Impossibility of spontaneous vector flavor symmetry breaking on the lattice

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I show that spontaneous breaking of vector flavor symmetry on the lattice is impossible in gauge theories with a positive functional-integral measure, for discretized Dirac operators linear in the quark masses, if the corresponding propagator and its commutator with the flavor symmetry generators can be bounded in norm independently of the gauge configuration and uniformly in the volume. Under these assumptions, any order parameter vanishes in the symmetric limit of fermions of equal masses. I show that these assumptions are satisfied by staggered, minimally doubled and Ginsparg-Wilson fermions for positive fermion mass, for any value of the lattice spacing, and so in the continuum limit if this exists. They are instead not satisfied by Wilson fermions, for which spontaneous vector flavor symmetry breaking is known to take place in the Aoki phase. The existence of regularizations unaffected by residual fermion doubling for which the symmetry cannot break spontaneously on the lattice establishes rigorously (at the physicist's level) the impossibility of its spontaneous breaking in the continuum for any number of flavors.

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I. INTRODUCTION

The importance of symmetries and of the way in which they are realized in quantum field theories can hardly be overemphasized. In the context of strong interactions and its microscopic theory, i.e., QCD, an important role is played by the approximate vector flavor symmetry involving the lightest two or three types ("flavors") of quarks, which holds exactly in the limit of quarks of equal masses; and by its enhancement to chiral flavor symmetry in the limit of massless quarks. Vector flavor symmetry and the pattern of its explicit breaking largely determine the structure of the hadronic spectrum; chiral flavor symmetry and its spontaneous breaking down to vector flavor symmetry explain the lightness of pions and their dynamics, as well as the absence of parity partners of hadrons. The full symmetry group at the classical level includes also the $\mathrm{U}(1)_B$ symmetry responsible for baryon number conservation, and the axial $\mathrm{U}(1)_A$ symmetry, that does not survive the quantization process and becomes anomalous in the quantum theory.

An interesting question is whether baryon number and vector flavor symmetry can break down spontaneously in general vector gauge theories, where the fermions' left-handed and right-handed chiralities are coupled in the same way to the gauge fields. This could in principle happen for exactly degenerate massive fermions, leading to the appearance of massless Goldstone bosons; and in the chiral limit of massless fermions it could lead to a different symmetry breaking pattern than the usual one, and so to a different set of Goldstone bosons. This question has been essentially answered in the negative by Vafa and Witten in a famous paper [1]. There they actually prove a stronger result, namely the impossibility of finding massless particles in the spectrum of a gauge theory with positive functional-integral measure that couple to operators with nonvanishing baryon number or transforming nontrivially under vector flavor transformations. This is done by deriving a bound on the fermion propagator that guarantees its exponential decay with the distance as long as the fermion mass is nonzero. Since massless bosons coupling to the operators mentioned above would appear in the spectrum as a consequence of Goldstone's theorem [2–4] if those symmetries were spontaneously broken, the impossibility of spontaneous breaking follows.

The elegant and powerful argument of Vafa and Witten is developed using the "mathematical fiction" of the functional integral formalism for interacting quantum field theories in continuum (Euclidean) spacetime. The crucial issue of the regularization of the functional integral, generally required to make it a mathematically well defined object, is discussed only briefly. In particular, the possibility of formulating the argument using a lattice regularization is mentioned, but not discussed in detail. The general validity of this statement is called into question by the

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existence of examples of spontaneous breaking of vector flavor symmetry on the lattice, namely in the Aoki phase [5–25] of lattice gauge theories with Wilson fermions [26]. While this is not in contradiction with the argument of Vafa and Witten in the continuum [22], it also makes clear that this argument does not trivially extend to the lattice in a general setting. It would then be desirable to identify conditions that guarantee the impossibility of baryon number and vector flavor symmetry breaking on the lattice, at least for small lattice spacing, which could help in putting Vafa and Witten's "theorem" on more solid ground.

The strategy of widest generality is to directly prove a lattice version of Vafa and Witten's bound on the propagator, which would allow one to recover all the conclusions of Ref. [1] in a rigorous way (under the tacit assumption of the existence of the continuum limit). This was done for staggered fermions [27–29] in Ref. [30], so excluding completely the possibility of breaking baryon number symmetry and the vector flavor symmetry of several staggered fields on the lattice using this discretization. However, in four dimensions one flavor of staggered fermions on the lattice describes four degenerate "tastes" of fermions in the continuum limit, and while the spontaneous breaking of the corresponding extended flavor symmetry is excluded by the result of Ref. [30], this limits the impossibility proof to a number of physical fermion species that is a multiple of four (and of $2^{[d/2]}$ in d dimensions). The extension to an arbitrary number of fermion species requires the "rooting trick" [31–33] to eliminate the taste degeneracy, a procedure that has been criticized in the past (see Refs. [34–39]). While both theoretical arguments and numerical evidence support the validity of the rooting procedure (see Refs. [40–46], the reviews [47–51], and references therein), its theoretical status is still not fully settled. It would then be nice to extend the proof of Ref. [30] or derive a similar bound also for other discretizations that describe a single fermion species. However, the proof makes essential use of the anti-Hermiticity and ultralocality of the operator: while it can probably be extended quite straightforwardly to other discretizations that share these properties, e.g., the minimally doubled fermions of Karsten and Wilczek [52,53] and of Creutz and Boriçi [54,55] (that are, however, still describing two fermion species in the continuum limit), it is not clear how to do so with discretizations that do not, e.g., Ginsparg-Wilson fermions [56–65].

A less general strategy, still sufficient to prove the impossibility of spontaneous symmetry breaking on the lattice, is to show that the corresponding order parameters must vanish. Partial results for vector flavor symmetry following this strategy are present in the literature. Already in Ref. [1] the authors show that vector flavor symmetry cannot be spontaneously broken by the formation of the simplest symmetry-breaking bilinear fermion condensate, when approaching the symmetric case of degenerate fermion masses starting from the nondegenerate case. Their argument works only for discretizations of the Dirac operator that are anti-Hermitian, so it applies again only to staggered and minimally doubled (and obviously to naive) fermions. In Ref. [66] the authors show that the simplest symmetry-breaking condensate must vanish also for Ginsparg-Wilson fermions. They do not add any symmetry-breaking term to the action, applying instead the formalism of probability distribution functions [67,68] to the relevant operator to show the absence of degenerate vacua. More precisely, their result shows that if degenerate nonsymmetric vacua are present, they cannot be distinguished by the (vanishing) expectation value of this operator.

In this paper I pursue this second strategy and present a simple argument that spontaneous vector flavor symmetry breaking is impossible on the lattice for gauge theories with a positive integration measure, as long as the discretization of the Dirac operator satisfies certain reasonable assumptions. More precisely, I show that any localized order parameter for vector flavor symmetry breaking must vanish in the symmetric limit of fermions of equal masses (taken of course after the thermodynamic limit), for massive lattice Dirac operators D_M that

(0.) are linear in the fermion masses, $\mathrm{D}_M = D^{(0)} + M\Delta D$, with $D^{(0)}$ and ΔD trivial in flavor space, and M a Hermitian mass matrix;

(1.) have inverse bounded in norm by a configuration- and volume-independent constant, finite in the symmetric limit;

(2.) have derivative with respect to the fermion masses, ΔD, also bounded in norm by a configuration- and volume-independent constant, finite in the symmetric limit.

Assumption (0.) is rather natural, and assumption (2.) is not really restrictive; both are satisfied by all common discretizations. Assumption (1.) is instead crucial, and it means that the propagator corresponding to D_M is bounded in norm for all configurations, uniformly in the volume. This may in general not be the case, for example if a finite density of near-zero modes of D_M develops in the thermodynamic limit, as it happens with Wilson fermions in the Aoki phase. For staggered [69], minimally doubled, and Ginsparg-Wilson fermions [70], assumption (1.) holds as long as the fermion masses are nonzero, and the functional-integration measure is positive for nonnegative fermion masses, so that for these discretizations the spontaneous breaking of vector flavor symmetry is impossible at finite positive fermion mass.

My argument is clearly of narrower scope than the one in Ref. [1] and its counterpart for staggered fermions in Ref. [30], and limited to quadratic fermion actions with the usual symmetry-breaking terms. On the other hand, it is mathematically rigorous for a physicist's standard, leaving little room for loopholes, and applies to more general discretizations than staggered fermions. The strategy of proof is standard: one starts from the explicitly broken case with fermions of different masses, and shows that observables related by a vector flavor transformation have the same expectation value in the symmetric limit of equal masses, taken after the infinite-volume limit. This is achieved by proving two rather elementary bounds on the fermion propagator and on its commutator with the generators of the vector flavor symmetry group, that hold independently of the lattice size under assumptions (0.)–(2.). This results in the magnitude of the difference between the expectation values of observables related by a vector flavor transformation obeying a bound proportional to the spread in mass of the fermions, uniformly in the volume. In the symmetric limit such expectation values are then equal, and any order parameter for symmetry breaking must therefore vanish. A few remarks are in order.

(i) The geometry of the lattice, the boundary conditions imposed on the fields, the type of gauge action, the temperature of the system, and the value of the lattice spacing and of the other parameters of the theory play no role as long as positivity of the integration measure and the boundedness assumptions (1.) and (2.) (or more generally the derived bounds on the propagator and on its commutator with the symmetry generators) hold.

(ii) The restriction to localized observables is natural, as Goldstone's theorem involves observables that are localized in spacetime, and in space in the finite temperature case [4]. Their counterparts on a finite lattice involve lattice fields associated with a finite number of lattice sites or edges (links), that remains unchanged as the system size grows. In particular, this means that they are polynomial in the fermion fields, of degree independent of the lattice size.

(iii) If assumptions (0.)–(2.) hold for any lattice spacing, or at least for any sufficiently small spacing, then all the relevant order parameters vanish in the symmetric infinite-volume theory also in the continuum limit, if this exists (notice the order of limits: thermodynamic first, then symmetric, continuum last). Vector flavor symmetry will then be realized in the continuum. For staggered, minimally doubled, and Ginsparg-Wilson fermions this is the case for any positive fermion mass.

(iv) The fate of vector flavor symmetry in the chiral limit, both on the lattice and in the continuum, can be discussed following the argument presented in Ref. [1]: barring accidental degeneracies of the ground states, vector flavor symmetry must remain unbroken.

(v) The restriction to quadratic actions is not a limitation as far as the eventual continuum limit is concerned. Renormalizable higher-order operators with the right global and local symmetries are available only in dimension lower than or equal to two, where spontaneous breaking of a continuous symmetry is forbidden [71–73]. The inclusion of symmetry-breaking nonrenormalizable operators in the action may lead to spontaneously broken phases on the lattice, but does not affect the long-distance physics in the continuum limit. Since lattice discretizations exist that guarantee the realization of vector flavor symmetry in the continuum limit, any hypothetical phase where it is spontaneously broken on the lattice should shrink as this limit is approached. This is the case also for the spontaneously broken phases possibly appearing on the lattice for discretizations that do not satisfy the assumptions of this paper, e.g., the Aoki phase found with Wilson fermions.

(vi) The existence of regularizations unaffected by residual fermion doubling in the continuum limit for which the symmetry cannot break spontaneously on the lattice at any spacing (e.g., Ginsparg-Wilson fermions) establishes rigorously (at the physicist's level of rigor) the impossibility of its spontaneous breaking in continuum gauge theories for any number of physical fermion species.

The plan of the paper is the following. After briefly reviewing gauge theories on the lattice to set up the notation in Sec. II, and vector flavor symmetry in Sec. III, I derive the relevant bounds and prove the main statement in Sec. IV. The cases of staggered, Ginsparg-Wilson, Wilson, and minimally doubled fermions are discussed in Sec. V. A brief summary is given in Sec. VI. A few technical details are given in the Appendix.

II. GAUGE THEORIES ON THE LATTICE

I will consider d-dimensional vector gauge theories with N_f flavors of fermions, all transforming in the same N_c-dimensional representation of a compact gauge group, discretized on a finite lattice containing $\mathcal{V}$ sites. Suitable boundary conditions are assumed on the gauge and fermion fields. The shape of the lattice and the boundary conditions play no distinctive role in the following; in particular, the discussion applies to systems both at zero and finite temperature. The partition function and the expectation values of the theory are given by

$$Z \equiv \int [\mathrm{D}U] \int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{G}}[U]-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]},$$
$$\langle \mathcal{O} \rangle \equiv \frac{1}{Z} \int [\mathrm{D}U] \int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{G}}[U]-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} \mathcal{O}[\psi,\bar{\psi},U], \tag{1}$$

where $[\mathrm{D}U] = \prod_\ell dU_\ell$ is the product of the Haar measures associated with the gauge variables U_ℓ attached to the lattice links ℓ, and $[\mathrm{D}\psi\mathrm{D}\bar{\psi}] = \prod_{xfa\alpha} d\psi_{fa\alpha}(x) d\bar{\psi}_{fa\alpha}(x)$ is the Berezin integration measure associated with the Grassmann variables $\psi_{fa\alpha}(x)$ and $\bar{\psi}_{fa\alpha}(x)$ attached to the lattice sites x. Here f and a are the discrete indices associated with the flavor and color (i.e., gauge group) degrees of freedom, $f = 1, \ldots, N_f$, $a = 1, \ldots, N_c$, and α is the Dirac index, typically $\alpha = 1, \ldots, 2^{[d/2]}$, but possibly absent altogether (e.g., for staggered fermions). The full set of discrete indices will be collectively denoted as $A = fa\alpha$; when needed, the color and Dirac indices will be denoted together as $A_\star = a\alpha$. Finally, $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\mathrm{F}}$ denote the gauge and fermionic parts of the action. The fermionic action is taken to be of the form

$$\mathcal{S}_{\mathrm{F}}[\psi, \bar{\psi}, U] = \sum_{x,y,A,B} \bar{\psi}_A(x)(\mathrm{D}_M[U])_{AB}(x,y)\psi_B(y) = \bar{\psi}\mathrm{D}_M[U]\psi, \tag{2}$$

where in the last passage I introduced the matrix notation that will be used repeatedly. Here D_M is the massive Dirac operator, whose dependence on the gauge links U_ℓ will be often omitted for simplicity. Expectation values are computed in two steps. For a generic observable $\mathcal{O}[\psi, \bar{\psi}, U]$, integration over Grassmann variables yields

$$\langle \mathcal{O} \rangle_{\mathrm{F}} \equiv \frac{\int [\mathrm{D}\psi\mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} \mathcal{O}[\psi, \bar{\psi}, U]}{\int [\mathrm{D}\psi\mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]}} = \frac{\int [\mathrm{D}\psi\mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} \mathcal{O}[\psi, \bar{\psi}, U]}{\det \mathrm{D}_M[U]}. \tag{3}$$

The expectation value of $\mathcal{O}$ is then obtained from Eq. (3) by averaging over gauge fields, $\langle \mathcal{O} \rangle = \langle \langle \mathcal{O} \rangle_{\mathrm{F}} \rangle_{\mathrm{G}}$, where for a purely gluonic observable $\tilde{\mathcal{O}}[U]$

$$\langle \tilde{\mathcal{O}} \rangle_{\mathrm{G}} \equiv \frac{\int [\mathrm{D}U] e^{-\mathcal{S}_{\mathrm{G}}[U]} \det \mathrm{D}_M[U] \tilde{\mathcal{O}}[U]}{\int [\mathrm{D}U] e^{-\mathcal{S}_{\mathrm{G}}[U]} \det \mathrm{D}_M[U]} = \frac{1}{Z} \int [\mathrm{D}U] e^{-\mathcal{S}_{\mathrm{G}}[U]} \det \mathrm{D}_M[U] \tilde{\mathcal{O}}[U]. \tag{4}$$

I assume that the full gluonic integration measure $d\mu_{\mathrm{G}} = [\mathrm{D}U] e^{-\mathcal{S}_{\mathrm{G}}[U]} \det \mathrm{D}_M[U]$ is nonnegative, i.e., $e^{-\mathcal{S}_{\mathrm{G}}[U]} \det \mathrm{D}_M[U] \geq 0$, and not identically zero. For brevity, I will refer to this assumption simply as positivity of the integration measure. The gluonic action is otherwise unspecified, besides its being gauge-invariant. I consider massive Dirac operators of the form

$$\mathrm{D}_M = \mathbf{1}_F D^{(0)} + M \Delta D, \tag{5}$$

with M a constant Hermitian matrix carrying only flavor indices and independent of coordinates and gauge links. The symbol $\mathbf{1}_F$, and similarly $\mathbf{1}_C$ and $\mathbf{1}_D$, denote the identity in flavor (F), color (C), and Dirac (D) space; $\mathbf{1}$ will denote the identity in the full flavor, color, Dirac and coordinate space. The operators $D^{(0)}$ and ΔD carry only color, Dirac, and coordinate indices, i.e., $(D^{(0)})_{A_\star B_\star}(x,y)$ and $(\Delta D)_{A_\star B_\star}(x,y)$. Since one can diagonalize M with a unitary transformation, and reabsorb this into a redefinition of the fermion fields that does not affect the Berezin integration measure, one can consider a diagonal mass matrix $M = \mathrm{diag}(m_1, \ldots, m_{N_f})$ without loss of generality, and write

$$\begin{aligned} \mathrm{D}_M &= \mathrm{diag}(D^{(m_1)}, \ldots, D^{(m_{N_f})}), \\ D^{(m)} &\equiv D^{(0)} + m\Delta D, \\ (\mathrm{D}_M)_{fA_\star gB_\star}(x,y) &= \delta_{fg} (D^{(m_f)})_{A_\star B_\star}(x,y). \end{aligned} \tag{6}$$

The fermion propagator is then

$$\begin{aligned} \mathrm{S}_M &= \mathrm{D}_M^{-1} = \mathrm{diag}(S^{(m_1)}, \ldots, S^{(m_{N_f})}), \\ S^{(m)} &\equiv (D^{(m)})^{-1}, \\ (\mathrm{S}_M)_{fA_\star gB_\star}(x,y) &= \delta_{fg} S^{(m_f)}_{A_\star B_\star}(x,y), \end{aligned} \tag{7}$$

and the fermion determinant is

$$\det \mathrm{D}_M = \prod_{f=1}^{N_f} \det D^{(m_f)}. \tag{8}$$

The trace over all indices, i.e., flavor, color, Dirac, and coordinates, will be denoted by Tr. The trace over one or more of the discrete indices will be denoted by tr with one or more of the subscripts F, C, D, indicating which indices are being traced. Matrix multiplication is understood not to involve the indices displayed explicitly: for matrices P, Q carrying all indices, and matrices P, Q carrying all but flavor indices,

$$\begin{aligned} (\mathrm{PQ})_{AB}(x,y) &= \sum_{z,C} \mathrm{P}_{AC}(x,z)\mathrm{Q}_{CB}(z,y), \\ (\mathrm{P}(x,z)\mathrm{Q}(z,y))_{AB} &= \sum_C \mathrm{P}_{AC}(x,z)\mathrm{Q}_{CB}(z,y), \\ (PQ)_{A_\star B_\star}(x,y) &= \sum_{C_\star,z} P_{A_\star C_\star}(x,z) Q_{C_\star B_\star}(z,y), \\ (P(x,z)Q(z,y))_{A_\star B_\star} &= \sum_{C_\star} P_{A_\star C_\star}(x,z) Q_{C_\star B_\star}(z,y). \end{aligned} \tag{9}$$

A similar convention applies to Hermitian conjugation,

$$\begin{aligned} (\mathrm{P}^\dagger)_{AB}(x,y) &= \mathrm{P}_{BA}(y,x)^*, \\ (\mathrm{P}(x,y)^\dagger)_{AB} &= \mathrm{P}_{BA}(x,y)^*, \\ (P^\dagger)_{A_\star B_\star}(x,y) &= P_{B_\star A_\star}(y,x)^*, \\ (P(x,y)^\dagger)_{A_\star B_\star} &= P_{B_\star A_\star}(x,y)^*. \end{aligned} \tag{10}$$

At a certain point I will assume that the propagator and the operator ΔD are suitably bounded in norm. In the finite-dimensional case, the operator norm $\|\mathrm{A}\|$ of an operator A equals the largest of the eigenvalues a_n^2 of the positive Hermitian operator $\mathrm{A}^\dagger\mathrm{A}$,

$$\|\mathrm{A}\|^2 = \sup_{\psi\neq 0}\frac{(\mathrm{A}\psi,\mathrm{A}\psi)}{(\psi,\psi)} = \sup_{\psi\neq 0}\frac{(\psi,\mathrm{A}^\dagger\mathrm{A}\psi)}{(\psi,\psi)} = \max_n a_n^2, \quad (11)$$

where (ψ,ϕ) denotes the standard Hermitian inner product. I will assume that

(1.) $\|\mathrm{S}_M\| \leq m_0^{-1} < \infty$, with m_0 independent of the gauge configuration and of the lattice volume $\mathcal{V}$, and finite in the limit of equal fermion masses;

(2.) $\|\Delta D\| \leq \Delta D_{\max} < \infty$, with $\Delta D_{\max}$ independent of the gauge configuration and of the lattice volume, and finite in the limit of equal fermion masses.

For the purposes of this paper it suffices to consider the most general localized gauge-invariant observable, so polynomial in the fermion fields and dependent on finitely many link variables, with fermion number zero [74]. For notational purposes it is convenient to write it with its discrete indices contracted with the most general matrix carrying flavor, color and Dirac indices, dependent on the link variables (and possibly also explicitly on the lattice coordinates, and on the parameters of the theory), and having the right transformation properties under gauge transformations to make the observable gauge invariant. I will then consider

$$\begin{aligned}
&\mathcal{O}_{\mathcal{M}}[\psi,\bar\psi,U]\\
&\quad\equiv \sum_{\substack{A_1,\ldots,A_n,\\ B_1,\ldots,B_n}} (\mathcal{M}[U])_{A_1\ldots A_n B_1\ldots B_n}(x_1,\ldots,x_n,y_1,\ldots,y_n)\\
&\qquad\times \prod_{i=1}^n \psi_{B_i}(y_i)\bar\psi_{A_i}(x_i)\\
&\quad= \sum_{\mathbf{A},\mathbf{B}} (\mathcal{M}[U])_{\mathbf{AB}}(\mathbf{x},\mathbf{y}) \prod_{i=1}^n \psi_{B_i}(y_i)\bar\psi_{A_i}(x_i). \qquad (12)
\end{aligned}$$

For brevity I will write $\mathcal{M}[U]_{\mathbf{AB}}(\mathbf{x},\mathbf{y})$, using bold typeface to denote collectively a set of indices or variables. I will generally omit the dependence on U when unimportant. The transformation properties of $\mathcal{M}$ under gauge transformations are easily obtained from those of the fermionic fields, and do not play any role in the following. Notably, the quantity

$$\mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y}) \equiv \mathrm{tr}_{FCD}\{\mathcal{M}(\mathbf{x},\mathbf{y})\mathcal{M}(\mathbf{x},\mathbf{y})^\dagger\} \quad (13)$$

is gauge invariant. The expectation value $\langle\mathcal{O}_{\mathcal{M}}\rangle = \langle\langle\mathcal{O}_{\mathcal{M}}\rangle_{\mathrm{F}}\rangle_{\mathrm{G}}$ is obtained averaging $\langle\mathcal{O}_{\mathcal{M}}\rangle_{\mathrm{F}}$ over gauge fields using Eq. (4). From Eq. (3) one finds using Wick's theorem

$$\begin{aligned}
\langle\mathcal{O}_{\mathcal{M}}\rangle_{\mathrm{F}} &= \sum_{\mathbf{A},\mathbf{B}} \mathcal{M}[U]_{\mathbf{AB}}(\mathbf{x},\mathbf{y}) \left\langle \prod_{i=1}^n \psi_{B_i}(y_i)\bar\psi_{A_i}(x_i)\right\rangle_{\mathrm{F}}\\
&= \sum_{\mathbf{A},\mathbf{B}} \mathcal{M}[U]_{\mathbf{AB}}(\mathbf{x},\mathbf{y})\\
&\quad\times \sum_{\mathrm{P}\in\mathrm{S}_n} \sigma_{\mathrm{P}} \prod_{i=1}^n \mathrm{S}_M[U]_{B_i A_{\mathrm{P}(i)}}(y_i, x_{\mathrm{P}(i)}), \qquad (14)
\end{aligned}$$

with P a permutation of n elements and $\sigma_{\mathrm{P}} = \pm 1$ its signature.

Restrictions on $\mathcal{M}$ are required in order for the integration over link variables to yield finite results for $\langle\mathcal{O}_{\mathcal{M}}\rangle$. In the physically relevant cases $\mathcal{M}$ is a product of Wilson lines, suitably connecting the fermion fields to achieve gauge invariance, and so polynomial in the link variables. Imposing that $\mathcal{M}$ be polynomial or, more generally, continuous in the link variables guarantees that in a finite volume $\mathcal{K}_{\mathcal{M}}$ is bounded from above by its maximum on the compact integration manifold. The thermodynamic limit is taken while keeping $\mathcal{M}$ fixed as a function of the link variables (in particular, its possible dependence on the lattice coordinates is unchanged and cannot cause convergence problems), and so the bound on $\mathcal{K}_{\mathcal{M}}$ is independent of the volume. Assumption (1.) on the propagator then suffices to show convergence of $\langle\mathcal{O}_{\mathcal{M}}\rangle$, both in a finite volume and in the infinite-volume limit, see below in Sec. IV. Finally, any possible dependence of $\mathcal{M}$ on the fermion masses is assumed to be continuous, at least in the symmetric limit of equal fermion masses. This guarantees that $\mathcal{K}_{\mathcal{M}}$ is bounded in a neighborhood of the symmetric point, which is all that is needed to prove the results of Sec. IV. In fact, the assumption of continuity of $\mathcal{M}$ in the link variables and in the fermion masses can be relaxed, without changing the arguments in Sec. IV, to the weaker assumption that $\mathcal{K}_{\mathcal{M}}$ be bounded from above, independently of the link configuration (and therefore of the volume), in a neighborhood of the symmetric point. The results of this paper can be proved also if one further relaxes the requirement of continuity or boundedness to absolute integrability of the entries $\mathcal{M}_{\mathbf{AB}}$: this is discussed in the Appendix.

III. VECTOR FLAVOR TRANSFORMATIONS

Vector flavor transformations are defined by

$$\begin{aligned}
\psi_{fA_\star}(x) &\to \sum_g V_{fg}\psi_{gA_\star}(x),\\
\bar\psi_{fA_\star}(x) &\to \sum_g \bar\psi_{gA_\star}(x)V^\dagger_{gf}, \qquad (15)
\end{aligned}$$

where V_{fg} are the entries of a unitary unimodular $N_f \times N_f$ matrix $V \in \mathrm{SU}(N_f)$. This can be written as $V = e^{i\theta_a t^a} \equiv e^{i\boldsymbol{\theta}\cdot\boldsymbol{t}} \equiv V(\boldsymbol{\theta})$, with $\theta_a \in \mathbb{R}$ and with t^a the Hermitian and traceless generators of $\mathrm{SU}(N_f)$, taken with the standard normalization $2\mathrm{tr}_F t^a t^b = \delta^{ab}$. The task is to show that any localized observable $\mathcal{O} = \mathcal{O}[\psi, \bar{\psi}, U]$ and its transformed $\mathcal{O}^{\boldsymbol{\theta}}$,

$$\mathcal{O}^{\boldsymbol{\theta}}[\psi, \bar{\psi}, U] \equiv \mathcal{O}[V(\boldsymbol{\theta})\psi, \bar{\psi}V(\boldsymbol{\theta})^\dagger, U], \tag{16}$$

have the same expectation value in the infinite-volume theory in the symmetric limit $M \to m\mathbf{1}_F$, i.e.,

$$\lim_{M\to m\mathbf{1}_F} \lim_{\mathcal{V}\to\infty} \langle \mathcal{O}^{\boldsymbol{\theta}} - \mathcal{O}\rangle = 0. \tag{17}$$

On a finite lattice all observables are obviously localized. In the thermodynamic limit $\mathcal{V} \to \infty$, every localized observable is a linear combination of finitely many of the $\mathcal{O}_{\mathcal{M}}$ discussed above, Eq. (12), where $\mathcal{M}$ is understood to be a fixed function of the link variables, independent of $\mathcal{V}$. Moreover, since $\mathrm{SU}(N_f)$ is a Lie group, any finite transformation can be obtained by composition of infinitesimal ones. It suffices then to consider observables $\mathcal{O}_{\mathcal{M}}$ and transformations with $\theta_a \ll 1$ in Eq. (17), i.e., one has to show

$$\lim_{M\to m\mathbf{1}_F} \lim_{\mathcal{V}\to\infty} \frac{\partial}{\partial\theta_a} \langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}}\rangle\Big|_{\boldsymbol{\theta}=0} = 0. \tag{18}$$

An explicit proof that Eq. (18) implies Eq. (17) is given in Appendix A 1. In Appendix A 2 I show that Eq. (17) implies that order parameters for vector flavor symmetry, i.e., expectation values of observables that transform nontrivially under $\mathrm{SU}(N_f)$, must vanish in the symmetric limit.

To efficiently study the effect of a vector flavor transformation on the expectation value of an arbitrary observable, it is convenient to make use of the corresponding well-known integrated Ward-Takahashi identity, derived for completeness in Appendix A 1,

$$i\frac{\partial}{\partial\theta_a}\langle\mathcal{O}^{\boldsymbol{\theta}}\rangle\Big|_{\boldsymbol{\theta}=0} = \langle \mathcal{C}^a\mathcal{O}\rangle,$$
$$\mathcal{C}^a[\psi, \bar{\psi}, U] \equiv \bar{\psi}[t^a, \mathrm{D}_M[U]]\psi. \tag{19}$$

Integrating fermions out one finds for a generic observable $\mathcal{O}$

$$i\frac{\partial}{\partial\theta_a}\langle\mathcal{O}^{\boldsymbol{\theta}}\rangle_{\mathrm{F}}\Big|_{\boldsymbol{\theta}=0} = \langle \mathcal{C}^a\mathcal{O}\rangle_{\mathrm{F}} = \langle \mathcal{C}^a\rangle_{\mathrm{F}}\langle\mathcal{O}\rangle_{\mathrm{F}} - \mathrm{Tr}\{F^a O\}, \tag{20}$$

with

$$O_{AB}(x,y) \equiv \left\langle \frac{\partial}{\partial_L\psi_B(y)}\frac{\partial}{\partial_L\bar{\psi}_A(x)}\mathcal{O}\right\rangle_{\mathrm{F}}, \tag{21}$$

where ∂_L denotes the usual left derivative with respect to Grassmann variables, and

$$F^a \equiv \mathrm{S}_M[t^a, \mathrm{D}_M]\mathrm{S}_M = [\mathrm{S}_M, t^a], \tag{22}$$

i.e., the commutator of the propagator with the generators of $\mathrm{SU}(N_f)$. The first term in Eq. (20) vanishes since

$$\langle\mathcal{C}^a\rangle_{\mathrm{F}} = -\mathrm{Tr}\{[t^a, \mathrm{D}_M]\mathrm{S}_M\} = -\mathrm{Tr}\{t^a[\mathrm{D}_M, \mathrm{S}_M]\} = 0. \tag{23}$$

Specializing now to observables of the form Eq. (12), one finds, for $i, j = 1, \ldots, n$,

$$\frac{\partial}{\partial_L\psi_{B_j}(y_j)}\frac{\partial}{\partial_L\bar{\psi}_{A_i}(x_i)}\mathcal{O}_{\mathcal{M}}[\psi,\bar{\psi},U]$$
$$= -s_{ij}(\mathcal{M}[U])_{\mathbf{AB}}(\mathbf{x},\mathbf{y})\prod_{k=1}^{n-1}\psi_{B_k^{(j)}}(y_k^{(j)})\bar{\psi}_{A_k^{(i)}}(x_k^{(i)}). \tag{24}$$

Here the superscript (i) means that the ith element is omitted from the set of indices while keeping their ordering unchanged, i.e., $\mathbf{A}^{(i)} = \{A_1^{(i)}, \ldots, A_{n-1}^{(i)}\} = \{A_1, \ldots, A_{i-1}\} \cup \{A_{i+1}, \ldots, A_n\}$, and similarly for the other sets. The sign factor $-s_{ij} = (-1)^{i-j-1}$ appears when reordering the Grassmann variables to be in the same form as in Eq. (12). Using now Eq. (14) one finds explicitly

$$O_{\mathcal{M}A_iB_j}(\mathbf{x},\mathbf{y}) \equiv \left\langle \frac{\partial}{\partial_L\psi_{B_j}(y_j)}\frac{\partial}{\partial_L\bar{\psi}_{A_i}(x_i)}\mathcal{O}_{\mathcal{M}}\right\rangle_{\mathrm{F}} = -s_{ij}\sum_{\mathrm{P}\in\mathrm{S}_{n-1}}\sigma_{\mathrm{P}}\sum_{\mathbf{A}^{(i)},\mathbf{B}^{(j)}}\mathcal{M}_{\mathbf{AB}}(\mathbf{x},\mathbf{y})\prod_{k=1}^{n-1}(\mathrm{S}_M)_{B_k^{(j)}A_{\mathrm{P}(k)}^{(i)}}(y_k^{(j)}, x_{\mathrm{P}(k)}^{(i)}), \tag{25}$$

where P is now a permutation of $n-1$ elements and σ_{P} its signature. Notice that A_i and B_j are not contracted in Eq. (25), and that the dependence of $O_{\mathcal{M}A_iB_j}$ on x_i, y_j is only through that of $\mathcal{M}$. One can finally write

$$i\frac{\partial}{\partial\theta_a}\langle\mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}}\rangle\Big|_{\boldsymbol{\theta}=0} = \langle\mathcal{C}^a\mathcal{O}_{\mathcal{M}}\rangle = \sum_{i,j=1}^{n} s_{ij}\sum_{\mathrm{P}\in\mathrm{S}_{n-1}}\sigma_{\mathrm{P}}\left\langle \sum\nolimits_{\mathbf{A},\mathbf{B}}\mathcal{M}_{\mathbf{AB}}(\mathbf{x},\mathbf{y})F^a_{B_jA_i}(y_j,x_i)\prod_{k=1}^{n-1}(\mathrm{S}_M)_{B_k^{(j)}A_{\mathrm{P}(k)}^{(i)}}(y_k^{(j)}, x_{\mathrm{P}(k)}^{(i)})\right\rangle_{\mathrm{G}}. \tag{26}$$

This relation could have been obtained also directly from Eq. (14) by noticing that a vector flavor transformation $V(\boldsymbol{\theta})$ can be seen as a transformation of the matrix $\mathcal{M}$, $\mathcal{M} \to \mathcal{M}^{\boldsymbol{\theta}}$, at fixed fermion fields, i.e., $\mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}^{\boldsymbol{\theta}}}$ [see Eq. (A7)]. In practice, $\langle \mathcal{O}_{\mathcal{M}^{\boldsymbol{\theta}}} \rangle_{\mathrm{F}}$ is obtained from $\langle \mathcal{O}_{\mathcal{M}} \rangle_{\mathrm{F}}$ by replacing each propagator with $\mathrm{S}_M \to V(\boldsymbol{\theta}) \mathrm{S}_M V(\boldsymbol{\theta})^\dagger$. Expanding to first order in $\boldsymbol{\theta}$, Eq. (26) follows.

IV. PROOF OF THE MAIN RESULT

Equation (26) is the starting point for establishing a bound on the magnitude of the variation of the expectation value of $\mathcal{O}_{\mathcal{M}}$ under an infinitesimal vector flavor transformation. The argument is quite elementary, and should be understandable in spite of my best attempts at obscuring it with cumbersome notation. As shown above, after performing the Wick contractions the variation of $\langle \mathcal{O}_{\mathcal{M}} \rangle$ is written as a sum of terms of the form $\langle \mathcal{M} F \prod \mathrm{S} \rangle_{\mathrm{G}}$, differing by permutations of the indices. By standard techniques, if the integration measure is positive this is bounded in magnitude by a sum of terms of the form $\langle (\mathrm{tr}\{\mathcal{M}\mathcal{M}^\dagger\} \mathrm{tr}\{FF^\dagger\} \prod \mathrm{tr}\{\mathrm{S}\mathrm{S}^\dagger\})^{\frac{1}{2}} \rangle_{\mathrm{G}}$, which under the assumptions of Sec. II on the Dirac operator and on the continuity (or boundedness) of $\mathcal{M}$ can be bounded uniformly in the volume by a constant times the spread in mass of the fermions. In the symmetric limit one then finds that the variation of $\langle \mathcal{O}_{\mathcal{M}} \rangle$ under any infinitesimal vector flavor transformation vanishes. By a similar argument one can show that $\langle \mathcal{O}_{\mathcal{M}} \rangle$ is finite, independently of the fermion masses, also in the thermodynamic limit.

I now present the detailed proof. Using standard inequalities for the absolute value, the assumed positivity of the integration measure, and the Cauchy-Schwarz inequality for inner products, one has

$$\begin{aligned} \left| \frac{\partial}{\partial \theta_a} \langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} \rangle \Big|_{\boldsymbol{\theta}=0} \right| &\le \sum_{i,j=1}^{n} \sum_{\mathrm{P} \in \mathrm{S}_{n-1}} \left| \left\langle \sum\nolimits_{\mathbf{A},\mathbf{B}} \mathcal{M}_{\mathbf{AB}}(\mathbf{x},\mathbf{y}) F^a_{B_j A_i}(y_j, x_i) \prod_{k=1}^{n-1} (\mathrm{S}_M)_{B^{(j)}_k A^{(i)}_{\mathrm{P}(k)}} (y^{(j)}_k, x^{(i)}_{\mathrm{P}(k)}) \right\rangle_{\mathrm{G}} \right| \\ &\le \sum_{i,j=1}^{n} \sum_{\mathrm{P} \in \mathrm{S}_{n-1}} \left\langle \left| \sum\nolimits_{\mathbf{A},\mathbf{B}} \mathcal{M}_{\mathbf{AB}}(\mathbf{x},\mathbf{y}) F^a_{B_j A_i}(y_j, x_i) \prod_{k=1}^{n-1} (\mathrm{S}_M)_{B^{(j)}_k A^{(i)}_{\mathrm{P}(k)}} (y^{(j)}_k, x^{(i)}_{\mathrm{P}(k)}) \right| \right\rangle_{\mathrm{G}} \\ &\le \sum_{i,j=1}^{n} \sum_{\mathrm{P} \in S_{n-1}} \left\langle \left(\mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y}) \mathcal{F}^a(y_j, x_i) \prod_{k=1}^{n-1} \mathcal{S}(y^{(j)}_k, x^{(i)}_{\mathrm{P}(k)}) \right)^{\frac{1}{2}} \right\rangle_{\mathrm{G}}, \end{aligned} \tag{27}$$

where $\mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})$ is defined in Eq. (13) and

$$\begin{aligned} \mathcal{F}^a(y,x) &\equiv \mathrm{tr}_{FCD}\{F^a(y,x) F^a(y,x)^\dagger\}, \\ \mathcal{S}(y,x) &\equiv \mathrm{tr}_{FCD}\{\mathrm{S}_M(y,x) \mathrm{S}_M(y,x)^\dagger\}. \end{aligned} \tag{28}$$

The quantity $\mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})$ is a positive gauge-invariant function of the link variables and their Hermitian conjugates, polynomial (or more generally continuous) if $\mathcal{M}$ is polynomial (continuous), defined on the compact domain given by the direct product of finitely many compact gauge-group manifolds. It is therefore bounded from above in magnitude by its maximum, which depends on the details of $\mathcal{M}$ but is otherwise a configuration- and volume-independent quantity,

$$\mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y}) = \mathrm{tr}_{FCD}\{\mathcal{M}(\mathbf{x},\mathbf{y}) \mathcal{M}(\mathbf{x},\mathbf{y})^\dagger\} \le C_{\mathcal{M}}. \tag{29}$$

One can actually relax the request of continuity of $\mathcal{M}$ to assuming that this bound on $\mathcal{K}_{\mathcal{M}}$ holds, without changing the argument below. Moreover, since $D^{(0)}$ and ΔD are trivial and M is diagonal in flavor space, one has $[t^a, \mathrm{D}_M] = [t^a, M] \Delta D$,

$$\begin{aligned} F^a_{gB_\star fA_\star}(y,x) &= (\mathrm{S}_M [t^a, M] \Delta D \mathrm{S}_M)_{gB_\star fA_\star}(y,x) \\ &= t^a_{gf}(m_f - m_g)(S^{(m_g)} \Delta D S^{(m_f)})_{B_\star A_\star}(y,x), \end{aligned} \tag{30}$$

and so

$$\begin{aligned} \mathcal{F}^a(y,x) = \sum_{f,g} |t^a_{gf}|^2 (m_f - m_g)^2 \mathrm{tr}_{CD}\{ &(S^{(m_g)} \Delta D S^{(m_f)})(y,x) \\ &\times (S^{(m_g)} \Delta D S^{(m_f)})(y,x)^\dagger\}. \end{aligned} \tag{31}$$

The partial traces appearing in Eqs. (28) and (31) can be bounded using an elementary lemma, proved in Appendix A 3: given a multi-indexed matrix A, the partial trace of $\mathrm{A}^\dagger \mathrm{A}$ over a subset of its indices is bounded by the dimension of the corresponding space times the square of the operator norm of A. One has then the following bound on the propagator [see Eq. (A15)], valid for arbitrary lattice coordinates x and y,

$$\mathcal{S}(y,x) \le N_f N_c N_D \|\mathrm{S}_M\|^2, \tag{32}$$

from which one obtains the bound

$$\prod_{k=1}^{n-1} \mathcal{S}(y_k^{(j)}, x_{\mathrm{P}(k)}^{(i)}) \le (N_f N_c N_D \|\mathrm{S}_M\|^2)^{n-1}. \qquad (33)$$

Since the bound Eq. (32) is independent of the coordinates, the bound Eq. (33) is independent of the particular choice of i and j and of the permutation P. Using again Eq. (A15) one finds

$$\begin{aligned} &\mathrm{tr}_{CD}\{(S^{(m_g)}\Delta D S^{(m_f)})(y,x)(S^{(m_g)}\Delta D S^{(m_f)})(y,x)^\dagger\} \\ &\quad \le N_c N_D \|S^{(m_g)}\Delta D S^{(m_f)}\|^2 \\ &\quad \le N_c N_D \|S^{(m_g)}\|^2 \|\Delta D\|^2 \|S^{(m_f)}\|^2 \\ &\quad \le N_c N_D \|\mathrm{S}_M\|^4 \|\Delta D\|^2, \end{aligned} \qquad (34)$$

where I used the well-known inequality $\|AB\| \le \|A\|\|B\|$, and the obvious fact that $\|S^{(m_f)}\| \le \max_{f'} \|S^{(m_{f'})}\| = \|\mathrm{S}_M\|$. From Eqs. (31) and (34) one obtains the following bound on the commutator of the propagator with the $\mathrm{SU}(N_f)$ generators,

$$\begin{aligned} \mathcal{F}^a(y,x) &\le \sum_{f,g} |t^a_{gf}|^2 (m_f - m_g)^2 N_c N_D \|\mathrm{S}_M\|^4 \|\Delta D\|^2 \\ &\le \frac{1}{2}(\delta m)^2 N_c N_D \|\mathrm{S}_M\|^4 \|\Delta D\|^2, \end{aligned} \qquad (35)$$

where I denoted with $\delta m \equiv \max_{f,g} |m_f - m_g|$ the spread in mass of the fermions, and I used $\mathrm{tr}_F (t^a)^2 = \frac{1}{2}$. Also this bound is independent of the coordinates, and so each of the $n^2(n-1)!$ terms appearing in the sums over i, j and P in Eq. (27) obeys the same bound. Collecting now Eqs. (29), (33), and (35), one finds

$$\left| \frac{\partial}{\partial \theta_a} \langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} \rangle \Big|_{\boldsymbol{\theta}=0} \right| \le \delta m \tilde{C}_{\mathcal{M}} \langle \|\mathrm{S}_M\|^{n+1} \|\Delta D\| \rangle_{\mathrm{G}}, \qquad (36)$$

having set $\tilde{C}_{\mathcal{M}} \equiv n n! (N_f N_c N_D)^{\frac{n}{2}} (\frac{C_{\mathcal{M}}}{2N_f})^{\frac{1}{2}}$. I now make the assumptions that $\|\mathrm{S}_M\| \le m_0^{-1} < \infty$ and $\|\Delta D\| \le \Delta D_{\max} < \infty$, with m_0 and $\Delta D_{\max}$ independent of the gauge configuration. With these assumptions one concludes

$$|\langle \mathcal{C}^a \mathcal{O}_{\mathcal{M}} \rangle| = \left| \frac{\partial}{\partial \theta_a} \langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} \rangle \Big|_{\boldsymbol{\theta}=0} \right| \le \delta m \frac{\tilde{C}_{\mathcal{M}} \Delta D_{\max}}{m_0^{n+1}}. \qquad (37)$$

Using also the assumption that m_0 and $\Delta D_{\max}$ are independent of the lattice size $\mathcal{V}$, the bound Eq. (37) is volume-independent and therefore holds also in the thermodynamic limit; using the continuity in mass of $C_{\mathcal{M}}$ (or its boundedness near the symmetric point) and the assumed finiteness of m_0 and $\Delta D_{\max}$ in the symmetric limit one concludes that

$$\lim_{\delta m \to 0} \lim_{\mathcal{V} \to \infty} \frac{\partial}{\partial \theta_a} \langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} \rangle \Big|_{\boldsymbol{\theta}=0} = 0, \qquad (38)$$

which is what had to be proved [see Eq. (18)].

Notice that under the same assumptions used above one can show that the expectation values $\langle \mathcal{O}_{\mathcal{M}} \rangle$ are indeed finite in the thermodynamic limit, independently of the choice of masses. In fact, a bound similar to Eq. (27) is obtained for $\langle \mathcal{O}_{\mathcal{M}} \rangle$ starting from Eq. (14),

$$|\langle \mathcal{O}_{\mathcal{M}} \rangle| \le \sum_{\mathrm{P} \in \mathrm{S}_n} \left\langle \left(\mathcal{K}_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \prod_{k=1}^{n} \mathcal{S}(y_k, x_{\mathrm{P}(k)}) \right)^{\frac{1}{2}} \right\rangle_{\mathrm{G}}. \qquad (39)$$

Under the boundedness assumptions on $\|\mathrm{S}_M\|$ and the continuity assumption on $\mathcal{M}$ (or the boundedness assumption on $\mathcal{K}_{\mathcal{M}}$)

$$\begin{aligned} |\langle \mathcal{O}_{\mathcal{M}} \rangle| &\le n! (N_f N_c N_D)^{\frac{n}{2}} \langle \mathcal{K}_{\mathcal{M}}(\mathbf{x}, \mathbf{y})^{\frac{1}{2}} \|\mathrm{S}_M\|^n \rangle_{\mathrm{G}} \\ &\le \frac{n! (N_f N_c N_D)^{\frac{n}{2}} C_{\mathcal{M}}^{\frac{1}{2}}}{m_0^n}, \end{aligned} \qquad (40)$$

which is a finite bound, independent of $\mathcal{V}$. The extension of this result and of Eq. (38) to the case of absolutely integrable $\mathcal{M}_{\mathbf{AB}}$ is discussed in Appendix A 4.

V. APPLICATION TO SPECIFIC DISCRETIZATIONS

In this section, I discuss explicitly several lattice discretizations of the single-flavor Dirac operator. A superscript is used to distinguish them and the corresponding propagators, i.e., $D^{\mathrm{X}(m)} = D^{\mathrm{X}(0)} + m \Delta D^{\mathrm{X}}$, $\mathrm{S}_M^{\mathrm{X}} = \mathrm{diag}(S^{\mathrm{X}(m_1)}, \ldots, S^{\mathrm{X}(m_{N_f})})$, $S^{\mathrm{X}(m)} = (D^{\mathrm{X}(m)})^{-1}$. I discuss in particular staggered fermions (S), Ginsparg-Wilson fermions (GW), Wilson fermions (W), and minimally doubled fermions (KW, BC), on hypercubic lattices. Lattice sites are labeled by coordinates $x_\mu = 0, \ldots, L_\mu - 1$, where L_μ is the linear size in direction μ, with $\mu = 1, \ldots, d$. The lattice oriented edges connect x and $x + \hat{\mu}$, with $\hat{\mu}$ the unit vector in direction μ; the associated link variables are denoted with $U(x, x + \hat{\mu})$, and $U(x, x - \hat{\mu}) \equiv U(x - \hat{\mu}, x)^\dagger$ denotes the link variable associated with the oppositely oriented edge. Finally, $\delta(x, y) = \prod_{\mu=1}^{d} \delta_{x_\mu y_\mu}$. Standard boundary conditions (periodic for link variables and periodic/antiperiodic in space/time for Grassmann variables) are understood, although they do not play any particular role.

A. Staggered fermions

The case of staggered fermions [27–29] is the most straightforward. The corresponding discretization of the lattice Dirac operator carries no Dirac index, and reads

$$D^{S(0)}(x,y) = \frac{1}{2}\sum_{\mu=1}^{d} U(x,y)(\eta_\mu(x)\delta(x+\hat{\mu},y) - \eta_\mu(y)\delta(x-\hat{\mu},y)),$$
$$\Delta D^{S}(x,y) = \mathbf{1}_C \delta(x,y), \tag{41}$$

where $\eta_\mu(x) = (-1)^{\sum_{\alpha<\mu} x_\alpha}$. Notice that L_μ must be an even number for every μ. The staggered operator is anti-Hermitian, and obviously commutes with ΔD^S. Let $i\lambda_n$, $\lambda_n \in \mathbb{R}$, be its purely imaginary eigenvalues. Since $D^{S(0)}$ has the chiral property $\{\varepsilon, D^{S(0)}\} = 0$, where $\varepsilon_{ab}(x,y) = (-1)^{\sum_\alpha x_\alpha}\delta_{ab}\delta(x,y)$, these come in complex conjugate pairs $\pm i\lambda_n$ or vanish, and so

$$\det D^{S(m)} = m^{\mathcal{N}_0^S} \prod_{n,\lambda_n>0} (\lambda_n^2 + m^2), \tag{42}$$

where $\mathcal{N}_0^S$ is the number of exact zero modes, which must be an even number. The integration measure $d\mu_G$ is therefore positive for any choice of m_f. For the propagator one has

$$S^{S(m)\dagger} S^{S(m)} = (m^2 - D^{S(0)2})^{-1},$$
$$\|S^{S(m)}\|^2 = \frac{1}{m^2 + \min_n \lambda_n^2} \le \frac{1}{m^2}, \tag{43}$$

and so

$$\|S_M^S\|^2 = \max_f \|S^{S(m_f)}\|^2 = \frac{1}{\min_f m_f^2 + \min_n \lambda_n^2}$$
$$\le \frac{1}{\min_f m_f^2} \equiv \frac{1}{m_0^2}. \tag{44}$$

Obviously $\|\Delta D^S\| = 1$. All the assumptions used in Sec. IV hold, and so vector flavor symmetry cannot be spontaneously broken, as long as the common fermion mass in the symmetric limit is nonzero. This was already known [30]. The result holds at any lattice spacing, and remains true for any choice of boundary conditions or inclusion of external fields (e.g., an imaginary chemical potential or a magnetic field) that preserves the anti-Hermiticity and the chiral property of $D^{S(0)}$.

The result still holds also for improved staggered operators as long as they retain these properties. In particular, this is the case if in Eq. (41) one replaces the "thin links" U with "fat links" obtained by some smearing procedure; and if one improves the lattice approximation of the covariant derivative by including terms that only couple even and odd lattice sites (i.e., sites with $\sum_\alpha x_\alpha$ even or odd), e.g., the Naik term [75]. This covers all the commonly used improved operators (e.g., ASQTAD [76], stout smeared [77], HISQ [78]).

Extending the result to rooted staggered fermions is not entirely straightforward. The expectation values of localized observables $\mathcal{O}_\mathcal{M}$ in the rooted theory with N_f flavors of staggered quarks are obtained by replacing the fermionic determinant for each flavor with its positive fourth root (so keeping the integration measure positive), and by including suitable counting factors for the various permutations appearing in the fermionic expectation value $\langle \mathcal{O}_\mathcal{M} \rangle_F$, Eq. (14). Defining vector flavor transformations on these observables directly as transformations of $\mathcal{M}$, i.e., $\mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}^{\boldsymbol{\theta}} = \mathcal{O}_{\mathcal{M}^{\boldsymbol{\theta}}}$ [see Eq. (A7) and comment after Eq. (26)], one can still show that the variation of $\langle \mathcal{O}_\mathcal{M} \rangle$ for infinitesimal $\boldsymbol{\theta}$ is bounded by the mass spread δm times a constant, hence it vanishes in the symmetric limit. In fact, such a variation remains of the form Eq. (26) up to inclusion of the counting factors; the rest of the argument is unchanged. However, one should still check that the flavor transformations defined above reduce in the continuum limit to the correct transformations of the physical subset of fermionic degrees of freedom. To this end, one may use the blocking transformations and the reweighted actions of Refs. [40,43], introduced to argue the validity of the rooting procedure (I thank an anonymous referee for pointing these references out). Such an analysis is, however, beyond the scope of this paper. It should be noted that if one accepts the validity of the rooting procedure, then the uniform bound of Ref. [30] on the staggered propagator in a gauge field background suffices to prove the absence of massless particles in the spectrum of the continuum theory for any nonzero common fermion mass, implying the impossibility of spontaneous flavor symmetry breaking.

B. Ginsparg-Wilson fermions

Massless Ginsparg-Wilson fermions are characterized by the relation [56]

$$\{D^{GW(0)}, \gamma_5\} = 2D^{GW(0)} R \gamma^5 D^{GW(0)}, \tag{45}$$

with R a local operator, satisfied by the corresponding lattice discretization $D^{GW(0)}$ of the Dirac operator. Most of the known examples [57–65] satisfy this relation with $2R = \mathbf{1}$, and moreover are γ_5-Hermitian,

$$\gamma_5 D^{GW(0)} \gamma_5 = D^{GW(0)\dagger}. \tag{46}$$

If these extra assumptions hold it is easy to show that

$$(D^{GW(0)} - \mathbf{1})(D^{GW(0)} - \mathbf{1})^\dagger = \mathbf{1}, \tag{47}$$

i.e., $D^{GW(0)} = \mathbf{1} + \mathcal{U}$ with $\mathcal{U}$ unitary. For massive Ginsparg-Wilson fermions one uses

$$\Delta D^{GW} = \mathbf{1} - \frac{1}{2} D^{GW(0)}, \tag{48}$$

and so

$$D^{\mathrm{GW}(m)} = \left(1+\frac{m}{2}\right)\mathbf{1} + \left(1-\frac{m}{2}\right)\mathcal{U}. \qquad (49)$$

This is a normal operator with spectrum lying on a circle of radius $|1-\frac{m}{2}|$ centered at $1+\frac{m}{2}$, so its eigenvalues are bounded in magnitude from below by the square root of

$$\min_{\varphi}\left|\left(1+\frac{m}{2}\right)+\left(1-\frac{m}{2}\right)e^{i\varphi}\right|^2 = \min(m^2,4). \qquad (50)$$

It follows that the propagator obeys $\|\mathrm{S}_M^{\mathrm{GW}}\| \leq \max\left(\frac{1}{\min_f|m_f|},\frac{1}{2}\right)$, which is a finite bound if $m_f \neq 0\ \forall\ f$. For ΔD^{GW} one has

$$\|\Delta D^{\mathrm{GW}}\|^2 \leq \frac{1}{2}\max_{\varphi}|1-e^{i\varphi}|^2 = 2 \equiv \Delta D^2_{\max} < \infty. \qquad (51)$$

As a consequence of Eq. (46), $\gamma_5\mathcal{U}\gamma_5 = \mathcal{U}^\dagger$, and so if ψ_n is a common eigenvector of $D^{\mathrm{GW}(0)}$ and $D^{\mathrm{GW}(0)\dagger}$ with eigenvalues $\mu_n = 1+e^{i\varphi_n}$ and $\mu_n^* = 1+e^{-i\varphi_n}$, respectively, then $D^{\mathrm{GW}(0)}\gamma_5\psi_n = \gamma_5 D^{\mathrm{GW}(0)\dagger}\psi_n = \mu_n^*\gamma_5\psi_n$. It follows that complex eigenvalues come in complex-conjugate pairs; for the real eigenvalues $\mu_n = \mu_n^* = 0, 2$, one can instead choose chiral eigenvectors $\psi_\pm$, satisfying $\gamma_5\psi_\pm = \pm\psi_\pm$. For the determinant of $D^{\mathrm{GW}(m)}$ one finds

$$\det D^{\mathrm{GW}(m)} = m^{\mathcal{N}_0^{\mathrm{GW}}} 2^{\mathcal{N}_2^{\mathrm{GW}}} \prod_{n,\sin\varphi_n>0}\left[\left(2\cos\frac{\varphi_n}{2}\right)^2 + \left(m\sin\frac{\varphi_n}{2}\right)^2\right], \qquad (52)$$

with $\mathcal{N}_{0,2}^{\mathrm{GW}}$ the degeneracies of the two real eigenvalues. It follows that the integration measure $d\mu_{\mathrm{G}}$ is positive if $m_f \geq 0\ \forall\ f$, and more generally for an even number of negative masses. Vector flavor symmetry cannot be spontaneously broken in the symmetric limit as long as the common fermion mass is positive, or just nonzero if N_f is even, at any value of the lattice spacing. The use of different boundary conditions or the inclusion of external fields in $D^{\mathrm{GW}(0)}$ does not change this result, as long as the operator remains of the form $D^{\mathrm{GW}(0)} = \mathbf{1}+\mathcal{U}$ and the γ_5-Hermiticity property Eq. (46) holds.

C. Wilson fermions

For Wilson fermions [26] the massless operator $D^{\mathrm{W}(0)} = D^{\mathrm{n}(0)} + R^{\mathrm{W}}$ is obtained adding the naive discretization $D^{\mathrm{n}(0)}$ of the massless Dirac operator and the Wilson term R^{W}, while ΔD^{W} is the identity in color, Dirac, and coordinate space,

$$\begin{aligned} D^{\mathrm{n}(0)}(x,y) &= \frac{1}{2}\sum_{\mu=1}^{d} U(x,y)\gamma_\mu(\delta(x+\hat{\mu},y)-\delta(x-\hat{\mu},y)),\\ R^{\mathrm{W}}(x,y) &= -\frac{r}{2}\mathbf{1}_D\sum_{\mu=1}^{d}(U(x,y)(\delta(x+\hat{\mu},y) + \delta(x-\hat{\mu},y)) - 2\mathbf{1}_C\delta(x,y)),\\ \Delta D^{\mathrm{W}}(x,y) &= \mathbf{1}_C\mathbf{1}_D\delta(x,y), \end{aligned} \qquad (53)$$

with r a nonzero real parameter. This operator is not anti-Hermitian and not even normal, satisfying only the γ_5-Hermiticity condition $\gamma_5 D^{\mathrm{W}(m)}\gamma_5 = D^{\mathrm{W}(m)\dagger}$. The spectrum of $D^{\mathrm{W}(m)}$ is generally complex, and while γ_5-Hermiticity guarantees that $\det D^{\mathrm{W}(m)}$ is real, one is not guaranteed to find a positive integration measure $d\mu_{\mathrm{G}}$, unless an even number of fermions with the same mass is present. Moreover, while ΔD^{W} is obviously bounded, no general lower bound applies to the spectrum of $D^{\mathrm{W}(m)\dagger}D^{\mathrm{W}(m)}$, even in the massive case, and so no uniform upper bound on the norm of the propagator is available. The result of the previous section therefore does not apply to Wilson fermions. This is not surprising since it is known that vector flavor symmetry is spontaneously broken in the Aoki phase [5–25]. One can, however, refine the discussion and see more precisely how things fail for Wilson fermions.

Presumably, for an even number of flavors and sufficiently small δm the sign problem of the integration measure affects only a set of gauge configurations of zero measure. If so, in this case the integration measure would effectively be positive, and so one could follow the derivation of the previous section up to Eq. (36), obtaining

$$\begin{aligned} \left|\frac{\partial}{\partial\theta_a}\langle\mathcal{O}_{\mathcal{M}}^{\theta}\rangle\Big|_{\theta=0}\right| &\leq \delta m\tilde{C}_{\mathcal{M}}\langle\|\mathrm{S}_M^{\mathrm{W}}\|^{n+1}\rangle_{\mathrm{G}}\\ &= \delta m\tilde{C}_{\mathcal{M}}\int_0^\infty ds\, p(s)s^{n+1}, \end{aligned} \qquad (54)$$

where

$$p(s) \equiv \langle\delta(s-\|\mathrm{S}_M^{\mathrm{W}}\|)\rangle_{\mathrm{G}}. \qquad (55)$$

One could then still exclude the spontaneous breaking of vector flavor symmetry if $p(s)$ vanished faster than any polynomial as $s\to\infty$, for example if $p(s)=0$ for $s>s_0$ for some s_0, or if it vanished exponentially. If $p(s)$ vanished only as a power law $p(s)\sim s^{-n_0}$, or not at all, the argument above would not provide a viable bound for $n > n_0-2$, and spontaneous breaking could not be excluded. In fact, in the Aoki phase one finds a finite spectral density of near-zero modes of the Hermitian operator $H = \gamma_5 D^{\mathrm{W}(m)}$ [7,20,21]. Since

$$\|S^{W(m)}\|^2 = \sup_{\psi\neq 0} \frac{(\psi, (D^{W(m)}D^{W(m)\dagger})^{-1}\psi)}{(\psi,\psi)} = \sup_{\psi\neq 0} \frac{(\gamma_5\psi, (H^2)^{-1}\gamma_5\psi)}{(\gamma_5\psi,\gamma_5\psi)} = \frac{1}{\min_n h_n^2}, \tag{56}$$

where $h_n \in \mathbb{R}$ are the eigenvalues of H, one finds that as the lattice volume grows and the lowest mode of H on typical configurations tends to zero $p(s)$ becomes more and more peaked at a larger and larger value of s, eventually tending to infinity in the thermodynamic limit, and the right-hand side of Eq. (54) blows up, making the bound useless. Outside the Aoki phase the spectrum of H is gapped around the origin, the right-hand side of Eq. (54) is finite, and the bound prevents spontaneous flavor symmetry breaking.

D. Minimally doubled fermions

For the minimally doubled fermions of Karsten and Wilczek (KW) [52,53], and of Creutz and Boriçi (BC) [54,55], the massless Dirac operator is of the form $D^{X(0)} = D^{n(0)} + R^X$, $X = \mathrm{KW}, \mathrm{BC}$, where the naive operator $D^{n(0)}$ is defined in Eq. (53), and the inclusion of the terms

$$\begin{aligned} R^{\mathrm{KW}}(x,y) &= -\frac{ir}{2}\gamma_d \sum_{\mu=1}^{d-1} (U(x,y)(\delta(x+\hat{\mu},y)+\delta(x-\hat{\mu},y)) \\ &\quad -2\mathbf{1}_C\delta(x,y)), \\ R^{\mathrm{BC}}(x,y) &= -\frac{ir}{2}\sum_{\mu=1}^{d} \gamma'_\mu (U(x,y)(\delta(x+\hat{\mu},y)+\delta(x-\hat{\mu},y)) \\ &\quad -2\mathbf{1}_C\delta(x,y)), \end{aligned} \tag{57}$$

where $\gamma'_\mu \equiv \Gamma\gamma_\mu\Gamma$ and $\Gamma \equiv \frac{1}{\sqrt{d}}\sum_{\nu=1}^d \gamma_\nu$, reduces the number of doublers to two when $r = 1$. The massive operator is obtained in both cases using the trivial mass term $\Delta D^{\mathrm{KW}} = \Delta D^{\mathrm{BC}} = \Delta D^{\mathrm{W}}$, see again Eq. (53). For both types of fermions the massless operator is anti-Hermitian and chiral, $\{D^{X(0)}, \gamma_5\} = 0$, and obviously commutes with the mass term. One can then diagonalize $D^{X(0)}$ obtaining purely imaginary eigenvalues $i\lambda_n^X$ and a symmetric spectrum, and so the single-flavor propagators $S^{X(m)}$ obey

$$\|S^{X(m)}\|^2 = \frac{1}{m^2 + \min_n \lambda_n^{X\,2}} \leq \frac{1}{m^2}, \quad X = \mathrm{KW}, \mathrm{BC}. \tag{58}$$

The fermionic determinant reads

$$\det D^{X(m)} = m^{\mathcal{N}_0^X} \prod_{n,\lambda_n>0} (\lambda_n^{X\,2} + m^2), \quad X = \mathrm{KW}, \mathrm{BC}, \tag{59}$$

with $\mathcal{N}_0^X$ the number of exact zero modes, so it is positive for nonnegative fermion masses, and for an even number of negative masses. The same argument therefore applies as with staggered fermions, and vector flavor symmetry cannot break spontaneously as long as the common fermion mass is positive (or just nonzero if N_f is even) in the symmetric limit, independently of the lattice spacing.

VI. CONCLUSIONS

In this paper I have shown that under quite general assumptions on the discretization of the Dirac operator D_M, one can rigorously exclude the possibility of spontaneous breaking of vector flavor symmetry on the lattice, in gauge theories with a positive functional-integral measure. These assumptions are

(0.) D_M is linear in the fermion masses, $\mathrm{D}_M = D^{(0)} + M\Delta D$, with $D^{(0)}$ and ΔD trivial in flavor space, and M a Hermitian mass matrix;

(1.) the norm of the propagator D_M^{-1} can be bounded by a configuration- and volume-independent quantity, that remains finite in the symmetric limit of fermions of equal masses, $M \to m\mathbf{1}_F$;

(2.) the norm of the derivative of D_M with respect to the fermion masses, ΔD, can be bounded by a configuration- and volume-independent quantity, that remains finite as $M \to m\mathbf{1}_F$.

The impossibility of spontaneous flavor symmetry breaking on the lattice is proved by showing that any localized order parameter must vanish in the symmetric limit, taken after the thermodynamic limit. If the assumptions above hold for any (or at least for sufficiently small) lattice spacing, this result remains true also in the continuum limit, if this exists. My argument applies in particular to staggered fermions [27–29]; to the minimally doubled fermions of Karsten and Wilczek [52,53] and of Creutz and Boriçi [54,55]; and to Ginsparg-Wilson fermions [56–65] that are γ_5-Hermitian and satisfy the Ginsparg-Wilson relation with $2R = \mathbf{1}$ [see Eq. (45)]. For these discretizations one can exclude spontaneous breaking of vector flavor symmetry on the lattice for any spacing, and so in the continuum limit as well, for any positive common fermion mass m (and for any nonzero m for an even number of flavors). Quite unsurprisingly, the argument fails in the case of Wilson fermions [26], where such a spontaneous breaking is known to happen in the Aoki phase [5–25]. While for staggered fermions spontaneous breaking of vector flavor symmetry (as well as of baryon number symmetry) was already completely excluded by the results of Ref. [30], for Ginsparg-Wilson fermions only partial results were previously available [66].

My result is clearly not as powerful as that obtained by Vafa and Witten working with the continuum functional integral in Ref. [1], and by Aloisio *et al.* in Ref. [30] working with staggered fermions on the lattice. In particular, although it excludes the possibility of Goldstone bosons appearing in the spectrum due to spontaneous flavor symmetry breaking, it cannot exclude completely the presence of massless bosons, as Refs. [1,30] do. On the other hand, the use of a properly regularized functional integral rather than the continuum one used in Ref. [1] makes the present argument mathematically fully rigorous. Since the ultralocality and anti-Hermiticity of the staggered operator are not used, as they are in Ref. [30], my argument works also for more general discretizations, in particular allowing one to treat the case of an arbitrary number of physical fermion flavors in the continuum limit without resorting to the "rooting trick."

The bound on the variation of expectation values under a vector flavor transformation [see Eq. (37)] proved here to derive the main result is probably far less than optimal, as it does not take into account the cancellations present in fermionic observables due to the oscillating sign of the contributions of the various field contractions. The bound on the propagator [see Eq. (32)] is also likely to be suboptimal, and one suspects that a lattice analog of the Vafa-Witten bound could be obtained also for more general discretizations than staggered fermions, for which it was proved in Ref. [30]. A direct extension of the proof of Ref. [30] to minimally doubled fermions seems feasible, while a different approach is probably needed for Ginsparg-Wilson fermions. It is worth noting, however, that a global, coordinate-independent bound like Eq. (32) suffices to prove the impossibility of vector flavor symmetry breaking, without the need to bound the long-distance behavior of the propagator as in Refs. [1,30].

The present argument does not rule out the appearance of phases with spontaneously broken vector flavor symmetry on the lattice if terms of order higher than quadratic are included in the fermionic action, even if the quadratic terms satisfy assumptions (0.)–(2.). Nonsymmetric vacua may in fact exist, degenerate with the symmetric one in the symmetric limit, but with ground state energy increased by the standard symmetry-breaking term used here. These vacua could not be reached with the procedure used here, and would require the addition of different symmetry-breaking terms to the symmetric action in order to select them. This possibility is of limited interest in the physical case of QCD, since in this theory vector flavor symmetry is broken explicitly precisely by the differences in the quark masses, and the symmetric limit of interest where one should investigate the possibility of its spontaneous breaking is the one considered in this paper. More generally, while such spontaneously broken phases on the lattice could be problematic for numerical simulations, they should be unphysical and not survive the continuum limit.

The restriction to a quadratic lattice action is in fact not really a limitation as far as the usual continuum limit is concerned. For continuum gauge theories in dimension $d > 2$ (in $d \leq 2$ the spontaneous breaking of a continuous symmetry is forbidden [71–73]) there are no perturbatively renormalizable fermionic operators with the right global and local symmetries other than the quadratic ones, approximated on the lattice by the action used here. The inclusion of higher order terms in the lattice action only adds perturbatively nonrenormalizable interactions that do not affect the long-distance physics in the usual continuum limit. Hypothetical spontaneously broken phases on the lattice should then shrink as the continuum limit is approached, with vector flavor symmetry being realized in the continuum theory. Phases with spontaneously broken vector flavor symmetry may still be found in the continuum if unconventional continuum limits exist, but this would concern a different type of continuum theories. Universality of the continuum limit also implies that the spontaneously broken phases potentially found on the lattice for quadratic actions not satisfying assumptions (0.)–(2.) should shrink in the usual continuum limit, as is the case for the Aoki phase of Wilson fermions.

In conclusion, the existence of lattice discretizations of the Dirac operator, free of doublers, for which spontaneous vector flavor symmetry breaking for finite positive fermion mass is impossible at any lattice spacing (i.e., the Ginsparg-Wilson fermions discussed above) implies the same impossibility in the continuum limit, if this exists, for an arbitrary number of fermion species. This settles the issue of spontaneous vector flavor symmetry breaking in a rigorous manner (for a physicist's standard of rigor).

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APPENDIX: TECHNICAL DETAILS

1. Integrated Ward-Takahashi identity and finite flavor transformations

Since the Berezin integration measure is invariant under vector flavor transformations, Eq. (15), one has after changing variables [see Eq. (16) for the notation]

$$\int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} \mathcal{O}^{\boldsymbol{\theta}}[\psi,\bar{\psi},U] = \int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\bar{\psi} V(\boldsymbol{\theta}) \mathrm{D}_M[U] V(\boldsymbol{\theta})^{\dagger} \psi} \mathcal{O}[\psi,\bar{\psi},U]. \tag{A1}$$

For infinitesimal θ_a, expanding both sides of the equation to leading order in θ_a, one finds

$$\int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} \left(\mathcal{O}[\psi,\bar{\psi},U] + \sum_a \theta_a \frac{\partial}{\partial \theta_a} \mathcal{O}^{\boldsymbol{\theta}}[\psi,\bar{\psi},U] \Big|_{\boldsymbol{\theta}=0} + O(\theta^2) \right)$$
$$= \int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} (\mathcal{O}[\psi,\bar{\psi},U] - i(\bar{\psi}[\boldsymbol{\theta}\cdot \boldsymbol{t}, \mathrm{D}_M[U]]\psi)\mathcal{O}[\psi,\bar{\psi},U] + O(\theta^2)), \tag{A2}$$

from which Eq. (19) follows,

$$i \frac{\partial}{\partial \theta_a} \langle \mathcal{O}^{\boldsymbol{\theta}} \rangle \Big|_{\boldsymbol{\theta}=0} = \langle (\bar{\psi}[t^a, \mathrm{D}_M]\psi)\mathcal{O} \rangle = \langle \mathcal{C}^a \mathcal{O} \rangle, \qquad \mathcal{C}^a[\psi,\bar{\psi},U] \equiv \bar{\psi}[t^a, \mathrm{D}_M[U]]\psi. \tag{A3}$$

For finite transformations, since $V(\boldsymbol{\theta})$ is analytic in θ_a one can write the trivial identity

$$\int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} (\mathcal{O}^{\boldsymbol{\theta}}[\psi,\bar{\psi},U] - \mathcal{O}[\psi,\bar{\psi},U]) = \int_0^1 d\alpha \frac{d}{d\alpha} \int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} \mathcal{O}^{\alpha\boldsymbol{\theta}}[\psi,\bar{\psi},U]. \tag{A4}$$

Using now Eq. (A1) one finds the following result for the change of the expectation value of an observable under a finite vector flavor transformation,

$$\int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} (\mathcal{O}^{\boldsymbol{\theta}}[\psi,\bar{\psi},U] - \mathcal{O}[\psi,\bar{\psi},U])$$
$$= -\int_0^1 d\alpha \int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\bar{\psi} V(\alpha\boldsymbol{\theta}) \mathrm{D}_M[U] V(\alpha\boldsymbol{\theta})^\dagger \psi} \left(\bar{\psi} V(\alpha\boldsymbol{\theta}) \left[V(\alpha\boldsymbol{\theta})^\dagger \frac{dV(\alpha\boldsymbol{\theta})}{d\alpha}, \mathrm{D}_M[U] \right] V(\alpha\boldsymbol{\theta})^\dagger \psi \right) \mathcal{O}[\psi,\bar{\psi},U]$$
$$= -i \int_0^1 d\alpha \int [\mathrm{D}\psi \mathrm{D}\bar{\psi}] e^{-\mathcal{S}_{\mathrm{F}}[\psi,\bar{\psi},U]} (\bar{\psi}[\boldsymbol{\theta}\cdot \boldsymbol{t}, \mathrm{D}_M[U]]\psi) \mathcal{O}^{\alpha\boldsymbol{\theta}}[\psi,\bar{\psi},U]. \tag{A5}$$

This implies

$$\langle \mathcal{O}^{\boldsymbol{\theta}} - \mathcal{O} \rangle = -i \sum_a \theta_a \int_0^1 d\alpha \langle \mathcal{C}^a \mathcal{O}^{\alpha\boldsymbol{\theta}} \rangle. \tag{A6}$$

For the observables $\mathcal{O}_{\mathcal{M}}$ of interest, the effect of a vector flavor transformation $V(\boldsymbol{\theta})$ can be fully accounted for by replacing $\mathcal{M} \to \mathcal{M}^{\boldsymbol{\theta}}$, i.e., $\mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}^{\boldsymbol{\theta}}}$, with

$$\mathcal{M}^{\boldsymbol{\theta}}_{\mathbf{AB}} \equiv \mathcal{M}_{A'_1 \ldots A'_n B'_1 \ldots B'_n} V(\boldsymbol{\theta})^\dagger_{A_1 A'_1} \ldots V(\boldsymbol{\theta})^\dagger_{A_n A'_n} V(\boldsymbol{\theta})_{B'_1 B_1} \ldots V(\boldsymbol{\theta})_{B'_n B_n}, \tag{A7}$$

where $V_{A_i A'_i} = V_{f_i f'_i} \delta_{a_i a'_i} \delta_{\alpha_i \alpha'_i}$. Notice that

$$\mathcal{K}_{\mathcal{M}^{\boldsymbol{\theta}}}(\mathbf{x},\mathbf{y}) = \mathrm{tr}_{FCD}\{\mathcal{M}^{\boldsymbol{\theta}}(\mathbf{x},\mathbf{y})\mathcal{M}^{\boldsymbol{\theta}}(\mathbf{x},\mathbf{y})^\dagger\} = \mathrm{tr}_{FCD}\{\mathcal{M}(\mathbf{x},\mathbf{y})\mathcal{M}(\mathbf{x},\mathbf{y})^\dagger\} = \mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y}), \tag{A8}$$

which implies that the same upper bound applies to $\mathcal{M}$ and $\mathcal{M}^{\boldsymbol{\theta}}$ in Eq. (29), i.e., $C_{\mathcal{M}^{\boldsymbol{\theta}}} = C_{\mathcal{M}}$, and the same constant $\tilde{C}_{\mathcal{M}} = \tilde{C}_{\mathcal{M}^{\boldsymbol{\theta}}}$ appears in the bound Eq. (37). Using now Eq. (A6) one finds

$$\langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} - \mathcal{O}_{\mathcal{M}} \rangle = \langle \mathcal{O}_{\mathcal{M}^{\boldsymbol{\theta}}} - \mathcal{O}_{\mathcal{M}} \rangle$$
$$= -i \sum_a \theta_a \int_0^1 d\alpha \langle \mathcal{C}^a \mathcal{O}_{\mathcal{M}^{\alpha\boldsymbol{\theta}}} \rangle. \tag{A9}$$

The integrand on the right-hand side obeys the bound Eq. (37), and since $\tilde{C}_{\mathcal{M}^{\alpha\boldsymbol{\theta}}} = \tilde{C}_{\mathcal{M}}$ are independent of α one finds

$$|\langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} - \mathcal{O}_{\mathcal{M}} \rangle| \le \sum_a |\theta_a| \int_0^1 d\alpha |\langle \mathcal{C}^a \mathcal{O}_{\mathcal{M}^{\alpha\boldsymbol{\theta}}} \rangle|$$
$$\le \delta m \frac{\tilde{C}_{\mathcal{M}} \Delta D_{\max}}{m_0^{n+1}} \sum_a |\theta_a|. \tag{A10}$$

Since this bound is independent of $\mathcal{V}$ and remains finite in the symmetric limit, one has

$$\lim_{\delta m \to 0} \lim_{\mathcal{V} \to \infty} \langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} - \mathcal{O}_{\mathcal{M}} \rangle = 0. \tag{A11}$$

This implies $\langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} \rangle = \langle \mathcal{O}_{\mathcal{M}} \rangle$ in the thermodynamic and symmetric limit, for any $\boldsymbol{\theta}$, and so vector flavor symmetry cannot be spontaneously broken if the assumptions of Sec. II are satisfied.

2. Vanishing of order parameters

Under $\mathrm{SU}(N_f)$ vector flavor transformations, the matrix $\mathcal{M}$ transforms in the product representation with n fundamental and n antifundamental factors. This is a unitary representation that can be decomposed in the direct sum of unitary irreducible representations. The most general $\mathcal{M}$ is then the linear combination of matrices $\mathcal{M}_r^{(R)}(\mathbf{x},\mathbf{y})$,

$$(\mathcal{M}_r^{(R)})_{f_1A_{\star 1}\ldots f_nA_{\star n}g_1B_{\star 1}\ldots g_nB_{\star n}}(\mathbf{x},\mathbf{y}) = (\mathcal{T}_r^{(R)})_{f_1\ldots f_ng_1\ldots g_n}\tilde{\mathcal{M}}_{A_{\star 1}\ldots A_{\star n}B_{\star 1}\ldots B_{\star n}}(\mathbf{x},\mathbf{y}), \quad (A12)$$

where $\tilde{\mathcal{M}}$ is a matrix acting only on color and Dirac space that can depend on the gauge variables in a suitably gauge-covariant way, and where $\mathcal{T}_r^{(R)}$, $r=1,\ldots,d_R$, are constant tensors in flavor space, independent of the gauge variables and of any feature of the theory other than the number of flavors, that transform irreducibly under vector flavor transformations [see Eq. (A7) for the notation],

$$\mathcal{T}_r^{(R)\boldsymbol{\theta}} = \sum_{r'}\mathcal{T}_{r'}^{(R)}D^{(R)}(V(\boldsymbol{\theta}))_{r'r}, \quad (A13)$$

with $D^{(R)}(V)$ the representative of $V\in\mathrm{SU}(N_f)$ in the irreducible unitary representation R of dimension d_R. For these quantities one has in the thermodynamic and symmetric limit [see Eq. (A11)]

$$\begin{aligned} u_r^{(R)} \equiv \langle \mathcal{O}_{\mathcal{T}_r^{(R)}\tilde{\mathcal{M}}}\rangle &= \langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{T}_r^{(R)}\tilde{\mathcal{M}}}\rangle \\ &= \sum_{r'}\langle \mathcal{O}_{\mathcal{T}_{r'}^{(R)}\tilde{\mathcal{M}}}\rangle D^{(R)}(V(\boldsymbol{\theta}))_{r'r} \\ &= \sum_{r'}u_{r'}^{(R)}D^{(R)}(V(\boldsymbol{\theta}))_{r'r}, \end{aligned} \quad (A14)$$

which expresses the invariance of the vector $u^{(R)}$ with components $u_r^{(R)}$, $r=1\ldots,d_R$, under an arbitrary $\mathrm{SU}(N_f)$ transformation. No nonzero invariant vector exists if R is a nontrivial representation [79], so $u^{(R)}=0$ follows unless R is the one-dimensional trivial representation, and all order parameters vanish.

3. Bound on partial traces

Let A be a matrix with entries labeled by pairs of an arbitrary number of indices. Denote the subset of indices corresponding to an "internal" space $\mathcal{I}$ collectively by I, and denote the remaining ones collectively by x, i.e., the matrix entries are labeled as $\mathrm{A}_{I'I}(x',x)$. The partial trace over the internal space of the matrix $\mathrm{A}(x',x)^\dagger\mathrm{A}(x',x)$, where Hermitian conjugation and matrix multiplication are understood to apply only to the internal indices, satisfies the bound

$$\begin{aligned} \mathrm{tr}_{\mathcal{I}}\mathrm{A}(x',x)^\dagger\mathrm{A}(x',x) &= \mathrm{tr}_{\mathcal{I}}\mathrm{A}(x',x)\mathrm{A}(x',x)^\dagger \\ &\equiv \sum_{I,I'}\mathrm{A}_{I'I}(x',x)\mathrm{A}_{I'I}(x',x)^* \\ &\le \dim\mathcal{I}\|\mathrm{A}\|^2, \end{aligned} \quad (A15)$$

where $\|\mathrm{A}\|$ is the usual operator norm of A. This is equal to the square root of the largest eigenvalue of the matrix $\mathrm{A}^\dagger\mathrm{A}$, where Hermitian conjugation and matrix multiplication are understood to apply to all the indices [see Eqs. (9) and (10) for the notation].

Proof By the spectral theorem, the Hermitian matrix $\mathrm{A}^\dagger\mathrm{A}$ can be written as follows,

$$\begin{aligned} \mathrm{A}^\dagger\mathrm{A} &= \sum_n a_n^2\phi_n\phi_n^\dagger, \\ (\mathrm{A}^\dagger\mathrm{A})_{I'I}(x',x) &= \sum_n a_n^2\phi_{nI'}(x')\phi_{nI}(x)^*, \end{aligned} \quad (A16)$$

where a_n^2 are its real positive eigenvalues and ϕ_n are a complete set of orthonormal eigenvectors, $(\mathrm{A}^\dagger\mathrm{A})\phi_n = a_n^2\phi_n$, with

$$\begin{aligned} (\phi_{n'},\phi_n) &\equiv \sum_{x,I}\phi_{n'I}(x)^*\phi_{nI}(x) = \delta_{n'n}, \\ &\sum_n \phi_{nI'}(x')\phi_{nI}(x)^* = \delta_{I'I}\delta(x',x). \end{aligned} \quad (A17)$$

Since $\mathrm{tr}_{\mathcal{I}}\mathrm{A}(x',x)^\dagger\mathrm{A}(x',x)\ge 0$ for any x', one finds

$$\begin{aligned} \mathrm{tr}_{\mathcal{I}}\mathrm{A}(x',x)^\dagger\mathrm{A}(x',x) &= \mathrm{tr}_{\mathcal{I}}(\mathrm{A}^\dagger)(x,x')\mathrm{A}(x',x) \le \sum_{x'}\mathrm{tr}_{\mathcal{I}}(\mathrm{A}^\dagger)(x,x')\mathrm{A}(x',x) = \mathrm{tr}_{\mathcal{I}}(\mathrm{A}^\dagger\mathrm{A})(x,x) \\ &= \sum_{I,n}a_n^2\phi_{nI}(x)\phi_{nI}(x)^* \le (\max_{n'}a_{n'}^2)\sum_{I,n}\phi_{nI}(x)\phi_{nI}(x)^* = \|\mathrm{A}\|^2\sum_I\delta_{II}\delta(x,x) \\ &= \dim\mathcal{I}\|\mathrm{A}\|^2. \end{aligned} \quad (A18)$$

4. Extension to absolutely integrable $\mathcal{M}_{\mathbf{AB}}$

The requirement of continuity of $\mathcal{M}_{\mathbf{AB}}$, used in Sec. IV to bound $\mathcal{K}_{\mathcal{M}}$ independently of the gauge configuration and uniformly in the volume, can be relaxed to the requirement that the entries $\mathcal{M}_{\mathbf{AB}}$ be absolutely integrable, a condition conveniently expressed as

$$\langle \mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})^{\frac{1}{2}}\rangle_{\mathrm{G}} = \langle (\mathrm{tr}_{FCD}\{\mathcal{M}(\mathbf{x},\mathbf{y})\mathcal{M}(\mathbf{x},\mathbf{y})^{\dagger}\})^{\frac{1}{2}}\rangle_{\mathrm{G}} < \infty. \tag{A19}$$

Since $|X_1| \le \sqrt{\sum_i |X_i|^2} \le \sum_i |X_i|$, this condition is equivalent to $\langle |\mathcal{M}_{\mathbf{AB}}(\mathbf{x},\mathbf{y})|\rangle_{\mathrm{G}} < \infty\ \forall\ \mathbf{A},\mathbf{B}$. It is further assumed that Eq. (A19) remains true in the thermodynamic limit.

Under the assumption of a positive integration measure, and under assumption (1.) in Sec. II on the norm of the propagator, this suffices to prove convergence of $\langle \mathcal{O}_{\mathcal{M}}\rangle$. Using this assumption in the first inequality in Eq. (40) one finds

$$\begin{aligned} |\langle \mathcal{O}_{\mathcal{M}}\rangle| &\le n!(N_f N_c N_D)^{\frac{n}{2}} \langle \mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})^{\frac{1}{2}} \|\mathrm{S}_M\|^n\rangle_{\mathrm{G}} \\ &\le \frac{n!(N_f N_c N_D)^{\frac{n}{2}}}{m_0^n} \langle \mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})^{\frac{1}{2}}\rangle_{\mathrm{G}}, \end{aligned} \tag{A20}$$

which under the absolute-integrability condition discussed above is a finite bound that remains so as $\mathcal{V} \to \infty$.

Using also assumption (2.) in Sec. II one can prove the impossibility of vector flavor symmetry breaking. Starting from Eq. (27), and using Eqs. (33) and (35) and the assumptions on $\|\mathrm{S}_M\|$ and $\|\Delta D\|$, one finds

$$\begin{aligned} \left|\frac{\partial}{\partial\theta_a}\langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}}\rangle\Big|_{\boldsymbol{\theta}=0}\right| &\le \delta m \bar{C}_n \langle \mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})^{\frac{1}{2}} \|\mathrm{S}_M\|^{n+1}\|\Delta D\|\rangle_{\mathrm{G}} \\ &\le \delta m \frac{\bar{C}_n \Delta D_{\max}}{m_0^{n+1}} \langle \mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})^{\frac{1}{2}}\rangle_{\mathrm{G}}, \end{aligned} \tag{A21}$$

where $\bar{C}_n \equiv n n!(N_f N_c N_D)^{\frac{n}{2}}(\frac{1}{2N_f})^{\frac{1}{2}}$. Since the last factor is finite with a finite thermodynamic limit thanks to the assumption of absolute integrability, the desired result, Eq. (38), follows.

The case of finite transformations is obtained by a straightforward extension of the argument of Appendix A 1: since the right-hand side of Eq. (A21) is unchanged when replacing $\mathcal{M} \to \mathcal{M}^{\alpha\boldsymbol{\theta}}$, one has

$$\begin{aligned} |\langle \mathcal{O}^{\boldsymbol{\theta}}_{\mathcal{M}} - \mathcal{O}_{\mathcal{M}}\rangle| &\le \sum_a |\theta_a| \int_0^1 d\alpha |\langle \mathcal{C}^a \mathcal{O}_{\mathcal{M}^{\alpha\boldsymbol{\theta}}}\rangle| \\ &\le \delta m \frac{\bar{C}_n \Delta D_{\max}}{m_0^{n+1}} \langle \mathcal{K}_{\mathcal{M}}(\mathbf{x},\mathbf{y})^{\frac{1}{2}}\rangle_{\mathrm{G}} \sum_a |\theta_a|, \end{aligned} \tag{A22}$$

from which Eq. (A11) follows under the assumption of absolute integrability.

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