$(1+1)$D QCD with heavy adjoint quarks

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In this paper, we determine, at weak coupling, the nonrelativistic n-body Schrödinger equation that describes the low-lying color singlet bound states of two-dimensional adjoint quantum chromodynamics with heavy quarks. In the case of three adjoint quarks, we show that the three-body equation reduces equivalently to the Schrödinger equation that describes a point electric dipole in an electric field in a plane angular sector. We conjecture that the three-body problem is exactly solvable. We show that the eigenstates are given in terms of the triconfluent Heun functions. Equivalently, our conjecture also implies a bound state of three adjoint quarks is described by a particle in two dimensions confined in a Cornell potential. We expect the n-parton problem also to be solvable in a similar approach. We also comment on the Hamiltonian that describes the associated classical n-parton system.

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I. INTRODUCTION

Quantum chromodynamics (QCD) is the fundamental theory that describes quarks and gluons (in four spacetime dimensions). In particle (accelerators and) detectors, the quarks and gluons are always observed bound together into hadrons. Thus, at low energy, the theory is believed to exhibit confinement. The main goal in QCD [and in general in Yang-Mills (YM) theory] has been to understand confinement (and/or the existence of a mass gap). However, a complete understanding of the phenomenon is still missing. In part, this is because the phenomenon is nonperturbative, and the theory is in general complex, for example, in terms of the number of dynamical degrees of freedom it contains and the phenomena it describes.

In two spacetime dimensions, adjoint QCD is a relatively simple and tractable theory that exhibits, among some other common properties, confinement [1–4] (and at finite temperature deconfinement [5]).[1] Therefore, it is useful to study this simple model to gain insights into confinement and other essential phenomena. A better understanding of the theory will be also useful in constructing a string world sheet realization of QCD strings. It is believed that at low energy, the properties of QCD might be reproduced by an effective theory of interacting long strings [6,7]. In this paper, we consider this model with these perspectives in mind.

In two spacetime dimensions, a gluon has no propagating degrees of freedom since there are no transverse spatial dimensions. Therefore, it cannot form a color singlet bound state with a matter quanta.[2] In adjoint QCD, thus, the quantum states are color singlet states of adjoint quarks bound together by nondynamical, stringlike, color flux tubes that confine the color gauge potential lines. The color singlet or gauge invariant bound states can contain two or more number of adjoint quarks. Thus, a color singlet bound state can be viewed as a chain of adjoint quarks on a closed string. However, depending on whether the number of the adjoint quarks is even or odd, the bound state is either a bosonic or fermionic state.

In two-dimensional QCD with fundamental quarks, all the meson states consist a quark and an antiquark pair, and they are arranged in a single Regge trajectory [8]. In adjoint QCD, on the other hand, it is expected that the states are grouped into separate multiple Regge trajectories [9]. See also [5].

As suggested by ‘t Hooft [10], considering the large N limit (where N is the rank of the gauge group) simplifies the theory. In this limit, there exists a systematic expansion in powers of $1/N$. This is easy to see since, in general, the theory can be obtained by dimensional reduction from higher dimensional gauge theories [9,11]. The theory, however, despite being two-dimensional and/or relatively simple, in the sense that the gluons are, for example, nondynamical, has not been solved completely, even in the

[1]Two-dimensional fundamental QCD does not exhibit a deconfinement transition [5].

[2]However, it mediates a nonlocal Coulomb force between the quarks.

large N limit. This is mainly because, in this limit, pair production and pair annihilation are not suppressed [9,11]. Therefore, the Hamiltonian relates states with different number of adjoint quarks or partons. This makes the computation of the exact spectrum analytically difficult.

The adjoint spectrum has been computed, however, approximately in the large N limit in [9,11] and recently, for finite values of N in [12]. In these papers, the authors use discrete light cone quantization, and they numerically diagonalize the light cone Hamiltonian. In this approach, the light cone momentum and the momentum fraction carried by a quark are discretized. Thus, since the light cone Hamiltonian and momentum commutes, for a given total momentum, the (approximate) truncated space of states is finite dimensional, and therefore, diagonalizing the Hamiltonian is relatively tractable. More recently, the low energy approximate spectrum has been also computed by diagonalizing the Hamiltonian in a set of states created by operators with dimensions below a certain cutoff [13]. The main point is that the two point functions of low dimension operators with a high dimension operator goes to zero exponentially fast. Therefore, the high dimension operators decouple from the low mass spectrum [14].[3] As a result, they can be ignored in the approximation with a small error. The error depends on the cutoff. Also more recently, a candidate relativistic Hamiltonian describing the high energy asymptotics of confining string has been obtained from effective long string world sheet theory [7,15,16]. The Hamiltonian equivalently describes a one-dimensional chain of ordered massless particles with nearest neighbor interaction. The interaction potential is related to the potential in Toda lattice (in certain limit) [17], and the Hamiltonian has been shown to be integrable.

The low-lying bound states of heavy quarks are believed to be described by a nonrelativistic Schrödinger equation. In this paper, we determine the nonrelativistic Schrödinger equation that describes the low-lying color singlet bound states of the two-dimensional adjoint QCD with heavy quarks. We work in the large N or planar limit. We keep the ('t Hooft) coupling parametrically small and fixed. We use the method employed in the papers [18–22]. In the paper [20], the authors obtained at weak coupling the nonrelativistic Schrödinger equation that describes the 't Hooft model [8] in the limit of heavy quarks and large number of colors. In this model, the quarks are in the fundamental representation of the gauge group. They also computed (at weak coupling) exactly the eigenstates and the spectrum. Interestingly, the nonrelativistic limit of 't Hooft model was actually discussed, and the same results were obtained earlier in [18,19].[4]

[3]The high dimension operators correspond to heavy states, and therefore, they decouple from the low energy physics. In this case, the Hamiltonian is truncated.

[4]I thank Igor Klebanov for bringing to my attention these interesting earlier works.

In Sec. II, we review in detail the method discussed in the papers [18–22]. We also discuss the results obtained in the papers [18–20] by applying the method to the two-dimensional 't Hooft model [8]. In Sec. III, using the same method, we derive, at weak coupling, the equation that describes the low-lying bound states of the two-dimensional adjoint QCD with heavy quarks. We find that the equation equivalently describes a particle confined to the surface of an inverted t-gonal pyramid potential in n dimensions. For a bound state with three constituent quarks, we conjecture that the corresponding equation is exactly solvable.[5] We show that the eigenstates are given in terms of the triconfluent Heun functions. We discuss our approach and the spectrum of the bound states of two and three adjoint quarks in Sec. IV. On general grounds, we expect the n-body problem also to be solvable in a similar approach.

We provided in Appendix A representative plots of closed periodic orbits in the associated classical system of the three quarks system. We note that the classical dynamics is sensitive to initial conditions. On general grounds, we also expect sensitivity to initial conditions in the general case. Chaotic dynamical systems are in particular known to exhibit such behavior. However, in general, sensitivity to initial conditions alone does not necessarily imply chaos. Thus, the general n-body classical system might be of interest to gain insights into chaos theory. We hope to study the system in a similar way to that of [23] in a separate paper in the future. In Appendix B, we collected some interesting intermediate results and useful equivalence relations.

II. THE LARGE MASS LIMIT OF THE 'T HOOFT MODEL

In this section, we summarize the facts about the 't Hooft model of two-dimensional QCD [8,10] with gauge group $U(N)$ and fundamental fermions in the large constituent quark mass limit. In the next sections, we will generalize this discussion to the case of two-dimensional adjoint QCD. We will use the discussion presented in [20], but we will take here the quark masses to be equal, $m_1 = m_2 = m$. See also [18] for a similar discussion.

The 't Hooft equation [8,10] involves the wave function of a meson (a bound state of quark and antiquark pair), $\phi(\xi)$. Here, $0 \leq \xi \leq 1$ is the fraction of the light cone momentum carried by one of the two quarks in the meson. Of course, the fraction carried by the other is $1-\xi$. The equation takes the form,

$$\mu^2\phi(\xi) = \alpha\left(\frac{1}{\xi}+\frac{1}{1-\xi}\right)\phi(\xi) - \text{p.v.}\int_0^1 d\xi' \frac{\phi(\xi')}{(\xi'-\xi)^2}, \tag{2.1}$$

[5]In the sense that one can write down a closed analytic expression.

where μ is a dimensionless[6] measure of the meson mass M,

$$M^2 = \frac{g^2 N}{\pi}\mu^2, \tag{2.2}$$

and

$$\alpha = \frac{\pi m^2}{g^2 N} - 1 \tag{2.3}$$

is a dimensionless measure of the size of the ‘t Hooft coupling or equivalently, the size of the coupling at the scale of the quark mass m. Large α corresponds to weak coupling. p.v. in (2.1) stands for principal value (see [18,20]).

We are interested in studying this system in the limit $\alpha \gg 1$. Loosely speaking, the first term gives a large contribution, of order α, to μ^2, and the second term gives a small correction. Also, the first term can be thought of as the contribution of the masses of the quarks to the mass of the meson. For $g = 0$, the second term, which is what gives confinement, is absent, and we get a continuum of values of μ^2, starting from the minimal value obtained when $\xi = \frac{1}{2}$,

$$\mu_0^2 = 4\alpha, \tag{2.4}$$

or using (2.2), (2.3), $M^2 = (2m)^2$. This is precisely what one would expect for a state of two quarks of mass m. As ξ deviates from $\frac{1}{2}$, the order α contribution to μ^2 grows. Thus, if we want μ^2 to be 4α plus a small correction, we want the wave function $\phi(\xi)$ to be sharply peaked around $\xi = \frac{1}{2}$ (see also [18]).

Now, suppose we want to turn on the coupling g, while keeping the ratio α very large. In the notation of [20], we take $a_1 = a_2 = 1$, so $\alpha_1 = \alpha_2 = \alpha$, $k_1 = k_2 = \frac{1}{2}$, and write

$$\xi = \frac{1}{2} + \omega. \tag{2.5}$$

The ‘t Hooft equation (2.1) takes the form,

$$\mu^2\phi(\omega) = \alpha\left(\frac{1}{\frac{1}{2}+\omega} + \frac{1}{\frac{1}{2}-\omega}\right)\phi(\omega) - \text{p.v.}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\omega' \frac{\phi(\omega')}{(\omega'-\omega)^2}. \tag{2.6}$$

As mentioned above, we are looking for states whose μ^2 is of the form,

$$\mu^2 = \mu_0^2 + \gamma, \tag{2.7}$$

where μ_0 is given by (2.4), and γ grows slower than α at large α, *i.e.*, $\lim_{\alpha\to\infty} \frac{\gamma}{\alpha} = 0$. Substituting (2.7) into (2.6), we get a ‘t Hooft type equation for γ,

$$\gamma\phi(\omega) = \frac{4\alpha\omega^2}{\frac{1}{4}-\omega^2}\phi(\omega) - \text{p.v.}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\omega' \frac{\phi(\omega')}{(\omega'-\omega)^2}. \tag{2.8}$$

We are looking for solutions to this equation in which $\gamma \ll \alpha$. This means that the wave function $\phi(w)$ is sharply peaked around $w = 0$. Thus, we can neglect the ω^2 in the denominator on the right-hand side (rhs) of (2.8), so it takes the form,

$$\gamma\phi(\omega) = 16\alpha\omega^2\phi(\omega) - \text{p.v.}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\omega' \frac{\phi(\omega')}{(\omega'-\omega)^2}. \tag{2.9}$$

To formalize the requirement that for large α, the wave function $\phi(\omega)$ is sharply peaked at $\omega = 0$, we demand that if we rescale ω by a factor $t = t(\alpha)$, that we need to determine; *i.e.*, we write

$$\omega = st, \tag{2.10}$$

then the wave function,[7]

$$\phi(\omega) = \phi(st) = f(s), \tag{2.11}$$

where f is a function that is not sensitive to α. Plugging this ansätz into (2.9) and demanding that the two terms on the rhs scale in the same way with α as $\alpha \to \infty$, we find that we must take

$$t = \alpha^{-\frac{1}{3}}, \tag{2.12}$$

and, if we take this value for t, then γ on the left-hand side (lhs) behaves like $\gamma \sim \alpha^{\frac{1}{3}}$. Thus, it is convenient to define

$$\gamma = \bar{\gamma}\alpha^{\frac{1}{3}}, \tag{2.13}$$

in terms of which the ‘t Hooft equation (2.9) takes the form,

$$\bar{\gamma}f(s) = 16s^2 f(s) - \text{p.v.}\int_{-\infty}^{\infty} ds' \frac{f(s')}{(s'-s)^2}. \tag{2.14}$$

A number of things to note at this point:

(1) Since $\bar{\gamma}$ is obtained by solving a problem, (2.14), which does not contain the expansion parameter α, it does not depend on α. Therefore, the solution for γ, (2.13), grows slower with α than the leading term in (2.7), in agreement with the assumptions that went into the analysis.

(2) In going from (2.9) to (2.14), we extended the range of integration. In fact, the correct range of integration in (2.14) should have been taken to be $-1/2t$ to $+1/2t$, with t given by (2.12). In the limit $\alpha \to \infty$, the boundaries of the integral go to infinity, so we expect the mistake in extending them to be small.

[6]The gauge coupling in two dimensions is dimensionful.

[7]Note that the Eq. (2.9) is linear in ϕ.

How small depends on the behavior of the solution $f(s)$ for large values of the integrand. We comment on this later in the section.

The variable s in (2.14) is a momentum type variable—it is related via (2.5), (2.10), (2.12), to the light cone momentum fraction ξ carried by a quark.[8] To solve (2.14), it is useful to Fourier transform it to position space, as done in [18,20]: we define

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds f(s) e^{isx}, \tag{2.15}$$

and write (2.14) as an equation for $\hat{f}(x)$,

$$\bar{\gamma}\hat{f}(x) = -16\hat{f}''(x) + \pi|x|\hat{f}(x). \tag{2.16}$$

The lhs and the first term on the rhs are obvious, and the second term on the rhs relies on the definition of the principal value [see, *e.g.*, Eq. (4) in [20] and Eq. (3.57) in [18]].

Comments:

(1) Equation (2.16) is interesting: it is the Schrödinger equation for a particle in the potential $|x|$.[9] This is basically the confining Coulomb potential in one spatial dimension. An interesting fact is that the treatment of the pole at zero momentum exchange in (2.14) (the $i\epsilon$ prescription associated with the principal value in that equation) is directly related to the fact that the potential rises both for positive and for negative x.

(2) Of course, the momentum s is lightlike momentum, and the conjugate position variable x is thus the light cone separation of the two quarks. Nevertheless, we get a compelling picture of the meson as a pair of quarks separated by the amount x in a lightlike direction, with the energy of the pair growing linearly with their separation. We will make use of this picture later, in the adjoint case.

The solution of (2.16) is an Airy function [18,20]. This is easy to see as follows. Consider first, the region $x > 0$. In this region, the Schrödinger equation (2.16) can be written as

$$\hat{f}(x) = g(y), \tag{2.17}$$

where $g(y)$ is a solution of the equation,

$$g''(y) = yg(y), \tag{2.18}$$

and

$$y = a(x - b), \qquad a = \left(\frac{\pi}{16}\right)^{\frac{1}{3}}, \qquad b = \frac{\bar{\gamma}}{\pi}. \tag{2.19}$$

This is in agreement with Eqs. (18), (19) in [20] and Eqs. (3.59), (3.60) in [18].

The solution of (2.18) is $g(y) = \mathrm{Ai}(y)$. The reason we need the Ai Airy function rather than the Bi is the usual: we need the solution to go to zero as $x, y \to \infty$, and the Ai function indeed goes to zero at infinity, while Bi blows up exponentially.

Thus, for $x > 0$, the solution to the Schrödinger equation (2.16) is $\hat{f}(x) = \mathrm{Ai}(y)$. What about negative x? Since the problem (2.16) is symmetric under $x \to -x$, there are two kinds of eigenstates, symmetric and antisymmetric under $x \to -x$. As usual, we will label the bound states by an integer n, with $n = 0, 2, 4, \ldots$ corresponding to the symmetric solutions, and $n = 1, 3, 5, \ldots$ corresponding to the antisymmetric ones.

Let us start with the antisymmetric ones. These must vanish at the origin, $\hat{f}_n(x = 0) = 0$, which means that

$$\mathrm{Ai}(-ab_n) = 0, \qquad b_n = \frac{\bar{\gamma}_n}{\pi}. \tag{2.20}$$

So, $-ab_n$ must be zeros of the Airy function Ai.

Similarly, for the symmetric wave functions, the derivative of the wave function must vanish at $x = 0$. Therefore, for the symmetric ones, $-ab_n$ must be zeros of the derivative of the Airy function Ai′.

For highly excited states, the authors [20] assert that the values $\bar{\gamma}_n$ have the asymptotic behavior (see also [24]),

$$\bar{\gamma}_n \simeq \left[3\pi^2\left(n + \frac{1}{2}\right)\right]^{\frac{2}{3}}. \tag{2.21}$$

We show this by applying semiclassical quantization to classical periodic orbits later in Sec. IV.

Another interesting question is, what is the momentum space wave function $f(s)$ (2.15) ? To compute it, we need to do the inverse Fourier transform,

$$f(s) = \int_{-\infty}^{\infty} dx \hat{f}(x) e^{-isx}. \tag{2.22}$$

We start by breaking the integral (2.22) into two parts,

$$\begin{aligned} f_n(s) &= \int_{-\infty}^{\infty} \hat{f}_n(x) e^{-isx} dx \\ &= \int_{-\infty}^{0} \hat{f}_n(x) e^{-isx} dx + \int_{0}^{\infty} \hat{f}_n(x) e^{-isx} dx, \end{aligned} \tag{2.23}$$

where n is a positive integer and labels the zeros $\bar{\gamma}_n$. For even n, since $\hat{f}_n$ is invariant under parity, we have

[8] One can think of s as follows. In the center of mass frame, the two quarks have energy E and momentum $\pm p$. s is proportional to p, and the wave function $f(s)$ is the momentum space wave function of the bound state.

[9] This is also the equation that governs a point electric dipole on a line with electric field proportional to x. As we will see in Sec. IV, viewing it in this picture is more useful.

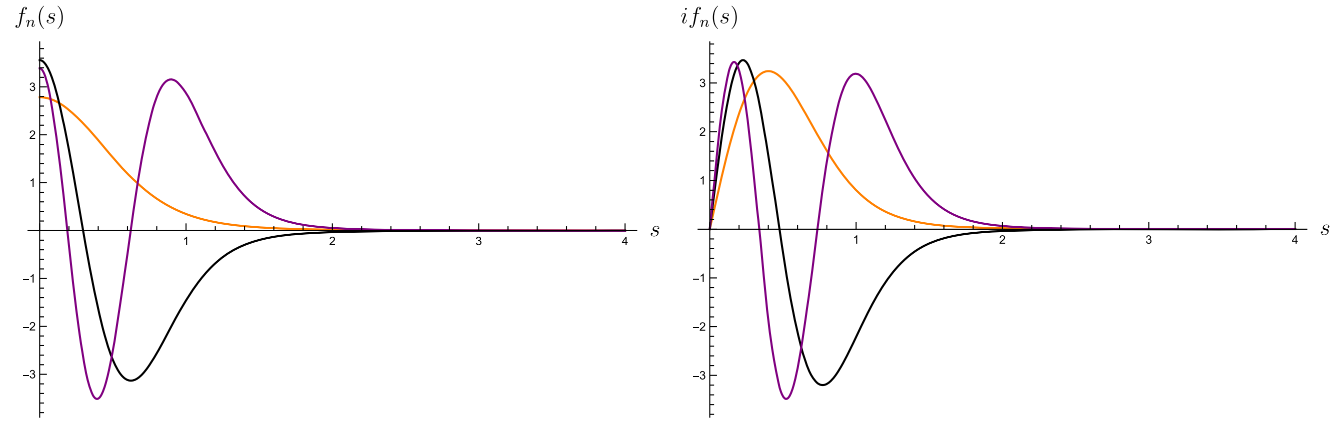

FIG. 1. Numerical plots of the momentum space wave function $f_n(s)$. On the left-hand side, we have $f_n(s)$ for $n = 0$ (orange), $n = 2$ (black), and $n = 4$ (purple). On the right-hand side, we have $if_n(s)$ for $n = 1$ (orange), $n = 3$ (black), and $n = 5$ (purple). We note that the wave functions go to zero for large s.

$$f_{2n}(s) = 2\int_0^\infty \hat{f}_{2n}(x)\cos(sx)dx, \tag{2.24}$$

and for odd n, $\hat{f}_n$ picks a minus sign under parity, and thus, we have

$$-if_{2n+1}(s) = -2\int_0^\infty \hat{f}_{2n+1}(x)\sin(sx)dx. \tag{2.25}$$

The integral that we need to evaluate, therefore, using (2.19) and (2.17), is given by

$$\begin{aligned} I_n(s) &\coloneqq 2\int_0^\infty \hat{f}_n(x)e^{-isx}dx \\ &= 2\int_0^\infty dz\mathrm{Ai}(az - ab_n)e^{-isz}. \end{aligned} \tag{2.26}$$

The real part of I_{2n} gives (2.24), and the imaginary part of I_{2n+1} gives (2.25).

The Airy function $\mathrm{Ai}(x)$ is an entire function with zeros located on the negative real axis. Therefore, it can be written as

$$\mathrm{Ai}(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{2.27}$$

where c_k is a constant. Using this and performing a term by term integration, we find[10]

$$\begin{aligned} I_n &= -2ie^{-isb_n}\sum_{k=0}^{\infty} c_k(ia)^k \frac{d^k}{ds^k}\frac{1}{s}(e^{isb_n} - \delta_{s,0}) \\ &= -2ie^{-isb_n}\mathrm{Ai}\left(ia\frac{d}{ds}\right)\frac{1}{s}(e^{isb_n} - \delta_{s,0}). \end{aligned} \tag{2.28}$$

[10]In general, we cannot exchange integration and sum unless the sum I_n exists.

We use this result shortly. See Fig. 1 for numerical plots of the momentum space wave function $f_n(s)$, *i.e.*, (2.24) and (2.25), for $n = 0$, 1, 2, 3, 4, and 5.

We now estimate the order of the error that we earlier introduced in (2.14) while taking the interval of integration length from $1/t$ to infinity. To estimate the order of the error, therefore, we only need the asymptotic behavior of $f(s)$ for large s.

In the large s limit, we have

$$\begin{aligned} I_n &= -\frac{2i}{s}\sum_{m=0}^{\infty}\left(\frac{ai}{s}\right)^m \mathrm{Ai}^{(m)}(-ab_n) \\ &= -\frac{2i\mathrm{Ai}(-ab_n)}{s} + \frac{2a\mathrm{Ai}'(-ab_n)}{s^2} + \mathcal{O}(s^{-3}), \end{aligned} \tag{2.29}$$

where $\mathrm{Ai}^{(m)}$ is the mth derivative of Ai. Thus, the term that we ignored in (2.8) in taking the limits of integration to infinity, for odd n, is of order

$$\begin{aligned} \int_{\frac{1}{t}}^{\infty}\frac{dsf_n(s)}{s^2} &\approx -ia^4\mathrm{Ai}^{(4)}(-ab_n)\int_{\frac{1}{t}}^{\infty}\frac{ds}{s^7} \\ &\approx -ia^4\mathrm{Ai}^{(4)}(-ab_n)t^6. \end{aligned} \tag{2.30}$$

Similarly, for even n, the error is of order

$$\begin{aligned} \int_{\frac{1}{t}}^{\infty}\frac{dsf_n(s)}{s^2} &\approx a^3\mathrm{Ai}^{(3)}(-ab_n)\int_{\frac{1}{t}}^{\infty}\frac{ds}{s^6} \\ &\approx a^3\mathrm{Ai}^{(3)}(-ab_n)t^5. \end{aligned} \tag{2.31}$$

We note from (2.8) that the contribution from odd n however cancels since the integrand is odd under $s \to -s$. Thus, the error we introduced by extending the integration limit to infinity comes only from even n, and it is of order t^5. This is in agreement with [21].

Note that we also ignored the ω^2 in the denominator of the first term on the rhs of (2.8), which is of the order t^2. Therefore, we are only considering the order t correction. As a result, at order t, we can freely extend the limit of integration to infinity.

We next apply the above method to the two-dimensional adjoint QCD.

III. THE LARGE MASS LIMIT OF 2D ADJOINT QCD

The theory is described by the action [5,9,11],

$$S = \int d^2x\,\mathrm{tr}\left(i\bar{q}\gamma^\alpha D_\alpha q - m\bar{q}q - \frac{1}{4g^2}F_{\alpha\beta}F^{\alpha\beta}\right), \quad (3.1)$$

where the matrices $\gamma^0 = \eta^{00}\gamma_0 = \gamma_0$, $\gamma^1 = \eta^{11}\gamma_1 = -\gamma_1$, $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}I_{2\times 2}$ are the 2×2 Dirac matrices in the Majorana representation, the field-strength tensor $F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]} + iA_{[\alpha}A_{\beta]}$, the covariant derivative $D_\alpha = \partial_\alpha + i[A_\alpha, \cdot]$, and the fermion q is a two component (Majorana–Weyl) spinor in the adjoint representation. We denote its top component as ψ and bottom component as $\bar{\psi}$. The fermions ψ and $\bar{\psi}$ are $N\times N$ Hermitian traceless matrices. The gauge potential A_α is an $N\times N$ Hermitian traceless matrix. m is the bare fermion mass,[11] and g is the gauge coupling.[12] We work in the ordinary vacuum of the theory [25–27].

It is very convenient to use light cone quantization [28–30]. We introduce the light cone coordinates by the definitions,

$$x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}}. \quad (3.2)$$

We treat x^+ as the time variable. A useful gauge is $A_- = 0$. In this gauge, we find

$$S = \int dx^+ dx^- \mathrm{tr}\left(i\psi\partial_+\psi + i\bar{\psi}\partial_-\bar{\psi} - i\sqrt{2}m\bar{\psi}\psi + \frac{1}{2g^2}(\partial_- A_+)^2 + A_+ J^+\right), \quad (3.3)$$

where

$$J^+_{ij} = 2\psi_{ik}\psi_{kj} \quad (3.4)$$

is an $SU(N)$ current. The gauge potential A_+ and the left moving fermion $\bar{\psi}$ are nondynamical and can be eliminated using their equations of motion. We write the gauge potential $A_+ = A_{+,0} + \bar{A}_+$, where $A_{+,0}$ is the zero mode. Using the variational principle of least action, we find

$$\int dx^- J^+ = 0, \quad \partial_-^2 \bar{A}_+ - g^2 J^+ = 0, \quad \sqrt{2}\partial_-\bar{\psi} - m\psi = 0. \quad (3.5)$$

Using these, the light cone momentum and energy are given by

$$P^+ = \int dx^- \mathrm{tr}(i\psi\partial_-\psi), \quad (3.6)$$

$$P^- = \frac{1}{2}\int dx^- \mathrm{tr}\left(im^2\psi\frac{1}{\partial_-}\psi - g^2 J^+ \frac{1}{\partial_-^2} J^+\right). \quad (3.7)$$

We now quantize the theory at $x^+ = 0$. We write the fermions as

$$\psi_{ij}(x^-) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} dk\psi_{ij}(k)e^{-ikx^-}. \quad (3.8)$$

The modes $\psi_{ij}(k)$ with $k < 0$ are creation operators, and the modes $\psi_{ij}(k)$ with $k \geq 0$ are annihilation operators.

The fermion modes satisfy the canonical anticommutation relation given by

$$\{\psi_{ab}(k), \psi_{cd}(k')\} = \delta(k+k')\left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right). \quad (3.9)$$

In terms of the modes, the translation generators takes the form,

$$\begin{aligned} P^+ &= \int_0^\infty dk k\psi_{ab}(-k)\psi_{ba}(k), \\ P^- &= \frac{1}{2}m^2\int_0^\infty \frac{dk}{k}\psi_{ab}(-k)\psi_{ba}(k) \\ &\quad + \frac{1}{2}g^2\int_0^\infty \frac{dk}{k^2}J^+_{ab}(-k)J^+_{ba}(k), \end{aligned} \quad (3.10)$$

where the current Fourier transform is given by

$$J^+(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx^- J^+(x^-)e^{-ikx^-}. \quad (3.11)$$

Upon writing the current in terms of the modes, we find, in the large N limit, that the light cone Hamiltonian operator is given by[13]

[11] See [1–4] for a discussion on the massless case.

[12] Note that in two dimensions, the gauge coupling g is dimensionful.

[13] In general, there are additional string breaking terms, which are suppressed by a factor of $1/N^2$. These are terms given by product of traces.

$$P^- = \frac{1}{2}m^2 \int_0^\infty \frac{dk}{k}\psi_{ji}(-k)\psi_{ij}(k) + \frac{g^2 N}{2\pi}\int_0^\infty \frac{dk}{k}C(k)\psi_{ji}(-k)\psi_{ij}(k) - \frac{g^2}{2\pi}\int_0^\infty dk_1 dk_2 dk_3 dk_4 A(k_1,k_2,k_3,k_4)$$
$$\times \delta(k_1+k_2-k_3-k_4)\psi_{ij}(-k_4)\psi_{jk}(-k_3)\psi_{kl}(k_1)\psi_{li}(k_2) + \frac{g^2}{2\pi}\int_0^\infty dk_1 dk_2 dk_3 dk_4 B(k_1,k_2,k_3,k_4)$$
$$\times \delta(k_1+k_2+k_3-k_4)\cdot[\psi_{jk}(-k_4)\psi_{kl}(k_1)\psi_{li}(k_2)\psi_{ij}(k_3) + \psi_{il}(-k_3)\psi_{lj}(-k_2)\psi_{jk}(-k_1)\psi_{ki}(k_4)], \quad (3.12)$$

where

$$A(k_1,k_2,k_3,k_4) = \frac{1}{(k_4-k_2)^2} - \frac{1}{(k_1+k_2)^2}, \quad (3.13)$$

$$B(k_1,k_2,k_3,k_4) = \frac{1}{(k_2+k_3)^2} - \frac{1}{(k_1+k_2)^2}, \quad (3.14)$$

$$C(k) = \int_0^\infty dp\left[\frac{k}{(p-k)^2} - \frac{k}{(p+k)^2}\right]. \quad (3.15)$$

We simplify $C(k)$ further as

$$\begin{aligned} C(k) &= \int_0^\infty dp\left[\frac{k}{(p-k)^2} - \frac{k}{(p+k)^2}\right], \\ &= \lim_{\epsilon\to 0}\left(\int_0^k dp\frac{k}{(p-k-\epsilon)^2} + \int_k^\infty dp\frac{k}{(p-k+\epsilon)^2}\right) \\ &\quad - \int_0^\infty dp\frac{k}{(p+k)^2}, \\ &= 2\int_0^k dp\frac{k}{(p-k)^2} = \text{p.v.}2\int_0^k dp\frac{k}{(p-k)^2}. \end{aligned} \quad (3.16)$$

Note the integral is understood in the principal value sense.

The light cone vacuum $|0\rangle$ is the ground state of P^- with eigenvalue zero.[14] All the physical states $|\chi\rangle$ must satisfy the zero charge constraint,

$$\int dx^- J^+|\chi\rangle = 0. \quad (3.17)$$

The Hilbert space that the translation generators are taken to act on, in the large N limit, is the space spanned by states of the form,

$$\text{tr}[\psi(-k_1)\psi(-k_2)\cdots\psi(-k_n)]|0\rangle. \quad (3.18)$$

These states satisfy the zero charge constraint. From the first line in (3.10), we see that the total P^+ of a state of the form (3.18), k^+, is

$$k^+ = \sum_{i=1}^n k_i. \quad (3.19)$$

It is diagonal on the states (3.18). To solve the theory, we need also to diagonalize the light cone Hamiltonian, P^- (3.10), on these states. In general, this is hard, since P^- relates states with different values of the quark or parton number n (3.18). However, one may hope that this effect becomes less significant in the limit,

$$\lambda \equiv \frac{g^2 N}{m^2} \to 0. \quad (3.20)$$

This limit is the weak coupling limit of the theory. Indeed, one can think of λ as the size of the (‘t Hooft) coupling at the scale m, which is the scale associated with the bound states in this theory.[15]

Let us start with the free theory, *i.e.*, $\lambda = 0$. In that case, P^- (3.10) is also diagonal on the states (3.18), and we can compute its value, k^-,

$$k^- = \frac{m^2}{2}\sum_{i=1}^n \frac{1}{k_i}. \quad (3.21)$$

It is useful to define the variables x_i via

$$k_i = x_i k^+. \quad (3.22)$$

These variables take value in (0,1) and can be thought of as the light cone momentum fraction carried by the i’th parton. Obviously, one has [from (3.19)]

$$\sum_{i=1}^n x_i = 1. \quad (3.23)$$

In terms of x_i, (3.21) can be written as

$$M^2 = 2k^+k^- = m^2\sum_{i=1}^n \frac{1}{x_i}. \quad (3.24)$$

The smallest value this quantity can take is $M = mn$, which is obtained by setting all x_i to be equal to $1/n$. Moving away from this value, M^2 increases, and it diverges when any of the $x_i \to 0$. Thus, in the free theory, *i.e.*, $\lambda = 0$, we find a continuum of masses starting at mn, precisely as we

[14] This should not cause confusion in the massive case.

[15] And with the process of pair creation of the adjoint quarks.

would expect for states of n free particles. Now, we would like to turn on the leading effect of the interaction in (3.10).

Consider, as an example, bound states consisting of two quarks. We can write these states in general as

$$|\phi\rangle = \int_0^1 dx\phi(x)\mathrm{tr}[\psi(-xk^+)\psi(-(1-x)k^+)]|0\rangle, \quad (3.25)$$

where $\phi(x)$ is the wave function associated with the state. We saw earlier that for $\lambda = 0$ the states that minimize the energy correspond to wave functions that are very sharply peaked around $x = 1/2$. Such states have mass $M \sim 2m$, the mass of a state of two free quarks. In general, we will choose the wave function $\phi(x)$ to satisfy the boundary condition,

$$\phi(0) = 0, \quad (3.26)$$

this is consistent with our definition of the modes. Note that the wave function is by definition antisymmetric under $x \to 1-x$,

$$\phi(1-x) = -\phi(x). \quad (3.27)$$

Thus, (3.26) also implies vanishing of the wave function at $x = 1$. The inner product between two states of the form (3.25), $|\phi\rangle$ and $|\phi'\rangle$, is given by

$$\langle\phi'|\phi\rangle = \frac{2N^2}{k^+}\delta(k^+ - k^{+\prime})\int_0^1 dx\phi(x)\phi'(x). \quad (3.28)$$

In particular, the norm $\langle\phi|\phi\rangle$, is positive definite, as expected.

Similarly, we can define a general n partons gauge invariant bound state as

$$|\phi\rangle := \int_0^{k^+} dk_1\cdots dk_n\delta\left(\sum_{i=1}^n k_i - k^+\right)\phi_n(k_1,\ldots,k_n) \times \mathrm{tr}[\psi(-k_1)\cdots\psi(-k_n)]|0\rangle, \quad (3.29)$$

where ϕ_n is the wave function associated with the state $|\phi\rangle$. Therefore, for even number of partons, the state is bosonic and for odd number of partons, the state is fermionic. Note that by definition the wave function has the property,

$$\phi_n(k_1,k_2,\ldots,k_{n-1},k_n) = (-1)^{n-1}\phi_n(k_2,k_3,\ldots,k_n,k_1). \quad (3.30)$$

We will choose the wave function, in general, to satisfy the condition,

$$\phi_n(0,k_2,\ldots,k_{n-1},k_n) = 0, \quad (3.31)$$

this is consistent with our definition of the modes.

It is important that we are working in a given sector in which P^+ is kept fixed. This is because in general there is no sense in which one can study nonrelativistic limit in light cone coordinates if P^+ is not fixed. For a given P^+ eigenvalue, the light cone quantization looks very much like a nonrelativistic theory [18,31]. Therefore, there is a sense in which nonrelativistic limit is sensible.

Acting with the light cone Hamiltonian P^- (3.12) on the state $|\phi\rangle$ gives the following equation for the M_n^2 of the state:

$$\begin{aligned} M_n^2\phi_n(x_1,\ldots,x_n) = {} & m^2\sum_{i=1}^n\frac{1}{x_i}\phi_n(x_1,\ldots,x_n) + \frac{2g^2N}{\pi}\sum_{i=1}^n\phi_n(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)\cdot\int_0^{x_i}dy\frac{1}{(y-x_i)^2} \\ & + \frac{g^2N}{\pi}\sum_{i=1}^n\frac{1}{(x_i+x_{i+1})^2}\int_0^{x_i+x_{i+1}}dy\phi_n(x_1,\ldots,x_{i-1},y,x_i+x_{i+1}-y,x_{i+2},\ldots,x_n) \\ & - \frac{g^2N}{\pi}\sum_{i=1}^n\int_0^{x_i+x_{i+1}}dy\phi_n(x_1,\ldots,x_{i-1},y,x_i+x_{i+1}-y,x_{i+2},\ldots,x_n)\cdot\frac{1}{(y-x_i)^2} \\ & + \frac{g^2N}{\pi}\sum_{i=1}^n\int_0^{x_i}dy\int_0^{x_i-y}dz\phi_{n+2}(x_1,\ldots,x_{i-1},y,z,x_i-y-z,x_{i+1},\ldots,x_n) \\ & \cdot\left[\frac{1}{(y+z)^2}-\frac{1}{(x_i-y)^2}\right] + \frac{g^2N}{\pi}\sum_{i=1}^n\phi_{n-2}(x_1,\ldots,x_{i-1},x_i+x_{i+1}+x_{i+2},x_{i+3},\ldots,x_n) \\ & \cdot\left[\frac{1}{(x_i+x_{i+1})^2}-\frac{1}{(x_{i+1}+x_{i+2})^2}\right], \qquad x_1 = x_{n+1}, \qquad \sum_{i=1}^n x_i = 1. \end{aligned} \quad (3.32)$$

We note that for even values of n, the equation only involves bosonic states, and similarly, for odd values of n, it only involves fermionic states. Thus, it does not mix bosonic and fermionic states. We also note that the equation relates or mixes states with different partons number n, $n \pm 2$. This is the main reason why solving this equation analytically and exactly, even in the planar limit, has been difficult. We rewrite this equation using the redefinitions,

$$\pi M_n^2 = g^2 N \mu_n^2, \qquad \alpha = \frac{m^2 \pi}{g^2 N}, \tag{3.33}$$

as

$$\begin{aligned}
\mu_n^2 \phi_n(x_1, \ldots, x_n) = {} & \alpha \sum_{i=1}^n \frac{1}{x_i} \phi_n(x_1, \ldots, x_n) + \sum_{i=1}^n \int_0^{x_i} dy \int_0^{x_i - y} dz \phi_{n+2}(x_1, \ldots, x_{i-1}, y, z, x_i - y - z, x_{i+1}, \ldots, x_n) \\
& \times \left[\frac{1}{(y+z)^2} - \frac{1}{(x_i - y)^2}\right] + \sum_{i=1}^n \frac{1}{(x_i + x_{i+1})^2} \int_0^{x_i + x_{i+1}} dy \phi_n(x_1, \ldots, x_{i-1}, y, x_i + x_{i+1} - y, x_{i+2}, \ldots, x_n) \\
& + \sum_{i=1}^n \int_0^{x_i + x_{i+1}} \frac{dy}{(y - x_i)^2} [\phi_n(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - \phi_n(x_1, \ldots, x_{i-1}, y, x_i + x_{i+1} - y, x_{i+2}, \ldots, x_n)] \\
& + \sum_{i=1}^n \phi_{n-2}(x_1, \ldots, x_{i-1}, x_i + x_{i+1} + x_{i+2}, x_{i+3}, \ldots, x_n) \left[\frac{1}{(x_i + x_{i+1})^2} - \frac{1}{(x_{i+1} + x_{i+2})^2}\right], \\
x_1 = x_{n+1}, \qquad & \sum_{i=1}^n x_i = 1.
\end{aligned} \tag{3.34}$$

Here, we have used the identity,

$$\int_{x_i}^{x_i + x_{i+1}} dy \frac{1}{(y - x_i)^2} = \int_0^{x_{i+1}} dy \frac{1}{(y - x_{i+1})^2}. \tag{3.35}$$

We now write as we did in the previous section,

$$x_i = \frac{1}{n} + \omega_i, \qquad \sum_{i=1}^n \omega_i = 0. \tag{3.36}$$

In the following analysis, we will assume $n\omega_i \ll 1$. Therefore, the states are sharply peaked around $x_1 = x_2 = \cdots = x_n = 1/n$. We also define

$$\mu_n^2 = n^2 \alpha + \gamma_n. \tag{3.37}$$

We are interested in the large α limit such that

$$\lim_{\alpha \to \infty} \frac{\gamma_n}{n^2 \alpha} \to 0. \tag{3.38}$$

Using the above redefinitions, the lhs of (3.34) becomes

$$n^2 \alpha \phi_n(x_1, \ldots, x_n) + \gamma_n \phi_n(x_1, \ldots, x_n). \tag{3.39}$$

We next look the rhs of the Eq. (3.34) term by term.

From the first term, we have

$$\begin{aligned}
\alpha \sum_{i=1}^n \frac{1}{x_i} \phi_n(x_1, \ldots, x_n) &= \alpha \sum_{i=1}^n \frac{n}{1 + n\omega_i} \phi_n(x_1, \ldots, x_n) \\
&= n^2 \alpha \phi_n + n^3 \alpha \sum_{i=1}^n \omega_i^2 \phi_n \\
&\quad + \mathcal{O}(n^4 \alpha \omega_i^3) \phi_n.
\end{aligned} \tag{3.40}$$

From the second term with

$$y = \frac{1}{n} + \omega_y, \qquad z = \frac{1}{n} + \omega_z, \qquad n\omega_y \ll 1, \qquad n\omega_z \ll 1, \tag{3.41}$$

we have

$$\sum_{i=1}^n \int_0^{x_i} dy \int_0^{x_i - y} dz \phi_{n+2} \left[\frac{1}{(y+z)^2} - \frac{1}{(x_i - y)^2}\right] = \sum_{i=1}^n \int_{-\frac{1}{n}}^{\omega_i} d\omega_y \int_{-\frac{1}{n}}^{-\frac{1}{n} + \omega_i - \omega_y} d\omega_z \phi_{n+2} \left[\frac{n^2}{4} - \frac{1}{(\omega_i - \omega_y)^2} + \mathcal{O}(n\omega_y + n\omega_z)\right]. \tag{3.42}$$

From the third term, we find

$$\sum_{i=1}^{n}\frac{1}{(x_i+x_{i+1})^2}\int_0^{x_i+x_{i+1}}dy\phi_n=\sum_{i=1}^{n}\frac{1}{(\frac{2}{n}+\omega_i+\omega_{i+1})^2}\int_{-\frac{1}{n}}^{\frac{1}{n}+\omega_i+\omega_{i+1}}d\omega_y\phi_n,$$
$$=\sum_{i=1}^{n}\left(\frac{n^2}{4}-\frac{n^3}{4}(\omega_i+\omega_{i+1})+\mathcal{O}(\omega_i^2)\right)\int_{-\frac{1}{n}}^{\frac{1}{n}+\omega_i+\omega_{i+1}}d\omega_y\phi_n. \tag{3.43}$$

From the fourth term, we get

$$\sum_{i=1}^{n}\int_0^{x_i+x_{i+1}}\frac{dy}{(y-x_i)^2}\cdot[\phi_n(x_1,\ldots,x_n)-\phi_n(x_1,\ldots,y,x_i+x_{i+1}-y,\ldots,x_n)]$$
$$=\sum_{i=1}^{n}\int_{-\frac{1}{n}}^{\frac{1}{n}+\omega_i+\omega_{i+1}}\frac{d\omega_y}{(\omega_y-\omega_i)^2}\cdot[\phi_n(\omega_1,\ldots,\omega_n)-\phi_n(\omega_1,\ldots,\omega_y,\omega_i+\omega_{i+1}-\omega_y,\ldots,\omega_n)]. \tag{3.44}$$

From the last term, we get

$$\sum_{i=1}^{n}\phi_{n-2}\left[\frac{1}{(x_i+x_{i+1})^2}-\frac{1}{(x_{i+1}+x_{i+2})^2}\right]=\sum_{i=1}^{n}\phi_{n-2}\left[\frac{1}{(\frac{2}{n}+\omega_i+\omega_{i+1})^2}-\frac{1}{(\frac{2}{n}+\omega_{i+1}+\omega_{i+2})^2}\right],$$
$$=\sum_{i=1}^{n}\phi_{n-2}\cdot\frac{n^2}{4}\left[n(\omega_{i+2}-\omega_i)-\frac{3n^2}{4}(\omega_{i+2}-\omega_i)(\omega_i+2\omega_{i+1}+\omega_{i+2})+\mathcal{O}(\omega_i^3)\right]. \tag{3.45}$$

We next rescale the ω_i's as

$$\omega_i=s_it,\qquad \omega_y=s_yt,\qquad \omega_z=s_zt. \tag{3.46}$$

As we did in the previous section, we assume that the wave functions,

$$\phi_n(\omega_1,\ldots,\omega_i,\ldots,\omega_n)\coloneqq\phi_n(s_1,\ldots,s_i,\ldots,s_n), \tag{3.47}$$

do not depend on t. That is, the wave functions are sharply peaked around $x_1=x_2=\cdots=x_n=1/n$. We take

$$t=\alpha^{-\frac{1}{3}}, \tag{3.48}$$

and redefine γ_n as

$$\bar{\gamma}_n=t\gamma_n. \tag{3.49}$$

In the large α limit, we then get

$$\bar{\gamma}_n\phi_n=n^3\sum_{i=1}^{n}s_i^2\phi_n-tn^4\sum_{i=1}^{n}s_i^3\phi_n+\mathcal{O}(t^2)\phi_n+t\sum_{i=1}^{n}\int_{-\frac{1}{nt}}^{s_i}ds_y\int_{-\frac{1}{nt}}^{-\frac{1}{nt}+s_i-s_y}ds_z\phi_{n+2}\left[\frac{n^2t^2}{4}-\frac{1}{(s_i-s_y)^2}+\mathcal{O}(t^3)\right]$$
$$+t^2\sum_{i=1}^{n}\left(\frac{n^2}{4}-\frac{n^3t}{4}(s_i+s_{i+1})+\mathcal{O}(t^2)\right)\int_{-\frac{1}{nt}}^{\frac{1}{nt}+s_i+s_{i+1}}ds_y\phi_n$$
$$+\sum_{i=1}^{n}\int_{-\frac{1}{nt}}^{\frac{1}{nt}+s_i+s_{i+1}}\frac{ds_y}{(s_y-s_i)^2}\cdot[\phi_n(s_1,\ldots,s_n)-\phi_n(s_1,\ldots,s_y,s_i+s_{i+1}-s_y,\ldots,s_n)]$$
$$+t^2\sum_{i=1}^{n}\phi_{n-2}\cdot\frac{n^2}{4}\left[n(s_{i+2}-s_i)-\frac{3n^2t}{4}(s_{i+2}-s_i)(s_i+2s_{i+1}+s_{i+2})+\mathcal{O}(t^2)\right]. \tag{3.50}$$

Therefore, to leading order, we have the mass squared equation,

$$\bar{\gamma}_n\phi_n=n^3\sum_{i=1}^{n}s_i^2\phi_n+\sum_{i=1}^{n}\int_{-\infty}^{\infty}\frac{ds_y}{(s_y-s_i)^2}\cdot[\phi_n(s_1,\ldots,s_n)-\phi_n(s_1,\ldots,s_y,s_i+s_{i+1}-s_y,\ldots,s_n)]-tn^4\sum_{i=1}^{n}s_i^3\phi_n+\mathcal{O}(t^2). \tag{3.51}$$

Note that at this order, *i.e.*, $\mathcal{O}(t)$, only ϕ_n contributes to the mass squared equation. Thus, for the low-lying states, there is no pair production or annihilation, as expected. This was noted already in [9,11], and there is also recent numerical evidence that suggests this is the case for the low-lying states even at moderate values of the coupling [32]. Note also that, at this order, we see using (3.34) that (3.51) is equivalent to the n-parton 't Hooft equation,

$$\mu_n^2\phi_n(x_1,\ldots,x_n) = \alpha\sum_{i=1}^n \frac{1}{x_i}\phi_n(x_1,\ldots,x_n) + \sum_{i=1}^n \int_0^{x_i+x_{i+1}} \frac{dy}{(y-x_i)^2}\cdot[\phi_n(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) - \phi_n(x_1,\ldots,x_{i-1},y,x_i+x_{i+1}-y,x_{i+2},\ldots,x_n)]. \tag{3.52}$$

Therefore, the goal is to solve this equation in the region in which the momentum fractions $x_1 = \cdots = x_n$ are near $1/n$. In particular, for $n = 2$, we have

$$\mu_2^2\phi_2(x) = \frac{\alpha}{x(1-x)}\phi_2(x) - 2\int_0^1 \frac{dy}{(y-x)^2}\phi_2(y). \tag{3.53}$$

Note that the integral is defined in the principal value sense; see (3.16). This is the 't Hooft equation (2.1).[16] The source of the extra factor 2 will be discussed shortly.

We write the Fourier transform of the wave function ϕ_n as

$$\hat{\phi}_n(\vec{x}) \coloneqq \frac{1}{(2\pi)^n}\int e^{i\vec{x}\cdot\vec{s}}\cdot\delta(s_1+\cdots+s_n)\phi_n(\vec{s})d\vec{s}, \tag{3.54}$$

equivalently,

$$\delta(s_1+\cdots+s_n)\phi_n(\vec{s}) = \int e^{-i\vec{x}\cdot\vec{s}}\hat{\phi}_n(\vec{x})d\vec{x}. \tag{3.55}$$

To do the Fourier transform of the mass squared equation (3.51), we need the value of the integral,

$$\int_{-\infty}^{\infty} \frac{e^{-i(x_i-x_{i+1})s_y}}{(s_y-s_i)^2}ds_y. \tag{3.56}$$

As in the t' Hooft model, the integral is defined by a principal value prescription. We assume the following integration prescription[17]:

$$\text{p.v.}\int\frac{f(s)}{(s-s_0)^2}ds = \frac{1}{2}\int\frac{f(s)}{(s-s_0+i\epsilon)^2}ds + \frac{1}{2}\int\frac{f(s)}{(s-s_0-i\epsilon)^2}ds. \tag{3.57}$$

Using this prescription, we get

$$\int_{-\infty}^{\infty} \frac{e^{-i(x_i-x_{i+1})s_y}}{(s_y-s_i)^2}ds_y = -\pi|x_i-x_{i+1}|e^{-i(x_i-x_{i+1})s_i}. \tag{3.58}$$

Using the above result, we see that

$$\int_{-\infty}^{\infty}\frac{ds_y}{(s_y-s_i)^2}\delta(s_1+\cdots+s_n)\phi_n(s_1,\ldots,s_y,s_i+s_{i+1}-s_y,\ldots,s_n) = -\pi\int|y_i-y_{i+1}|e^{-i\vec{y}\cdot\vec{s}}\hat{\phi}_n(\vec{y})d\vec{y}. \tag{3.59}$$

Therefore, to order $\mathcal{O}(t)$, the Fourier transform of Eq. (3.51) becomes

$$\bar{\gamma}_n\hat{\phi}_n(\vec{x}) = -n^3\sum_{i=1}^n\partial_{x_i}^2\hat{\phi}_n(\vec{x}) + \pi\sum_{i=1}^n|x_i-x_{i+1}|\hat{\phi}_n(\vec{x}) + \mathcal{O}(t). \tag{3.60}$$

The wave function $\hat{\phi}_n$ has the following symmetries:

$$\hat{\phi}_n(x_1+c,\ldots,x_n+c) = \hat{\phi}_n(x_1,\ldots,x_n), \hat{\phi}_n(x_1,x_2,\ldots,x_{n-1},x_n) = (-1)^{n-1}\hat{\phi}_n(x_2,x_3,\ldots,x_n,x_1). \tag{3.61}$$

Here, c is a constant.

[16]See also [33] for a similar equation obtained using a formulation of 2D fundamental QCD in terms of bilocal fields and the method of coadjoint orbits.

[17]This is similar to (3.16).

We note that the n-parton bound state potential is given by a pairwise sum of two-parton potentials. This can be also seen directly from the n-parton 't Hooft equation (3.52). The doubling of the strength of the coulomb interaction or potential for $n = 2$ is due to the two color flux tubes connecting a pair of partons (in a quark antiquark pair there is only one flux tube) (see, for example, [11]). For three adjoint quarks, the potential V is given by

$$V(x_1, x_2, x_3) = |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1| + \mathcal{O}(t). \tag{3.62}$$

In $(1+1)$D fundamental QCD, a similar expression was obtained in [18] for a baryon, which is a bound state of three quarks, in the heavy-quark limit; see Sec. (3.9) of the paper.[18] In $(1+3)$D QCD, there are two ansätzes regarding the three quraks potential. They are known as the Δ and Y ansätzes. In the Δ ansätz, the potential is given by (3.62). There is no clear answer however regarding the correct three quarks static potential. For recent discussions on three quarks potential in phenomenological models of QCD, see [34–37].

In the next section, we discuss the cases $n = 2$ and $n = 3$. These cases can be easily generalized to the $n \geq 4$ cases in a similar manner.

IV. DISCUSSION

We now discuss the n-body nonrelativistic Schrödinger equation,

$$\bar{\gamma}_n \hat{\phi}_n(\vec{x}) = -n^3 \sum_{i=1}^{n} \partial_{x_i}^2 \hat{\phi}_n(\vec{x}) + \pi \sum_{i=1}^{n} |x_i - x_{i+1}| \hat{\phi}_n(\vec{x}),$$
$$x_{n+1} = x_1, \tag{4.1}$$

with the (boundary) conditions or constraints,

$$\hat{\phi}_n(x_1 + c, \ldots, x_n + c) = \hat{\phi}_n(x_1, \ldots, x_n),$$
$$\hat{\phi}_n(x_1, x_2, \ldots, x_{n-1}, x_n) = (-1)^{n-1} \hat{\phi}_n(x_2, x_3, \ldots, x_n, x_1), \tag{4.2}$$

for the cases where the partons number n is 2 and 3. We begin our discussion with $n = 2$.

The $n = 2$ case is very similar to the (fundamental) 't Hooft model. See also [27] for a related result. In this case, the Schrödinger equation is

$$\bar{\gamma}_2 \hat{\phi}_2(x_1, x_2) = -16(\partial_{x_1}^2 + \partial_{x_2}^2) \hat{\phi}_2(x_1, x_2) + 2\pi |x_1 - x_2| \hat{\phi}_2(x_1, x_2), \tag{4.3}$$

and

$$\hat{\phi}_2(x_1, x_2) = -\hat{\phi}_2(x_2, x_1). \tag{4.4}$$

It is very convenient to introduce the Jacobi coordinates,

$$z_1 = x_1 - x_2, \qquad z_2 = \frac{x_1 + x_2}{2}, \tag{4.5}$$

in terms of which the equation becomes

$$\bar{\gamma}_2 \hat{\phi}_2 = -8\left(2\partial_{z_1}^2 + \frac{1}{2}\partial_{z_2}^2\right)\hat{\phi}_2 + 2\pi |z_1| \hat{\phi}_2. \tag{4.6}$$

Since we are interested on bound states, we set the center of mass coordinate z_2, using the translation symmetry, to zero. Therefore, the relative motion of the quarks is described by

$$\bar{\gamma}_2 \hat{\phi}_2 = -16 \frac{d^2 \hat{\phi}_2}{dz_1^2} + 2\pi |z_1| \hat{\phi}_2, \qquad \hat{\phi}_2(z_1) = -\hat{\phi}_2(-z_1). \tag{4.7}$$

After rescaling the coordinates, this can be put into the more familiar form,

$$\gamma \phi = -\frac{1}{2}\frac{d^2\phi}{dz^2} + |z|\phi, \qquad \gamma = \frac{\bar{\gamma}_2}{2\pi}\left(\frac{\pi}{2\cdot 8}\right)^{\frac{1}{3}},$$
$$\phi(z) = -\phi(-z). \tag{4.8}$$

This is the Airy equation, and its solutions are discussed in detail in Sec. II. The wave function $\hat{\phi}_2$ in the adjoint case is given by

$$\hat{\phi}_2^{(l)}(z) = \begin{cases} \mathrm{Ai}\left((\frac{\pi}{8})^{\frac{1}{3}}\left(z - \frac{\bar{\gamma}_2^{(l)}}{2\pi}\right)\right), & z > 0, \\ -\mathrm{Ai}\left((\frac{\pi}{8})^{\frac{1}{3}}\left(-z - \frac{\bar{\gamma}_2^{(l)}}{2\pi}\right)\right), & z < 0, \end{cases} \tag{4.9}$$

where $\bar{\gamma}_2^{(l)}$ are given by the equations,

$$\mathrm{Ai}(-\bar{\gamma}_2^{(l)}/2\pi(8\pi^2)^{1/3}) = 0, \quad l = 1, 2, 3, \ldots. \tag{4.10}$$

Therefore, the masses are given by

$$M_{(2,l)}^2 = m^2\left(4 + \lambda^{\frac{2}{3}}\bar{\gamma}_2^{(l)}\right), \qquad \lambda := \frac{g^2 N}{m^2 \pi}, \quad l = 1, 2, 3, \ldots. \tag{4.11}$$

The quantum spectrum for the highly excited bound states can be computed by considering the periodic orbits of the corresponding classical Hamiltonian. The classical Hamiltonian in this case is

$$H = \frac{p^2}{2} + |z|. \tag{4.12}$$

[18] I thank Igor Klebanov for bringing to my attention this result.

A typical periodic motion in this system is described by

$$z(t) = \begin{cases} -t(t-t_2), & 0 \le t \le t_2, \\ (t-t_2)(t-2t_2), & t_2 \le t \le 2t_2. \end{cases} \tag{4.13}$$

Here, $T = 2t_2$ is the period. We now apply the Einstein-Brillouin-Keller (EBK) quantization. We first evaluate the action integral,

$$2\int_0^{\frac{T}{2}} p^2 dt = \frac{T^3}{24} = \frac{8}{3}E^{\frac{3}{2}}, \tag{4.14}$$

where E is the energy of the system along the orbit. This gives making use of the EBK quantization condition the spectrum,

$$E_n = \left[\frac{3}{4}\pi\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}. \tag{4.15}$$

From this, it follows that

$$\bar{\gamma}_2^{(n)} = 2\pi\left(\frac{2\cdot 8}{\pi}\right)^{\frac{1}{3}} E_n = 2\left[3\pi^2\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}. \tag{4.16}$$

Note the factor of 2 due to the two flux tubes. In one dimension, EBK is similar to Wentzel-Kramers-Brillouin (WKB) approximation. Putting all together, we have for the highly excited states,

$$M^2_{(2,n)} = m^2\left\{4 + 2\cdot\lambda^{2/3}\left[3\pi^2\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}\right\}, \tag{4.17}$$

where n is odd and large integer and λ is the ‘t Hooft coupling (4.11) at the scale of the constituent quark mass m.

We now consider the three partons case. In this case, the Schrödinger equation takes the form,

$$\bar{\gamma}_3\hat{\phi}_3(\vec{x}) = -27(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3})\hat{\phi}_3(\vec{x}) + \pi(|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|)\hat{\phi}_3(\vec{x}), \tag{4.18}$$

with the constrains on the wave function,

$$\begin{aligned}\hat{\phi}_3(x_1 + c, \dots, x_3 + c) &= \hat{\phi}_3(x_1, \dots, x_3), \\ \hat{\phi}_3(x_1, x_2, x_3) &= \hat{\phi}_3(x_2, x_3, x_1) \\ &= \hat{\phi}_3(x_3, x_1, x_2).\end{aligned} \tag{4.19}$$

We next conjecture that this equation is solvable. In particular, after making a change of coordinates, we conjecture that it can be solved using the method of separation of variables. It is important that one makes a change to parabolic coordinates to solve the problem.[19] We expect that this generalizes to a n-parton state.

We write (4.18) as

$$\begin{aligned}\gamma\hat{\phi}_3 &= -\frac{1}{2}(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3})\hat{\phi}_3 + (|x_1 - x_2| + |x_2 - x_3| \\ &\quad + |x_3 - x_1|)\hat{\phi}_3, \\ \gamma &= \frac{\bar{\gamma}_3}{\pi}\left(\frac{\pi}{2\cdot 27}\right)^{\frac{1}{3}}.\end{aligned} \tag{4.20}$$

Since we are interested in the relative motion of the quarks, we introduce the Jacobi coordinates,

$$z_1 = \frac{x_1 + x_2 + x_3}{3}, \qquad z_2 = \frac{x_2 - x_1}{\sqrt{2}}, \qquad z_3 = \sqrt{\frac{2}{3}}\left(x_3 - \frac{x_1 + x_2}{2}\right). \tag{4.21}$$

We note that

$$x_1 - x_2 = -\sqrt{2}z_2, \qquad x_2 - x_3 = \frac{1}{\sqrt{2}}(z_2 - \sqrt{3}z_3), \qquad x_3 - x_1 = \frac{1}{\sqrt{2}}(z_2 + \sqrt{3}z_3), \tag{4.22}$$

The relative motion of the quarks then becomes

$$\gamma\psi = -\frac{1}{2}(\partial^2_{z_2} + \partial^2_{z_3})\psi + V(z_2, z_3)\psi, \tag{4.23}$$

where the potential V is given by

$$V(z_2, z_3) = \sqrt{2}|z_2| + \frac{1}{\sqrt{2}}|z_2 - \sqrt{3}z_3| + \frac{1}{\sqrt{2}}|z_2 + \sqrt{3}z_3|. \tag{4.24}$$

The equation can be written in a more familiar and useful form using the polar coordinates. We define

$$z_2 = -r\sin\phi, \qquad z_3 = -r\cos\phi, \tag{4.25}$$

where

$$0 \le r < \infty, \qquad 0 \le \phi < 2\pi. \tag{4.26}$$

The lightlike separations of the partons are given in terms of the polar coordinates by

[19] I would like to mention that a similar equation to (4.18) was previously obtained in [18] by K. Hornbostel for a baryon. I thank Igor Klebanov for bringing this result to my attention. However, the equation was not solved. The author is not aware of any other work.

$$x_1 - x_2 = -\sqrt{2}z_2 = \sqrt{2}r\sin\phi,$$
$$x_2 - x_3 = \frac{1}{\sqrt{2}}(z_2 - \sqrt{3}z_3) = \sqrt{2}r\sin\left(\phi + \frac{2}{3}\pi\right),$$
$$x_3 - x_1 = \frac{1}{\sqrt{2}}(z_2 + \sqrt{3}z_3) = \sqrt{2}r\sin\left(\phi + \frac{4}{3}\pi\right). \quad (4.27)$$

This follows from (4.25) and (4.22). The different sectors or orderings in the original and new coordinates are related as follows:

$$\begin{aligned} &x_1 > x_2 > x_3, \quad \text{i.e.,} \quad 0 < \phi < \frac{1}{3}\pi, \\ &x_1 > x_3 > x_2, \quad \text{i.e.,} \quad \frac{1}{3}\pi < \phi < \frac{2}{3}\pi, \\ &x_3 > x_1 > x_2, \quad \text{i.e.,} \quad \frac{2}{3}\pi < \phi < \pi, \\ &x_3 > x_2 > x_1, \quad \text{i.e.,} \quad \pi < \phi < \frac{4}{3}\pi, \\ &x_2 > x_3 > x_1, \quad \text{i.e.,} \quad \frac{4}{3}\pi < \phi < \frac{5}{3}\pi, \\ &x_2 > x_1 > x_3, \quad \text{i.e.,} \quad \frac{5}{3}\pi < \phi < 2\pi. \end{aligned} \quad (4.28)$$

In terms of the polar coordinates, the Schrödinger equation now becomes

$$H\psi = \gamma\psi, \quad (4.29)$$

where H is the Hamiltonian,

$$\begin{aligned} H = &-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right) \\ &+ \sqrt{2}r\left(|\sin\phi| + \left|\sin\left(\phi + \frac{2\pi}{3}\right)\right| \right. \\ &\left. + \left|\sin\left(\phi + \frac{4\pi}{3}\right)\right|\right), \end{aligned} \quad (4.30)$$

and the wave function satisfy the symmetry,[20]

$$\psi(r,\phi) = -\psi\left(r, \phi + \frac{\pi}{3}\right), \quad (4.31)$$

which also implies

$$\psi(r,\phi) = \psi(r, \phi + 2\pi). \quad (4.32)$$

Note also that the equation is invariant under the parity $\phi \to -\phi$. We will later show that this implies a Neumann boundary condition on the wave function. Thus, we only need to consider the sector,

$$\left(\frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\phi^2 + 2\sqrt{2}r\,\sin\phi - \gamma\right)\psi(r,\phi) = 0,$$
$$\frac{\pi}{3} < \phi < \frac{2\pi}{3}, \quad (4.33)$$

with the antiperiodic boundary condition,

$$\psi\left(r, \frac{\pi}{3}\right) = -\psi\left(r, \frac{2\pi}{3}\right), \quad (4.34)$$

and the usual boundary conditions at $r = 0$ (i.e., the wave function must be finite at the origin and it must be also single valued as we approach the origin from different angular directions) and $r = \infty$ (i.e., the wave function must be normalizable),

$$\psi(0,\phi) = 0, \qquad \psi(\infty,\phi) = 0, \quad (4.35)$$

where

$$p_r^2 = \bar{p}_r^2 - \frac{1}{4r^2}, \quad \bar{p}_r = \frac{-i}{r^{\frac{1}{2}}}\frac{\partial}{\partial r}r^{\frac{1}{2}}, \quad p_\phi = -i\frac{\partial}{\partial\phi}, \quad (4.36)$$

are the (generalized) radial and angular momenta operators.

We slightly rewrite (4.33) and (4.34) in the following form:

$$\left(\frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\phi^2 + pr\,\cos(\phi - \phi_0) - \gamma\right)\psi(r,\phi) = 0,$$
$$\phi_0 - \frac{\pi}{6} < \phi < \phi_0 + \frac{\pi}{6}, \quad (4.37)$$

where

$$\psi(r, \phi_0 - \pi/6) = -\psi(r, \phi_0 + \pi/6), \quad (4.38)$$

and the constant $p = 2\sqrt{2}$. The phase ϕ_0 takes different values depending on which sector or domain ϕ belongs to. It only takes the values $\pm\pi/6, \pm\pi/2, \pm 5\pi/6$ since there are only six domains in total. In (4.33), $\phi_0 = \pi/2$. Also note that (4.37) and (4.38) are invariant under $\phi \to 2\phi_0 - \phi$. Thus, the equation has a $\mathbf{Z}_2$ reflection symmetry.

We note that this equation together with the cyclic constraint (4.31) describes a particle confined in (or to the surface of) an inverted hexagonal pyramid potential; see Fig. 2. The hexagonal base is at infinity. That is, it is open upwards and unbounded. The potential has the same symmetry group as the base. In general, the general equation (4.1) describes a particle confined to the surface of an inverted t-gonal pyramid potential in n dimensions. We saw that $t = 1$ for $n = 2$, and $t = n!$ for $n = 3$.

[20]Recall that the fermions are Hermitian. Also, in two dimensions, we can impose simultaneously both Weyl and Majorana conditions.

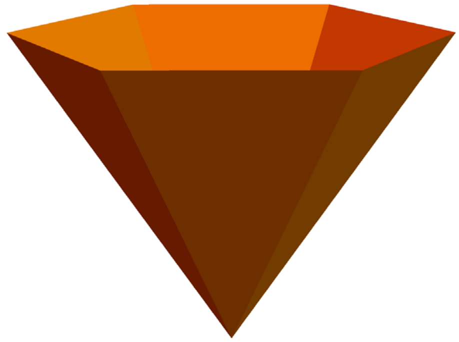

FIG. 2. The confining potential that a bound state of three quarks sees.

We also note that the Eq. (4.37) describes a nonrelativistic point electric dipole in a plane angular sector with electric field proportional to r in an appropriate unit.[21] The plane angular sector has (wedge) angle $\pi/3$. The quantity p is the magnitude of the electric dipole moment in an appropriate unit, and the phase ϕ_0 is the electric field angular direction. In general, we expect such interpretation to arise in the general case (4.1).

The trajectories of charged particles in the presence of electric field are parabolic. Thus, it is very convenient to use parabolic coordinates to simplify the equation further. We introduce the parabolic coordinates with the definitions,

$$r\sin(\phi-\phi_0)=\xi\sigma\tau, \qquad r\cos(\phi-\phi_0)=\frac{1}{2}(\tau^2-\sigma^2), \tag{4.39}$$

where $\xi=\pm1$ is introduced for convenience. We assume, without loss of generality, $\sigma\geq0$. Note that at $\sigma=0$, i.e., $\phi=\phi_0$, one can choose either $\tau\geq0$ or $\tau\leq0$. In what follows, we assume $\xi\tau\geq0$ at $\sigma=0$. In these coordinates, the Eq. (4.37) now becomes

$$-\frac{1}{2}\frac{1}{(\sigma^2+\tau^2)}(\partial_\sigma^2+\partial_\tau^2)\psi+\frac{1}{2}p(\tau^2-\sigma^2)\psi-\gamma\psi=0,$$
$$|\tau|\geq\chi\sigma, \qquad \chi=2+\sqrt{3}. \tag{4.40}$$

The equation has decomposed into two parts. We note that

$$-\frac{1}{2}\partial_\sigma^2\psi-\frac{1}{2}p\sigma^4\psi-\gamma\sigma^2\psi=-\left(-\frac{1}{2}\partial_\tau^2\psi+\frac{1}{2}p\tau^4\psi-\gamma\tau^2\psi\right),$$
$$|\tau|\geq\chi\sigma. \tag{4.41}$$

Therefore, this equation can naturally be solved using the separation of variables method. We write the wave function as a product of two functions as

$$\psi(\tau,\sigma)=T(\tau)S(\sigma). \tag{4.42}$$

This ansätz leads to the equation,

$$\frac{S''}{S}+p\sigma^4+2\gamma\sigma^2=-\left(\frac{T''}{T}-p\tau^4+2\gamma\tau^2\right), \qquad |\tau|\geq\chi\sigma. \tag{4.43}$$

Thus, for this equation to hold for all values of τ and σ, we need to demand

$$-\frac{d^2T}{d\tau^2}+(p\tau^4-2\gamma\tau^2-l)T=0,$$
$$-\frac{d^2S}{d\sigma^2}-(p\sigma^4+2\gamma\sigma^2-l)S=0, \tag{4.44}$$

where l is a constant and $|\tau|\geq\chi\sigma$. The separation constant l is determined by imposing the appropriate boundary conditions on T and S. Note that $S\equiv T(i\sigma)$. Thus, we only need to solve the first equation.

The conditions (4.35) and (4.38) on the wave function are now given by

$$T(\pm\infty)=0, \qquad T(\tau)=-T(-\tau). \tag{4.45}$$

Consider the case $\tau>0$. We write T as

$$T(\tau)=e^{-a\tau^3-b\tau}H(\tau), \qquad a=\frac{p^{\frac{1}{2}}}{3}=\frac{2^{\frac{3}{4}}}{3},$$
$$b=-\frac{\gamma}{p^{\frac{1}{2}}}=-\frac{\gamma}{2^{\frac{3}{4}}}. \tag{4.46}$$

Plugging this into the equation for T, we get

$$\frac{d^2H}{dz^2}-(3z^2+\xi)\frac{dH}{dz}-(3z-\delta)H=0, \tag{4.47}$$

where

$$z=\eta^{\frac{1}{3}}\tau, \qquad \eta=2a, \qquad \xi=2b/\eta^{\frac{1}{3}}, \qquad \delta=(b^2+l)/\eta^{\frac{2}{3}}. \tag{4.48}$$

The function $H(\delta,0,\xi;z)$ is the triconfluent Heun function [38]. The triconfluent Heun function $H(\alpha,\beta,\nu;z)$ satisfies the equation,

$$\frac{d^2H}{dz^2}-(3z^2+\nu)\frac{dH}{dz}-((-\beta+3)z-\alpha)H=0. \tag{4.49}$$

Since (4.37) is the two-dimensional generalization of (4.8), the triconfluent Heun function can be considered as the generalization of Airy function.

[21] Alternatively, it describes a unit point charge in a constant electric field background. The unit charge can be treated as a dipole with dipole moment proportional to $\vec{r}$. The constant electric field has the angular direction ϕ_0.

The solution for T is then given by

$$T(\tau) = \begin{cases} \exp\left[-\left(\frac{2^{3/4}}{3}\tau^2 - 2^{-3/4}\gamma\right)\tau\right] H(\delta, 0, \xi; \eta^{1/3}\tau), & \tau > 0, \\ -\exp\left[\left(\frac{2^{3/4}}{3}\tau^2 - 2^{-3/4}\gamma\right)\tau\right] H(\delta, 0, \xi; -\eta^{1/3}\tau), & \tau < 0. \end{cases} \tag{4.50}$$

We need to impose a boundary condition at $\phi = \phi_0$, *i.e.*, at $\sigma = 0$. In general, $S(\sigma)$ has a definite parity. This corresponds to the following two possible boundary conditions. One boundary condition is

$$S'(\sigma)|_{\sigma=0} = 0. \tag{4.51}$$

The other boundary condition is

$$S(\sigma)|_{\sigma=0} = 0. \tag{4.52}$$

However, we note that $\phi = \phi_0$ is the fixed point of the $\mathbf{Z}_2$ reflection symmetry mentioned above. Thus, at $\sigma = 0$, $S(\sigma)$ must vanish. That is, $S(\sigma)$ is an odd function. Therefore, $S(\sigma) = cT(i\sigma)$ for some constant c.

The wave function should be also continuous at $\phi - \phi_0 = -\pi/6$ and $\phi - \phi_0' = \pi/6$, where $\phi_0' = \phi_0 - \pi/3$. We next show that indeed it is continuous. Let $\psi_\xi^{\phi_0}(\tau, \sigma)$ denotes the wave function in the sector ϕ_0 with $\xi = +1$ or $\xi = -1$. We thus have from (4.31) that

$$\psi_{\xi'}^{\phi_0'}(\tau, \sigma) = -\psi_\xi^{\phi_0}(\tau, \sigma). \tag{4.53}$$

We also have from (4.42) and (4.50) with $\xi = +1$ that

$$\psi_{+1}^{\phi_0}(\tau, \sigma) = \begin{cases} T(\tau)S(\sigma), & \tau > \sigma > 0, \text{ i.e., } 0 < \phi - \phi_0 < \pi/6, \\ -T(-\tau)S(\sigma), & -\tau > \sigma > 0, \text{ i.e., } -\pi/6 < \phi - \phi_0 < 0, \end{cases} \tag{4.54}$$

where $T(0) = 0$, $T(\infty) = 0$, and $S(\sigma) = cT(i\sigma)$. Therefore, we observe that with $\xi' = -\xi$, the wave function is continuous.

$$\psi_{-1}^{\phi_0'}(\tau, \sigma) = \begin{cases} -T(-\tau)S(\sigma), & -\tau > \sigma > 0, \text{ i.e., } 0 < \phi - \phi_0' < \pi/6, \\ T(\tau)S(\sigma), & \tau > \sigma > 0, \text{ i.e., } -\pi/6 < \phi - \phi_0' < 0. \end{cases} \tag{4.55}$$

This also implies the wave function $\psi(r, \phi)$ is even under the symmetry $\phi \to -\phi$. Therefore, the first derivative of ψ with respect to ϕ at $\phi = 0$ must vanish. That is,

$$(\tau\partial_\sigma - \sigma\partial_\tau)\psi_{+1}^{\phi_0}(\tau, \sigma)|_{\tau=\chi\sigma} = 0. \tag{4.56}$$

This further constrains the wave function. In general, solving the Neumann boundary condition (4.56) is difficult since it requires some knowledge of the properties of the Heun function. However, I will next show indirectly that it leads to a reasonable answer.

Comments:

(1) The constraint (4.56) ensures that the wave function matches smoothly across the boundaries of the different sectors. The Neumann boundary condition should be viewed as a constraint on l. This will become evident as we go along. It is trivially satisfied at $\sigma = 0$, $\tau = 0$.

(2) For complex $z \coloneqq \tau + i\sigma$, the anti-Stokes lines for $T(z)$ divide the complex plane into six domains. Interestingly, the lines are given by the directions ϕ_0. They trace their origin from the cyclic symmetry.

(3) For small σ and large τ, i.e., near $\phi = \phi_0$ and far away from the origin $r \gg 1$, we have $\phi - \phi_0 = 2\sigma/\tau$ and $r = \tau^2/2$. Interestingly, in this limit, the Eq. (4.44) reduces to

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} - \frac{l}{4r} + pr - \gamma\right)R(r) = 0, \quad S(\sigma) = c \cdot \sigma, \tag{4.57}$$

where $R(r) = r^{1/4}T(r)$ and c is some constant. Note the appearance of the Cornell potential [39,40]. This is reasonable since it is known to describe well confinement and/or bound states of heavy quarks in $(1+3)$D [41]. In fact, the first equation in (4.44) in general describes, interestingly, a particle in two dimensions in a Cornell potential. It reduces to

$$\left(\frac{1}{2}p_r^2 + \frac{p_\phi^2}{2r^2} - \frac{l}{4r} + pr - \gamma\right)\chi(r, \phi) = 0, \tag{4.58}$$

where p_r and p_ϕ are the radial and angular momenta (4.36), and the ϕ dependence of $\chi(r, \phi)$ is $\exp(im\phi)$

for some known m; see Appendix B for the details. The equation for R reduces in the strict large r limit to Airy equation. The wave function is $\psi_0 \approx (\phi - \phi_0) \cdot r^{1/4} \cdot \mathrm{Ai}$. Thus, the spectrum is given by (4.15) and (4.20) with $p = 2\sqrt{2}$,

$$\bar{\gamma}_3^{(j,l)} = (2 \cdot 27 \cdot \pi^2)^{\frac{1}{3}}\gamma = 3\left[3\pi^2\left(j + \frac{1}{2}\right) + \cdots\right]^{\frac{2}{3}} + \cdots, \tag{4.59}$$

where j is large integer and $\cdots$ denotes corrections that involve j and l. The factor 3 is due to the three flux tubes. In general, we expect a factor of n. n is the number of flux tubes or adjoint quarks. This is the case since the quarks are on a closed string, and each quark is connected to two flux tubes.

The spectrum γ in general is determined by the boundary conditions at $\tau = 0$ and $\tau = \infty$, and the smoothness condition (4.56). We stress that the ansätz (4.33) and thus the conjecture holds provided we find nontrivial and real values for the spectrum γ that are consistent with the boundary conditions and smoothness of the wave function (4.56).

In this paper, we studied analytically the mass spectrum of the low-lying bound states with n constituent adjoint quarks of equal mass m in two-dimensional adjoint QCD in the case $n = 3$. In the absence of the interaction term or $\lambda = 0$, in general, the mass spectrum M is a smooth surface and symmetric about the minimum value given by $M_0 = nm$. We studied the spectrum around this minimum value at weak coupling or heavy quarks limit $\alpha \gg 1$. In the $n = 3$ case, we obtained the result (4.59) for the spectrum and, to the best of my knowledge, this result has not been obtained before. We also determined the nonrelativistic Schrödinger equation that describes the low-lying bound states for any n in the heavy quarks limit. Also this general equation, to the best of my knowledge, has not been reported elsewhere in the literature before.

We hope to provide a detailed analysis of the spectrum γ beyond the approximate answer (4.59) and related quantities in a future paper. We also hope to study the $n \geq 4$ cases in a separate paper in the future.

The same analysis can be also done for the case where the fermions have different masses. We hope to study this and extend the discussion to higher orders in the coupling in the future.

In Appendix A, we provided representative plots of periodic motions in the associated classical system.

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APPENDIX A: 3-PARTON CLASSICAL DYNAMICS

The associated classical system to the n-body quantum system (4.1) is described by the Hamiltonian,[22]

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, \ldots, q_n), \tag{A1}$$

where the potential V is given by

$$V(q_1, \ldots, q_n) = \sum_{i=1}^{n} |q_i - q_{i+1}|, \qquad q_{n+1} \coloneqq q_1. \tag{A2}$$

We hope to study the large n limit of the Hamiltonian (A1) by applying the collective field method [42] in the future. We also hope to study entanglement entropy for coupled mesons in a similar approach to that of [43] in a future paper.

In this appendix, we provide representative plots of classical periodic motions that possibly correspond bound states in the 3-parton quantum system. We will choose the center of mass position to be zero; thus, $z_1(t) = 0$. The classical Hamiltonian in this case is given by

$$H = \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2 + V(z_2, z_3), \tag{A3}$$

where the potential $V(z_2, z_3)$ is given by (4.24). The equations of motion are given by Hamilton's equations,

$$\dot{z}_2 = p_2, \qquad \dot{z}_3 = p_3, \qquad -\dot{p}_2 = \frac{\partial H}{\partial z_2}, \qquad -\dot{p}_3 = \frac{\partial H}{\partial z_3}. \tag{A4}$$

In this classical system, there are two classes of closed periodic orbits, depending on initial conditions. We hope to discuss their semiclassical quantization in relation to the spectrum of the quantum system in a future work.

In the first class, the trajectories of the three quarks, i.e., $x_1(t)$, $x_2(t)$, and $x_3(t)$, meet together only at zero position. In terms of $z_2(t)$ and $z_3(t)$, this implies, $z_2(t)$ and $z_3(t)$ meet or cross each other only at the origin. Thus, there is no exchange of momentum. The energies along the trajectories $z_2(t)$ and $z_3(t)$ are conserved independently. A typical plot is given in Fig. 3.

[22] At the quantum level, $H\phi_n^{(m)} = E_n^{(m)}\phi_n^{(m)}$, $E_n^{(m)} = \bar{\gamma}_n^{(m)}/(2\pi^2)^{1/3}n$.

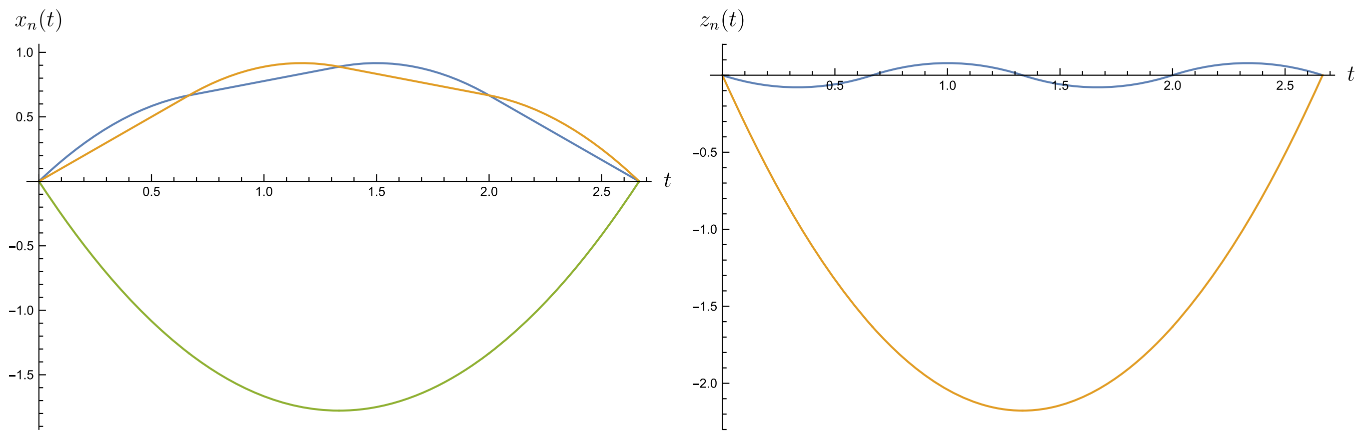

FIG. 3. All the masses are taken to be one in mass unit. On the left side, we have the trajectories $x_1(t)$(blue), $x_2(t)$(orange), and x_3(t)(green). On the right side, we have $z_2(t) = (x_2(t) - x_1(t))/\sqrt{2}$ (blue) and $z_3(t) = \sqrt{3/2}x_3(t)$ (orange). At $t = 0$, $\dot{x}_1 = (5/3)\dot{x}_2$, $\dot{x}_2 = 1$, $\dot{x}_3 = -\dot{x}_1 - \dot{x}_2$. Here, the plot is for a half period.

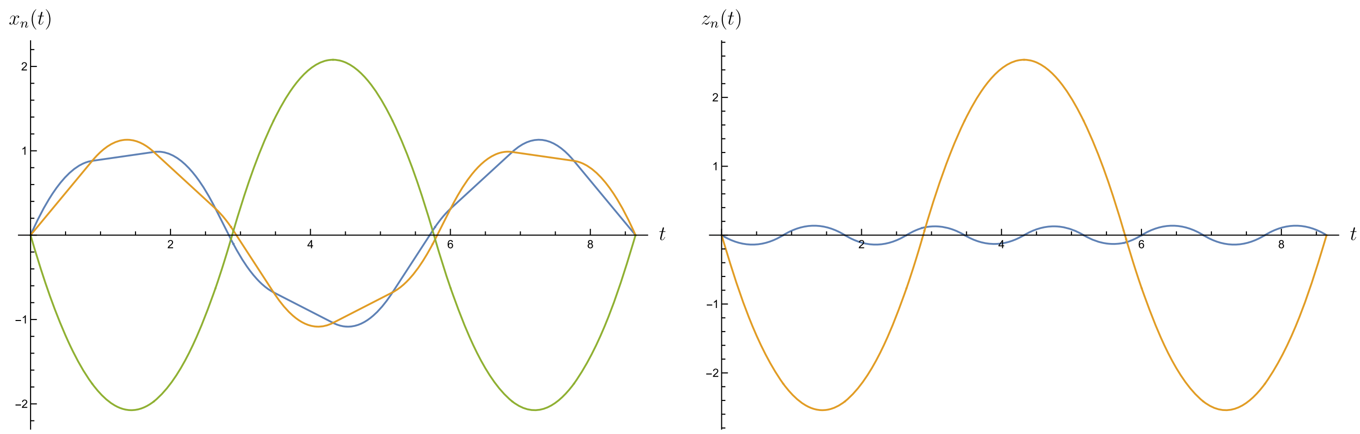

FIG. 4. All the masses are taken to be one in mass unit. On the left side, we have the trajectories $x_1(t)$(blue), $x_2(t)$(orange), and $x_3(t)$(green). On the right side, we have $z_2(t) = (x_2(t) - x_1(t))/\sqrt{2}$ (blue) and $z_3(t) = \sqrt{3/2}x_3(t)$ (orange). At $t = 0$, $\dot{x}_1 = \alpha\dot{x}_2$, $\dot{x}_2 = 1$, $\dot{x}_3 = -\dot{x}_1 - \dot{x}_2$, $\alpha \approx 1.880810$. α is given by the real solution of a polynomial of degree 8. Here, the plot is for a half period.

In the second class of closed periodic orbits, the trajectories $z_2(t)$ and $z_3(t)$ meet at least once away from zero position before they both meet again at the origin for the first nonzero time. In this case, there is an exchange of momentum between z_2 and z_3. However, the total energy is conserved. A typical plot is given in Fig. 4.

APPENDIX B: THE QUARTIC ANHARMONIC OSCILLATOR

The quartic anharmonic oscillator is described by the Eq. (4.44),

$$-\frac{d^2T}{d\tau^2} + (p\tau^4 - 2\gamma\tau^2 - q)T(\tau) = 0, \qquad p > 0. \quad \text{(B1)}$$

Here, q is the energy of the oscillator and thus, positive, γ is the quadratic coupling and p is the quartic coupling.

We first note that the radial part of the equation,

$$\left(\frac{1}{2}p_r^2 + \frac{p_\phi^2}{2r} + pr - \gamma\right)\psi(r,\phi) = 0, \quad \text{(B2)}$$

where

$$p_r^2 = -\frac{1}{r^{2l}}\frac{\partial}{\partial r}r^{2l}\frac{\partial}{\partial r} = -\frac{1}{r^l}\frac{\partial^2}{\partial r^2}r^l + l(l-1)\frac{1}{r^2},$$
$$p_\phi^2 = -\frac{\partial^2}{\partial\phi^2}, \quad \text{(B3)}$$

is equivalent to (B1) with $l = 1/4$. To see this, we write

$$r = \frac{1}{2}\tau^2, \qquad \psi(r,\phi) = T(r)e^{\sqrt{\frac{q}{2}}\phi}. \tag{B4}$$

This gives precisely (B1).

We write (B2) as

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l-1)}{2r^2} - \frac{q}{4r} + pr - \gamma\right)R(r) = 0, \tag{B5}$$

where $l = 1/4$ and $R(r) = r^l T(r) = r^{1/4}T(r)$. As it is clear from (B3), the $1/r^2$ term comes from the radial momentum operator. We note that (B5) can be put in the form,

$$\left(\frac{1}{2}p_r^2 + \frac{l^2}{2r^2} - \frac{q}{4r} + pr - \gamma\right)\chi(r) = 0, \tag{B6}$$

where $\chi(r) = r^{-1/2}R(r)$, p_r is the two-dimensional radial momentum operator (4.36), and l can be thought of as the effective orbital angular momentum. Note that the effective potential is convex everywhere; see (B5) [44]. Therefore, the quartic oscillator (4.44) equivalently describes a particle in two dimensions in a Cornell potential,

$$V(r) = -\frac{q}{4r} + pr. \tag{B7}$$

In the case $q \neq 0$, the large r limit of (B5) gives

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} - \frac{q}{4r} + pr - \gamma\right)R(r) = 0. \tag{B8}$$

Note we kept the $1/r$ term to account for the q dependence.

We also note that for small r (B5) is the (radial part of the) hydrogen problem. In this limit, the equation reduces to

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} - \frac{q}{4r} + \frac{l(l-1)}{2r^2} - \gamma\right)R(r) = 0. \tag{B9}$$

Here, the coupling γ is the energy of the particle, and the energy q measures its charge. The solutions are given by Whittaker functions.

Note also (B9) is equivalent to setting $p = 0$ in (B1). Thus (depending on the sign of γ), it also describes the harmonic oscillator.

We note that in the case in which $q = 0$ the Eq. (B5) reduces to

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l-1)}{2r^2} + pr - \gamma\right)R(r) = 0. \tag{B10}$$

The solutions give a relation between the couplings p and γ which corresponds to a state in (B1) with zero energy. A semiclassical calculation [24][23] gives

$$\gamma_{(n,l)} = \left[\frac{3}{4}\pi \cdot p\left(n + \frac{l}{2}\right)\right]^{\frac{2}{3}}. \tag{B11}$$

In the large r limit, the Eq. (B10) further reduces to

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + pr - \gamma\right)R(r) = 0. \tag{B12}$$

In this limit, the solutions are given by Airy functions. γ is given by (B11) with $l = 0$ and large n.

[23] In the paper l is assumed to be large.

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