Radiative meson and glueball decays in the Witten-Sakai-Sugimoto model

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We calculate radiative decay rates of mesons and glueballs in the top-down holographic Witten-Sakai-Sugimoto model with finite quark masses. After assessing to what extent this model agrees or disagrees with experimental data, we present its predictions for so far undetermined decay channels. Contrary to widespread expectations, we obtain sizeable two-photon widths of scalar, tensor, and pseudoscalar glueballs, suggesting in particular that the observed two-photon rate of the glueball candidate $f_0(1710)$ is not too large to permit a glueball interpretation, but could be even much higher. We also discuss the so-called exotic scalar glueball, which in the Witten-Sakai-Sugimoto model is too broad to match either of the main glueball candidates $f_0(1500)$ and $f_0(1710)$ but might be of interest with regard to the alternative scenario of the so-called fragmented scalar glueball. Employing the exotic scalar glueball for the latter, much smaller two-photon rates are predicted for the ground-state glueball despite a larger total width; relatively large two-photon rates would then apply to the excited scalar glueball described by the predominantly dilatonic scalar glueball. In either case, the resulting contributions to the muon $g-2$ from hadronic light-by-light scattering involving glueball exchanges are small compared to other single meson exchanges, of the order of $\lesssim 10^{-12}$.

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I. INTRODUCTION AND SUMMARY

Glueballs, bound states of gluons without valence quarks, have been proposed as a consequence of QCD from the start [1–4], but it is still a widely open question how they manifest themselves in the hadron spectrum [5–9]. Lattice QCD [10–14], mostly in the quenched approximation, provides more or less clear predictions for the spectrum, with a lightest glueball being a scalar, followed by a tensor glueball with an important role as the lightest state associated with the pomeron [15], a pseudoscalar glueball participating in the manifestation of the $\mathrm{U}(1)_A$ anomaly responsible for the large mass of the η' meson [16], and towers of states with arbitrary integer spin as well as parity. However, it has turned out to be difficult to discriminate glueball states from bound states of quarks with the same quantum numbers with which they can mix, since the various available phenomenological models give strongly divergent pictures, in particular for the lightest glueballs. For the ground-state scalar glueball, the initially favored scenario that the isoscalar meson $f_0(1500)$ contains the most glue content while being strongly mixed with quarkonia [17–19] is contested by models which identify the $f_0(1710)$ as a glueball candidate [20–22] with more dominant glue content. The latter also appears favored by its larger production rate in supposedly gluon-rich radiative J/ψ decays [23], but there it was proposed that the glue content might rather be distributed over several scalars involving a new meson $f_0(1770)$ previously lumped together with the established $f_0(1710)$ [8,24,25].

In order to clarify the situation, dynamical information on decay patterns is required from first principles, which is difficult to extract from Euclidean lattice QCD. Analytical approaches always involve uncontrollable approximations, albeit recently interesting progress has been made using Schwinger-Dyson equations [26].

In this work we continue the analytical explorations made using gauge/gravity duality, which has been employed for studying glueball spectra in strongly coupled non-Abelian theories shortly after the discovery of the AdS/CFT correspondence [27–31], inspiring phenomenological "bottom-up" model building for glueball physics [32–38]. Of particular interest here is the top-down construction of a dual to low-energy QCD in the large-N_c limit from type-IIA string theory by Witten [39], where the glueball spectrum has been obtained in [40,41]. Sakai and Sugimoto [42,43] have extended this model by a D-brane construction introducing N_f chiral quarks in the 't Hooft limit $N_c \gg N_f$, which turns out to reproduce many features of low-energy QCD and chiral effective theory, not only qualitatively, but often semi-quantitatively, while having a minimal set of free parameters.

Glueball decay patterns were first studied in the Witten-Sakai-Sugimoto (WSS) model for the scalar glueball in [44] and revisited and extended in [45]. This involves a so-called exotic scalar glueball [40] for which it is unclear whether it should be identified with the ground-state glueball in QCD or instead be discarded together with the other states that more evidently do not relate to states in QCD.

Assuming that the ground-state scalar glueball corresponds to the predominantly dilatonic bulk metric fluctuations which do not involve polarizations in the extra Kaluza-Klein dimension employed for supersymmetry breaking, [46,47] found that the resulting decay pattern could match remarkably well the one of the $f_0(1710)$ meson when effects of finite quark masses are included (or $f_0(1770)$ when this is split off from a tetraquark $f_0(1710)$ [24]). Instead of the chiral suppression postulated for flavor asymmetries of scalar glueball decay [48], a nonchiral enhancement of decays into heavier pseudoscalars was obtained, which is correlated with a reduction of the $\eta\eta'$ decay mode [47]. This mechanism of flavor symmetry violation is absent for the tensor glueball, whose hadronic decays have been worked out also in [45]; hadronic decays of pseudoscalar and pseudovector glueballs have been studied in [49–51].

In the present paper, we revisit and extend the study of glueball decay patterns of [45–47] to also include radiative decays. As discussed already in [43], the WSS model naturally incorporates vector meson dominance (VMD), crucially involving an infinite tower of vector mesons. After assessing the predictions of the WSS model with regard to radiative decays of ordinary pseudoscalar and (axial) vector mesons, we analyze its corresponding results for glueballs.

Contrary to widespread expectations, the WSS model predicts that glueballs can have sizeable radiative decay widths in the keV range, exceeding even the claimed observation of two-photon rates for $f_0(1710)$ by the BESIII Collaboration [52], which was taken as evidence against its glueball nature.

In this context we also reconsider the exotic scalar glueball, which differs from the dilatonic one in that it has smaller couplings to vector mesons as well as photons, while having a total width in excess of the one of either $f_0(1500)$ or $f_0(1710)$, when its mass is suitably adjusted. As such it may instead be a candidate for the so-called fragmented scalar glueball proposed in [8,24,25], which is a wider resonance distributed over $f_0(1710)$, a novel $f_0(1770)$, $f_0(2020)$, and $f_0(2100)$, without showing up as an identifiable meson on its own.

In the case of the tensor glueball, where the WSS model is unequivocal in identifying the ground state, even though its mass also needs correction, we find again two-photon rates in the keV region, larger than the old predictions of Kada *et al.* [53], but comparable to those obtained by Cotanch and Williams [54] using VMD. (The latter have obtained even larger two-photon rates for the scalar glueball, which are an order of magnitude above the WSS results.)

The next heavier glueball, the pseudoscalar glueball, which plays an important rule in the realization of the $\mathrm{U}(1)_A$ anomaly [50], is also found to have two-photon rates in the keV region.

Because of their sizeable two-photon coupling in the WSS model, we consider also the effect the lightest three glueballs may have as single-meson contributions to hadronic light-by-light scattering, which is an important ingredient of the Standard Model prediction of the anomalous magnetic moment of the muon [55] $a_\mu = (g-2)_\mu/2$. With the dilatonic scalar glueball as ground state, we find results of $a_\mu^G = -(1\ldots1.6)\times10^{-12}$, and one order of magnitude smaller when the exotic scalar glueball is used instead with mass raised to the value of the fragmented glueball of [24]. With its larger mass and comparable two-photon rate, the tensor glueball is bound to contribute less than the dilatonic scalar glueball. The pseudoscalar glueball, which contributes with a different sign, yields $a_\mu^{G_{PS}} = +(0.2\ldots0.4)\times10^{-12}$ depending on its actual mass. All of these results are thus safely smaller than the current uncertainties in the hadronic light-by-light scattering contributions to a_μ.

II. THE WITTEN-SAKAI-SUGIMOTO MODEL AUGMENTED BY QUARK MASSES

The Witten-Sakai-Sugimoto (WSS) model [42,43] is constructed by placing a stack of N_f flavor probe D8 and $\overline{\mathrm{D8}}$-branes into the near-horizon double Wick rotated black D4-brane background proposed in [39] as a supergravity dual of four-dimensional $\mathrm{U}(N_c\to\infty)$ Yang-Mills (YM) theory at low energies, where supersymmetry and conformal symmetry are broken by compactifications. It thus serves as a model for the low-energy limit of large N_c QCD with $N_f \ll N_c$, corresponding to a quenched approximation when extrapolated to $N_f = N_c = 3$. The background geometry is given by the metric

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{U}{R_{\mathrm{D4}}}\right)^{3/2}[\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + f(U)\mathrm{d}\tau^2] \\ &\quad + \left(\frac{R_{\mathrm{D4}}}{U}\right)^{3/2}\left[\frac{\mathrm{d}U^2}{f(U)} + U^2\mathrm{d}\Omega_4^2\right], \\ e^\phi &= g_s\left(\frac{U}{R_{\mathrm{D4}}}\right)^{3/4}, \qquad F_4 = \mathrm{d}C_3 = \frac{(2\pi l_s)^3 N_c}{V_4}\epsilon_4, \\ f(U) &= 1 - \frac{U_{\mathrm{KK}}^3}{U^3}, \end{aligned} \tag{2.1}$$

with dilaton ϕ and Ramond-Ramond three-form field C_3, a solution of type IIA supergravity, whose bosonic part of the action reads

$$S_{\mathrm{grav}} = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g} \times \left[e^{-2\phi}(R+4(\nabla\phi)^2) - \frac{1}{2}|F_4|^2\right]. \tag{2.2}$$

The N_c D4-branes extend along the directions parametrized by the coordinates x^μ, $\mu = 0, 1, 2, 3$ and another spatial dimension with coordinate τ, while U corresponds to the radial (holographic) direction transverse to the D4-brane. The remaining four transverse coordinates span a unit S^4 with line element $\mathrm{d}\Omega_4^2$, volume form ϵ_4 and volume $V_4 = 8\pi^2/3$. The τ-direction is compactified to a supersymmetry breaking S^1, whose period is chosen as

$$\tau \simeq \tau + \delta\tau = \tau + 2\pi M_{\mathrm{KK}}^{-1}, \qquad M_{\mathrm{KK}} = \frac{3}{2}\frac{U_{\mathrm{KK}}^{1/2}}{R_{\mathrm{D4}}^{3/2}}, \tag{2.3}$$

to avoid a conical singularity at $U = U_{\mathrm{KK}}$. The radius R_{D4} is related to the string coupling g_s and the string length l_s through $R_{\mathrm{D4}}^3 = \pi g_s N_c l_s^3$, and the 't Hooft coupling of the dual four-dimensional Yang-Mills theory is given by

$$\lambda = g_{\mathrm{YM}}^2 N_c = \frac{g_5^2}{\delta\tau}N_c = 2\pi g_s l_s M_{\mathrm{KK}} N_c. \tag{2.4}$$

The flavor D8 and $\overline{\mathrm{D8}}$-branes extend along x^μ, U, and the S^4. They are placed antipodally on the τ-circle to join at U_{KK}. In adopting the probe approximation, i.e. $N_c \gg N_f$ for the N_f D8-branes, one can ignore backreactions from the D8-branes to the D4-brane background. The gauge fields on the D8-branes, which are dual to left and right chiral quark currents separated in the Kaluza-Klein (τ) direction, are governed at leading order by a Dirac-Born-Infeld (DBI) plus Chern-Simons (CS) action

$$S_{\mathrm{DBI}} = -T_8\int \mathrm{d}^9x e^{-\phi}\mathrm{Tr}\sqrt{-\det(g_{MN} + 2\pi\alpha' F_{MN})},$$
$$S_{\mathrm{CS}} = T_8\int_{D8} C \wedge \mathrm{Tr}\left[\exp\left\{\frac{F}{2\pi}\right\}\right]\sqrt{\hat{A}(\mathcal{R})}, \tag{2.5}$$

where $\hat{A}(\mathcal{R})$ is the so-called A-roof genus [56,57].

Considering only SO(5)-invariant excitations and restricting to terms quadratic in the field strength, the nine-dimensional DBI action can be reduced to a five-dimensional Yang-Mills theory with action [42,43][1]

$$S_{\mathrm{D8}}^{\mathrm{DBI}} = -\kappa\int \mathrm{d}^4x\mathrm{d}z\mathrm{Tr}\left[\frac{1}{2}K^{-1/3}F_{\mu\nu}^2 + M_{\mathrm{KK}}^2 K F_{\mu z}^2\right], \tag{2.6}$$

with

$$\kappa \equiv \frac{\lambda N_c}{216\pi^3}, \qquad K(z) \equiv 1 + z^2 = U^3/U_{\mathrm{KK}}^3. \tag{2.7}$$

To identify the four-dimensional meson fields, we make the ansatz

$$A_\mu(x^\mu, z) = \sum_{n=1}^{\infty} B_\mu^{(n)}(x^\mu)\psi_n(z),$$
$$A_z(x^\mu, z) = \sum_{n=0}^{\infty}\varphi^{(n)}(x^\mu)\phi_n(z) \tag{2.8}$$

for the five-dimensional gauge field using the complete sets $\{\psi_n(z)\}_{n\geq 1}$ and $\{\phi_n(z)\}_{n\geq 0}$ of normalizable functions of z with normalization conditions

$$\kappa\int \mathrm{d}z K^{-1/3}\psi_m\psi_n = \delta_{mn},$$
$$\kappa\int \mathrm{d}z K\phi_m\phi_n = \delta_{mn}, \tag{2.9}$$

satisfying the completeness relations

$$\kappa\sum_n K^{-1/3}\psi_n(z)\psi_n(z') = \delta(z - z'),$$
$$\kappa\sum_n K\phi_n(z)\phi_n(z') = \delta(z - z'). \tag{2.10}$$

With this ansatz, the fields $B_\mu^{(n)}$ and $\varphi^{(n)}$ have canonical kinetic terms; the eigenvalue equation

$$-K^{-1/3}\partial_z(K\partial_z\psi_n) = \lambda_n\psi_n, \tag{2.11}$$

which can be used to relate the two complete sets via $\phi_n(z) \propto \partial_z\psi_n(z)$ for $(n \geq 1)$, yields a mass term for $B_\mu^{(n)}$. The remaining massless mode is given by $\phi_0(z) = 1/(\sqrt{\pi\kappa}M_{\mathrm{KK}}K(z))$.

Inserting the separation ansatz (2.8) into the DBI action (2.6) and integrating over z, we obtain

$$S_{\mathrm{D8}}^{\mathrm{DBI}} = -\mathrm{Tr}\int \mathrm{d}^4x\left[(\partial_\mu\varphi^{(0)})^2 + \sum_{n=1}^{\infty}\left(\frac{1}{2}(\partial_\mu B_\nu^{(n)} - \partial_\nu B_\mu^{(n)})^2 + m_n^2(B_\mu^{(n)} - m_n^{-1}\partial_\mu\varphi^{(n)})^2\right)\right] + (\text{interaction terms}). \tag{2.12}$$

The scalar fields $\varphi^{(n)}$ with $(n \geq 1)$ can be absorbed by the fields $B_\mu^{(n)}$, which are interpreted as (axial) vector meson fields, with masses $m_n = \sqrt{\lambda_n}M_{\mathrm{KK}}$ determined by the eigenvalue equation for the normalizable modes (2.11).

[1]Note that in (2.6) one uses the Minkowski metric $\eta_{\mu\nu}$, in the mostly plus convention, to contract the four-dimensional space-time indices.

The lightest vector mesons, identified with the rho and omega mesons, have $m_\rho = m_1 = \sqrt{0.669314}M_{\rm KK}$, with the traditional value [42,43] of $M_{\rm KK} = 949$ MeV corresponding to $m_\rho = 776.4$ MeV.

The remaining field $\varphi^{(0)}$ is identified as the multiplet of massless pion fields produced by chiral symmetry breaking, which is realized geometrically by D8 and $\overline{\rm D8}$-branes joining at $z = 0$, with the $\mathrm{U}(N_f)$-valued Goldstone boson field given by the holonomy

$$U(x) = e^{i\Pi^a(x)\lambda^a/f_\pi} = \mathrm{P}\exp i\int_{-\infty}^{\infty} \mathrm{d}z A_z(z,x), \qquad (2.13)$$

where $\lambda^a = 2T^a$ are Gell-Mann matrices including $\lambda^0 = \sqrt{2/N_f}\mathbf{1}$. For $N_f = 3$ we have

$$\Pi(x) \equiv \Pi^a(x)T^a = \frac{1}{2}\begin{pmatrix} \pi^0 + \eta^8/\sqrt{3} + \eta^0\sqrt{2/3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta^8/\sqrt{3} + \eta^0\sqrt{2/3} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\eta^8/\sqrt{3} + \eta^0\sqrt{2/3} \end{pmatrix}. \qquad (2.14)$$

The pion decay constant is determined by

$$f_\pi^2 = \frac{\lambda N_c M_{\rm KK}^2}{54\pi^4}; \qquad (2.15)$$

with the choice $f_\pi \approx 92.4$ MeV one obtains $\lambda \approx 16.63$. Following [45], we shall also consider the smaller value $\lambda \approx 12.55$ obtained by matching the large-N_c lattice result for the string tension obtained in Ref. [58] (resulting in $f_\pi \approx 80.3$ MeV). A smaller 't Hooft coupling has also been argued for in Ref. [59] from studies of the spectrum of higher-spin mesons in the WSS model. The downward variation of $\lambda \approx 16.63\ldots 12.55$ will thus be used as an estimate of the variability of the predictions of this model.

A. Pseudoscalar masses

In the WSS model, the $\mathrm{U}(1)_A$ flavor symmetry is broken by an anomalous contribution of order $1/N_c$ due to the C_1 Ramond-Ramond field, which gives rise to a Witten-Veneziano [60,61] mass term for the singlet η_0 pseudoscalar with [42]

$$m_0^2 = \frac{2N_f}{f_\pi^2}\chi_g = \frac{N_f}{27\pi^2 N_c}\lambda^2 M_{\rm KK}^2, \qquad (2.16)$$

where χ_g is the topological susceptibility.

For $N_f = N_c = 3$, one has $m_0 = 967\ldots 730$ MeV for $\lambda = 16.63\ldots 12.55$, which is indeed a phenomenologically interesting ballpark when finite quark masses are added to the model by the addition of an effective Lagrangian

$$\mathcal{L}_m^{\mathcal{M}} \propto \mathrm{Tr}(\mathcal{M}U(x) + \mathrm{H.c.}),$$
$$\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s). \qquad (2.17)$$

This deformation can be generated by either worldsheet instantons [62,63] or nonnormalizable modes of bifundamental fields corresponding to open-string tachyons [64–67]. Assuming for simplicity isospin symmetry, $m_u = m_d = \hat{m}$, this leads to masses [47]

$$m_{\eta,\eta'}^2 = \frac{1}{2}m_0^2 + m_K^2 \mp \sqrt{\frac{m_0^4}{4} - \frac{1}{3}m_0^2(m_K^2 - m_\pi^2) + (m_K^2 - m_\pi^2)^2} \qquad (2.18)$$

for the mass eigenstates

$$\eta = \eta_8\cos\theta_P - \eta_0\sin\theta_P,$$
$$\eta' = \eta_8\sin\theta_P + \eta_0\cos\theta_P, \qquad (2.19)$$

with mixing angle

$$\theta_P = \frac{1}{2}\arctan\frac{2\sqrt{2}}{1 - \frac{3}{2}m_0^2/(m_K^2 - m_\pi^2)}. \qquad (2.20)$$

Using $m_\pi^2 = m_{\pi_0}^2 \approx (135\ \mathrm{MeV})^2$ and

$$m_K^2 = \frac{1}{2}(m_{K_\pm}^2 + m_{K_0}^2) - \frac{1}{2}(m_{\pi_\pm}^2 - m_{\pi_0}^2) \approx (495\ \mathrm{MeV})^2 \qquad (2.21)$$

as isospin symmetric parameters, the WSS result $m_0 \approx 967\ldots 730$ MeV for $\lambda = 16.63\ldots 12.55$ leads to $\theta_P \approx -14°\ldots -24°$ and $m_\eta \approx 520\ldots 470$, $m_{\eta'} \approx 1080\ldots 890$ MeV. In the following we shall consider this range of mixing angles in conjunction with the variation of λ, but we shall fix m_η and $m_{\eta'}$ to their experimental values when evaluating phase space integrals. In the radiative decay rates considered below, the explicit quark masses will not modify the (chiral) results for the couplings; they only appear in phase space factors.

B. Hadronic vector and axial vector meson decays

Vertices for the hadronic decays of vector and axial vector meson involving pseudoscalar mesons are contained in the second term of the DBI action (2.6). For the ρ meson, this contains the term (with indices restricted to the first two quark flavors)

$$L_{\rho\pi\pi} = -g_{\rho\pi\pi}\varepsilon_{abc}(\partial_\mu \pi^a)\rho^{b\mu}\pi^c,$$
$$g_{\rho\pi\pi} = \int \mathrm{d}z \frac{1}{\pi K}\psi_1 = 33.98\lambda^{-\frac{1}{2}}N_c^{-\frac{1}{2}}, \qquad (2.22)$$

yielding $\Gamma_{\rho\to\pi\pi} = 98.0\ldots130$ MeV for $\lambda = 16.63\ldots12.55$, which somewhat underestimates the experimental result of $\approx$150 MeV.

There is also a vertex involving one vector, one axial vector, and one pseudoscalar meson, which for the ground-state isotriplet mesons reads

$$L_{a_1\rho\pi} = g_{a_1\rho\pi}\varepsilon_{abc}a^a_\mu\rho^{b\mu}\pi^c,$$
$$g_{a_1\rho\pi} = 2M_{\mathrm{KK}}\sqrt{\frac{\kappa}{\pi}}\int \mathrm{d}z\psi_2'\psi_1 = -34.43\lambda^{-\frac{1}{2}}N_c^{-\frac{1}{2}}M_{\mathrm{KK}}. \qquad (2.23)$$

In the WSS model, the predicted mass of the a_1 meson, 1186.5 MeV, is rather close to the experimental result [68] of 1230(40) MeV. The predicted width for $a_1 \to \rho\pi$ (already studied in [43]) is 425…563 MeV, which is within the experimental result for the total width of 250…600 MeV [average value 420(35) MeV], but according to [69] only 60% of the three-pion decays are due to S-wave $\rho\pi$ decays, whereas the latter saturate the hadronic decays in the WSS model.

For the light quark flavors, these results for the decay rates of ρ and a_1 seem to indicate that the WSS model is working quite well. When the mass of the strange quark is included, a shortcoming of the model, which is shared by many bottom-up holographic QCD models (see e.g. [70]), is that the ϕ meson remains degenerate with ρ and ω. In the following we shall nevertheless also consider K^* and ϕ mesons by simply raising their masses in the resulting phase space factors while keeping their vertices such as $g_{K^*K\pi} = g_{\phi KK} = g_{\rho\pi\pi}$ unchanged. The resulting widths, $\Gamma(K^* \to K\pi) = 28\ldots37$ MeV and $\Gamma(\phi \to K\bar{K}) = 2.12\ldots2.82$ MeV, are between 40% and 20% too small. These deviations are at least not dramatically larger than the one for the ρ width, which amounts to 33%…12%; all appear to remain in the range to be expected for a large-N approach.

III. RADIATIVE MESON DECAYS

Before considering radiative decays of the experimentally elusive glueballs, we shall evaluate the predictions of the WSS model with nonzero quark masses for radiative decay widths of regular mesons and compare with experimental data as far as available. As discussed extensively in the second paper of Sakai and Sugimoto [43], holographic QCD models naturally provide a realization of vector meson dominance [71–74] involving an infinite tower of vector mesons. There it was already observed that the chiral WSS model yields a result for $\Gamma(\omega \to \pi^0\gamma)$ which is roughly consistent with the experimental value. In the following we shall recapitulate the results of [43] and extend them to the WSS model including quark masses and the Witten-Veneziano mass term.

A. Vector meson dominance

According to the holographic principle, non-normalizable modes are interpreted as external sources. This permits one to study electromagnetic interactions to leading order by setting asymptotic values of the gauge field A_μ on the D8-branes according to [43]

$$\lim_{z\to\pm\infty} A_\mu(x,z) = A_{L,R\mu}(x) = eQA^{\mathrm{em}}_\mu(x), \qquad (3.1)$$

where e is the electromagnetic coupling constant and Q is the electric charge matrix, given as

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad (3.2)$$

for the $N_f = 3$ case. The ansatz (2.8) changes to

$$A_\mu(x^\mu,z) = A_{L\mu}(x^\mu)\psi_+(z) + A_{R\mu}(x^\mu)\psi_-(z) + \sum_{n=1}^{\infty} v^n_\mu(x^\mu)\psi_n(z), \qquad (3.3)$$

with the functions $\psi_\pm(z)$ defined as

$$\psi_\pm(z) \equiv \frac{1}{2}(1 \pm \psi_0(z)), \qquad \psi_0(z) \equiv \frac{2}{\pi}\arctan z. \qquad (3.4)$$

They satisfy (2.11) as non-normalizable zero modes, because $\partial_z\psi_\pm(z) \propto \phi_0(z) \propto 1/K(z)$.

To distinguish between vector and axial-vector fields we introduce the notation

$$\mathcal{V}_\mu \equiv \frac{1}{2}(A_{L\mu} + A_{R\mu}), \qquad \mathcal{A}_\mu \equiv \frac{1}{2}(A_{L\mu} - A_{R\mu}),$$
$$v^n_\mu \equiv B^{(2n-1)}_\mu, \qquad a^n_\mu \equiv B^{(2n)}_\mu, \qquad (3.5)$$

so that

$$A_\mu(x^\mu,z) = \mathcal{V}_\mu(x^\mu) + \mathcal{A}_\mu(x^\mu)\psi_0(z) + \sum_{n=1}^{\infty} v^n_\mu(x^\mu)\psi_{2n-1}(z) + \sum_{n=1}^{\infty} a^n_\mu(x^\mu)\psi_{2n}(z). \qquad (3.6)$$

The first term in (2.6) can then be expanded as

$$\frac{\kappa}{2}\int \mathrm{d}zK^{-1/3}F_{\mu\nu}^2 = \frac{a_{\mathcal{V}\mathcal{V}}}{2}\mathrm{tr}(\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu)^2 + \frac{a_{\mathcal{A}\mathcal{A}}}{2}\mathrm{tr}(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu)^2 + \frac{1}{2}\mathrm{tr}(\partial_\mu v_\nu^n - \partial_\nu v_\mu^n)^2 + \frac{1}{2}\mathrm{tr}(\partial_\mu a_\nu^n - \partial_\nu a_\mu^n)^2$$
$$+ a_{\mathcal{V}v^n}\mathrm{tr}((\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu)(\partial_\mu v_\nu^n - \partial_\nu v_\mu^n)) + a_{\mathcal{A}a^n}((\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu)(\partial_\mu a_\nu^n - \partial_\nu a_\mu^n))$$
$$+ (\text{interaction terms}), \tag{3.7}$$

with coupling constants

$$a_{\mathcal{V}v^n} = \kappa\int dzK^{-1/3}\psi_{2n-1}, \qquad a_{\mathcal{V}\mathcal{V}} = \kappa\int dzK^{-1/3},$$
$$a_{\mathcal{A}a^n} = \kappa\int dzK^{-1/3}\psi_{2n}\psi_0, \qquad a_{\mathcal{A}\mathcal{A}} = \kappa\int dzK^{-1/3}\psi_0^2, \tag{3.8}$$

mixing the photon field $\mathcal{V}$ with every vector meson v^n. The coefficients $a_{\mathcal{V}\mathcal{V}}$ and $a_{\mathcal{A}\mathcal{A}}$ are divergent, since the external fields correspond to non-normalizable modes in the radial direction, and need to be renormalized to canonical values. The photon field $\mathcal{V}$ does not appear in the interaction terms of this model and can only couple via the mixing (3.8), fully realizing VMD. Alternatively, it is possible to perform a field redefinition to diagonalize the action and to get rid of the mixing terms, thus producing new interaction terms coupling mesons to photons.

B. Radiative decays of pseudoscalars and vector mesons

The relevant vertices for radiative decays of pseudoscalars and (axial) vector mesons come from the Chern-Simons term

$$S_{CS} \supset T_8\int \mathrm{tr}(\exp(2\pi\alpha' F_2 + B_2)\wedge C_3)$$
$$\supset \frac{N_c}{96\pi^2}\epsilon^{\mu\nu\rho\sigma z}\int \mathrm{tr}(3A_z F_{\mu\nu}F_{\rho\sigma} - 4A_\mu\partial_z A_\nu F_{\rho\sigma}), \tag{3.9}$$

where we have used partial integration.

Inserting the mode expansion (3.6) and integrating over the radial coordinate we obtain for the interaction term involving two vectors and one pseudoscalar

$$\mathcal{L}_{\Pi v^m v^n} = \frac{N_c}{4\pi^2 f_\pi}c_{v^n v^m}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(\Pi\partial_\mu v_\nu^n\partial_\rho v_\sigma^m), \tag{3.10}$$

with coupling constants

$$c_{v^n v^m} = \frac{1}{\pi}\int \mathrm{d}zK^{-1}\psi_{2n-1}\psi_{2m-1} = \left\{\frac{1350.83}{\lambda N_c},\ldots\right\} \tag{3.11}$$

as studied in [43], where numerical results for the coefficients beyond $c_{v^1v^1}$ given above can be found.

1. Vector meson 1γ-decays

Using VMD, we can calculate the interaction term for the radiative decay of a vector meson into a pseudoscalar and one photon as

$$\mathcal{L}_{\Pi\mathcal{V}v^n} = \frac{N_c}{4\pi^2 f_\pi}c_{\mathcal{V}v^n}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(\Pi\partial_\mu v_\nu^n\partial_\rho\mathcal{V}_\sigma + \Pi\partial_\mu\mathcal{V}_\nu\partial_\rho v_\sigma^n), \tag{3.12}$$

with coupling

$$c_{\mathcal{V}v^n} = \sum_m c_{v^n v^m}a_{\mathcal{V}v^m} = \frac{1}{\pi}\int \mathrm{d}zK^{-1}\psi_{2n-1}$$
$$= \{33.9839,\ldots\}(N_c\lambda)^{-1/2}, \tag{3.13}$$

where we have used the completeness relation (2.10) to eliminate the summed-over modes.

Performing the polarization sums we get

$$|\mathcal{M}_{(v^n\to\Pi\mathcal{V})}|^2 = \sum_{(v^n)}\sum_{(\mathcal{V})}\frac{1}{3}\epsilon_\mu^{(v^n)}\epsilon_\nu^{(v^n)*}\epsilon_\rho^{(\mathcal{V})}\epsilon_\sigma^{(\mathcal{V})*}$$
$$\times\mathcal{M}^{\mu\rho}_{(\Pi v^n\mathcal{V})}\mathcal{M}^{\nu\sigma*}_{(\Pi v^n\mathcal{V})}$$
$$= \frac{c_{\mathcal{V}v^n}^2 e^2 N_c^2}{96\pi^4 f_\pi^2}(\mathrm{tr}(T_\Pi T_{v^n}Q) + \mathrm{tr}(T_\Pi Q T_{v^n}))^2$$
$$\times(m_\Pi^2 - m_{v^n}^2)^2.$$

The partial width then reads

$$\Gamma_{v^n\to\Pi\gamma} = \frac{1}{8\pi}|\mathcal{M}_{(v^n\to\Pi\mathcal{V})}|^2\frac{|\mathbf{p}_v|}{m_v^2}. \tag{3.14}$$

2. Pseudoscalar meson 2γ-decays

Employing VMD a second time, we can derive the interaction term for a decay of a pseudoscalar meson in two photons

TABLE I. Results for various radiative decay widths of pseudoscalar and vector mesons involving vector and pseudoscalar mesons, with 't Hooft coupling $\lambda = 16.63\ldots12.55$ ($\lambda = 16.63$ is the traditional [42,43] value matching $f_\pi = 92.4$ MeV; $\lambda = 12.55$ an alternative choice matching the large-N string tension at the expense of f_π). For nonmonotonic dependence on λ intermediate extremal values are also given. Ideal mixing is assumed for ω and ϕ, fixing the WSS result for $\phi \to \pi^0\gamma$ to zero. Experimental results are from the PDG [68] except for the π^0 width, which is from [75].

	$\Gamma^{\mathrm{exp.}}$[keV]	Γ^{WSS}[keV]
$\pi^0 \to 2\gamma$	0.00780(12)	0.00773…0.0102
$\eta \to 2\gamma$	0.515(18)	0.480…0.978
$\eta' \to 2\gamma$	4.34(14)	5.72…5.87…5.75
$\rho^0 \to \pi^0\gamma$	70(12)	56.2…98.6
$\rho^\pm \to \pi^\pm\gamma$	68(7)	56.2…98.6
$\rho^0 \to \eta\gamma$	45(3)	40.3…90.5
$\omega \to \pi^0\gamma$	725(34)	521…915
$\omega \to \eta\gamma$	3.9(4)	4.87…10.9
$\eta' \to \rho^0\gamma$	$55.4(1.9)^{\mathrm{fit}}, 68(7)^{\mathrm{av.}}$	54.1…59.2…58.5
$\eta' \to \omega\gamma$	$4.74(20)^{\mathrm{fit}}, 5.8(7)^{\mathrm{av.}}$	5.37…5.89…5.81
$\phi \to \pi^0\gamma$	5.6(2)	0
$\phi \to \eta\gamma$	55.3(1.2)	84.7…92.8…91.6
$\phi \to \eta'\gamma$	0.264(10)	0.525…1.18
$K^{*0} \to K^0\gamma$	116(10)	124…218
$K^{*\pm} \to K^\pm\gamma$	50(5)	31.0…54.5

$$\mathcal{L}_{\Pi\mathcal{V}\mathcal{V}} = -\frac{N_c}{4\pi^2 f_\pi} c_{\mathcal{V}\mathcal{V}} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(\Pi \partial_\mu \mathcal{V}_\nu \partial_\rho \mathcal{V}_\sigma), \tag{3.15}$$

where the sum over the entire tower of vector mesons yields

$$c_{\mathcal{V}\mathcal{V}} = \sum_m c_{\mathcal{V}v^m} a_{\mathcal{V}v^m} = \frac{1}{\pi}\int \mathrm{d}z K^{-1} = 1, \tag{3.16}$$

leading to the standard result

$$\Gamma_{\Pi\to\gamma\gamma} = \frac{1}{8\pi} |\mathcal{M}_{(\Pi\to\mathcal{V}\mathcal{V})}|^2 \frac{|\mathbf{p}_\gamma|}{m_\Pi^2}\frac{1}{2} \tag{3.17}$$

with

$$|\mathcal{M}_{(\Pi\to\mathcal{V}\mathcal{V})}|^2 = \frac{e^4 N_c^2}{4\pi^4 f_\pi^2} (\mathrm{tr}(T_\Pi Q^2))^2 m_\Pi^4. \tag{3.18}$$

The numeric results for the various radiative decays involving one pseudoscalar and two vector particles are summarized in Table I for $\lambda = 16.63\ldots12.55$. As mentioned above, $\lambda = 16.63$ is the traditional [42,43] value matching $f_\pi = 92.4$ MeV, whereas $\lambda = 12.55$ is an alternative choice matching the large-N string tension at the expense of f_π. The decay rate for π^0 is therefore close to the experimental value only for the first value of λ, but the partial widths of the decays ρ and ω into $\pi\gamma$ are reproduced by an intermediate value of λ.

In processes involving η and η', we have used the pseudoscalar mixing angle following from (2.20), which varies as $\theta_P \approx -14°\ldots-24°$ when $\lambda = 16.63\ldots12.55$, which enters the flavor matrix T_Π in (3.18). Here the dependence on λ is nonmonotonic, because also f_π in the prefactor depends on λ; Table I also gives the extremal values attained at intermediate values of λ.

The vector couplings in the WSS model augmented by quark masses according to (2.17) are flavor-symmetric, but we distinguish ϕ and ω mesons through their experimental masses. The undetermined mixing of ϕ and ω could be fixed by fitting for example the small ratio of the widths for their decays into $\pi^0\gamma$, 5.6/725, which yields a mixing angle close to ideal mixing, $\theta_V = \theta_V^{\mathrm{ideal}} + 3.32°$, as in [76]. However, here and in the following we shall assume completely ideal mixing for simplicity, which eliminates $\phi \to \pi\gamma$ but does not change the other partial widths of ϕ significantly. This gives generally good results for decays involving ω, but larger discrepancies with experiment for ϕ mesons irrespective of the precise value of θ_V.[2] Note that the standard value of $M_{\mathrm{KK}} = 949$ MeV, which we are using, is chosen such that the ρ mass is reproduced, which is rather close to the mass of the ω meson, but less suitable for the ϕ meson.

C. Radiative axial-vector decays

From the five-dimensional CS term (3.9) we can also extract a term including two vector mesons and one axial-vector meson

$$\mathcal{L}_{v^m v^n a^p} = -\frac{N_c}{4\pi^2} d_{v^m v^n a^p} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(v_\mu^m a_\nu^p \partial_\rho v_\sigma^n), \tag{3.19}$$

with

$$d_{v^m v^n a^p} = \int \mathrm{d}z \psi_{2m-1} \psi'_{2n-1} \psi_{2p}, \tag{3.20}$$

where we again made use of partial integration.

As noted already in [43] and observed before in other holographic models [77–79] as well as in the hidden local symmetry approach of [80], the vertex for the decay of an axial vector meson into a pseudoscalar and a photon, which would have to come from the DBI part of the action, vanishes,[3] even though there is a nonvanishing vertex for $a_1^\pm \to \pi^\pm\rho^0$, see (2.23). But the corresponding coupling for an on-shell photon is obtained by replacing ψ_1 therein by a unity, leading to

[2]With a $\phi - \omega$ mixing angle of 3.32° above ideal mixing [76], we would have $\Gamma^{\mathrm{WSS}}(\phi \to \pi\gamma) = 4\ldots7$ keV, consistent with experiment; the result for $\omega \to \eta\gamma$ would be somewhat closer to the experimental value, but the one for $\eta' \to \omega\gamma$ further off.

[3]In the hidden local symmetry approach, $a_1 \to \pi\gamma$ has been included by adding higher-derivative terms to the action [81].

$$g_{a_1\pi\mathcal{V}} = 2M_{\mathrm{KK}}\sqrt{\frac{\kappa}{\pi}}\int \mathrm{d}z\psi_2' = 0, \tag{3.21}$$

implying a cancellation between the contribution from the lowest vector meson and the remaining tower. Indeed, the experimental result for $a_1^{\pm} \to \pi^{\pm}\gamma$ is much smaller than expected from naive VMD [82].

1. Axial-vector 1γ decays

Employing VMD once we obtain for the interaction between one axial vector meson, one vector meson and one photon

$$\mathcal{L}_{\mathcal{V}v^n a^p} = -\frac{N_c}{4\pi^2} d_{\mathcal{V}v^n a^p}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(v^n_\mu a^p_\nu \partial_\rho \mathcal{V}_\sigma), \tag{3.22}$$

with

$$d_{\mathcal{V}v^n a^p} = \int \mathrm{d}z\psi'_{2n-1}\psi_{2p} = \{-2497.14, \ldots\}N_c^{-1}\lambda^{-1}, \tag{3.23}$$

where we had to sum over the radial mode without the derivative to get a nonvanishing result since the bulk-to-boundary propagator associated to an on-shell photon is constant. The amplitudes for the decay $a \to v\mathcal{V}$ for the different combinations of polarizations read

$$|\mathcal{M}^{a^p\to v^n\mathcal{V}}_{-101}| = \frac{d_{\mathcal{V}v^n a^p}(m^2_{a^p} - m^2_{v^n})N_c}{8m_{v^n}\pi^2}\mathrm{tr}(eQT_{a^p}T_{v^n}),$$
$$|\mathcal{M}^{a^p\to v^n\mathcal{V}}_{-110}| = \frac{d_{\mathcal{V}v^n a^p}(m^2_{a^p} - m^2_{v^n})N_c}{8m_{a^p}\pi^2}\mathrm{tr}(eQT_{a^p}T_{v^n}), \tag{3.24}$$

which yields

$$\begin{aligned}|\mathcal{M}_{a^p\to v^n\mathcal{V}}|^2 &= \frac{1}{3}(2|\mathcal{M}^{a^p\to v^n\mathcal{V}}_{-101}|^2 + 2|\mathcal{M}^{a^p\to v^n\mathcal{V}}_{-110}|^2) \\ &= \frac{d^2_{\mathcal{V}v^n a^p}(m^2_{a^p} - m^2_{v^n})^2(m^2_{a^p} + m^2_{v^n})N_c^2}{96\pi^4 m^2_{a^p}m^2_{v^n}} \\ &\quad\times(\mathrm{tr}(eQT_{a^p}T_{v^n}))^2.\end{aligned} \tag{3.25}$$

The decay width is given by

$$\Gamma_{a^p\to v^n\gamma} = \frac{1}{8\pi}|\mathcal{M}_{a^p\to v^n\mathcal{V}}|^2\frac{|\mathbf{p}_\mathcal{V}|}{m^2_{a^p}}, \tag{3.26}$$

and the numerical results are listed in Table II.

The PDG [68] gives experimental results only for the f_1 mesons, which in the WSS model have the same mass as the a_1 meson. Besides extrapolating to their experimental masses we consider also two possible values (motivated below) for the mixing angle for the f_1 and f_1' mesons using the convention

TABLE II. Radiative axial-vector meson decay with $\lambda = 16.63\ldots12.55$ and two values of the f_1 mixing angle $\theta_f = 20.4°|26.4°$. Experimental values are from the PDG [68] with the exception of the lower values for $f_1(1285)\to\rho\gamma$, which are from VES [83]; Zanke *et al.* [84] propose here an experimental average 950(280) keV.

	Γ^{exp}[keV]	Γ^{WSS}[keV]
$a_1(1260)\to\rho\gamma$		28.9…50.8
$a_1(1260)\to\omega\gamma$		247…434
$f_1(1285)\to\rho\gamma$	1380(300)…640(240)	295…518|270…473
$f_1(1285)\to\omega\gamma$		31.3…54.9|28.6…50.2
$f_1(1285)\to\phi\gamma$	17(7)	2.44…4.29|3.97…6.98
$f_1(1420)\to\rho\gamma$		73.0…128|119…209
$f_1(1420)\to\omega\gamma$		7.80…13.7|12.7…22.3
$f_1(1420)\to\phi\gamma$	164(55)	52.9…92.9|48.3…84.8

$$|f_1(1285)\rangle = \cos\theta_f|\bar{n}n\rangle - \sin\theta_f|\bar{s}s\rangle,$$
$$|f_1(1420)\rangle = \sin\theta_f|\bar{n}n\rangle + \cos\theta_f|\bar{s}s\rangle \tag{3.27}$$

so that ideal mixing corresponds to $\theta_f = 0$.

In Table II, the ϕ-ω mixing is again assumed to be ideal. A value a bit above ideal mixing increases somewhat the branching ratio of $\phi\gamma$ over $\omega\gamma$ for $f_1(1285)$, while decreasing it for $f_1(1420)$.

2. Axial-vector 2γ decays

As mentioned above, the radial derivative of the bulk-to-boundary propagator for a photon vanishes for on-shell photons, which implies that in accordance with the Landau-Yang theorem at least one photon in the two-photon decay of an axial vector meson has to be off-shell. Denoting the virtual photon by v^* we have

$$d_{\mathcal{V}v^*a^p} = \int \mathrm{d}z\mathcal{J}'\psi_{a^p}, \tag{3.28}$$

where we have introduced the (off-shell) bulk-to-boundary propagator $\mathcal{J}$ defined by

$$(1+z^2)^{1/3}\partial_z[(1+z^2)\partial_z\mathcal{J}] = \frac{Q^2}{M^2_{\mathrm{KK}}}\mathcal{J}. \tag{3.29}$$

Since we are only interested in the low Q regime we make the ansatz

$$\mathcal{J}(Q,z) = 1 + \frac{Q^2}{M^2_{\mathrm{KK}}}\alpha(z) + \mathcal{O}(Q^4) \tag{3.30}$$

satisfying

$$(1+z^2)^{1/3}\partial_z[(1+z^2)\partial_z\alpha] = 1. \tag{3.31}$$

With the solution

$$\partial_z\alpha = \frac{z}{(1+z^2)}\,{}_2F_1\left(\frac{1}{3},\frac{1}{2},\frac{3}{2},-z^2\right) \tag{3.32}$$

we obtain for the relevant coupling constant

$$\begin{aligned} d_{\mathcal{V}v^*a^p} &= \frac{Q^2}{M_{\mathrm{KK}}^2}\int \mathrm{d}z\alpha'\psi_{a^p} + \mathcal{O}(Q^4) \\ &= \frac{Q^2}{M_{\mathrm{KK}}^2} c_{\mathcal{V}v^*a^p} + \mathcal{O}(Q^4) \end{aligned} \tag{3.33}$$

with

$$c_{\mathcal{V}v^*a^p} = 101.309 N_c^{-1/2}\lambda^{-1/2}. \tag{3.34}$$

The decay widths then read

$$\begin{aligned} \Gamma(f(1285)\to\gamma_L^*\gamma_T) &= \frac{2}{3}\left(\frac{c_{\mathcal{V}v^*a} m_a^2 N_c}{8\pi^2 M_{\mathrm{KK}}^2}\right)^2 \frac{1}{8\pi}\frac{|\mathbf{p}|}{m_a^2} \\ &\quad\times\left(\frac{5e^2}{18}\cos\theta_f - \frac{e^2}{9\sqrt{2}}\sin\theta_f\right)^2 Q^2 \\ &\quad + \mathcal{O}(Q^4), \\ \Gamma(f(1285)\to\gamma_T^*\gamma_T) &= \mathcal{O}(Q^6). \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} \Gamma(f(1420)\to\gamma_L^*\gamma_T) &= \frac{2}{3}\left(\frac{c_{\mathcal{V}v^*a} m_a^2 N_c}{8\pi^2 M_{\mathrm{KK}}^2}\right)^2 \frac{1}{8\pi}\frac{|\mathbf{p}|}{m_a^2}\left(\frac{5e^2}{18}\sin\theta_f\right. \\ &\quad\left. + \frac{e^2}{9\sqrt{2}}\cos\theta_f\right)^2 Q^2 + \mathcal{O}(Q^4). \end{aligned} \tag{3.36}$$

In the literature one usually finds the values for the so-called equivalent photon rate

$$\tilde{\Gamma}_{\gamma\gamma} = \lim_{Q^2\to 0}\frac{m_a^2}{Q^2}\frac{1}{2}\Gamma(a\to\gamma_L^*\gamma_T), \tag{3.37}$$

which are listed in Table III.

The mixing angle is inferred from

$$\tan^2\left(\theta_f - \arctan\frac{\sqrt{2}}{5}\right) = \left(\frac{m_{f_1}}{m_{f_1'}}\right)^{1+\xi}\frac{\tilde{\Gamma}_{\gamma\gamma}^{f_1'\,\mathrm{exp}}}{\tilde{\Gamma}_{\gamma\gamma}^{f_1\,\mathrm{exp}}}, \tag{3.38}$$

where the usual assumption of $\xi = 0$ leads to $\theta_f = 26.4°$, corresponding to the central value of $\theta_A = 62(5)°$ in [84]. However, in the WSS the coupling $d_{\mathcal{V}v^*a^p}$ is proportional to $1/M_{\mathrm{KK}}^2$, which leads to a scaling of $\tilde{\Gamma}_{\gamma\gamma}$ with four additional powers of m_a, i.e. $\xi = 4$, resulting in $\theta_f = 20.4°$.

In Tables II and III we consider two possible extrapolations to axial vector mesons with realistic masses. In the first we keep the parameters of the theory unchanged in the expressions for the couplings and use the measured masses only in kinematical factors, which leads to $\xi = 4$ and $\theta_f = 20.4°$; in the second we rescale M_{KK} proportional to $m_a^{\mathrm{exp}}/m_a^{\mathrm{WSS}}$ such that $\xi = 0$ and $\theta_f = 26.4°$.

TABLE III. Equivalent photon rates of axial vector mesons for two values of the f_1 mixing angle $\theta_f = 20.4°|26.4°$ [in the latter case with M_{KK} rescaled such that m_a is raised to the experimental value which reduces ξ in (3.38) to zero]; the range denoted by dots corresponds again to $\lambda = 16.63\ldots12.55$, where only the first value is matching the axial anomaly exactly. Experimental values from L3 [85,86], see also [84].

	$\tilde{\Gamma}_{\gamma\gamma}^{\mathrm{exp}}$[keV]	$\tilde{\Gamma}_{\gamma\gamma}^{\mathrm{WSS}}$[keV]
$a_1(1260)$		1.60…2.12|1.39…1.85
$f_1(1285)$	3.5(8)	3.84…5.09|2.39…3.17
$f_1(1420)$	3.2(9)	3.50…4.64|2.19…2.90

While the predictions for the equivalent photon rate for the f_1 mesons (shown in Table III) agree well with the experimental result for the standard choice of $\lambda = 16.63$ and $\theta_f = 20.4°$, the 1γ decay rates are significantly underestimated. In contrast to the radiative decays of vector mesons, lowering λ does not increase the rates sufficiently to cover the experimental results. Unfortunately no experimental results are available for isotriplet axial vector mesons, where the WSS model is generally performing best.

IV. GLUEBALLS IN THE WITTEN-SAKAI-SUGIMOTO MODEL

Glueballs are realized in the WSS model as fluctuations of the background in which the probe D8-branes are placed, where certain superselection rules are applied. In particular states with odd parity in the extra circle along τ are discarded, as well as Kaluza-Klein modes of the compact S^1 and S^4 subspaces. The resulting glueball spectrum was discussed in [41], where the lift of (2.1) to 11-dimensional supergravity is used. In the following we shall consider scalar, tensor, and pseudoscalar glueballs, for which hadronic decays have been worked out in the WSS model in [44–47,50] and which we review and update in Appendix A in some detail for the scalar and tensor glueballs.

The lift of a type IIA string-frame metric to 11-dimensional supergravity is given by the relation

$$\begin{aligned} \mathrm{d}s^2 &= G_{MN}\mathrm{d}x^M\mathrm{d}x^N \\ &= e^{-2\phi/3}g_{AB}\mathrm{d}x^A\mathrm{d}x^B + e^{4\phi/3}(\mathrm{d}x^{11} + A_B\mathrm{d}x^B)^2, \end{aligned} \tag{4.1}$$

with $M, N = 0,\ldots10$ and $A, B = 0,\ldots9$, omitting the eleventh index. By introducing the radial coordinate r related to U by $U = \frac{r^2}{2L}$, we get the lifted metric

$$\begin{aligned} \mathrm{d}s^2 &= \frac{r^2}{L^2}[f(r)\mathrm{d}x_4^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}x_{11}^2] \\ &\quad + \frac{L^2}{r^2}\frac{\mathrm{d}r^2}{f(r)} + \frac{L^2}{4}\mathrm{d}\Omega_4^2, \end{aligned} \tag{4.2}$$

and the field strength $F_{\alpha\beta\gamma\delta} = \frac{6}{L}\sqrt{g_{S^4}}\epsilon_{\alpha\beta\gamma\delta}$, which are solutions to the equations of motion following from the unique supergravity action

$$\begin{aligned} 2\kappa_{11}^2 S_{11} &= \int \mathrm{d}^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) \\ &\quad - \frac{1}{3!}\int A_3 \wedge F_4 \wedge F_4. \end{aligned} \tag{4.3}$$

Scalar and tensor glueball modes appear as normalizable modes of metric fluctuations δG, which translate to perturbations of the type-IIA string metric and dilaton through

$$\begin{aligned} g_{\mu\nu} &= \frac{r^3}{L^3}\left[\left(1 + \frac{L^2}{2r^2}\delta G_{11,11}\right)\eta_{\mu\nu} + \frac{L^2}{r^2}\delta G_{\mu\nu}\right], \\ g_{44} &= \frac{r^3 f}{L^3}\left[1 + \frac{L^2}{2r^2}\delta G_{11,11} + \frac{L^2}{r^2 f}\delta G_{44}\right], \\ g_{rr} &= \frac{L}{rf}\left[1 + \frac{L^2}{2r^2}\delta G_{11,11} + \frac{r^2 f}{L^2}\delta G_{rr}\right], \\ g_{r\mu} &= \frac{r}{L}\delta G_{r\mu}, \\ g_{\Omega\Omega} &= \frac{r}{L}\left(\frac{L}{2}\right)^2\left(1 + \frac{L^2}{2r^2}\delta G_{11,11}\right), \\ e^{4\phi/3} &= \frac{r^2}{L^2}\left(1 + \frac{L^2}{r^2}\delta G_{11,11}\right). \end{aligned} \tag{4.4}$$

Inducing these metric fluctuations to the world volume of the D8-brane system described by the action (2.5), Ref. [44] calculated interaction vertices of the lightest scalar glueball with mesons, which was revisited and extended in [45].

Pseudoscalar, vector, and pseudovector glueballs appear as fluctuations of the type-IIA form fields; glueballs with higher spin would need a stringy description beyond the supergravity approximation [87].

A. Exotic and dilatonic scalar glueballs

Superficially, the emerging glueball spectrum resembles the one found in lattice calculations (see Fig. 1 in [47]), containing a lightest scalar glueball with a mass below that of the tensor glueball, whereas most other holographic models have the scalar glueball degenerate with the tensor. This is achieved by an "exotic" polarization of the bulk metric involving the extra compact dimension (τ) separating the D8-branes,

$$\begin{aligned} \delta G_{\tau\tau} &= -\frac{r^2}{\mathcal{N}_E L^2}f(r)S_4(r)G_E(x^\sigma), \\ \delta G_{\mu\nu} &= \frac{r^2}{\mathcal{N}_E L^2}S_4(r)\left[\frac{1}{4}\eta_{\mu\nu} - \left(\frac{1}{4} + \frac{3r_{\mathrm{KK}}^6}{5r^6 - 2r_{\mathrm{KK}}^6}\right)\frac{\partial_\mu\partial_\nu}{M^2}\right] \\ &\quad \times G_E(x^\sigma), \\ \delta G_{11,11} &= \frac{r^2}{\mathcal{N}_E 4L^2}S_4(r)G_E(x^\sigma), \\ \delta G_{rr} &= -\frac{L^2}{\mathcal{N}_E r^2 f(r)}\frac{3r_{\mathrm{KK}}^6}{5r^6 - 2r_{\mathrm{KK}}^6}S_4(r)G_E(x^\sigma), \\ \delta G_{r\mu} &= \delta G_{\mu r} = \frac{90r^7 r_{\mathrm{KK}}^6}{\mathcal{N}_E M^2 L^2(5r^6 - 2r_{\mathrm{KK}}^6)^2}S_4(r)\partial_\mu G_E(x^\sigma), \end{aligned} \tag{4.5}$$

with eigenvalue equation [41]

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}(r^7 - rr_{KK}^6)\frac{\mathrm{d}}{\mathrm{d}r}S_4(r) \\ &\quad + \left(L^4 M_E^2 r^3 + \frac{432r^5 r_{\mathrm{KK}}^{12}}{(5r^6 - 2r_{KK}^6)^2}\right)S_4(r) = 0. \end{aligned} \tag{4.6}$$

However, with $M_E = 855$ MeV its mass is only a bit higher than that of the ρ meson, whereas the predominantly dilatonic mode that is the ground state of another tower of scalar modes with respect to $3+1$ dimensions is only a little lighter than the traditional glueball candidates $f_0(1500)$ and $f_0(1710)$. This mode is degenerate with the tensor mode and involves only metric fluctuations $\delta G_{11,11}$ and $\delta G_{\mu\nu}$, see (A1).

The exotic scalar glueball, denoted by G_E in the following, turns out [45] to have a relative width Γ/M that is much higher than that of the predominantly dilatonic scalar glueball (G_D), but only the latter has a Γ/M in the ballpark of $f_0(1500)$ and $f_0(1710)$.

It was therefore proposed in [45] to discard G_E from the spectrum of glueballs of the WSS model as a spurious mode that perhaps disappears in the inaccessible limit $M_{\mathrm{KK}} \to \infty$, where the supergravity approximation breaks down. Already in [40] it was speculated that only one of the two scalar glueball towers might correspond to the glueballs in QCD. Since it appears somewhat unnatural that an excited scalar glueball should have a smaller width than the ground-state scalar glueball, Ref. [45] preferred the dilatonic scalar glueball as a candidate for the actual ground state.

Indeed, the dilatonic scalar glueball turns out to have a decay pattern that can match surprisingly well the glueball candidate $f_0(1710)$, in particular when including additional couplings associated with the quark mass term [46,47]. This may actually apply instead to $f_0(1770)$, which was proposed originally in [88] as an additional f_0 resonance between 1700 and 1800 MeV and more recently in [24] in

TABLE IV. Total decay widths of the exotic and the dilatonic scalar glueball G_E and G_D with original masses of 855 and 1487 MeV, respectively, and also with extrapolations to the masses of the glueball candidates $f_0(1510)$ or $f_0(1710)$ and the fragmented glueball of [8,24,25], for two choices of the extra coupling parameter x associated with the quark mass term as defined in [47]. The range of results corresponds again to $\lambda = 16.63\ldots12.55$. In addition to the two-body decays reviewed in Appendix A, the decays $G_D \to \rho\pi\pi \to 4\pi$ which interfere destructively with $G_D \to \rho\rho \to 4\pi$ have been taken into account here.

M_E	$\Gamma^{x=0}_{G_E}$[MeV]	$\Gamma^{x=1}_{G_E}$[MeV]
855	72…96	85…113
1506	286…383	430…570
1712	351…469	483…640
1865	398…530	521…691

M_D	$\Gamma^{x=0}_{G_D}$[MeV]	$\Gamma^{x=1}_{G_D}$[MeV]
1487	19…26	80…106
1506	19…27	80…106
1712	88…113	139…180
1865	151…197	198…259

radiative J/ψ decays, where it appears dominantly as the most glue-rich resonance.[4]

The fact that the ratio $\Gamma_{f_0\to K\bar{K}}/\Gamma_{f_0\to\pi\pi}$ is significantly higher for $f_0(1710)$ [68] (or for $f_0(1770)$ according to [24]) than expected from a flavor-symmetric glueball coupling can be attributed to the fact that dilaton fluctuations couple naturally to quark mass terms, similar to, but more pronounced than, in a model by Ellis and Lanik [89]. There is therefore no need to invoke the previous conjecture of chiral suppression of scalar glueball decay [48,90,91], which was questioned in [92].

In the following we shall mainly explore the consequences of this identification of the scalar glueball. In the radiative decay rates considered here, the explicit quark masses will however not modify the (chiral) results for the couplings; they are only included in phase space factors.

We shall however need to make assumptions on how to extrapolate to realistic glueball masses, which we describe in more detail below. While the mass of $f_0(1710)$ is not too much above the original mass of G_D in the WSS model, larger extrapolations are required for the tensor and pseudoscalar glueballs when comparing to the various glueball candidates or lattice results.

As an alternative scenario, we shall also consider the option of keeping the exotic scalar glueball mode G_E, whose relative decay width Γ/M is much too large to be identified with the traditional glueball candidates $f_0(1510)$ or $f_0(1710)$ with total width 112(9) MeV and 128(18) MeV, respectively, see Table IV. It would in fact fit better to the proposal in [8,24,25] of a relatively broad fragmented glueball of mass 1865 MeV and a width of 370 MeV that does not show up as a separate meson but only as admixture in the mesons $f_0(1710)$, a novel $f_0(1770)$, $f_0(2020)$, and $f_0(2100)$. Of course, this requires a drastic rise of the original mass of G_E by a factor of over 2, but also the mass of the tensor mode G_T would have to be raised by a factor of 1.6 to match the expectation of $m_T \sim 2400$ MeV from lattice QCD; the mass of G_D, which would then be identified with the first excited scalar glueball, would need to be raised somewhat more, as lattice results point to a mass above the tensor glueball, from around 2670 MeV [11] to around 3760 MeV [13].

[4]The next (2023) update of the PDG [68] will in fact include $f_0(1770)$ as a separate resonance (C. Amsler, private communication).

B. Extrapolations to realistic glueball masses

In the WSS model, the masses of glueballs are given by pure numbers times M_{KK}, which is also the case for the (axial) vector mesons. However, when M_{KK} has been fixed by the mass of the ρ meson, the glueball masses appear to be too small compared to lattice QCD results.

In order to predict decay rates for different glueball candidates we manually change the masses of glueball modes in amplitudes and phase space integrals, which could be viewed as assuming a different scale M_{KK} for the glueball sector. The coupling constants involving glueballs are all inversely proportional to M_{KK} and we interpret this appearance of M_{KK} to be tied to the mass scale of glueballs, which shows up also in their normalization factors $\mathcal{N}$, whereas explicit appearances of M_{KK} in the DBI action of the D-brane are considered as being fixed like the mass of the ρ meson. When upscaling glueball masses, we have therefore correspondingly reduced the dimensionful glueball-meson/photon coupling constants. [Without such a rescaling, the results for all glueball decay rates and the glueball contributions to a_μ presented in Sec. V extrapolated to some mass M_G would be simply larger by a factor $(M_G/M_G^{\mathrm{WSS}})^2$.]

We consider this rescaling plausible in that the overlap integrals of glueball and meson holographic profiles should become smaller when glueball and meson modes are separated further in energy. It may well be, however, that this reduction is only insufficiently accounted for by the overall change of the mass scale in the glueball coupling constants; thus our numerical results should be considered as somewhat rough estimates.

V. RADIATIVE GLUEBALL DECAYS

In the following we shall concentrate on glueball interactions involving vector mesons which through VMD also give rise to glueball-photon vertices. Other hadronic interactions of glueballs are reviewed in Appendix A.

We shall consider the first three lightest glueball states, scalar, tensor, and pseudoscalar in turn, choosing the dominantly dilatonic scalar glueball over the exotic scalar glueball, since the former has been found to match remarkably well to the decay pattern of the glueball candidate $f_0(1710)$. The more unwieldy results for the exotic scalar glueball are worked out in Appendixes A and B.

A. Dilatonic scalar glueball decays

Inducing the fluctuation (A1) in the D8-brane action (2.5) we obtain the interaction terms of the dilatonic scalar glueball with two vector mesons as

$$\mathcal{L}_{G_D v^m v^n} = \mathrm{tr}\int \mathrm{d}^4x (d_3^{mn}\eta^{\rho\sigma}F^m_{\mu\rho}F^n_{\nu\sigma} + d_2^{mn}M^2_{\mathrm{KK}}v^m_\mu v^n_\nu) \times \left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)G_D, \tag{5.1}$$

where the coupling constants are given by

$$d_2^{mn} = \kappa\int \mathrm{d}zK\psi'_{2n-1}\psi'_{2m-1}H_D = \{4.3714,\ldots\}\frac{1}{\lambda^{\frac{1}{2}}N_cM_{\mathrm{KK}}},$$
$$d_3^{mn} = \kappa\int \mathrm{d}zK^{-1/3}\psi_{2n-1}\psi_{2m-1}H_D = \{18.873,\ldots\}\frac{1}{\lambda^{\frac{1}{2}}N_cM_{\mathrm{KK}}}. \tag{5.2}$$

Restricting to the ground-state vector mesons ($m = n = 1$), the amplitudes for the decay of the dilatonic scalar glueball into vector mesons with transverse and longitudinal polarizations read

$$|\mathcal{M}_T^{(G_D\to v^1v^1)}| = \left[d_3^{11}\left(2m^2_{v^1} - \frac{3M_D^2}{4}\right) - d_2^{11}M^2_{\mathrm{KK}}\right],$$
$$|\mathcal{M}_L^{(G_D\to v^1v^1)}| = \left[\frac{d_2^{11}M_D^2M^2_{\mathrm{KK}}}{4m^2_{v^1}} + d_3^{11}m^2_{v^1}\right], \tag{5.3}$$

in terms of which the partial decay width is given by

$$\Gamma_{G_D\to v^{1,a}v^{1,b}} = \frac{1}{S}(2|\mathcal{M}_T^{(G_D\to v^1v^1)}|^2 + |\mathcal{M}_L^{(G_D\to v^1v^1)}|^2)\frac{|\mathbf{p}_{v^1}|}{8\pi M_D^2}, \tag{5.4}$$

where S equals 2 for identical particles ($a = b$) and 1 otherwise.

In the narrow-resonance approximation, this vanishes for the WSS model mass $M_D = 1487$ MeV, which is below the threshold of two ρ mesons. However, when M_D is manually adjusted to the mass of $f_0(1710)$, which we assume as 1712 MeV (the average of the T-matrix pole results of [93,94]), the decay $G_D \to \rho\rho$ becomes the largest channel, exceeding even the dominant pseudoscalar channel $G_D \to KK$ (see Appendix A, Table VIII).

As discussed in [45], the holographic prediction for the total rate $G_D \to 4\pi$ is somewhat reduced by a destructive interference from $G_D \to \rho\pi\pi$, rendering the partial width of $G_D \to 4\pi$ similar to and slightly less than $G_D \to KK$ [46]. Remarkably, data from radiative J/ψ decays [95] for $f_0(1740)$ (or $f_0(1770)$ in [24]) seem to be fairly consistent with this result.

1. Dilatonic scalar glueball 1γ decays

From the interaction terms (5.1) we can also derive the interactions including photons by using VMD. Replacing one vector meson by a photon we find

$$\mathcal{L}_{G_D\mathcal{V}v^m} = 2d_3^{m\mathcal{V}}\eta^{\rho\sigma}\mathrm{tr}(F^m_{\mu\rho}F^{\mathcal{V}}_{\nu\sigma})\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)G_D, \tag{5.5}$$

with

$$d_3^{m\mathcal{V}} \equiv \kappa\int \mathrm{d}zK^{-1/3}\psi_{2m-1}H_D = \{0.46895,\ldots\}\frac{1}{M_{\mathrm{KK}}\sqrt{N_c}}. \tag{5.6}$$

The other coupling $d_2^{m\mathcal{V}}$ vanishes for an on-shell photon, since at zero virtuality its radial mode is constant and drops out in the replacement $\psi' \to \mathcal{J}' = 0$.

In radiative decays, only the transverse amplitude remains, which reads

$$|\mathcal{M}_T^{(G_D\to\mathcal{V}v^m)}| = \frac{d_3^{m\mathcal{V}}(m_v^4 - 4m_v^2M_D^2 + 3M_D^4)}{2M_D^2}\mathrm{tr}(eQT_v), \tag{5.7}$$

yielding

$$\Gamma_{G_D\to v^m\gamma} = 2|\mathcal{M}_T^{(G_D\to\mathcal{V}v^m)}|^2\frac{|\mathbf{p}_{\mathcal{V}}|}{8\pi M_D^2}. \tag{5.8}$$

The results are displayed in Table V for two mass parameters corresponding to $f_0(1500)$ and $f_0(1710)$, where ideal mixing was assumed for the ω and ϕ mesons.

TABLE V. Radiative scalar glueball decay with G_D identified alternatively with $f_0(1500)$ and $f_0(1710)$ with masses 1506 and 1712 MeV, respectively, for $\lambda = 16.63\ldots12.55$.

	Γ_{G_D}[keV]
$f_0(1500) \to \rho\gamma$	184
$f_0(1500) \to \omega\gamma$	19.9
$f_0(1500) \to \phi\gamma$	14.1
$f_0(1500) \to \gamma\gamma$	1.74…1.32
$f_0(1710) \to \rho\rho$	$(53.5\ldots71.0) \times 10^3$
$f_0(1710) \to \omega\omega$	$(16.6\ldots22.0) \times 10^3$
$f_0(1710) \to \rho\gamma$	276
$f_0(1710) \to \omega\gamma$	30.1
$f_0(1710) \to \phi\gamma$	29.4
$f_0(1710) \to \gamma\gamma$	1.98…1.50

The latter implies that $\rho\gamma$ and $\omega\gamma$ decay rates are very close to the ratio 9:1. The ratio of decay rates $\phi\gamma$ and $\omega\gamma$, which would be 2:1 with equal masses, is, however, significantly reduced by the larger ϕ mass.[5]

2. Dilatonic scalar glueball 2γ decays

Replacing the second vector meson by a photon by means of VMD, we obtain the 2γ interactions

$$\mathcal{L}_{G_D\mathcal{V}\mathcal{V}} = d_3^{\mathcal{V}\mathcal{V}} \eta^{\rho\sigma} \mathrm{tr}(F^{\mathcal{V}}_{\mu\rho} F^{\mathcal{V}}_{\nu\sigma}) \left(\eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box}\right) G_D, \tag{5.9}$$

with

$$d_3^{\mathcal{V}\mathcal{V}} \equiv \kappa \int \mathrm{d}z K^{-1/3} H_D = 0.0130195 \lambda^{1/2} M_{\mathrm{KK}}^{-1} \tag{5.10}$$

which gives

$$|\mathcal{M}_T^{(G_D\to\mathcal{V}\mathcal{V})}| = \frac{3}{2} d_3^{\mathcal{V}\mathcal{V}} M_D^2 \mathrm{tr}(e^2 Q^2). \tag{5.11}$$

The resulting width

$$\Gamma_{G_D\to\gamma\gamma} = \frac{1}{8\pi} |\mathcal{M}_T^{(G_D\to\mathcal{V}\mathcal{V})}|^2 \frac{|\mathbf{p}_\mathcal{V}|}{M_D^2} \tag{5.12}$$

is again displayed in Table V for the two mass parameters corresponding to $f_0(1500)$ and $f_0(1710)$, which in both cases is above 1 keV.

This is larger than the old prediction by Kada *et al.* [53], but an order of magnitude smaller than the VMD based result of Cotanch and Williams [54], who obtained 15.1 keV for a scalar glueball with mass 1700 MeV after correcting their previous result of 2.6 keV in [96] (note that the corresponding preprint has erroneously 2.6 eV instead). Also all other radiative decay rates obtained in [53] are about an order of magnitude larger than ours (not uniformly so, however, but varying between a factor of 7 to 26, thereby deviating from the ratios discussed at the end of Sec. V A 1).

On the other hand, the two-vector meson decay rates obtained in [96] (44.4 MeV for $\rho\rho$ and 34.6 MeV for $\omega\omega$) are not very far from our results. In fact, our holographic prediction for $f_0(1710) \to \omega\omega$ with $f_0(1710)$ as a (predominantly dilaton) glueball appears to be in the right ballpark considering the measured branching ratios of radiative J/ψ decays in $\gamma f_0(1710) \to \gamma K\bar{K}$ and $\gamma f_0(1710) \to \gamma\omega\omega$ [68] [which according to [24] may be instead $f_0(1770)$]. The PDG [68] quotes two results for $\mathcal{B}(K\bar{K})$: a BNL measurement [97] from 1986 with $\mathcal{B}(K\bar{K}) = 0.38^{+0.09}_{-0.19}$ and a phenomenological analysis [98] concluding 0.36 (12), which both are consistent with the WSS result obtained in [46] as approximately 0.35. Using $\mathcal{B}(K\bar{K}) = 0.36(12)$ and the total decay width of $f_0(1710)$ [68] of 123(18) MeV lead to a partial decay width for $f_0(1710) \to \omega\omega$ of about 15(8) MeV, for which the holographic prediction from G_D amounts to 16.6…22.0 MeV.

No experimental results for single-photon decays of $f_0(1710)$ appear to be available, but in [52] the BELLE Collaboration reports a measurement of $f_0(1710) \to \gamma\gamma$ with the result $\Gamma_{\gamma\gamma}\mathcal{B}(K\bar{K}) = 12^{+3+227}_{-2-8}$ eV, with the stated conclusion that the $f_0(1710)$ meson was unlikely to be a glueball because of a width larger than that expected ("much less than 1 eV") for a pure glueball state. However the holographic prediction for $\Gamma_{\gamma\gamma}\mathcal{B}(K\bar{K}) \approx 690…520$ eV is $3-2\sigma$ above the upper limit of the BELLE result.[6] Ironically, the BELLE result for the two-photon rate appears to be rather too small for a pure (predominantly dilaton) glueball interpretation of $f_0(1710)$ within the WSS model.[7] The central value of the BELLE result for $\Gamma_{\gamma\gamma}$ of only a few tens of eV would thus seem to indicate that VMD does not apply for radiative decays of $f_0(1710)$.

In Appendix B we also evaluate radiative decays of the exotic glueball of the WSS model. The two-photon decay width of G_E is considerably smaller than that of G_D, 87…65 eV, when the mass of G_E is extrapolated to that of $f_0(1710)$. However, the decay pattern of G_E does not fit to either $f_0(1500)$ or $f_0(1710)$ when extrapolating to their masses.

B. Tensor glueball decays

The holographic mode functions associated with tensor glueballs are reviewed in Appendix A 3 together with the results of hadronic two-body decays.

Radiative decays of tensor glueballs can be derived from the interaction terms with two vector mesons, which are given by

$$\mathcal{L}_{G_T v^m v^n} = \mathrm{tr}[t_2 M_{\mathrm{KK}}^2 v_\mu^m v_\nu^n G_T^{\mu\nu} + t_3 F_{\mu\rho}^m F_\nu^{n\rho} G_T^{\mu\nu}], \tag{5.13}$$

with

$$t_2^{mn} = \int \mathrm{d}z K \psi'_{2m-1} \psi'_{2n-1} T = 2\sqrt{3} d_2^{mn}, \tag{5.14}$$

$$t_3^{mn} = \int \mathrm{d}z K^{-1/3} \psi_{2m-1} \psi_{2n-1} T = 2\sqrt{3} d_3^{mn}, \tag{5.15}$$

[5] A more realistic value for the ϕ-ω mixing angle of 3.32° above ideal mixing [76] increases the partial width for $\omega\gamma$ by about 17% and decreases the one for $\phi\gamma$ by about 8.5%. This also holds true for all the other glueball decay widths below.

[6] Older upper limits for $\Gamma_{\gamma\gamma}\mathcal{B}(K\bar{K})$ are 480 eV from ARGUS [99], 200 eV from CELLO [100], and 560 eV from TASSO [101]. (The latter two are quoted by the PDG [68] with lower values, 110 and 280 eV, respectively, corresponding however to the assumption of helicity 2 which leads to smaller upper limits.)

[7] Assuming a tensor glueball $f_2(1720)$, [53] predicted $\Gamma_{\gamma\gamma}\mathcal{B}(K\bar{K}) \approx 95$ eV.

and $d_{2,3}^{mn}$ as given in (5.2). (Note that due to a different normalization of the tensor field, the tensor coupling constants differ from those in [45] by a factor $\sqrt{2}$; all other glueball coupling constants are defined as in [45].)

The decay rate of a tensor glueball into two vector mesons reads

$$\Gamma_{G_T\to vv} = \frac{1}{S}\left\{\frac{t_2^2}{120}\frac{M_{\mathrm{KK}}^4}{m_v^4}(M_G^4+12m_v^2M_G^2+56m_v^4) + \frac{2}{3}t_2t_3M_{\mathrm{KK}}^2(M_G^2-m_v^2) + \frac{t_3^2}{10}(M_G^4-3m_v^2M_G^2+6m_v^4)\right\}\frac{|\mathbf{p}_v|}{8\pi M_G^2}, \quad (5.16)$$

where S is again the symmetry factor for identical particles.

1. Tensor glueball 1γ decays

Through VMD (5.13) leads to a coupling of the tensor glueball with one photon and one vector meson with interaction Lagrangian

$$\mathcal{L}_{G_T v^n \mathcal{V}} = 2t_3^{\mathcal{V}n} G_T^{\mu\nu}\eta^{\rho\sigma}\mathrm{tr}(F_{\mu\rho}^{\mathcal{V}}F_{\nu\sigma}^n), \quad (5.17)$$

with

$$t_3^{\mathcal{V}n} = \int \mathrm{d}z K^{-1/3}\psi_{2n-1}T = 2\sqrt{3}d_3^{\mathcal{V}n} \quad (5.18)$$

and $d_3^{\mathcal{V}n}$ as given in (5.6).

This yields

$$\Gamma_{G_T\to v\gamma} = \frac{1}{15M_G^4}(\mathrm{tr}(eQT_v))^2(M_G^2-m_v^2)^2(6M_G^4+3M_G^2m_v^2 + 2m_v^4)\frac{|\mathbf{p}_v|}{8\pi M_G^2}. \quad (5.19)$$

2. Tensor glueball 2γ decays

Similarly (5.13) leads to

$$\mathcal{L}_{G_T\mathcal{V}\mathcal{V}} = t_3^{\mathcal{V}\mathcal{V}} G_T^{\mu\nu}\eta^{\rho\sigma}\mathrm{tr}(F_{\mu\rho}^{\mathcal{V}}F_{\nu\sigma}^{\mathcal{V}}), \quad (5.20)$$

with

$$t_3^{\mathcal{V}\mathcal{V}} = \int \mathrm{d}z K^{-1/3}T = 2\sqrt{3}d_3^{\mathcal{V}\mathcal{V}} \quad (5.21)$$

and $d_3^{\mathcal{V}\mathcal{V}}$ as given in (5.10).

The resulting two-photon decay width of the tensor glueball is given by

$$\Gamma_{G_T\to\gamma\gamma} = \frac{1}{5}[t_3^{\mathcal{V}\mathcal{V}}M_G^2\mathrm{tr}(e^2Q^2)]^2\frac{|\mathbf{p}_\gamma|}{8\pi M_G^2}. \quad (5.22)$$

TABLE VI. Radiative tensor glueball decays and decays into two vector mesons for $\lambda = 16.63\ldots12.55$. Besides the pristine results for the WSS model mass of 1487 MeV, their extrapolations to glueball masses of 2000 and 2400 MeV are given.

	$\Gamma_{G_T^{\mathrm{WSS}}}$ [keV]	$\Gamma_{G_T(2000)}$ [keV]	$\Gamma_{G_T(2400)}$ [keV]
$G_T\to\rho\rho$		$(270\ldots358)\times10^3$	$(382\ldots507)\times10^3$
$G_T\to\omega\omega$		$(88.2\ldots117)\times10^3$	$(127\ldots169)\times10^3$
$G_T\to K^*K^*$		$(240\ldots318)\times10^3$	$(417\ldots552)\times10^3$
$G_T\to\phi\phi$			$(76.7\ldots102)\times10^3$
$G_T\to\rho\gamma$	260	522	716
$G_T\to\omega\gamma$	28.3	57.5	79.1
$G_T\to\phi\gamma$	24.7	81.1	127
$G_T\to\gamma\gamma$	1.84…1.39	2.47…1.86	2.97…2.24

The resulting partial widths are listed in Table VI for three values of the mass of the tensor glueball, the unrealistically small WSS model mass value 1487 MeV as well as two higher values motivated by pomeron physics [15][8] and QCD lattice studies [11], respectively, assuming ideal mixing of ω and ϕ mesons. With increasing mass of the glueball, the partial decay widths for $\rho\gamma$, $\omega\gamma$, and $\phi\gamma$ gradually approach the ratios 9∶1∶2 for degenerate vector meson masses; again, a more realistic value of θ_V changes the $\omega\gamma$ and $\phi\gamma$ results only slightly (cf. footnote 5).

The radiative decay widths obtained for the tensor glueball turn out to be comparable with those for the dilatonic scalar glueball for equal glueball mass, rising approximately linear with glueball mass (due to the rescaling described in Sec. IV B).

Our prediction of the two-photon width of ∼2–3 keV is significantly larger than the old prediction of Kada *et al.* [53] who have values in the range of hundreds of eV, and also higher than the more recent prediction in [103], where $\Gamma_{f_2(1950)\to\gamma\gamma} = 960(50)$ eV was obtained. Cotanch and Williams [54], on the other hand, have also results above 1 keV, $\Gamma_{G_T(2010)\to\gamma\gamma} = 1.72$ keV and $\Gamma_{G_T(2300)\to\gamma\gamma} = 1.96$ keV, by using VMD. Also their results for single-photon decays are comparable with ours, even though their results for decays into two vector mesons are significantly smaller than ours. A particular point of disagreement is their result for a relatively large

[8]A candidate for a tensor glueball around 2000 MeV is the broad resonance $f_2(1950)$, which has recently also been argued for in [102] on the basis of a chiral hadronic model. The latter turns out to yield a dominance of the decay modes into two vector mesons, in qualitative agreement with the WSS model, which in fact predicts a very broad tensor glueball (see Appendix A 3).

$\omega\phi$ decay mode, which in the WSS model is absent. As noted in [104], this is possible only by allowing for a rather strong deviation from the large-N_c limit.

C. Pseudoscalar glueball decays

In the WSS model, the pseudoscalar glueball is represented by a Ramond-Ramond 1-form field C_1, which has a kinetic mixing with the singlet η_0 given by [50]

$$\eta_0 \to \eta_0 + \zeta_2 G_{PS} = \eta_0 + 0.01118\sqrt{N_f/N_c}\lambda G_{PS}, \quad (5.23)$$

with G_{PS} remaining unchanged to leading order in $\sqrt{N_f/N_c}$ (formally treated as a small quantity because of the probe brane approximation). In contrast to the conventional mixing scenarios of Refs. [16,105] mass mixing is absent here, while the mass of the pseudoscalar glueball is raised by a factor $(1+\zeta_2^2)$ from 1789 MeV to $(1819.7\ldots1806.5)$ MeV for $\lambda = 16.63\ldots12.55$. Lattice QCD (in the quenched approximation), however, typically finds values around 2600 MeV, so we also consider the latter in our extrapolations.[9]

Through (5.23) the pseudoscalar glueball acquires the same interactions as η_0, and the same form of transition form factors, only with correspondingly modified coupling constants. Thus the formulas given in Sec. III B for the decays of pseudoscalars in vector mesons or photons remain essentially unchanged, but the higher mass of the pseudoscalar glueball permits also decays into pairs of vector mesons.

The resulting interaction Lagrangian reads

$$\begin{aligned}\mathcal{L}_{G_{\rm PS}vv/v\mathcal{V}/\mathcal{V}\mathcal{V}} = G_{\rm PS}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}[&k_1^{vv}\partial_\mu v_\nu \partial_\rho v_\sigma \\ &+ 2k_1^{v\mathcal{V}}\partial_\mu v_\nu \partial_\rho \mathcal{V}_\sigma + k_1^{\mathcal{V}\mathcal{V}}\partial_\mu \mathcal{V}_\nu \partial_\rho \mathcal{V}_\sigma]\end{aligned} \quad (5.24)$$

with[10]

$$k_1^{v^1v^1} = 19.6184 N_c^{-1}\lambda^{-1/2}M_{\rm KK}^{-1}, \quad (5.25)$$

$$k_1^{v^1\mathcal{V}} = 0.493557 N_c^{-1/2}M_{\rm KK}^{-1}, \quad (5.26)$$

$$k_1^{\mathcal{V}\mathcal{V}} = 0.0145232\lambda^{1/2}M_{\rm KK}^{-1}. \quad (5.27)$$

The various resulting partial widths are listed in Table VII.

In the WSS model, all other hadronic decay channels of the pseudoscalar glueball, such as those considered in [109,110], turn out to be very weak compared to two-vector-meson decays [50]. The relative strength of the latter entails correspondingly important radiative decay modes, and a two-photon partial width in the keV range. Note, however, that these results have been obtained from the first term in a formal expansion in $\sqrt{N_f/N_c}$, which is not a small parameter in real QCD. It might nevertheless be meaningful, since the parameter ζ_2 in (5.23) is reasonably small, $0.19\ldots0.14$ for $\lambda = 16.63\ldots12.55$.

TABLE VII. Radiative pseudoscalar glueball decay and decays into two vector mesons $\lambda = 16.63\ldots12.55$. Besides the WSS model result for the pseudoscalar mass, $M_G = 1813 \pm 7$ MeV, an extrapolation to 2600 MeV (motivated by lattice results) is considered.

	$\Gamma_{G_{\rm PS}^{\rm WSS}}$[keV]	$\Gamma_{G_{\rm PS}(2600)}$[keV]
$G_{\rm PS}\to\rho\rho$	$(36.8\ldots45.0)\times10^3$	$(190\ldots248)\times10^3$
$G_{\rm PS}\to\omega\omega$	$(11.3\ldots13.8)\times10^3$	$(62.2\ldots81.3)\times10^3$
$G_{PS}\to\phi\phi$		$(29.2\ldots38.2)\times10^3$
$G_{PS}\to K^*K^*$	$(2.69\ldots1.81)\times10^3$	$(188\ldots246)\times10^3$
$G_{\rm PS}\to\rho\gamma$	$272\ldots263$	$536\ldots528$
$G_{\rm PS}\to\omega\gamma$	$29.8\ldots28.9$	$59.2\ldots58.3$
$G_{\rm PS}\to\phi\gamma$	$35.6\ldots34.1$	$95.4\ldots94.0$
$G_{\rm PS}\to\gamma\gamma$	$1.75\ldots1.30$	$2.49\ldots1.86$

VI. GLUEBALL CONTRIBUTIONS TO HADRONIC LIGHT-BY-LIGHT SCATTERING AND THE MUON $g-2$

In order to calculate the contribution of the glueball exchange diagram in the light-by-light scattering amplitude, which enters the muon-photon vertex at two loop order, the above results for the vertices of a glueball with two on-shell photons need to be generalized to nonzero photon virtualities.

In the case of the dilatonic scalar glueball G_D, this involves two interaction terms that are obtained by replacing v_μ in (5.1) by $eQA_\mu^{e.m.}$ and the holographic profile functions $\psi(z)$ in (5.2) by the bulk-to-boundary propagator $\mathcal{J}(Q,z)$ defined in (3.29), yielding two form factors,

$$\begin{aligned}d_2^{\mathcal{V}\mathcal{V}}(Q_1^2,Q_2^2) &\equiv \kappa\int dz K\partial_z\mathcal{J}(Q_1,z)\partial_z\mathcal{J}(Q_2,z)H_D(z),\\ d_3^{\mathcal{V}\mathcal{V}}(Q_1^2,Q_2^2) &\equiv \kappa\int dz K^{-1/3}\mathcal{J}(Q_1,z)\mathcal{J}(Q_2,z)H_D(z),\end{aligned} \quad (6.1)$$

in place of the coupling constants d_2 and d_3.

The exotic scalar glueball G_E has more complicated interactions with two vector fields, written out in (B1), with five coupling constants (B2). The latter are generalized in a completely analogous manner to form

[9] Note that historically the pseudoscalar glueball was expected to be the lightest glueball, with $\eta(1405)$ a prominent candidate after $\iota(1440)$ [106] was split into $\eta(1405)$ and $\eta(1475)$. This is still occasionally considered a possibility, see for example [107,108].

[10] The couplings differ by a factor of 2 from [50] since we use SU(3) generators $T^a = \lambda^a/2$.

factors $c_i^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)$ with $i = 2, 3, 4$, and $\breve{c}_j^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)$ with $j = 2, 3$.

Following the notation of [111], the result for the matrix element of a scalar glueball with two electromagnetic currents $j^\mu_{\rm em}(x)$ can be written in terms of two transition form factors $\mathcal{F}^S_{1,2}$ defined by

$$\begin{aligned}\mathcal{M}^{\mu\nu}(p \to q_1, q_2) &= i \int d^4x e^{iq_1\cdot x} \langle 0 | j^\mu_{\rm em}(x) j^\nu_{\rm em}(0) | G_S(p) \rangle \\ &= \frac{\mathcal{F}^S_1(q_1^2, q_2^2)}{M_S} T_1^{\mu\nu} + \frac{\mathcal{F}^S_2(q_1^2, q_2^2)}{M_S^3} T_2^{\mu\nu}\end{aligned} \tag{6.2}$$

with

$$\begin{aligned}T_1^{\mu\nu} &= q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu, \\ T_2^{\mu\nu} &= q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu.\end{aligned} \tag{6.3}$$

For the dilatonic scalar glueball we obtain

$$\begin{aligned}\mathcal{F}_1^D = &-2\frac{d_3^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2){\rm tr}Q^2}{M_D}[(q_1^2 + q_2^2) + (q_1 \cdot q_2) + 2M_D^2)] \\ &- \frac{d_2^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)M_{\rm KK}^2 {\rm tr}Q^2}{M_D},\end{aligned} \tag{6.4}$$

$$\begin{aligned}\mathcal{F}_2^D = &-2d_3^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2){\rm tr}Q^2 M_D + \frac{d_2(Q_1^2, Q_2^2)M_{\rm KK}^2 {\rm tr}Q^2 M_D}{q_1^2 q_2^2} \\ &\times [(q_1 \cdot q_2) + M_D^2)],\end{aligned} \tag{6.5}$$

and for the exotic scalar glueball

$$\begin{aligned}\mathcal{F}_1^E = -2\frac{{\rm tr}Q^2}{M_E}\Big[&c_3^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)((q_1^2 + q_2^2) + (q_1 \cdot q_2) + M_E^2) - \breve{c}_3^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)M_E^2 \\ &+ c_2^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)M_{\rm KK}^2 - \frac{3}{2}(c_4^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2) + c_4^{\mathcal{V}\mathcal{V}}(Q_2^2, Q_1^2))\Big],\end{aligned} \tag{6.6}$$

$$\begin{aligned}\mathcal{F}_2^E = -2{\rm tr}Q^2 M_E\Big[&c_3^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2) - c_2^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)M_{\rm KK}^2 \frac{(q_1 \cdot q_2)}{q_1^2 q_2^2} \\ &+ \breve{c}_2^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)\frac{M_E^2 M_{\rm KK}^2}{q_1^2 q_2^2} - \frac{3}{2}M_{\rm KK}^2 \frac{c_4^{\mathcal{V}\mathcal{V}}(Q_1^2, Q_2^2)q_1^2 + c_4^{\mathcal{V}\mathcal{V}}(Q_2^2, Q_1^2)q_2^2}{q_1^2 q_2^2}\Big],\end{aligned} \tag{6.7}$$

where $q_1 \cdot q_2 = -\frac{1}{2}(q_1^2 + q_2^2 M_{D/E})$ and ${\rm tr}Q^2 = 2/3$ for $N_f = 3$.

We have used these results to estimate the glueball contribution to the muon anomalous magnetic moment $a_\mu = (g-2)_\mu/2$ in a narrow-width approximation by inserting the above expressions in the two-loop expression for the muon-photon vertex.

In the scenario where the exotic scalar glueball is discarded from the spectrum and G_D is identified with the ground-state scalar glueball, we obtain for $M_D = 1506$ MeV and $M_D = 1712$ MeV corresponding to the glueball candidates $f_0(1500)$ and $f_0(1710)$

$$\begin{aligned}a_\mu^{G_D(1506)} &= -1.62 \times 10^{-12}, \\ a_\mu^{G_D(1712)} &= -1.01 \times 10^{-12}.\end{aligned} \tag{6.8}$$

While the former result is approximately identical to the unmodified WSS result, since $M_D^{\rm WSS} = 1487$ MeV, the latter depends on the specific extrapolations laid out in Sec. IV B. Had we only raised the mass, it would have been somewhat larger, -1.35×10^{-12}, but in this case the rather good agreement of the hadronic decay pattern obtained for $G_D(1712)$ with the experimental results for the glueball candidate $f_0(1710)$ (or $f_0(1770)$ according to [24]) would have deteriorated.

If the exotic scalar glueball is not discarded from the spectrum but identified with the ground-state scalar glueball, its mass needs to be raised substantially to match the predictions from lattice QCD. Its decay pattern and in particular its large width then does not fit to either $f_0(1500)$ and $f_0(1710)$; it might instead be identified with the broad "fragmented" glueball $G(1865)$ proposed in [8,24,25]. Raising the mass of G_E artificially to this glueball, we obtain for its a_μ contribution

$$a_\mu^{G_E(1865)} = -0.10 \times 10^{-12}, \tag{6.9}$$

which is an order of magnitude smaller in accordance with the much smaller two-photon rate of G_E. Since in this case the narrow-width approximation is rather questionable, we have also considered the spacelike Breit-Wigner function proposed in [112]. However, this changes the result (6.9) only by about 2%.

In [112] the authors consider scalar resonances including $f_0(1500)$, which is assumed to have a sizeable photon coupling while being a glueball-like state, with a coupling constant similar to the one obtained for $f_0(980)$, leading to

$\Gamma^{f_0(1500)\to\gamma\gamma} \approx 0.79$ keV. The assumed transition form factors therein yield $a_\mu = -(1.3\ldots2) \times 10^{-12}$. This is comparable to our results, even though the two-photon rate obtained with G_D is about twice as large.

In the WSS model, tensor glueballs have two-photon decay rates comparable to G_D with similar values of $\Gamma_{\gamma\gamma}/M_G$. We have not evaluated their contribution to a_μ, but we expect that they will be smaller than those of G_D by some power of the ratio M_{G_T}/M_{G_D}.

We have however evaluated the contribution of pseudoscalar glueballs, which contribute with a positive sign. With the WSS model mass of 1789 MeV we find $a_\mu^{G_{\rm PS}^{\rm WSS}} = 0.39 \times 10^{-12}$, and when extrapolated to a value typically found in quenched lattice QCD calculations of 2600 MeV this reduces to

$$a_\mu^{G_{\rm PS}(2600)} = 0.19 \times 10^{-12}. \tag{6.10}$$

This is about an order of magnitude smaller than the pseudoscalar contribution called G/η'' in the bottom-up holographic model of [113], $a_\mu^{\eta''} \approx 2 \times 10^{-12}$. In this more realistic model, the pseudoscalar glueball mixes not only with η_0 but also with excited $\eta(')$ mesons (which are absent in our simple extension of the WSS model to massive pseudoscalars).

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APPENDIX A: HADRONIC DECAYS OF THE SCALAR AND TENSOR GLUEBALLS

In the following we review the hadronic decays of scalar and tensor glueballs in the WSS model as worked out in [45–47], including additional subdominant decay channels neglected therein, in particular $G \to a_1\pi$. The latter has been emphasized in the phenomenological analysis of [114], where it was providing the largest partial decay width of a pure glueball (177 MeV for a glueball mass of 1600 MeV). While their results for decays of a scalar glueball into two vector mesons are remarkably compatible with the WSS result for G_D when the mass is raised to 1500–1700 MeV, the WSS prediction for $G \to a_1\pi$ turns out to be fairly small, $\lesssim$1 MeV, in stark contrast to the model of [114].[11]

We also review the dependence on the so far unconstrained extra coupling to be associated with the quark mass term that we have added to the chiral WSS model (parametrized by x in Table IV). As discussed in [47], this correlates the flavor asymmetries in the decay pattern in two pseudoscalars with the $\eta\eta'$ partial width. Good agreement of the decay pattern of G_D with $f_0(1710)$ [or $f_0(1770)$] is obtained only for small or vanishing $\eta\eta'$ decay rates. Here a new experimental result has been published in [116]: $\mathcal{B}(f_0(1710) \to \eta\eta')/\mathcal{B}(f_0(1710) \to \pi\pi) < 1.61 \times 10^{-3}$, contradicting [24,25] where this ratio is ~1 for $f_0(1710)$ and ~0.1 for $f_0(1770)$.

1. Dilatonic scalar glueball

The scalar glueball fluctuation which in [45] is referred to as (predominantly) dilatonic scalar glueball, reads

$$\delta G_{\mu\nu} = \frac{r^2}{\mathcal{N}_D L^2} T_4(r) \left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) G_D(x^\sigma),$$
$$\delta G_{11,11} = -3\frac{r^2}{\mathcal{N}_D L^2} T_4(r) G_D(x^\sigma), \tag{A1}$$

with an undetermined normalization parameter $\mathcal{N}_D$. To be a solution of the Einstein equations, the radial function $T_4(r)$ has to satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^7 - r r_{KK}^6)\frac{\mathrm{d}}{\mathrm{d}r} T_4(r) + L^4 M_D^2 r^3 T_4(r) = 0, \tag{A2}$$

with boundary conditions $T_4(r_{\rm KK}) = 1$ and $T_4'(r_{\rm KK}) = 0$, and therefore is normalizable for a discrete set of mass eigenvalues M_D. In the following, we will only consider the lightest mode with $M_D = 1.567 M_{\rm KK} = 1487$ MeV.

The kinetic and mass term for G_D reads

$$\mathcal{L}_4|_{G_D^2} = \mathcal{C} \int \mathrm{d}r \frac{3r^3 T_4(r)^2}{L^3 \mathcal{N}_D^2} G_D(\Box - M_D^2) G_D \tag{A3}$$

with the constant

$$\mathcal{C} = \left(\frac{L}{2}\right)^4 \Omega_4 \frac{1}{2\kappa_{11}^2} (2\pi)^2 R_4 R_{11}. \tag{A4}$$

The radial integration for the lightest mode yields the constant

$$\int \mathrm{d}r \frac{r^3 T_4(r)^2}{L^3} = 0.22547 [T_4(r_{\rm KK})]^2 \frac{r_{\rm KK}^4}{L^3}. \tag{A5}$$

To get a canonically normalized kinetic term

$$\mathcal{L}_4|_{G_D^2} = \frac{1}{2} G_D(\Box - M_D^2) G_D, \tag{A6}$$

we have to set

[11] For $f_0(1500)$ the experimental value from [115] is 12(5)% of $\Gamma_{4\pi}$, i.e., ~7 MeV; for $f_0(1710)$ no corresponding experimental results seem to be available.

$$\mathcal{N}_D = 0.0335879\lambda^{\frac{1}{2}} N_C M_{\mathrm{KK}}. \tag{A7}$$

Inducing the fluctuation (A1) in the D8-brane action (2.5) we obtain the derivative coupling of two pseudoscalar mesons to G_D as

$$\mathcal{L}_{G_D\Pi\Pi} = d_1 \mathrm{tr}\partial_\mu \Pi \partial_\nu \Pi \left(\eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box}\right) G_D \tag{A8}$$

where

$$d_1 = \frac{17.2261}{\sqrt{\lambda} M_{\mathrm{KK}} N_c} \tag{A9}$$

(see [45] for further couplings).

Already in the chiral WSS model, a mass term arises for the singlet component of Π through the $U(1)_A$ anomaly [42]. The latter requires a redefinition of the Ramond-Ramond 2-form field strength F_2 which is associated with a θ term. The bulk action is thus given by

$$S_{C_1} = -\frac{1}{4\pi(2\pi l_s)^6} \int \mathrm{d}^{10}x \sqrt{-g} |\tilde{F}_2|^2, \tag{A10}$$

where

$$\tilde{F}_2 = \frac{6\pi U_{\mathrm{KK}}}{U^4 M_{\mathrm{KK}}} \left(\theta + \frac{\sqrt{2N_f}}{f_\pi} \eta_0\right) \mathrm{d}U \wedge dx^4, \tag{A11}$$

from which one obtains the Witten-Veneziano mass as [42]

$$m_0^2 = \frac{N_f}{27\pi^2 N_c} \lambda^2 M_{\mathrm{KK}}. \tag{A12}$$

Inducing the metric fluctuations gives rise to an additional coupling between the scalar glueballs and η_0. For the dilatonic glueball it is given by [46,47]

$$\mathcal{L}_{\eta_0} \supset \frac{3}{2} m_0^2 \eta_0^2 d_0 G_D, \tag{A13}$$

with ($H_D \equiv T_4/\mathcal{N}_D$)

$$d_0 = 3U_{\mathrm{KK}}^3 \int_{U_{\mathrm{KK}}}^\infty \mathrm{d}U H_D(U) U^{-4} \approx \frac{17.915}{\sqrt{\lambda} N_c M_{\mathrm{KK}}}. \tag{A14}$$

Massive quarks can be introduced by worldsheet instantons [62,63,117] or tachyon condensation [65,66,118], which give

$$\mathcal{L}_m^{\mathcal{M}} \propto \int \mathrm{d}^4x \mathrm{Tr}(\mathcal{M} U(x) + \mathrm{H.c.}), \tag{A15}$$

where

$$U(x) = \mathrm{P}\exp i \int \mathrm{d}z A_z(z,x) = e^{i\Pi^a \lambda^a / f_\pi}. \tag{A16}$$

Expanding the mass term with $\mathcal{M} = \mathrm{diag}(\hat{m}, \hat{m}, m_s)$ leads to

$$\begin{aligned}\mathcal{L}_m^{\mathcal{M}} = &-\frac{1}{2} m_\pi^2 \pi_0^2 - m_\pi^2 \pi^+ \pi^- - m_K^2 (K_0 \bar{K}_0 + K_+ K_-) \\ &-\frac{1}{2} m_1^2 \eta_0^2 - \frac{1}{2} m_8^2 \eta_8 + \frac{2\sqrt{2}}{3}(m_K^2 - m_\pi^2)\eta_0 \eta_8,\end{aligned} \tag{A17}$$

with

$$\begin{aligned} m_\pi^2 &= 2\hat{m}\mu, \qquad m_K^2 = (\hat{m} + m_s)\mu, \\ m_1^2 &= \frac{2}{3} m_K^2 + \frac{1}{3} m_\pi^2, \qquad m_8^2 = \frac{4}{3} m_K^2 - \frac{1}{3} m_\pi^2, \end{aligned} \tag{A18}$$

and μ being the overall scale. We also note a sign error in the $\eta_0\eta_8$ mixing term in [46]. With

$$\eta = \eta_8 \cos\theta_P - \eta_0 \sin\theta_P, \qquad \eta' = \eta_8 \sin\theta_P + \eta_0 \cos\theta_P, \tag{A19}$$

the mass term is diagonalized by

$$\theta_P = \frac{1}{2} \arctan \frac{2\sqrt{2}}{1 - \frac{3}{2} m_0^2/(m_K^2 - m_\pi^2)} \tag{A20}$$

leading to

$$\begin{aligned} m_{\eta,\eta'}^2 = &\frac{1}{2} m_0^2 + m_K^2 \\ &\mp \sqrt{\frac{m_0^4}{4} - \frac{1}{3} m_0^2 (m_K^2 - m_\pi^2) + (m_K^2 - m_\pi^2)^2} \end{aligned} \tag{A21}$$

for the η and η' meson, respectively.

As in [46,47], we assume a scalar glueball coupling to the quark mass terms of the form (correcting a typo in [47])

$$\mathcal{L}_{G_D q\bar{q}} = -3 d_m G_D \mathcal{L}_m^{\mathcal{M}} \tag{A22}$$

with d_m being of the same order as d_0, i.e.

$$d_m = x d_0, \qquad x = \mathcal{O}(1). \tag{A23}$$

This leads to a $G_D \eta\eta'$ interaction given by

$$\mathcal{L}_{G_D\eta\eta'} = -\frac{3}{2}(1-x) d_0 \sin(2\theta_P) m_0^2 G_D \eta\eta'. \tag{A24}$$

With these modifications we obtain the coupling of the dilaton glueball to $\eta\eta$ as

TABLE VIII. Hadronic two-body decays of the dilatonic scalar glueball G_D with WSS model mass and extrapolated to the masses of $f_0(1500)$, $f_0(1710)$, and $M = 1865$ MeV, for $\lambda = 16.63\ldots12.55$. In decays into two pseudoscalar mesons, the two sets of values correspond to $x = 0$ and $x = 1$ in the coupling to the quark mass term (A22).

	$\Gamma^{WSS}_{G_D}$[MeV]	$\Gamma_{G_D(1506)}$[MeV]	$\Gamma_{G_D(1712)}$[MeV]	$\Gamma_{G_D(1865)}$[MeV]
$G_D \to \pi\pi$	12.4…16.5|15.2…20.1	12.6…16.7|15.4…20.4	14.6…19.3|17.0…22.5	16.1…21.3|18.3…24.2
$G_D \to KK$	4.16…5.51|50.5…67.0	4.43…5.87|50.4…66.8	7.49…9.93|49.4…65.4	9.87…13.1|48.8…64.7
$G_D \to \eta\eta$	1.85…3.71|14.1…18.7	1.93…3.82|14.1…18.7	2.77…4.96|13.9…18.4	3.38…5.75|13.7…18.1
$G_D \to \eta\eta'$		0.29…0.30|0	4.35…4.54|0	4.19…4.38|0
$G_D \to a_1\pi$	0.14…0.18	0.17…0.23	0.66…0.87	1.08…1.43
$G_D \to \rho\rho$			53.5…71.0	90.1…119
$G_D \to \omega\omega$			16.6…22.0	28.7…38.1
$G_D \to K^*K^*$				42.6…56.4
Sum	18.6…25.9|79.9…106	19.4…26.9|80.0…106	100…133|151…200	196…260|243…322

$$\mathcal{L}_{G_D\eta\eta} = \frac{3}{2}d_0 m_0^2(1-x)\sin\theta_P^2 G_D\eta\eta + \frac{3}{2}d_0 x m_\eta^2 G_D\eta\eta + \frac{d_1}{2}\partial_\mu\eta\partial_\nu\eta\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\mu}{\Box}\right)G_D. \tag{A25}$$

For the coupling to the η' meson we get $\cos\theta_P^2$ instead of $\sin\theta_P^2$.

The partial decay width for G_D decaying into two identical pseudoscalar mesons becomes

$$\Gamma_{G_D\to PP} = \frac{n_P d_1^2 M_D^3}{512\pi}\left(1 - 4\frac{m_P^2}{M_D^2}\right)^{1/2}\left(1 + \alpha\frac{m_P^2}{M_D^2}\right)^2, \tag{A26}$$

where P refers to pions ($n_P = 3$), kaons ($n_P = 4$) or $\eta^{(\prime)}$ ($n_P = 1$) mesons, and

$$\alpha = 4\left(3\frac{d_0}{d_1}x - 1\right) \tag{A27}$$

for pions and kaons, and

$$\alpha = 4\left[3\frac{d_0}{d_1}\left(x + \frac{m_0^2}{m_P^2}\sin^2\theta_P(1-x)\right) - 1\right] \tag{A28}$$

for $\eta\eta$, and with the replacement $\sin\theta_P \to \cos\theta_P$ for $\eta'\eta'$.

There is also a trilinear coupling of a dilatonic scalar glueball with one axial vector and one pseudoscalar meson, which has been neglected in [45], given by

$$\mathcal{L}_{G_D\Pi a^m} = -2d_6^m M_{\mathrm{KK}}\mathrm{tr}(\partial_\mu\Pi a_\nu^m)\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)G_D, \tag{A29}$$

with

$$d_6^m \equiv \sqrt{\frac{\kappa}{\pi}}\int \mathrm{d}z\psi'_{2m}H_D = \{11.768, 7.809, 2.350, ...\}\frac{1}{M_{\mathrm{KK}}N_c\sqrt{\lambda}}. \tag{A30}$$

Restricting ourselves to two-body decays, for which the relevant vertices for vector mesons are given in Sec. V A, the resulting partial decay widths are collected in Table VIII.

2. Exotic scalar glueball

The lighter exotic scalar glueball fluctuation with mass $M_E = 0.901M_{\mathrm{KK}} = 855$ MeV, which we have discarded from the spectrum when identifying the dilatonic scalar glueball with the ground-state glueball of QCD, is given by (4.5) with eigenvalue equation (4.6). This mode involves the metric component $h_{\tau\tau}$, which has no analogue in other holographic QCD models, and has therefore been termed “exotic” in [40]. Its canonical normalization is obtained from

$$\mathcal{L}_4|_{G_E^2} = \mathcal{C}\int \mathrm{d}r\frac{r^3 S_4(r)^2}{2L^3\mathcal{N}_E^2}\frac{5}{8}G_E(\Box - M_E^2)G_E = \frac{1}{2}G_E(\Box - M_E^2)G_E, \tag{A31}$$

with

$$\int \mathrm{d}r\frac{r^3 S_4(r)^2}{L^3} = 0.09183[S_4(r_{\mathrm{KK}})]^2\frac{r_{\mathrm{KK}}^4}{L^3} \tag{A32}$$

and

$$\mathcal{N}_E = 0.008751\lambda^{\frac{1}{2}}N_C M_{\mathrm{KK}}. \tag{A33}$$

Derivative couplings of pseudoscalars to G_E are given by

$$\mathcal{L}_{G_E} \supset -\mathrm{tr}\left\{c_1\left[\partial_\mu\Pi\partial_\nu\Pi\frac{\partial^\mu\partial^\nu}{M_E^2}G_E + \frac{1}{2}(\partial_\mu\Pi)^2\left(1 - \frac{\Box}{M_E^2}\right)G_E\right] + \breve{c}_1\partial_\mu\Pi\partial^\mu\Pi G_E\right\} \tag{A34}$$

with c_1 and $\breve{c}_1$ as in [45].

TABLE IX. Hadronic two-body decays of the exotic scalar glueball G_E with WSS model mass 855 MeV and extrapolated to the masses of $f_0(1500)$, $f_0(1710)$, and the scalar glueball at 1865 MeV proposed in [24], for $\lambda = 16.63\ldots12.55$. In decays into two pseudoscalar mesons, the two sets of values correspond to $x = 0$ and $x = 1$ in the coupling to the quark mass term (A37).

	$\Gamma^{WSS}_{G_E}$[MeV]	$\Gamma_{G_E(1506)}$[MeV]	$\Gamma_{G_E(1712)}$[MeV]	$\Gamma_{G_E(1865)}$[MeV]
$G_E \to \pi\pi$	72.2…95.7|84.9…113	135…179|142…189	154…205|161…213	169…224|175…231
$G_E \to KK$		120…158|229…304	152…202|255…338	176…233|273…362
$G_E \to \eta\eta$		31.3…45.4|57.7…76.4	40.0…56.9|65.1…86.3	45.9…64.6|69.8…92.5
$G_E \to \eta\eta'$		0.21…0.22|0	3.12…3.26|0	3.01…3.14|0
$G_E \to a_1\pi$		0.06…0.08	0.55…0.73	1.36…1.80
$G_E \to \rho\rho$			0.77…1.02	2.91…3.86
$G_E \to \omega\omega$			0.19…0.26	0.84…1.12
$G_E \to K^*K^*$				0.15…0.20
Sum	72.2…95.7|84.9…113	286…383|430…570	351…469|483…640	398…530|521…691

In the Witten-Veneziano mass term for η_0^2, inducing the metric fluctuations leads to additional couplings between the scalar glueballs and η_0. For the exotic scalar glueball it is given by

$$\mathcal{L}_{\eta_0} \supset -\frac{5}{2} m_0^2 \eta_0^2 \check{c}_0 G_E, \tag{A35}$$

with ($H_E \equiv S_4/\mathcal{N}_E$)

$$\check{c}_0 = \frac{3}{4} U_{\mathrm{KK}}^3 \int_{U_{\mathrm{KK}}}^{\infty} \mathrm{d}U H_E(U) U^{-4} \approx \frac{15.829}{\sqrt{\lambda} N_c M_{\mathrm{KK}}} \tag{A36}$$

as previously studied in [47].

Assuming the coupling of the exotic scalar glueball to quark masses to be of the form

$$\mathcal{L}_{G_E q\bar{q}} = 5\check{c}_m G_E \mathcal{L}_m^{\mathcal{M}} \tag{A37}$$

with $\check{c}_m$ being of the same order as $\check{c}_0$, i.e.

$$\check{c}_m = x\check{c}_0, \qquad x = \mathcal{O}(1), \tag{A38}$$

we get

$$\mathcal{L}_{G_E\eta\eta'} = \frac{5}{2}(1-x)\check{c}_0 \sin(2\theta_P) m_0^2 G_E \eta\eta'. \tag{A39}$$

All together we obtain the coupling of the exotic scalar glueball to $\eta\eta$ as

$$\begin{aligned}\mathcal{L}_{G_E\eta\eta} = {}& \frac{5}{2}\check{c}_0 m_0^2 (x-1)\sin\theta_P^2 G_E\eta\eta - \frac{5}{2}\check{c}_0 x m_\eta^2 G_E \eta\eta \\ & -\frac{c_1}{2}\partial_\mu\eta\partial_\nu\eta\left(\frac{1}{2}\eta^{\mu\nu}\left(1-\frac{\Box}{M_E^2}\right)+\frac{\partial^\mu\partial^\nu}{M_E^2}\right)G_E \\ & -\frac{\check{c}_1}{2}\partial_\mu\eta\partial^\mu\eta G_E. \end{aligned}\tag{A40}$$

For pions and kaons we have

$$|\mathcal{M}_{G_E\to PP}| = \frac{1}{4}|20\check{c}_0 m_P^2 x + 2\check{c}_1(M_E^2 - 2m_P^2) + c_1 M_E^2| \tag{A41}$$

and for η

$$\begin{aligned}|\mathcal{M}_{G_E\to\eta\eta}| = {}& \frac{1}{4}|-20\check{c}_0 m_0^2(x-1)\sin\theta_P^2 + 20\check{c}_0 m_P^2 x \\ & + 2\check{c}_1(M_E^2 - 2m_P^2) + c_1 M_E^2|, \end{aligned}\tag{A42}$$

from which the η' amplitude is obtained by the replacement $\sin\theta_P \to \cos\theta_P$.

In both cases the decay width is given by

$$\Gamma_{G_E\to PP} = \frac{n_P}{2}\frac{1}{8\pi}|\mathcal{M}_{G_E\to PP}|^2 \frac{|\mathbf{p}_P|}{M_E^2}. \tag{A43}$$

The coupling of the exotic scalar glueball to one axial vector meson and one pseudoscalar meson is given by

$$\mathcal{L}_{G_E\Pi a^m} = 2c_6^m M_{\mathrm{KK}} \mathrm{tr}(\partial_\mu \Pi a_\nu^m)\frac{\partial^\mu\partial^\nu}{M_E^2} G_E, \tag{A44}$$

with

$$\begin{aligned}c_6^m &= \sqrt{\frac{\kappa}{\pi}}\int \mathrm{d}z \psi'_{2m}\left[\frac{1}{4}+\frac{3}{5K-2}\right]H_E \\ &= \{57.659, 72.057, 65.190, \ldots\}\frac{1}{M_{\mathrm{KK}} N_c \sqrt{\lambda}}. \end{aligned}\tag{A45}$$

Restricting ourselves to two-body decays, for which the relevant vertices for vector mesons are given separately in Appendix B, the resulting partial decay widths are collected in Table IX.

3. Tensor glueball

The tensor glueball fluctuations read

$$h_{\mu\nu} = q_{\mu\nu}\frac{r^2}{L^2\mathcal{N}_T} T_4(r) G_T(x^\sigma), \tag{A46}$$

where $q_{\mu\nu}$ is a symmetric transverse traceless polarization tensor, which we normalize such that $q_{\mu\nu}q^{\mu\nu} = 1$, differing from [45].

$T_4(r)$ satisfies the same eigenvalue equation as in the case of the dilatonic scalar glueball, (A2), but it acquires a different normalization. The Lagrangian reads

$$\mathcal{L}_4|_{G_T^2} = \mathcal{C}\int \mathrm{d}r \frac{r^3 T_4(r)^2}{4L^3\mathcal{N}_T^2} G_T(\Box - M^2)G_T = \frac{1}{2}G_T(\Box - M^2)G_T, \tag{A47}$$

with

$$\int \mathrm{d}r \frac{r^3 T_4(r)^2}{2L^3} = 0.112735[T_4(r_{\mathrm{KK}})]^2 \frac{r_{\mathrm{KK}}^4}{L^3} \tag{A48}$$

and

$$\mathcal{N}_T = 0.00969598\lambda^{\frac{1}{2}} N_C M_{\mathrm{KK}} = \frac{1}{2\sqrt{3}}\mathcal{N}_D. \tag{A49}$$

This leads to

$$\mathcal{L}_{G_T\Pi\Pi} = t_1 \mathrm{tr}(\partial_\mu \Pi \partial_\nu \Pi) G_T^{\mu\nu} \tag{A50}$$

with ($T \equiv T_4/\mathcal{N}_T$)

$$t_1 = \frac{1}{\pi}\int \mathrm{d}z K^{-1} T = \frac{59.6729}{\sqrt{\lambda} M_{\mathrm{KK}} N_c} = 2\sqrt{3} d_1. \tag{A51}$$

Here no additional couplings arise from the mass terms of the pseudoscalars, because the tensor glueball fluctuations are traceless.

There is also a coupling of the tensor glueball to one axial vector and one pseudoscalar meson,

$$\mathcal{L}_{G_T \Pi a^m} = -2t_6 M_{\mathrm{KK}} \mathrm{tr}(\partial_\mu \Pi a_\nu^m) G_T^{\mu\nu} \tag{A52}$$

with

$$t_6 = \sqrt{\frac{\kappa}{\pi}} \int \mathrm{d}z \psi'_{2m} T = \{40.764, 27.050, 8.140, \ldots\} \frac{1}{M_{\mathrm{KK}} N_c \sqrt{\lambda}}. \tag{A53}$$

TABLE X. Hadronic two-body decays of the tensor glueball G_T with WSS model 1487 MeV mass and extrapolated to masses of 2000 and 2400 MeV, for $\lambda = 16.63 \ldots 12.55$. In decays involving f_1 we additionally vary $\theta_f = 20.4° \ldots 26.4°$. Partial decay widths much smaller than 1 MeV are left out.

	$\Gamma_{G_T}^{WSS}$ [MeV]	$\Gamma_{G_T(2000)}$ [MeV]	$\Gamma_{G_T(2400)}$ [MeV]
$G_T \to \pi\pi$	19.9…26.3	27.7…36.8	33.8…44.7
$G_T \to KK$	6.66…8.83	19.2…25.4	29.2…38.6
$G_T \to \eta\eta$	1.02…1.35	3.97…5.26	6.48…8.58
$G_T \to a_1\pi$	0.53…0.71	5.12…6.78	8.00…10.6
$G_T \to \rho\rho$		270…358	382…507
$G_T \to \omega\omega$		88.2…117	127…169
$G_T \to K^*K^*$		240…318	417…552
$G_T \to f_1\eta$		0.98…1.71	3.97…6.89
$G_T \to \eta'\eta'$			0.92…1.22
$G_T \to \phi\phi$			76.7…102
Total	28.1…37.2	655…869	1084…1437

Restricting ourselves to two-body decays, for which the relevant vertices for vector mesons are given in Sec. V B, the resulting partial decay widths are collected in Table X.

Recently, Ref. [102] calculated branching ratios of tensor glueball decays in a chiral hadronic model, the so-called extended linear sigma model, where the ratios of all the decay modes of Table X can be obtained, although not their absolute magnitudes. In that model a similar dominance of decays into two vector mesons (when kinematically allowed) has been obtained, which is numerically even more pronounced.[12] The authors of Ref. [102] also gave a rough estimate of $\Gamma(G_T \to \pi\pi) \sim 15$ MeV, which turns out to be comparable with the WSS result.

APPENDIX B: RADIATIVE DECAYS OF THE EXOTIC SCALAR GLUEBALL

The exotic glueball interactions contain the vertices

$$\begin{aligned}\mathcal{L}_{G_E v^m v^n} = -\mathrm{tr}\Big\{ & c_2^{mn} M_{\mathrm{KK}}^2 \left[v_\mu^m v_\nu^n \frac{\partial^\mu \partial^\nu}{M_E^2} G_E + \frac{1}{2} v_\mu^m v^{n\mu}\left(1 - \frac{\Box}{M_E^2}\right) G_E \right] \\ & + c_3^{mn}\left[\eta^{\rho\sigma} F_{\mu\rho}^m F_{\nu\sigma}^n \frac{\partial^\mu \partial^\nu}{M_E^2} G_E - \frac{1}{4} F_{\mu\nu}^m F^{n\mu\nu}\left(1 + \frac{\Box}{M_E^2}\right) G_E\right] + 3c_4^{mn} \frac{M_{\mathrm{KK}}^2}{M_E^2} v_\mu^n F^{m\mu\nu} \partial_\nu G_E \\ & + \breve{c}_2^{mn} M_{\mathrm{KK}}^2 v_\mu^m v^{n\mu} G_E + \frac{1}{2}\breve{c}_3^{mn} F_{\mu\nu}^m F^{n\mu\nu} G_E, \Big\}\end{aligned} \tag{B1}$$

with coupling constants

[12]For example, while in the WSS model the branching ratio $\rho\rho : \pi\pi$ is around 10–11 for a tensor glueball mass between 2000 and 2400 MeV, in Ref. [102] it varies between 60 and 50. Also the branching ratio $\rho\rho : a_1\pi$ is 6–5 times larger there for this mass range.

$$c_2^{mn} = \kappa \int \mathrm{d}z K \psi'_{2m-1}\psi'_{2n-1}\bar{H}_E = \frac{\{7.116, ...\}}{M_{\mathrm{KK}} N_c \sqrt{\lambda}},$$
$$c_3^{mn} = \kappa \int \mathrm{d}z K^{-1/3} \psi_{2m-1}\psi_{2n-1}\bar{H}_E = \frac{\{69.769, ...\}}{M_{\mathrm{KK}} N_c \sqrt{\lambda}},$$
$$c_4^{mn} = \kappa \int \mathrm{d}z \frac{20zK}{(5K-2)^2} \psi_{2m-1}\psi'_{2n-1} H_E = \frac{\{-10.5798, ...\}}{M_{\mathrm{KK}} N_c \sqrt{\lambda}},$$
$$\breve{c}_2^{mn} = \frac{\kappa}{4} \int \mathrm{d}z K \psi'_{2m-1}\psi'_{2n-1} H_E = \frac{\{2.966, ...\}}{M_{\mathrm{KK}} N_c \sqrt{\lambda}},$$
$$\breve{c}_3^{mn} = \frac{\kappa}{4} \int \mathrm{d}z K^{-1/3} \psi_{2m-1}\psi_{2n-1} H_E = \frac{\{18.122, ...\}}{M_{\mathrm{KK}} N_c \sqrt{\lambda}}, \tag{B2}$$

where $\bar{H}_E = [\frac{1}{4} + \frac{3}{5K-2}] H_E$.

Calculating the amplitude for different polarizations we get

$$|\mathcal{M}_T^{(G_E \to v^1 v^1)}| = \frac{1}{2}[c_3(M_E^2 - 4m_v^2) - 6c_4 M_{\mathrm{KK}}^2 - 4\breve{c}_2 M_{\mathrm{KK}}^2 - 2\breve{c}_3(M_E^2 - 2m_v^2)]$$
$$|\mathcal{M}_L^{(G_E \to v^1 v^1)}| = \frac{c_2 M_{\mathrm{KK}}^2 (M_E^2 - 4m_v^2) + 2\breve{c}_2 M_{\mathrm{KK}}^2 (M_E^2 - 2m_v^2) + 6c_4 M_{\mathrm{KK}}^2 m_v^2 + 4\breve{c}_3 m_v^4}{2m_v^2}. \tag{B3}$$

1. Exotic scalar glueball 1γ decays

For the decay in one vector meson and one photon, we use

$$\mathcal{L}_{G_E \mathcal{V} v^m} = -\mathrm{tr}\Bigg\{ c_3^{m\mathcal{V}} \left[2\eta^{\rho\sigma} F^m_{\mu\rho} F^{\mathcal{V}}_{\nu\sigma} \frac{\partial^\mu \partial^\nu}{M_E^2} G_E - \frac{1}{2} F^m_{\mu\nu} F^{\mathcal{V}\mu\nu} \left(1 + \frac{\Box}{M_E^2}\right) G_E \right] + 3c_4^{\mathcal{V}n} \frac{M_{\mathrm{KK}}^2}{M_E^2} v^n_\mu F^{\mathcal{V}\mu\nu} \partial_\nu G_E + \breve{c}_3^{m\mathcal{V}} F^m_{\mu\nu} F^{\mathcal{V}\mu\nu} G_E \Bigg\}, \tag{B4}$$

with

$$c_3^{m\mathcal{V}} = \kappa \int \mathrm{d}z K^{-1/3} \psi_{2m-1} \bar{H}_E = \frac{\{1.551, ...\}}{M_{\mathrm{KK}} N_c^{\frac{1}{2}}},$$
$$c_4^{\mathcal{V}m} = \kappa \int \mathrm{d}z \frac{20ZK}{(5K-2)^2} \psi'_{2m-1} H_E = \frac{\{-0.262, ...\}}{M_{\mathrm{KK}} N_c^{\frac{1}{2}}},$$
$$\breve{c}_3^{m\mathcal{V}} = \frac{\kappa}{4} \int \mathrm{d}z K^{-1/3} \psi_{2m-1} H_E, = \frac{\{0.425, ...\}}{M_{\mathrm{KK}} N_c^{\frac{1}{2}}}$$

to obtain

$$|\mathcal{M}_T^{(G_E \to v^m \mathcal{V})}| = \frac{(M_E^2 - m_v^2)}{2M_E^2} |3c_4^{\mathcal{V}n} M_{\mathrm{KK}}^2 + 2\breve{c}_3^{m\mathcal{V}} M_E^2 + c_3^{m\mathcal{V}}(m_v^2 - M_E^2)| \mathrm{tr}(eQT_{v^m}). \tag{B5}$$

2. Exotic scalar glueball 2γ decays

The two-photon decay rate is obtained from

$$\mathcal{L}_{G_E \mathcal{V}\mathcal{V}} = -\mathrm{tr}\left\{ c_3^{\mathcal{V}\mathcal{V}} \left[F^{\mathcal{V}}_{\mu\rho} F^{\mathcal{V}\rho}_{\nu} \frac{\partial^\mu \partial^\nu}{M_E^2} G_E - \frac{1}{4} F^{\mathcal{V}}_{\mu\nu} F^{\mathcal{V}\mu\nu} \left(1 + \frac{\Box}{M_E^2}\right) G_E \right] + \frac{1}{2} \breve{c}_3^{\mathcal{V}\mathcal{V}} F^{\mathcal{V}}_{\mu\nu} F^{\mathcal{V}\mu\nu} G_E \right\} \tag{B6}$$

with

TABLE XI. Radiative and two-vector decays of the exotic scalar glueball G_E with WSS model mass 855 MeV and extrapolated to the masses of $f_0(1500)$, $f_0(1710)$ and the scalar glueball at 1865 MeV proposed in [24].

	$\Gamma_{G_E^{\rm WSS}}$ [keV]	$\Gamma_{G_E(1506)}$ [keV]	$\Gamma_{G_E(1712)}$ [keV]	$\Gamma_{G_E(1865)}$ [keV]
$G_E \to \rho\rho$			771…1022	2910…3857
$G_E \to \omega\omega$			194…257	843…1117
$G_E \to K^* K^*$				149…197
$G_E \to \rho\gamma$	0.047	13.4	20.7	26.4
$G_E \to \omega\gamma$	0.003	1.4	2.23	2.86
$G_E \to \phi\gamma$		0.30	0.98	1.72
$G_E \to \gamma\gamma$	0.043…0.033	0.076…0.058	0.087…0.066	0.095…0.071

$$c_3^{\mathcal{V}\mathcal{V}} = \kappa \int \mathrm{d}z K^{-1/3} \bar{H}_E = \frac{237.587\kappa}{M_{\rm KK} N_c \lambda^{1/2}} = 0.0355 \frac{\lambda^{\frac{1}{2}}}{M_{\rm KK}}, \tag{B7}$$

$$\breve{c}_3^{\mathcal{V}\mathcal{V}} = \frac{\kappa}{4} \int \mathrm{d}z K^{-1/3} H_E = \frac{71.18\kappa}{M_{\rm KK} N_c \lambda^{1/2}} = 0.0106 \frac{\lambda^{\frac{1}{2}}}{M_{\rm KK}}, \tag{B8}$$

yielding

$$|\mathcal{M}_T^{(G_E\to\mathcal{V}\mathcal{V})}| = \frac{M_E^2}{2} (c_3^{\mathcal{V}\mathcal{V}} - 2\breve{c}_3^{\mathcal{V}\mathcal{V}}) \mathrm{tr}(e^2 Q^2). \tag{B9}$$

In Table XI the results for the partial widths for the radiative and two-vector decays of the exotic scalar glueball are given when the above amplitudes are substituted in the respective formulas for the dilaton scalar glueball, (5.4), (5.8), and (5.12). Again, these are evaluated for the WSS model mass, which is only 855 MeV for the exotic scalar glueball, as well as for three higher masses, corresponding to the glueball candidates $f_0(1500)$, $f_0(1710)$, and the one proposed in [24]. While the total decay width of G_E is much larger than that of G_D at equal mass, see Table IV, the radiative and two-vector widths of G_E are much smaller than those of G_D, see Table V.

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