mu \gamma_5\right) u(\bar{P}, S_z) \Phi(x_i, k_{i\perp}), \quad (10)$$

and for Σ_Q^*, in which the u, d quarks are also considered to form a 1^+ diquark,

$$\Psi_1^{S=\frac{3}{2},S_z}(\tilde{p}_i, \lambda_i) = A_1' \bar{u}(p_3, \lambda_3)(\bar{\not{P}} + M_0)(\gamma^\mu - v^\mu) C \bar{u}^T(p_2, \lambda_2) \times \bar{u}(p_1, \lambda_1) u_\mu(\bar{P}, S_z) \Phi(x_i, k_{i\perp}), \quad (11)$$

where $v^\mu \equiv \bar{P}^\mu / M_0$ and Φ is the momentum wave function. The proofs of Eqs. (9)–(11) can be found in the Appendix.

With the normalization of the baryon state

$$\langle \mathcal{B}(P', S', S_z') | \mathcal{B}(P, S, S_z) \rangle = 2(2\pi)^3 P^+ \delta^3(\tilde{P}' - \tilde{P}) \times \delta_{S'S} \delta_{S_z' S_z} \quad (12)$$

and

$$\int \left(\prod_{i=1}^{3} \frac{dx_i d^2 k_{i\perp}}{2(2\pi)^3}\right) 2(2\pi)^3 \delta\left(1 - \sum x_i\right) \delta^2\left(\sum k_{i\perp}\right) \times |\Phi(x_i, k_{i\perp})|^2 = 1, \quad (13)$$

one can obtain

$$A_0 = A_1 = A_1' = \frac{1}{4\sqrt{M_0^3 (e_1 + m_1)(e_2 + m_2)(e_3 + m_3)}}. \quad (14)$$

The momentum wave function can be given by

$$\Phi(x_i, k_{i\perp}) = \sqrt{\frac{e_1 e_2 e_3}{x_1 x_2 x_3 M_0}} \varphi(\vec{k}_1, \beta_1) \varphi\left(\frac{\vec{k}_2 - \vec{k}_3}{2}, \beta_{23}\right), \quad (15)$$

where $\varphi(\vec{k}, \beta) \equiv 4(\frac{\pi}{\beta^2})^{3/4} \exp(\frac{-k_\perp^2 - k_z^2}{2\beta^2})$, and β_1 and β_{23} are the shape parameters.

Some important notes are given below:

(i) The definition of a baryon state is the most important part of LFQM, while the spin wave function is the most important part in the definition of a baryon state. It is worth pointing out that, when we arrive at Eqs. (9)–(11), we do not introduce any additional assumptions; for example, it does not assume heavy quark symmetry, nor does it depend on the coordinate system selection of LFQM (see below). Therefore, the spin wave functions in Eqs. (9)–(11) may not only apply to heavy flavor baryons but should also apply to light flavor baryons.

(ii) From the proof in the Appendix, one can clearly see that, the Lorentz boost between the rest frame of "diquark" and the rest frame of $\bar{P}$ plays a crucial role. The proof of Eq. (9) is relatively simple, and this is because the first case involves a scalar diquark, whose Lorentz boost is trivial. The proofs of Eqs. (10) and (11) are relatively complicated, because the latter two cases involve an axis-vector diquark, whose Lorentz boost is nontrivial.

(iii) If we only consider the spin coupling, in principle, we can choose any two quarks for spin coupling first. However, when we consider the flavor wave function, the two quarks that are coupled first are usually already determined. For example, for Λ_Q, whose flavor wave function is $(ud - du)Q/\sqrt{2}$, we couple the u and d quarks first; while for Ξ_{cc}^{++}, whose flavor wave function is just ccu, we couple the two charm quarks first.

(iv) If identical quarks are contained in the baryon state, some additional factors should be added. For example, for Ξ_{cc}^{++}, when we calculate

$\langle\mathcal{B}(P',S',S'_z)|\mathcal{B}(P,S,S_z)\rangle$ to normalize the baryon state, a factor of 2 appears because of two equivalent contractions. An additional factor $1/\sqrt{2}$ should be added in the definition of $|\Xi_{cc}^{++}\rangle$ in order to keep Eq. (12) unchanged.

B. To determine the shape parameters

The shape parameters in Eq. (15) characterize the momentum distribution inside the baryon. In this work, we suggest that the shape parameters can be determined by the pole residue of the baryon, whose numerical result can be taken from, for example, lattice QCD or QCD sum rules.

Taking Λ_Q as an example, let us focus on the matrix element $\langle 0|J_{\Lambda_Q}|\Lambda_Q\rangle$ with $J_{\Lambda_Q}=\epsilon_{abc}[u_a^T C\gamma_5 d_b]Q_c$. On the one hand, this matrix element can be calculated in LFQM,

$$\begin{aligned}&\langle 0|J_{\Lambda_Q}|\Lambda_Q(P,S_z)\rangle\\&\quad=\int\frac{dx_2d^2k_{2\perp}}{2(2\pi)^3}\frac{dx_3d^2k_{3\perp}}{2(2\pi)^3}\frac{1}{\sqrt{x_1x_2x_3}}\Phi(x_i,k_{i\perp})\sqrt{6}A_0\\&\qquad\times\mathrm{Tr}[C\gamma_5(\not{p}_3+m_3)(\bar{\not{P}}+M_0)(-\gamma_5)C(\not{p}_2+m_2)^T]\\&\qquad\times(\not{p}_1+m_1)u(\bar{P},S_z).\end{aligned}\tag{16}$$

On the other hand, the pole residue of baryon is defined by

$$\langle 0|J_{\Lambda_Q}|\Lambda_Q(P,S_z)\rangle=\lambda_{\Lambda_Q}u(P,S_z).\tag{17}$$

Respectively multiplying Eqs. (16) and (17) with $\sum_{S_z}\bar{u}(P,S_z)\gamma^+$ from the left, also noting that $\gamma^+u(P)=\gamma^+u(\bar{P})$, one can arrive at

$$\begin{aligned}&\int\frac{dx_2d^2k_{2\perp}}{2(2\pi)^3}\frac{dx_3d^2k_{3\perp}}{2(2\pi)^3}\frac{1}{\sqrt{x_1x_2x_3}}\Phi(x_i,k_{i\perp})\sqrt{6}A_0\\&\quad\times\mathrm{Tr}[C\gamma_5(\not{p}_3+m_3)(\bar{\not{P}}+M_0)(-\gamma_5)C(\not{p}_2+m_2)^T]\\&\quad\times\mathrm{Tr}[\gamma^+(\not{p}_1+m_1)(\bar{\not{P}}+M_0)]\end{aligned}\tag{18}$$

and

$$\lambda_{\Lambda_Q}\mathrm{Tr}[\gamma^+(\not{P}+M)].\tag{19}$$

Equating Eqs. (18) and (19), one can obtain the expression of pole residue in LFQM

$$\begin{aligned}\lambda_{\Lambda_Q}=&\frac{1}{\mathrm{Tr}[\gamma^+(\not{P}+M)]}\\&\times\int\frac{dx_2d^2k_{2\perp}}{2(2\pi)^3}\frac{dx_3d^2k_{3\perp}}{2(2\pi)^3}\frac{1}{\sqrt{x_1x_2x_3}}\Phi(x_i,k_{i\perp})\sqrt{6}A_0\\&\times\mathrm{Tr}[C\gamma_5(\not{p}_3+m_3)(\bar{\not{P}}+M_0)(-\gamma_5)C(\not{p}_2+m_2)^T]\\&\times\mathrm{Tr}[\gamma^+(\not{p}_1+m_1)(\bar{\not{P}}+M_0)],\end{aligned}\tag{20}$$

which can be used to determine the shape parameters in Φ provided the pole residue is known. Since there are two shape parameters in Φ, one more equation is desirable; at this time, use $\sum_{S_z}\bar{u}(P,S_z)\gamma^+\gamma^-$ to left multiply instead, and finally we have

$$\begin{aligned}\lambda_{\Lambda_Q}=&\frac{1}{\mathrm{Tr}[\gamma^+\gamma^-(\not{P}+M)]}\\&\times\int\frac{dx_2d^2k_{2\perp}}{2(2\pi)^3}\frac{dx_3d^2k_{3\perp}}{2(2\pi)^3}\frac{1}{\sqrt{x_1x_2x_3}}\Phi(x_i,k_{i\perp})\sqrt{6}A_0\\&\times\mathrm{Tr}[C\gamma_5(\not{p}_3+m_3)(\bar{\not{P}}+M_0)(-\gamma_5)C(\not{p}_2+m_2)^T]\\&\times\mathrm{Tr}[\gamma^+\gamma^-(\not{p}_1+m_1)(\bar{\not{P}}+M_0)].\end{aligned}\tag{21}$$

The expressions of pole residues of Σ_Q and Ξ_{cc} can also be obtained in a similar way.

In addition, it should be noted that the baryon mass M can in turn be extracted by equating Eqs. (20) and (21) once the shape parameters are fixed by, for example, global fitting.

C. Form factors of $\Lambda_b\to\Lambda_c$

On the one hand, the weak decay matrix element $\langle\Lambda_c|\bar{c}\gamma^\mu(1-\gamma_5)b|\Lambda_b\rangle$ can be parametrized in terms of form factors

$$\begin{aligned}&\langle\Lambda_c(P',S'_z)|\bar{c}\gamma^\mu(1-\gamma_5)b|\Lambda_b(P,S_z)\rangle\\&\quad=\bar{u}(P',S'_z)\left\{\left[\gamma^\mu f_1(q^2)+i\sigma^{\mu\nu}\frac{q_\nu}{M}f_2(q^2)+\frac{q^\mu}{M}f_3(q^2)\right]\right.\\&\qquad\left.-\left[\gamma^\mu g_1(q^2)+i\sigma^{\mu\nu}\frac{q_\nu}{M}g_2(q^2)+\frac{q^\mu}{M}g_3(q^2)\right]\gamma_5\right\}u(P,S_z),\end{aligned}\tag{22}$$

where $q=P-P'$ and f_i, g_i are the form factors. On the other hand, the matrix element can also be calculated in LFQM

$$\begin{aligned}&\langle\Lambda_c(P',S'_z)|\bar{c}\gamma^\mu(1-\gamma_5)b|\Lambda_b(P,S_z)\rangle\\&=\int\{d^3\tilde{p}_2\}\{d^3\tilde{p}_3\}\frac{A'_0A_0}{\sqrt{p_1'^+p_1^+P'^+P^+}}\Phi'^*(x'_i,k'_{i\perp})\Phi(x_i,k_{i\perp})\\&\quad\times\mathrm{Tr}[(\bar{\not{P}}+M_0)(-\gamma_5)C(\not{p}_2+m_2)^TC\gamma_5(\bar{\not{P}}'+M'_0)(\not{p}_3+m_3)]\\&\quad\times\bar{u}(\bar{P}',S'_z)(\not{p}'_1+m'_1)\gamma^\mu(1-\gamma_5)(\not{p}_1+m_1)u(\bar{P},S_z).\end{aligned}\tag{23}$$

Now we extract the form factors $f_{1,2}$ and $g_{1,2}$ in the following method. Respectively multiplying the "+" component of the vector current part of Eq. (22) by $\sum_{S_z,S'_z}\bar{u}(P,S_z)\gamma^+u(P',S'_z)$ and $\sum_{S_z,S'_z}\bar{u}(P,S_z)\times(\sum_{j=1}^2 i\sigma^{+j}q^j)u(P',S'_z)$ from the left, one can obtain

$$\mathrm{Tr}\left[(\not{P}+M)\gamma^+(\not{P}'+M')\left(f_1\gamma^+ + f_2 i\sigma^{+\nu}\frac{q_\nu}{M}\right)\right] = 8P^+P'^+f_1 \quad (24)$$

and

$$\mathrm{Tr}\left[(\not{P}+M)\left(\sum_{j=1}^{2} i\sigma^{+j}q^j\right)(\not{P}'+M') \times \left(f_1\gamma^+ + f_2 i\sigma^{+\nu}\frac{q_\nu}{M}\right)\right] = -8P^+P'^+\frac{q^2}{M}f_2. \quad (25)$$

Respectively multiplying the "+" component of the axial-vector current part of Eq. (22) by $\sum_{S_z,S'_z}\bar{u}(P,S_z)\times\gamma^+\gamma_5 u(P',S'_z)$ and $\sum_{S_z,S'_z}\bar{u}(P,S_z)(\sum_{j=1}^{2} i\sigma^{+j}q^j\gamma_5)\times u(P',S'_z)$ from the left, one can obtain

$$\mathrm{Tr}\left[(\not{P}+M)\gamma^+\gamma_5(\not{P}'+M')\left(g_1\gamma^+ + g_2 i\sigma^{+\nu}\frac{q_\nu}{M}\right)\gamma_5\right] = 8P^+P'^+g_1 \quad (26)$$

and

$$\mathrm{Tr}\left[(\not{P}+M)\left(\sum_{j=1}^{2} i\sigma^{+j}q^j\gamma_5\right)(\not{P}'+M') \times \left(g_1\gamma^+ + g_2 i\sigma^{+\nu}\frac{q_\nu}{M}\right)\gamma_5\right] = 8P^+P'^+\frac{q^2}{M}g_2. \quad (27)$$

Then doing the same thing for Eq. (23), one can obtain

$$\begin{aligned} f_1 = \frac{1}{8P^+P'^+}\int\{d^3\tilde{p}_2\}\{d^3\tilde{p}_3\}\frac{\Phi'^*\Phi}{\sqrt{p_1'^+p_1^+P'^+P^+}}A'_0A_0 \\ \times \mathrm{Tr}[(\bar{\not{P}}+M_0)(-\gamma_5)C(\not{p}_2+m_2)^TC\gamma_5(\bar{\not{P}}'+M'_0) \\ \times(\not{p}_3+m_3)] \\ \times \mathrm{Tr}[(\bar{\not{P}}+M_0)\Gamma_1(\bar{\not{P}}'+M'_0)(\not{p}'_1+m'_1)\Gamma_2(\not{p}_1+m_1)] \end{aligned} \quad (28)$$

with

$$\Gamma_1 = \gamma^+, \qquad \Gamma_2 = \gamma^+, \quad (29)$$

where we have used $\gamma^+u(P) = \gamma^+u(\bar{P})$. The same expressions can be obtained for f_2 and $g_{1,2}$, except that

(i) for f_2,

$$\Gamma_1 = -\frac{M}{q^2}\sum_{j=1}^{2} i\sigma^{+j}q^j, \qquad \Gamma_2 = \gamma^+; \quad (30)$$

(ii) for g_1,

$$\Gamma_1 = \gamma^+\gamma_5, \qquad \Gamma_2 = \gamma^+\gamma_5; \quad (31)$$

(iii) for g_2,

$$\Gamma_1 = \frac{M}{q^2}\sum_{j=1}^{2} i\sigma^{+j}q^j\gamma_5, \qquad \Gamma_2 = \gamma^+\gamma_5. \quad (32)$$

In practice, we choose the frame that satisfies $q^+ = 0$, that is,

$$P^+ - P'^+ = 0. \quad (33)$$

When calculating the weak decay matrix element in Eq. (23), one can find that the momenta of quark 2 and quark 3 remain unchanged from the initial state to the final state, from which one can obtain

$$\begin{aligned} x'_2 = x_2, \qquad k'_{2\perp} = k_{2\perp} + x_2 q_\perp, \\ x'_3 = x_3, \qquad k'_{3\perp} = k_{3\perp} + x_3 q_\perp, \end{aligned} \quad (34)$$

and furthermore,

$$x'_1 = x_1, \qquad k'_{1\perp} = k_{1\perp} - (1-x_1)q_\perp. \quad (35)$$

One comment. As pointed out in Ref. [32], the form factors f_3 and g_3 cannot be extracted for we have imposed the condition $q^+ = 0$. However, these two form factors do not contribute to the $1/2 \to 1/2$ semileptonic decays if the electron mass is neglected.

D. Form factors of $\Sigma_b \to \Sigma_c$

$\langle\Sigma_c|\bar{c}\gamma^\mu(1-\gamma_5)b|\Sigma_b\rangle$ can also be obtained in LFQM as

$$\begin{aligned} \langle\Sigma_c(P',S'_z)|\bar{c}\gamma^\mu(1-\gamma_5)b|\Sigma_b(P,S_z)\rangle = \int\{d^3\tilde{p}_2\}\{d^3\tilde{p}_3\}\frac{A'_1A_1}{\sqrt{p_1'^+p_1^+P'^+P^+}}\Phi'^*(x'_i,k'_{i\perp})\Phi(x_i,k_{i\perp}) \\ \times \mathrm{Tr}[(\bar{\not{P}}+M_0)(\gamma^\rho - v^\rho)C(\not{p}_2+m_2)^TC(\gamma^\sigma - v'^\sigma)(\bar{\not{P}}'+M'_0)(\not{p}_3+m_3)] \\ \times \bar{u}(\bar{P}',S'_z)\frac{1}{\sqrt{3}}\gamma_\sigma\gamma_5(\not{p}'_1+m'_1)\gamma^\mu(1-\gamma_5)(\not{p}_1+m_1)\frac{1}{\sqrt{3}}\gamma_\rho\gamma_5 u(\bar{P},S_z), \end{aligned} \quad (36)$$

where $v^\mu = \bar{P}^\mu/M_0$ and $v'^\mu = \bar{P}'^\mu/M'_0$. It turns out that

$$f_1 = \frac{1}{8P^+P'^+}\int\{d^3\tilde{p}_2\}\{d^3\tilde{p}_3\}\frac{\Phi'^*\Phi}{\sqrt{p_1'^+p_1^+P'^+P^+}}A_1'A_1 \times \mathrm{Tr}[(\bar{\not{P}}+M_0)(\gamma^\rho - v^\rho)C(\not{p}_2+m_2)^T C(\gamma^\sigma - v'^\sigma) \times(\bar{\not{P}}'+M_0')(\not{p}_3+m_3)] \times \mathrm{Tr}\left[(\bar{\not{P}}+M_0)\Gamma_1(\bar{\not{P}}'+M_0')\frac{1}{\sqrt{3}}\gamma_\sigma\gamma_5(\not{p}_1'+m_1')\Gamma_2 \times(\not{p}_1+m_1)\frac{1}{\sqrt{3}}\gamma_\rho\gamma_5\right] \tag{37}$$

with

$$\Gamma_1 = \gamma^+, \qquad \Gamma_2 = \gamma^+. \tag{38}$$

f_2 and $g_{1,2}$ can also be obtained by the same assignments to $\Gamma_{1,2}$ as those in Sec. II C.

E. Form factors of $\Xi_{cc} \to \Lambda_c$

$\langle\Lambda_c|\bar{d}\gamma^\mu(1-\gamma_5)c|\Xi_{cc}\rangle$ can also be obtained in LFQM as

$$\langle\Lambda_c(P',S_z')|\bar{d}\gamma^\mu(1-\gamma_5)c|\Xi_{cc}(P,S_z)\rangle = \int\{d^3\tilde{p}_2\}\{d^3\tilde{p}_3\}\frac{A_0'A_1}{\sqrt{p_1'^+p_1^+P'^+P^+}}\Phi'^*(x_i',k_{i\perp}')\Phi(x_i,k_{i\perp}) \times\frac{2}{\sqrt{2}}\bar{u}(\bar{P}',S_z')(\not{p}_2+m_2)(\bar{\not{P}}+M_0)(\gamma^\nu - v^\nu)C \times(\not{p}_1+m_1)^T(1-\gamma_5)^T\gamma^{\mu T}(\not{p}_1'+m_1')^T C\gamma_5 \times(\bar{\not{P}}'+M_0')(\not{p}_3+m_3)\frac{1}{\sqrt{3}}\gamma_\nu\gamma_5 u(\bar{P},S_z), \tag{39}$$

where the factor of 2 comes from the two equivalent contractions and the factor of $1/\sqrt{2}$ comes from the normalization of the Ξ_{cc}^{++} state, which has been pointed out in Sec. II A. It turns out that

$$f_1 = \frac{1}{8P^+P'^+}\int\{d^3\tilde{p}_2\}\{d^3\tilde{p}_3\}\frac{\Phi'^*\Phi}{\sqrt{p_1'^+p_1^+P'^+P^+}}A_0'A_1 \times\frac{2}{\sqrt{2}}\mathrm{Tr}\left[(\bar{\not{P}}+M_0)\Gamma_1(\bar{\not{P}}'+M_0')(\not{p}_2+m_2)(\bar{\not{P}}+M_0) \times(\gamma^\nu - v^\nu)(\not{p}_1-m_1)\Gamma_2(\not{p}_1'-m_1')\gamma_5(\bar{\not{P}}'+M_0') \times(\not{p}_3+m_3)\frac{1}{\sqrt{3}}\gamma_\nu\gamma_5\right] \tag{40}$$

with

$$\Gamma_1 = \gamma^+, \qquad \Gamma_2 = \gamma^+. \tag{41}$$

f_2 and $g_{1,2}$ can also be obtained by the same assignments to $\Gamma_{1,2}$ as those in Sec. II C except for the only one difference for $g_{1,2}$

$$\Gamma_2 = \gamma_5\gamma^+, \tag{42}$$

because we have performed a transpose in Eq. (39).

F. The relation between the two pictures

Denote the spin wave function in Eq. (9) as $\psi_0(321)$ and that in Eq. (10) as $\psi_1(321)$, i.e.,

$$\psi_0(321) \equiv \bar{u}(p_3,\lambda_3)(\bar{\not{P}}+M_0)(-\gamma_5)C\bar{u}^T(p_2,\lambda_2) \times\bar{u}(p_1,\lambda_1)u(\bar{P},S_z),$$
$$\psi_1(321) \equiv \bar{u}(p_3,\lambda_3)(\bar{\not{P}}+M_0)(\gamma^\mu - v^\mu)C\bar{u}^T(p_2,\lambda_2) \times\bar{u}(p_1,\lambda_1)\left(\frac{1}{\sqrt{3}}\gamma_\mu\gamma_5\right)u(\bar{P},S_z). \tag{43}$$

$\psi_{0,1}(321)$ have the same normalization factor as that in Eq. (14). Moreover, $\psi_{0,1}(321)$ are orthogonal

$$\sum_{\lambda_1\lambda_2\lambda_3}\psi_0^\dagger(321')\psi_1(321) = 0, \tag{44}$$

where quark $1'$ does not need to be the same as quark 1.

Now consider the relation between the three-quark picture and the diquark picture. The key observation is that in the three-quark picture, the first two quarks form a diquark in the diquark picture. Specifically, in $\psi_0(321)/\psi_1(321)$, quark 3 and quark 2 are considered to form a scalar/axial-vector diquark. In fact, $\psi_{0,1}(321)$ constitute a diquark basis.

One can easily check that

$$\psi_0(321) = -\psi_0(231),$$
$$\psi_1(321) = \psi_1(231). \tag{45}$$

In addition, it can be shown that

$$\begin{pmatrix}\psi_0(312)\\ \psi_1(312)\end{pmatrix} = T\begin{pmatrix}\psi_0(321)\\ \psi_1(321)\end{pmatrix}, \tag{46}$$

with the transition matrix

$$T = \begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}, \tag{47}$$

which satisfies $T^{-1} = T$, as expected.

Now we are ready to determine the overlap factors in the diquark picture. Take the process of $\Xi_{bc}^+(cbu) \to \Lambda_b(dbu)$ as an example, where the b, c quarks in the initial state are considered to form an axial-vector diquark, while the u, d quarks in the final state are considered to form a scalar diquark. The initial and final states can be rewritten as

$$\psi_1(bcu) = -\frac{\sqrt{3}}{2}\psi_0(buc) - \frac{1}{2}\psi_1(buc),$$
$$\psi_0(udb) = \frac{1}{2}\psi_0(ubd) - \frac{\sqrt{3}}{2}\psi_1(ubd) = -\frac{1}{2}\psi_0(bud) - \frac{\sqrt{3}}{2}\psi_1(bud). \quad (48)$$

Then it is easy to write the transition matrix element as

$$\langle\psi_0(udb)|\psi_1(bcu)\rangle = \frac{\sqrt{3}}{4}\langle\psi_0(bud)|\psi_0(buc)\rangle + \frac{\sqrt{3}}{4}\langle\psi_1(bud)|\psi_1(buc)\rangle, \quad (49)$$

from which one can read the two overlap factors

$$c_S = c_A = \frac{\sqrt{3}}{4}. \quad (50)$$

They are the same as those in the diquark picture in Ref. [7].

It is time to go one step further to derive the overlap factors of $\Xi_{cc}^{++}(ccu) \to \Lambda_c(dcu)$. To this end, notice that there is only one difference between this process and $\Xi_{bc}^{+}(cbu) \to \Lambda_b(dbu)$; that is, Ξ_{cc}^{++} contains two identical quarks. At this time, one can obtain the overlap factors for $\Xi_{cc}^{++} \to \Lambda_c$,

$$c_S = c_A = \frac{2}{\sqrt{2}} \times \frac{\sqrt{3}}{4} = \frac{\sqrt{6}}{4}, \quad (51)$$

where the factor $2/\sqrt{2}$ can be found in Eq. (39). These factors are also the same as those in the diquark picture in Ref. [7].

G. Improved definitions of interpolating currents

The definitions of interpolating currents are the starting point of lattice QCD and QCD sum rules. The following definitions are usually adopted for Λ_Q and Σ_Q in the literature [41–44]:

$$J_{\Lambda_Q} = \epsilon_{abc}[u_a^T C\gamma_5 d_b]Q_c,$$
$$J_{\Sigma_Q} = \epsilon_{abc}[u_a^T C\gamma^\mu d_b]\gamma_\mu\gamma_5 Q_c. \quad (52)$$

These interpolating currents of baryons were first given in Ref. [45], and then widely used to study the properties of baryons. They are obtained with the help of symmetry analysis; however, to our knowledge, there is no literature that provides a rigorous proof starting from quark spinors and Dirac matrices. This work may fill this gap.

Hermite conjugating Eq. (43), one can obtain the following improved definitions of interpolating currents for Λ_Q and Σ_Q:

$$J_{\Lambda_Q} = \epsilon_{abc}[u_a^T C\gamma_5(1+\not{v})d_b]Q_c,$$
$$J_{\Sigma_Q} = \epsilon_{abc}[u_a^T C(\gamma^\mu - v^\mu)(1+\not{v})d_b]\frac{1}{\sqrt{3}}\gamma_\mu\gamma_5 Q_c, \quad (53)$$

where $v^\mu \equiv p^\mu/\sqrt{p^2}$ with p the four-momentum of baryon. Some comments are in order.

(i) As can be seen in Eq. (56) that $\lambda_{\Sigma_Q} \approx 2\lambda_{\Lambda_Q}$, which are calculated in QCD sum rules using the definitions in Eq. (52) [42,43]. If the factor of $1/\sqrt{3}$ in Eq. (53) is considered, one would expect $\lambda_{\Sigma_Q} \approx \lambda_{\Lambda_Q}$.

(ii) It can be seen that we can even let $v \to 0$ in Eq. (53) to get the definitions in Eq. (52) if we temporarily forget the coefficient $1/\sqrt{3}$. However, we cannot do that, because v^μ, according to its definition, is $\mathcal{O}(1)$.

III. NUMERICAL RESULTS

The following quark masses are adopted:

$$m_u = m_d = 0.25 \text{ GeV},$$
$$m_c = 1.4 \text{ GeV}, \qquad m_b = 4.8 \text{ GeV}, \quad (54)$$

which are widely used in previous literature of LFQM [7–11,46–48]. Throughout this article, the following baryon masses are used [49]:

$$m_{\Lambda_b} = 5.620 \text{ GeV}, \qquad m_{\Lambda_c} = 2.286 \text{ GeV},$$
$$m_{\Sigma_b} = 5.811 \text{ GeV}, \qquad m_{\Sigma_c} = 2.453 \text{ GeV},$$
$$m_{\Xi_{cc}^{++}} = 3.622 \text{ GeV}. \quad (55)$$

A. The shape parameters

Using the pole residues of Λ_Q, Σ_Q, and Ξ_{cc} obtained in Refs. [42–44],

$$\lambda_{\Lambda_b} = 0.030 \pm 0.009, \qquad \lambda_{\Lambda_c} = 0.022 \pm 0.008,$$
$$\lambda_{\Sigma_b} = 0.062 \pm 0.018, \qquad \lambda_{\Sigma_c} = 0.045 \pm 0.015,$$
$$\lambda_{\Xi_{cc}} = 0.115 \pm 0.027, \quad (56)$$

which are calculated in QCD sum rules, one can obtain the following optimal shape parameters:

$$\beta_{b,[ud]} = 0.63 \pm 0.05 \text{ GeV}, \qquad \beta_{[ud]} = 0.27 \pm 0.03 \text{ GeV},$$
$$\beta_{c,[ud]} = 0.45 \pm 0.05 \text{ GeV};$$
$$\beta_{b,\{ud\}} = 0.66 \pm 0.04 \text{ GeV}, \qquad \beta_{\{ud\}} = 0.28 \pm 0.03 \text{ GeV},$$
$$\beta_{c,\{ud\}} = 0.49 \pm 0.04 \text{ GeV};$$
$$\beta_{u,\{cc\}} = 0.490 \pm 0.040 \text{ GeV},$$
$$\beta_{\{cc\}} = 0.400 \pm 0.025 \text{ GeV}. \quad (57)$$

Some comments are in order:

(i) Respectively denote the pole residues calculated in Eqs. (20) and (21) as λ_1 and λ_2, with the pole residue in Eq. (56) as $\lambda_{\rm QCDSR}$. By letting $\lambda_1 \approx \lambda_2 \approx \lambda_{\rm QCDSR}$, one can determine the shape parameters in Eq. (57), and the uncertainty comes from that of the pole residue.

(ii) It would be interesting to compare the shape parameters with those used in the diquark picture [8], and the latter are in fact the shape parameters of mesons. For example, numerically, our $\beta_{b,[ud]}$, $\beta_{c,[ud]}$, and $\beta_{[ud]}$ are, respectively, close to [in units of giga-electron volts (GeV)] $\beta_{b\bar{s}} = 0.623$, $\beta_{c\bar{s}} = 0.535$, and $\beta_{d\bar{s}} = 0.393$ in Ref. [8]. A significant difference is found between $\beta_{\{cc\}} = 0.400$ GeV in this work and $\beta_{c\bar{c}} = 0.753$ GeV in Ref. [8]. It seems that the charm quark and the anticharm quark in η_c are more energetic than the two charm quarks in Ξ_{cc}. It is worth noting that, when deriving the pole residue expressions for Ξ_{cc}, a factor of $2/\sqrt{2}$ also appears. Using the pole residue $\lambda_{\Xi_{cc}}$ in Eq. (56), and having considered this factor, one can obtain the much smaller shape parameter $\beta_{\{cc\}}$ together with $\beta_{u,\{cc\}}$.

B. The form factors and semileptonic decays

The following form factors at $q^2 = 0$ are obtained for $\Lambda_b \to \Lambda_c$:

$$f_1(0) = 0.469 \pm 0.029, \quad f_2(0) = -0.105 \pm 0.011,$$
$$g_1(0) = 0.461 \pm 0.027, \quad g_2(0) = 0.006 \pm 0.005; \tag{58}$$

for $\Sigma_b \to \Sigma_c$:

$$f_1(0) = 0.490 \pm 0.018, \quad f_2(0) = 0.467 \pm 0.006,$$
$$g_1(0) = -0.163 \pm 0.005, \quad g_2(0) = 0.007 \pm 0.001; \tag{59}$$

and for $\Xi_{cc} \to \Lambda_c$:

$$f_1(0) = 0.517 \pm 0.071, \quad f_2(0) = -0.036 \pm 0.007,$$
$$g_1(0) = 0.155 \pm 0.019, \quad g_2(0) = -0.072 \pm 0.012. \tag{60}$$

It can be seen that about 6%, 4%, 14% uncertainties are, respectively, introduced for the three groups of form factors because of the uncertainty of the pole residue.

To access the q^2 dependence of the form factors, we calculate the form factors in an interval $q^2 \in [-5, 0]$ GeV^2 for $\Lambda_b \to \Lambda_c$ and $\Sigma_b \to \Sigma_c$, and $q^2 \in [-0.5, 0]$ GeV^2 for $\Xi_{cc} \to \Lambda_c$, and fit the results with the following simplified z-expansion [50]:

$$f(q^2) = \frac{a + bz(q^2)}{1 - q^2/m_{\rm pole}^2}, \tag{61}$$

where $m_{\rm pole} = m_{B_c}$ for $\Lambda_b \to \Lambda_c$ and $\Sigma_b \to \Sigma_c$, and $m_{\rm pole} = m_D$ for $\Xi_{cc} \to \Lambda_c$,

$$z(q^2) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - q^2_{\rm max}}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - q^2_{\rm max}}} \tag{62}$$

with $t_+ = m^2_{\rm pole}$, $q^2_{\rm max} = (M_{\Lambda_b} - M_{\Lambda_c})^2$ for $\Lambda_b \to \Lambda_c$, $q^2_{\rm max} = (M_{\Sigma_b} - M_{\Sigma_c})^2$ for $\Sigma_b \to \Sigma_c$, and $q^2_{\rm max} = (M_{\Xi_{cc}} - M_{\Lambda_c})^2$ for $\Xi_{cc} \to \Lambda_c$. The fitted results of (a, b) for the three processes are given in Table I.

The obtained form factors are then applied to semi-leptonic decays, it turns out that the central values of decay widths and branching ratios are

$$\Gamma(\Lambda_b \to \Lambda_c e^- \bar{\nu}_e) = 2.54 \times 10^{-14}\ \text{GeV},$$
$$\mathcal{B}(\Lambda_b \to \Lambda_c e^- \bar{\nu}_e) = 5.68\%,$$
$$\Gamma(\Sigma_b \to \Sigma_c e^- \bar{\nu}_e) = 0.870 \times 10^{-14}\ \text{GeV},$$
$$\Gamma(\Xi_{cc} \to \Lambda_c e^+ \nu_e) = 0.755 \times 10^{-14}\ \text{GeV},$$
$$\mathcal{B}(\Xi_{cc} \to \Lambda_c e^+ \nu_e) = 0.294\%, \tag{63}$$

where $\tau_{\Lambda_b} = 1.471 \times 10^{-12}$ s, $|V_{cb}| = 0.0408$, and $\tau_{\Xi_{cc}^{++}} = 0.256 \times 10^{-12}$ s, $|V_{cd}| = 0.221$ have been used [49]. Considering the uncertainties of form factors in Eqs. (58)–(60), there are, respectively, about 13%, 7%, and 29% uncertainties in these phenomenological predictions.

C. Comparison with other results in the literature

In Tables II and III, we, respectively, compare our form factors and semileptonic decay widths with those in the literature. It can be seen that our predictions are comparable with other results; in addition, it seems that the diquark picture tends to give larger predictions. It is likely that

TABLE I. Fitted results of (a, b) for the form factors.

Transition	F	(a, b)	Transition	F	(a, b)	Transition	F	(a, b)
$\Lambda_b \to \Lambda_c$	f_1	(0.648, −2.177)	$\Sigma_b \to \Sigma_c$	f_1	(0.794, −3.617)	$\Xi_{cc} \to \Lambda_c$	f_1	(1.101, −3.312)
	f_2	(−0.162, 0.704)		f_2	(0.728, −3.103)		f_2	(−0.064, 0.162)
	g_1	(0.632, −2.068)		g_1	(−0.222, 0.697)		g_1	(0.280, −0.710)
	g_2	(0.011, −0.065)		g_2	(0.011, −0.053)		g_2	(−0.185, 0.644)

TABLE II. Our form factors are compared with other results in the literature.[a]

$\Lambda_b \to \Lambda_c$	$f_1(0)$	$f_2(0)$	$f_3(0)$	$g_1(0)$	$g_2(0)$	$g_3(0)$
This work	0.469 ± 0.029	-0.105 ± 0.011	⋯	0.461 ± 0.027	0.006 ± 0.005	⋯
Three-quark [32]	0.488	−0.180	⋯	0.470	−0.048	⋯
Diquark [8]	0.670	−0.132	⋯	0.656	−0.012	⋯
Diquark [51]	0.506	−0.099	⋯	0.501	−0.009	⋯
QCDSR [21]	0.431	−0.123	0.022	0.434	0.036	−0.160
LQCD [50]	0.418	−0.099	−0.075	0.390	−0.004	−0.206
$\Sigma_b \to \Sigma_c$	$f_1(0)$	$f_2(0)$	$f_3(0)$	$g_1(0)$	$g_2(0)$	$g_3(0)$
This work	0.490 ± 0.018	0.467 ± 0.006	⋯	-0.163 ± 0.005	0.007 ± 0.001	⋯
Three-quark [32]	0.494	0.407	⋯	−0.156	−0.053	⋯
Diquark [29]	0.466	0.736	⋯	−0.130	−0.090	⋯
$\Xi_{cc} \to \Lambda_c$	$f_1(0)$	$f_2(0)$	$f_3(0)$	$g_1(0)$	$g_2(0)$	$g_3(0)$
This work	0.517 ± 0.071	-0.036 ± 0.007	⋯	0.155 ± 0.019	-0.072 ± 0.012	⋯
Diquark [7]	0.790	−0.008	⋯	0.224	−0.050	⋯
QCDSR [19]*	0.63 ± 0.20	-0.05 ± 0.02	-0.81 ± 0.26	0.24 ± 0.08	-0.11 ± 0.03	-0.84 ± 0.30
LCSR [23]	0.81 ± 0.01	0.32 ± 0.01	-0.90 ± 0.07	1.09 ± 0.02	-0.86 ± 0.02	0.76 ± 0.01
NRQM [52]	0.36	0.14	0.08	0.20	0.01	−0.03
MBM [52]	0.45	0.01	−0.28	0.15	0.01	−0.70

[a]The asterisk on Ref. [19] indicates that, in this literature, we made a mistake in the sign for the axial-vector form factors, and here we have corrected it.

TABLE III. Our decay widths (in units of 10^{-14} GeV) are compared with other results in the literature.[a]

$\Lambda_b \to \Lambda_c e^- \bar{\nu}_e$	Decay width	$\Sigma_b \to \Sigma_c e^- \bar{\nu}_e$	Decay width	$\Xi_{cc} \to \Lambda_c e^+ \nu_e$	Decay width
This work	2.54 ± 0.33	This work	0.870 ± 0.061	This work	0.755 ± 0.219
Three-quark [32]	2.78	Three-quark [32]	1.03	Diquark [7]	1.05
Diquark [8]	3.96	Diquark [29]	0.908	QCDSR [19]*	0.76 ± 0.37
Diquark [51]	3.39			LCSR [23]	3.95 ± 0.21
QCDSR [21]	2.96 ± 0.48				
LQCD [50]	2.35 ± 0.15				

[a]The asterisk on Ref. [19] indicates that, in this literature, although we made a mistake in the sign for the axial-vector form factors, the prediction for the decay width is not affected.

larger shape parameters are used in the diquark picture, as pointed out in Sec. III A.

IV. CONCLUSIONS AND DISCUSSIONS

In this work, a three-quark picture is constructed using a bottom-up approach for baryons in a light-front quark model, where quark spinors and Dirac matrices act as building blocks. The shape parameters, which characterize the momentum distribution inside a baryon, are determined with the help of the pole residue of the baryon. Some semileptonic decays are investigated under this three-quark picture. The relation between the three-quark picture and the diquark picture is clarified. There is still a small flaw worth pointing out: that is, when determining the shape parameters, we demand that $\lambda_1 \approx \lambda_2 \approx \lambda_{\mathrm{QCDSR}}$, and some uncertainty can still be introduced. A better prescription we can think of is to do global fitting. Given that our main goal in this article is to develop a set of methods, such a more detailed consideration is left for our future work. Here are some prospects:

(i) At this point, we have constructed a relatively complete three-quark picture for baryons, which can be applied to study semileptonic, nonleptonic, strong, and electromagnetic decay processes of heavy flavor baryons in the future.

(ii) When building the model, we found that the Lorentz boost plays a crucial role. As can be seen in the Appendix, in spin space, we first couple quark 3 and quark 2 to form a diquark, and then couple this diquark to quark 1. Obviously, this bottom-up modeling approach can be generalized to multiquark states. We may establish a unified theoretical framework for describing multiquark states.

(iii) As a by-product of model construction, we can easily obtain an improved definition of baryon interpolating current. The hadron interpolating currents are the starting point of lattice QCD and QCD sum rules, and therefore are of great importance. These new interpolating currents can be applied to study more detailed problems, such as the $\Xi_Q - \Xi_Q'$ mixing [53].

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APPENDIX: SPIN WAVE FUNCTIONS

In this appendix, we will derive the spin wave functions of Λ_Q, Σ_Q, and Σ_Q^*. For spinor and Dirac matrix conventions, as well as some important equations, readers can refer to Ref. [30].

In Eq. (3), Ψ^{SS_z} is defined as

$$\begin{aligned}\Psi^{SS_z}(\tilde{p}_i,\lambda_i) &= \sum_{s_1s_2s_3}\langle\lambda_1|\mathcal{R}_M^\dagger(\tilde{p}_1,m_1)|s_1\rangle\langle\lambda_2|\mathcal{R}_M^\dagger(\tilde{p}_2,m_2)|s_2\rangle\\ &\quad\times\langle\lambda_3|\mathcal{R}_M^\dagger(\tilde{p}_3,m_3)|s_3\rangle\\ &\quad\times\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|S_{23}s_{23}\right\rangle\left\langle\frac{1}{2}S_{23};s_1s_{23}\middle|SS_z\right\rangle\\ &\quad\times\Phi(x_i,k_{i\perp}),\end{aligned}\tag{A1}$$

where S_{23} is the total spin of the diquark and $S_{23}=0$, 1, 1 for Λ_Q, Σ_Q, Σ_Q^*, respectively. An instant spinor $u_D(p,s)$ is Melosh transformed into a light-front spinor $u(p,s)$ by

$$\begin{aligned}&\sum_{s_i}\langle\lambda_i|\mathcal{R}_M^\dagger(\tilde{p}_i,m_i)|s_i\rangle\bar{u}_D(p_i,s_i)\\ &\quad=\sum_{s_i}\bar{u}(p_i,\lambda_i)\frac{u_D(p_i,s_i)\bar{u}_D(p_i,s_i)}{2m_i}=\bar{u}(p_i,\lambda_i).\end{aligned}\tag{A2}$$

Once the Clebsch-Gordan (CG) coefficients in Eq. (A1) are rewritten into the product of instant spinors and Dirac matrices, the instant spinors can then be transformed into the light-front ones using Eq. (A2). Therefore, in the following, we will focus on rewriting the CG coefficients in Eq. (A1) into the product of instant spinors and Dirac matrices. Note that the spinors appearing below are all instant spinors (we have omitted their subscript D), except $u(\bar{P},S_z)$ and $u_\mu(\bar{P},S_z)$. When one of these two spinors is involved, we always take the rest frame of $\bar{P}$, where its instant form and light-front form coincide.

It is worth pointing out that the proof given here does not introduce any additional assumptions; for example, it does not assume heavy quark symmetry, nor does it depend on the coordinate system selection of LFQM. In addition, one can clearly see that the Lorentz boost between the rest frame of the diquark and the rest frame of $\bar{P}$ plays a crucial role for the case involving an axial-vector diquark. (Of course, for the case involving a scalar diquark, the Lorentz boost is trivial.)

1. To derive the spin wave function of Λ_Q

Λ_Q has quark components udQ, in which ud are considered to form a 0^+ diquark.

(i) Step 1, couple the spins of quark 3 and quark 2 to form a scalar diquark

$$\begin{aligned}I&\equiv\bar{u}(p_3,s_3)\frac{(\bar{\not{P}}+M_0)}{2M_0}\gamma_5(-C)\bar{u}^T(p_2,s_2)\\ &=\sqrt{(e_2+m_2)(e_3+m_3)}\chi_{s_3}^\dagger(i\sigma_2)\chi_{s_2}^*\\ &=\sqrt{2(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|0s_{23}\right\rangle.\end{aligned}\tag{A3}$$

(ii) Step 2, calculate the trivial coupling

$$\begin{aligned}II&\equiv\bar{u}(p_1,s_1)u(\bar{P},S_z)\\ &=\sqrt{2M_0(e_1+m_1)}\chi_{s_1}^\dagger\chi_{S_z}\\ &=\sqrt{2M_0(e_1+m_1)}\left\langle\frac{1}{2}0;s_10\middle|\frac{1}{2}S_z\right\rangle.\end{aligned}\tag{A4}$$

Therefore, for Λ_Q, the CG coefficients in Eq. (A1) can be rewritten into

$$\begin{aligned}&\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|0s_{23}\right\rangle\left\langle\frac{1}{2}0;s_10\middle|\frac{1}{2}S_z\right\rangle\\ &\quad=A_0\bar{u}(p_3,s_3)(\bar{\not{P}}+M_0)(-\gamma_5)C\bar{u}^T(p_2,s_2)\\ &\quad\quad\times\bar{u}(p_1,s_1)u(\bar{P},S_z)\end{aligned}\tag{A5}$$

with

$$A_0=\frac{1}{4\sqrt{M_0^3(e_1+m_1)(e_2+m_2)(e_3+m_3)}}.\tag{A6}$$

2. To derive the spin wave function of Σ_Q

Σ_Q also has quark components udQ, in which ud are considered to form a 1^+ diquark. Technically, it is much more complicated to arrive at the spin wave function of Σ_Q.

(i) Step 1, couple the spins of quark 3 and quark 2 in the rest frame of p_{23} (defined below) to form an axial-vector diquark

$$\begin{aligned}I^\mu&\equiv\bar{u}(p_3,s_3)\frac{(\bar{\not{P}}+M_0)}{2M_0}\gamma_\perp^\mu(p_{23})\\ &\quad\times(-C)\bar{u}^T(p_2,s_2)\end{aligned}\tag{A7}$$

with

$$\gamma_\perp^\mu(p_{23}) = \gamma_\perp^\mu(\bar{P}) - \frac{M_0 p_{23}^\mu + m_{23}\bar{P}^\mu}{M_0(e_{23}+m_{23})}\frac{\gamma_\perp(\bar{P})\cdot p_{23}}{m_{23}},$$
$$p_{23} = p_2 + p_3, \qquad m_{23}^2 = p_{23}^2,$$
$$\gamma_\perp^\mu(\bar{P}) = \gamma^\mu - \not{v}v^\mu, \qquad v^\mu = \bar{P}^\mu/M_0. \tag{A8}$$

p_{23}^μ is the four-momentum of the diquark, e_{23} is its energy in the rest frame of $\bar{P}$, and m_{23} is its invariant mass. $\gamma_\perp^\mu(p_{23})$ and $\gamma_\perp^\mu(\bar{P})$ are related by a Lorentz boost. It can be shown that

$$I^\mu = \sqrt{2(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle \times \epsilon^{*\mu}(p_{23},s_{23}). \tag{A9}$$

Equation (A9) can be verified by considering three specific cases: $\mu=0$, $\mu=1$, 2, and $\mu=3$. Take the $\mu=0$ case as an example:

$$\begin{aligned} I^0 &= \frac{1}{\sqrt{e_3+m_3}}\begin{pmatrix}(e_3+m_3)\chi_{s_3}^\dagger & -\chi_{s_3}^\dagger\vec{\sigma}\cdot\vec{p}_3\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\left(-\frac{M_0e_{23}+m_{23}M_0}{M_0(e_{23}+m_{23})}\frac{-\gamma^3|\vec{p}_{23}|}{m_{23}}\right)\\ &\quad\times\begin{pmatrix}0 & i\sigma_2\\ i\sigma_2 & 0\end{pmatrix}\frac{1}{\sqrt{e_2+m_2}}\begin{pmatrix}(e_2+m_2)\chi_{s_2}^*\\ -(\vec{\sigma}\cdot\vec{p}_2)^T\chi_{s_2}^*\end{pmatrix}\\ &= \sqrt{(e_2+m_2)(e_3+m_3)}\frac{|\vec{p}_{23}|}{m_{23}}\chi_{s_3}^\dagger(i\sigma_3\sigma_2)\chi_{s_2}^*\\ &= \sqrt{2(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle\epsilon^{*0}(p_{23},s_{23}). \end{aligned} \tag{A10}$$

In Eq. (A10), in the rest frame of $\bar{P}$, we choose the diquark to move along the z-axis, and then its four-momentum $p_{23}^\mu = (e_{23},0,0,|\vec{p}_{23}|)$, polarization vector $\epsilon^\mu(0) = (|\vec{p}_{23}|,0,0,e_{23})/m_{23}$.

(ii) Step 2, couple the diquark to quark 1,

$$T \equiv I^\mu \times \bar{u}(p_1,s_1)\Gamma_{1,23\mu}u(\bar{P},S_z) \tag{A11}$$

with

$$\Gamma_{1,23\mu} = \frac{\gamma_5}{\sqrt{3}}\left(\gamma_\mu - \frac{M_0+m_1+m_{23}}{M_0(e_{23}+m_{23})}\bar{P}_\mu\right). \tag{A12}$$

It can be shown that

$$T = 2\sqrt{M_0(e_1+m_1)(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle\left\langle\frac{1}{2}1;s_1s_{23}\middle|\frac{1}{2}S_z\right\rangle. \tag{A13}$$

Equation (A13) can be verified by considering two specific cases: $s_{23}=0$ and $s_{23}=\pm$. Take $s_{23}=0$ as an example:

$$\begin{aligned} T &= \sqrt{2(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle\frac{1}{\sqrt{e_1+m_1}}\begin{pmatrix}(e_1+m_1)\chi_{s_1}^\dagger & -\chi_{s_1}^\dagger\vec{\sigma}\cdot\vec{p}_1\end{pmatrix}\\ &\quad\times\frac{1}{\sqrt{3}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\left[\frac{1}{m_{23}}(|\vec{p}_{23}|\gamma^0 - e_{23}\gamma^3) - \frac{M_0+m_1+m_{23}}{M_0(e_{23}+m_{23})}\frac{|\vec{p}_{23}|}{m_{23}}M_0\right]\sqrt{2M_0}\begin{pmatrix}\chi_{S_z}\\ 0\end{pmatrix}\\ &= \sqrt{2(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle\frac{1}{\sqrt{3}}\sqrt{e_1+m_1}\sqrt{2M_0}\chi_{s_1}^\dagger\sigma_3\chi_{S_z}\\ &= 2\sqrt{M_0(e_1+m_1)(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle\left\langle\frac{1}{2}1;s_1s_{23}\middle|\frac{1}{2}S_z\right\rangle. \end{aligned} \tag{A14}$$

(iii) Step 3, tensor simplify T

$$\begin{aligned}
&\bar{u}(p_3,s_3)(\bar{\not{P}}+M_0)\gamma_\perp^\mu(p_{23})(-C)\bar{u}^T(p_2,s_2)\bar{u}(p_1,s_1)\Gamma_{1,23\mu}u(\bar{P},S_z)\\
&\quad=\bar{u}(p_3,s_3)(\bar{\not{P}}+M_0)\gamma_\perp^\mu(\bar{P})(-C)\bar{u}^T(p_2,s_2)\bar{u}(p_1,s_1)\Gamma_{1,23\mu}u(\bar{P},S_z)\\
&\quad=\bar{u}(p_3,s_3)(\bar{\not{P}}+M_0)\gamma_\perp^\mu(\bar{P})(-C)\bar{u}^T(p_2,s_2)\bar{u}(p_1,s_1)\frac{\gamma_5}{\sqrt{3}}\gamma_\mu u(\bar{P},S_z)\\
&\quad=\bar{u}(p_3,s_3)(\bar{\not{P}}+M_0)(\gamma^\mu-v^\mu)(-C)\bar{u}^T(p_2,s_2)\bar{u}(p_1,s_1)\frac{\gamma_5}{\sqrt{3}}\gamma_\mu u(\bar{P},S_z)\\
&\quad=\bar{u}(p_3,s_3)(\bar{\not{P}}+M_0)(\gamma^\mu-v^\mu)C\bar{u}^T(p_2,s_2)\bar{u}(p_1,s_1)\left(\frac{1}{\sqrt{3}}\gamma_\mu\gamma_5\right)u(\bar{P},S_z),
\end{aligned}\tag{A15}$$

where, in the second step, we have used $\gamma_\perp^\mu(\bar{P})\cdot\bar{P}=0$.

Therefore, for Σ_Q, the CG coefficients in Eq. (A1) can be rewritten into

$$\begin{aligned}
&\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle\left\langle\frac{1}{2}1;s_1s_{23}\middle|\frac{1}{2}S_z\right\rangle\\
&\quad=A_1\bar{u}(p_3,s_3)(\bar{\not{P}}+M_0)(\gamma^\mu-v^\mu)C\bar{u}^T(p_2,s_2)\\
&\qquad\times\bar{u}(p_1,s_1)\left(\frac{1}{\sqrt{3}}\gamma_\mu\gamma_5\right)u(\bar{P},S_z)
\end{aligned}\tag{A16}$$

with

$$A_1=\frac{1}{4\sqrt{M_0^3(e_1+m_1)(e_2+m_2)(e_3+m_3)}}.\tag{A17}$$

3. To derive the spin wave function of Σ_Q^*

Σ_Q^* also has quark components udQ, in which ud are considered to form a 1^+ diquark.

(i) Step 1, same as that of Σ_Q.

(ii) Step 2, couple the diquark to quark 1

$$T'\equiv I^\mu\times\bar{u}(p_1,s_1)\Gamma_{1,23\mu\nu}u^\nu(\bar{P},S_z)\tag{A18}$$

with

$$\Gamma_{1,23\mu\nu}=-g_{\mu\nu}+\frac{\bar{P}_\mu(M_0p_{23\nu}+m_{23}\bar{P}_\nu)}{M_0^2(e_{23}+m_{23})},\tag{A19}$$

and the vectorial spinor $u_\mu(\bar{P},S_z)$ can be expressed by [9,54]

$$u_\mu(\bar{P},S_z)=\sum_{s_1,s_{23}}\left\langle\frac{1}{2}1;s_1s_{23}\middle|\frac{3}{2}S_z\right\rangle u(\bar{P},s_1)\epsilon_\mu(\bar{P},s_{23}).\tag{A20}$$

It can be shown that

$$\begin{aligned}
T'&=2\sqrt{M_0(e_1+m_1)(e_2+m_2)(e_3+m_3)}\\
&\quad\times\left\langle\frac{1}{2}\frac{1}{2};s_3s_2\middle|1s_{23}\right\rangle\left\langle\frac{1}{2}1;s_1s_{23}\middle|\frac{3}{2}S_z\right\rangle.
\end{aligned}\tag{A21}$$

Equation (A21) can be proved as follows:

$$\begin{aligned}
&\frac{T'}{\sqrt{2(e_2+m_2)(e_3+m_3)}\left\langle\frac{1}{2}\frac{1}{2};s_3s_2|1s_{23}\right\rangle}\\
&\quad=\bar{u}(p_1,s_1)\left[-\epsilon^{*\mu}(p_{23},s_{23})+\frac{M_0p_{23}^\mu+m_{23}\bar{P}^\mu}{M_0^2(e_{23}+m_{23})}\epsilon^*(p_{23},s_{23})\cdot\bar{P}\right]u_\mu(\bar{P},S_z)\\
&\quad=\frac{1}{\sqrt{e_1+m_1}}\left((e_1+m_1)\chi_{s_1}^\dagger\quad -\chi_{s_1}^\dagger\vec{\sigma}\cdot\vec{p}_1\right)(-\epsilon^{*\mu}(\bar{P},s_{23}))\sum_{\lambda_1\lambda_2}\left\langle\frac{1}{2}1;\lambda_1\lambda_2\middle|\frac{3}{2}S_z\right\rangle u(\bar{P},\lambda_1)\epsilon_\mu(\bar{P},\lambda_2)\\
&\quad=\frac{1}{\sqrt{e_1+m_1}}\left((e_1+m_1)\chi_{s_1}^\dagger\quad -\chi_{s_1}^\dagger\vec{\sigma}\cdot\vec{p}_1\right)\sum_{\lambda_1\lambda_2}\left\langle\frac{1}{2}1;\lambda_1\lambda_2\middle|\frac{3}{2}S_z\right\rangle\sqrt{2M_0}\begin{pmatrix}\chi_{\lambda_1}\\0\end{pmatrix}\delta_{s_{23},\lambda_2}\\
&\quad=\sqrt{e_1+m_1}\sqrt{2M_0}\sum_{\lambda_1\lambda_2}\left\langle\frac{1}{2}1;\lambda_1\lambda_2\middle|\frac{3}{2}S_z\right\rangle\delta_{s_1,\lambda_1}\delta_{s_{23},\lambda_2}\\
&\quad=\sqrt{2M_0(e_1+m_1)}\left\langle\frac{1}{2}1;s_1s_{23}\middle|\frac{3}{2}S_z\right\rangle.
\end{aligned}\tag{A22}$$

Here, we have used

$$\epsilon^\mu(p_{23}, s_{23}) - \frac{M_0 p_{23}^\mu + m_{23}\bar{P}^\mu}{M_0^2(e_{23}+m_{23})}\epsilon(p_{23}, s_{23})\cdot\bar{P} = \epsilon^\mu(\bar{P}, s_{23}) \tag{A23}$$

and

$$\epsilon^{*\mu}(\bar{P}, s_{23})\epsilon_\mu(\bar{P}, \lambda_2) = -\delta_{s_{23},\lambda_2}. \tag{A24}$$

(iii) Step 3, tensor simplify T'

$$\begin{aligned} &\bar{u}(p_3, s_3)(\bar{\not{P}} + M_0)\gamma_\perp^\mu(p_{23})(-C)\bar{u}^T(p_2, s_2)\bar{u}(p_1, s_1)\Gamma_{1,23\mu\nu}u^\nu(\bar{P}, S_z) \\ &= \bar{u}(p_3, s_3)(\bar{\not{P}} + M_0)\gamma_\perp^\mu(\bar{P})(-C)\bar{u}^T(p_2, s_2)\bar{u}(p_1, s_1)\Gamma_{1,23\mu\nu}u^\nu(\bar{P}, S_z) \\ &= \bar{u}(p_3, s_3)(\bar{\not{P}} + M_0)\gamma_\perp^\mu(\bar{P})(-C)\bar{u}^T(p_2, s_2)\bar{u}(p_1, s_1)(-g_{\mu\nu})u^\nu(\bar{P}, S_z) \\ &= \bar{u}(p_3, s_3)(\bar{\not{P}} + M_0)(\gamma^\mu - v^\mu)C\bar{u}^T(p_2, s_2)\bar{u}(p_1, s_1)u_\mu(\bar{P}, S_z). \end{aligned} \tag{A25}$$

Therefore, for Σ_Q^*, the CG coefficients in Eq. (A1) can be rewritten into

$$\left\langle \frac{1}{2}\frac{1}{2}; s_3 s_2 \middle| 1 s_{23} \right\rangle \left\langle \frac{1}{2} 1; s_1 s_{23} \middle| \frac{3}{2} S_z \right\rangle = A_1' \bar{u}(p_3, s_3)(\bar{\not{P}} + M_0)(\gamma^\mu - v^\mu)C\bar{u}^T(p_2, s_2)\bar{u}(p_1, s_1)u_\mu(\bar{P}, S_z) \tag{A26}$$

with

$$A_1' = \frac{1}{4\sqrt{M_0^3(e_1+m_1)(e_2+m_2)(e_3+m_3)}}. \tag{A27}$$

One can see that $A_0 = A_1 = A_1'$.

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