T-duality–shift–T-duality transformation versus light-cone reduction and the gravity dual of a nonrelativistic fluid

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We discuss two different approaches to synthesize holographic nonrelativistic fluid from its relativistic counterpart. In the first approach we obtain the nonrelativistic fluid by light-cone reduction of a relativistic conformal fluid. In the second approach we consider the bulk dual of the relativistic fluid, uplift the solution to 10 dimensions and perform TsT transformations on the bulk solution to change the asymptotic structure. Reducing the TsT transformed geometry over S^5 we find an effective 5 dimensional locally boosted solution. We then use the bulk-boundary dictionary to compute the nonrelativistic constitutive relations. We show that the nonrelativistic fluids obtained by these two methods are equivalent up to second order in derivative expansion. Our results also provide explicit expressions for different constitutive relations and transports of holographic $U(1)$ charged nonrelativistic fluids (both parity odd and even) up to second order in derivative expansion.

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I. INTRODUCTION

Finding holographic descriptions of nonrelativistic systems is an interesting area of research as they are realized in low energy experiments and various condensed matter systems. One possible way to construct such descriptions is to take nonrelativistic limits on both sides of the AdS/CFT duality [1–3].

There are various distinct ways to derive nonrelativistic systems from a given relativistic theory. Light-cone reduction (LCR) is one of such techniques. The technique is based on the fact that a relativistic system respects Poincaré invariance whereas a system is called nonrelativistic if it respects Galilean symmetry. It is well known that the Galilean algebra in d space dimensions can be obtained by reducing the Poincaré algebra $so(d+1,1)$ in $(d+1,1)$ dimensions ($d+1$ space directions, one time direction) on the light-cone directions $x^{\pm} \sim x^0 \pm x^{d+1}$, where x^0 is the relativistic time and x^{d+1} is a spatial coordinate. The light-cone reduced theory respects the Galilean symmetry while evolving along the light-cone time x^+. In the same way if we consider conformal algebra $so(d+2,2)$ in $(d+1,1)$ dimensions, under LCR it boils down to Schrödinger algebra [4–10] in d space dimensions.

In order to provide a holographic description of a theory one has to first check the isometries on both sides. A conformal theory in $(d+1,1)$ dimensions is dual to a gravity theory in AdS_{d+3} whose metric is given by,

$$ds^2 = -r^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dr^2}{r^2} \tag{1.1}$$

where η is a $d+1$, 1 dimensional Minkowski metric and $\mu,\nu = 0,\ldots,d+1$. The isometry group of AdS_{d+3} is $SO(d+2,2)$ which is the conformal group of the boundary theory. In a similar spirit, the holographic dual of a theory with Schrödinger isometry was first proposed in [11–14]. The bulk metric is given by[1]

$$ds^2 = r^2(-2dx^+dx^- - r^2(dx^+)^2 + d\vec{x}^2) + \frac{dr^2}{r^2} \tag{1.2}$$

where $\vec{x}$ is a d dimensional vector. This spacetime is known as Schrödinger spacetime, denoted by Sch_{d+3} and has the

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[1] In [12] the bulk metric was proposed for Galilean symmetry algebra with an arbitrary scaling z. $z=2$ corresponds to Schrödinger algebra with an extra generator corresponds to special conformal transformation in time direction. Here we consider the $z=2$ case.

Schrödinger algebra as a global symmetry algebra [15,16]. Although the Schrödinger algebra of the boundary theory can be derived from the conformal algebra by LCR, the Schrödinger metric (1.2) cannot be obtained from the AdS_{d+3} metric (1.1) by LCR. However, there is a way to get the Sch_{d+3} spacetime from the AdS geometry using the solution generating technique [17–20], know as *TsT* transformation [21] (or equivalently *null-Melvin twist* [17,22–24]). One can start with a AdS_5 geometry, uplift the metric to ten dimensions to embed in a type IIB solution. Identify two isometry directions in ten dimensions: x^- and ψ, say. The TsT transformation consists of three steps. A T-duality along x^- direction, followed by a shift in ψ: $\psi \to \psi + x^-$ and finally a further T-duality along x^- direction. As a result the reduced five dimensional spacetime is given by (1.2).

LCR on the boundary and TsT in the bulk are therefore consistent at the level of isometry, in a sense that the asymptotic symmetry of a TsT transformed bulk spacetime matches with the corresponding Schrödinger symmetry of the boundary theory. Our goal is to understand the consistency between the LCR and TsT transformation beyond the geometry, in particular at the hydrodynamic scale, which is considered to be a low energy or long wavelength departure from local thermal equilibrium. Our starting point is a hydrodynamic system with conformal invariance and its gravity dual. Since hydrodynamics is an effective description, its evolution is governed by the set of conservation equations (known as constitutive equations). The holographic description of a hydrodynamic system is given by locally boosted black brane geometry [25]. In the LCR method, we reduce the relativistic constitutive relations over the light-cone directions to obtain a set of equations consistent with Schrödinger isometry and thus we get a *nonrelativistic* hydrodynamic system with stress tensor, energy current, charge current etc. In the TsT method, we uplift the locally boosted black brane geometry to ten dimensions, perform the TsT transformation and get a reduced locally boosted five dimensional Schrödinger spacetime. Then we use the bulk-boundary dictionary of [26] to write down the conserved quantities (stress tensor, energy current, charge current etc) of the boundary nonrelativistic system from the reduced effective five dimensional effective action. The question is whether these two reduced systems are identical. We describe the problem with help of the following commutative diagram in Fig. 1.

A partial answer to this question was given by [27]. It was shown that both the paths (LCR and TsT) commute up to first order in derivative expansion. The first order calculations were somewhat trivial and it was not clear whether the commutativity would hold at higher orders. In this paper we address this question more rigorously. We consider second order $U(1)$ charged conformal hydrodynamics with both parity-odd and parity-even sectors and its gravity dual [28]. We show that the nonrelativistic fluid obtained via LCR and TsT transformation are *identical* order by order in derivative expansion up to second order in derivative expansion. As a byproduct of our calculation we gave a complete holographic description of nonrelativistic charged fluid with Schrödinger isometry up to second order in derivative expansion.

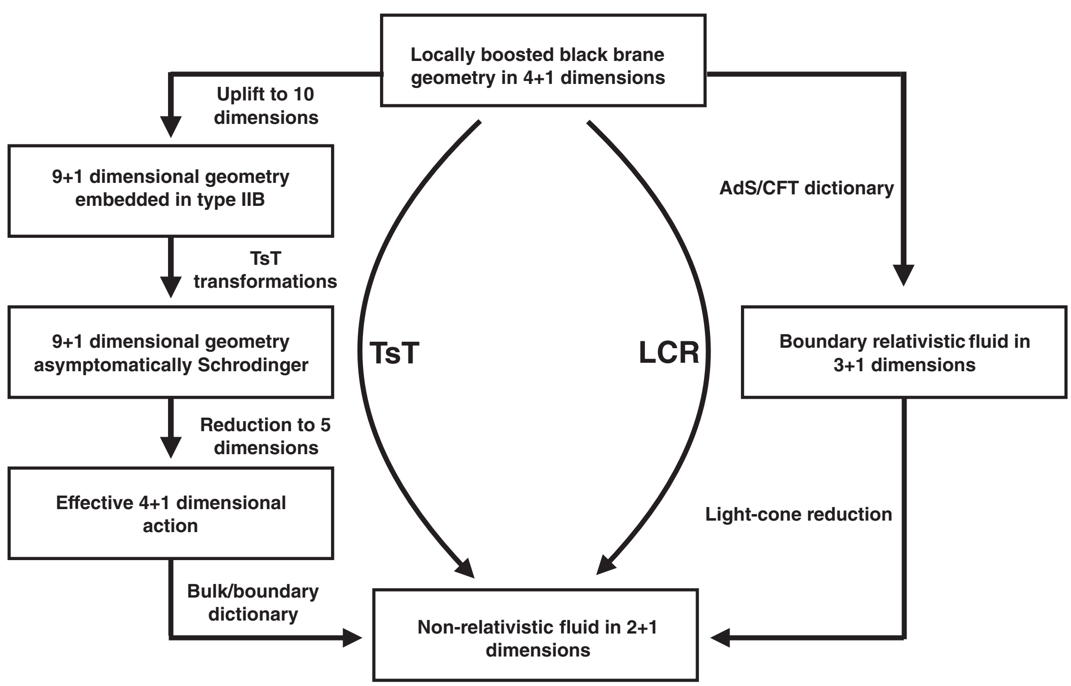

FIG. 1. Two different paths to obtain second order nonrelativistic fluids.

II. LOCALLY BOOSTED BLACK BRANE GEOMETRY AND THE RELATIVISTIC FLUID

We start with Einstein-Maxwell theory in AdS_5 with a Chern-Simons term

$$S = \int d^5x(R - 2\Lambda - \mathcal{G}_{ab}\mathcal{G}^{ab} - \frac{4\kappa}{3}\epsilon^{abcde}\mathcal{C}_a\mathcal{G}_{bc}\mathcal{G}_{de}) \quad (2.1)$$

where $\Lambda = -6/L^2$ is the five dimensional cosmological constant, $\mathcal{G}$ field strength of the $U(1)$ gauge field. The Einstein's and the Maxwell's equations obtained from this action admit the following solutions

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(r,m,q)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu$$
$$\mathcal{C}_\mu = \frac{\sqrt{3}q}{2r^2}u_\mu \quad (2.2)$$

where

$$x^\mu = \{v, x, y, z\} \quad (2.3)$$

are boundary coordinates, u^μ is a constant four velocity whose different components are given by

$$u_v = -\gamma, \qquad u_i = \gamma\beta_i, \qquad \gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}} \quad (2.4)$$

such that $u^2 = -1$ and

$$f(r,m,q) = 1 - \frac{m}{r^4} + \frac{q^2}{r^6}. \quad (2.5)$$

$$P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu, \qquad \eta_{\mu\nu} = \mathrm{diag}\{-1,1,1,1\}.$$

is the projection operator given by

$$P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu, \qquad \eta_{\mu\nu} = \mathrm{diag}\{-1,1,1,1\}. \quad (2.6)$$

The solutions (2.2) are parametrized by five constant parameters: three boosts $\vec{\beta}$, one mass parameter m and one charge parameter q and represent an electrically charged black hole with horizon at $r = R$, given by $f(R,m,q) = 0$. In the rest frame ($\vec{\beta} = 0$) the metric becomes the standard Reissner–Nordström metric written in Eddington–Finkelstein coordinates. We have taken the AdS radius $L = 1$.

We now consider the black hole parameters $\vec{\beta}$, m and q to be slowly varying functions of boundary coordinates x^μ

$$m \to m(x^\mu), \qquad \vec{\beta} \to \vec{\beta}(x^\mu), \qquad q \to q(x^\mu).$$

With such replacement the metric and gauge field (2.2) do not solve Einstein's as well as Maxwell's equations any more. In order to find the solutions with local parameters we have to supplement the metric and gauge field in (2.2) with extra terms proportional to the derivatives of the parameters. Assuming m, q and $\vec{\beta}$ are slowly varying functions of x^μ, one can solve Einstein's equations order by order in derivative expansion. Up to second order in derivative expansion the solutions are given by [28],

$$\begin{aligned} ds^2 = &-2u_\mu dx^\mu dr - r^2 f(r,m,q)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu + \frac{2r^2}{R}F_2\left(\frac{r}{R},\frac{m}{R^4}\right)\sigma_{\mu\nu}dx^\mu dx^\nu + \frac{2}{3}ru_\mu u_\nu(\partial u)dx^\mu dx^\nu \\ &- 2ru_\mu(u^\lambda\partial_\lambda u_\nu)dx^\mu dx^\nu - 2u_\mu\left(\frac{\sqrt{3}\kappa q^3}{mr^4}l_\nu + \frac{6qr^2}{R^7}P^\lambda_\nu\mathcal{D}_\lambda qF_1\left(\frac{r}{R},\frac{m}{R^4}\right)\right)dx^\mu dx^\nu + r^2\alpha_{\mu\nu}(r)dx^\mu dx^\nu \\ &+ 3h(r)u_\mu dx^\mu dr + r^2 h(r)P_{\mu\nu}dx^\mu dx^\nu - 12r^2 j_\alpha P^\alpha_\nu u_\mu dx^\mu dx^\nu + \frac{k(r)}{r^2}u_\mu u_\nu dx^\mu dx^\nu \end{aligned} \quad (2.7)$$

and

$$\mathcal{C}_\mu = \frac{\sqrt{3}q}{2r^2}u_\mu + \frac{\sqrt{3}w(r)}{2r^2}u_\mu + \frac{3\kappa q^2}{2mr^2}l_\mu - \frac{\sqrt{3}r^5}{2R^8}P^\lambda_\mu\mathcal{D}_\lambda qF_1^{(1,0)}\left(\frac{r}{R},\frac{m}{R^4}\right) + \frac{\sqrt{3}r^5}{2}g_\mu(r). \quad (2.8)$$

$\sigma^{\mu\nu}$, l^μ and the Weyl covariant derivative $\mathcal{D}_\mu$ are given by

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\partial_{(\alpha}u_{\beta)} - \frac{1}{3}P^{\mu\nu}\partial\cdot u, \qquad l^\mu = \epsilon^{\nu\lambda\sigma\mu}u_\nu\partial_\lambda u_\sigma, \qquad P^\lambda_\mu\mathcal{D}_\lambda q = P^\lambda_\mu\partial_\lambda q + 3(u\cdot\partial u_\mu)q. \quad (2.9)$$

The functions F_1 and F_2 are given by

$$F_1\left(\frac{r}{R},\frac{m}{R^4}\right)=\frac{1}{3}\left(1-\frac{m}{r^4}+\frac{q^2}{r^6}\right)\int_{\frac{r}{R}}^{\infty}dp\frac{1}{(1-\frac{m}{R^4p^4}+\frac{q^2}{R^6p^6})^2}\left(\frac{1}{p^8}-\frac{3}{4p^7}\left(1+\frac{R^4}{m}\right)\right)$$

$$F_2\left(\frac{r}{R},\frac{m}{R^4}\right)=\int_{\frac{r}{R}}^{\infty}dp\frac{p(p^2+p+1)}{(p+1)(p^4+p^2-\frac{m}{R^4}+1)}. \tag{2.10}$$

The parameters m, q, and R are not independent at the leading order, they are related by

$$q^2=R^2(m-R^4). \tag{2.11}$$

The exact form of various scalar, vector and tensor functions [$h(r)$, $k(r)$, $w(r)$, $j_\mu(r)$, $g_\mu(r)$, and $\alpha_{\mu\nu}(r)$] can be found in [28]. We have also presented these functions in light cone coordinates in appendix A.

The metric and gauge field given by Eqs. (2.7) and (2.8) provide holographic description of $U(1)$ charged conformal fluid in flat $3+1$ dimensions. One can use the AdS/CFT dictionary to compute the constitutive relations (stress-energy tensor and $U(1)$ current) of the relativistic fluid up to second order in derivative expansion. They are given by

$$T^{\mu\nu}=(E+P)u^\mu u^\nu+P\eta^{\mu\nu}-2\eta\sigma^{\mu\nu}+\mathcal{N}_1u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu}-2\mathcal{N}_2(\omega^\mu{}_\lambda\sigma^{\lambda\nu}+\omega^\nu{}_\lambda\sigma^{\lambda\mu})+\mathcal{N}_3\left(\sigma^\mu{}_\lambda\sigma^{\lambda\nu}-\frac{1}{3}P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta}\right)$$
$$+4\mathcal{N}_4\left(\omega^\mu{}_\lambda\omega^{\lambda\nu}+\frac{1}{3}P^{\mu\nu}\omega^{\alpha\beta}\omega_{\alpha\beta}\right)+\mathcal{N}_5q^{-1}\Pi^{\mu\nu\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta q+\mathcal{N}_6q^{-2}\Pi^{\mu\nu\alpha\beta}\mathcal{D}_\alpha q\mathcal{D}_\beta q+\mathcal{N}_7\Pi^{\mu\nu\alpha\beta}(\mathcal{D}_\alpha l_\beta+\mathcal{D}_\beta l_\alpha)$$
$$+\mathcal{N}_8q^{-1}\Pi^{\mu\nu\alpha\beta}l_\alpha\mathcal{D}_\beta q+\mathcal{N}_9q^{-1}\epsilon^{\alpha\beta\lambda(\mu}\sigma_\lambda^{\nu)}u_\alpha\mathcal{D}_\beta q \tag{2.12}$$

and

$$J^\mu=4\sqrt{3}qu^\mu-4\sqrt{3}\lambda_rP^{\mu\nu}\mathcal{D}_\nu q+\xi_rl^\mu+\gamma_1P^{\mu\nu}\mathcal{D}_\lambda\sigma_\nu^\lambda$$
$$+\gamma_2P^{\mu\nu}\mathcal{D}_\lambda\omega_\nu^\lambda+\gamma_3l^\lambda\sigma_\lambda^\mu$$
$$+\gamma_4q^{-1}\sigma^{\mu\lambda}\mathcal{D}_\lambda q+\gamma_5q^{-1}\omega_{\mu\lambda}\mathcal{D}_\lambda q. \tag{2.13}$$

The tensors $\omega^{\mu\nu}$ and $\Pi^{\alpha\beta\mu\nu}$ is given by

$$\omega^{\mu\nu}=\frac{1}{2}P^{\mu\alpha}P^{\nu\beta}(\partial_\alpha u_\beta-\partial_\beta u_\alpha) \tag{2.14}$$

$$\Pi^{\alpha\beta\mu\nu}=\frac{1}{2}\left(P^{\alpha\mu}P^{\beta\nu}+P^{\alpha\nu}P^{\beta\mu}-\frac{2}{3}P^{\alpha\beta}P^{\mu\nu}\right). \tag{2.15}$$

One can independently construct this stress tensor by demanding Weyl covariance [29]. The transport coefficients for such fluid in general depend on the temperature and charge density. The holographic values of different independent transport coefficients appear in (2.12) and (2.13) are given in Table I. m_0 and q_0 appearing in different transports are leading (unperturbed) values of m and q. Also we have set the leading value of the horizon radius R to be one.

TABLE I. Relativistic transports.

η	R^3
λ_r	$\frac{m+R^4}{4mR}$
ξ_r	$\frac{12\kappa q^2}{m}$
$\mathcal{N}_1$	$2+\frac{m_0}{\sqrt{4m_0-3}}\log\left(\frac{3-\sqrt{4m_0-3}}{3+\sqrt{4m_0-3}}\right)$
$\mathcal{N}_2$	$\frac{-1}{2}(\mathcal{N}_1-2)$
$\mathcal{N}_3$	2
$\mathcal{N}_4$	$-\frac{(m_0-1)}{m_0}(12(m_0-1)\kappa^2-m_0)$
$\mathcal{N}_5$	$-\frac{(m_0-1)}{2m_0}$
$\mathcal{N}_6$	$\frac{1}{2}(m_0-1)(3\log(2)-1)$
$\mathcal{N}_7$	$\frac{\sqrt{3}\kappa(m_0-1)^{\frac{3}{2}}}{m_0}$
$\mathcal{N}_8$, $\mathcal{N}_9$	0
γ_1	$\frac{3\sqrt{3}q_0}{m_0}$
γ_2	$\frac{2\sqrt{3}q_0^3\kappa^2}{m_0^2}$
γ_3	$-\frac{12q_0^2\kappa}{m_0^2}$
γ_4	$2\sqrt{3}q_0\log 2$
γ_5	$-\frac{\sqrt{3}q_0(m_0^2-48\kappa^2q_0^2+3)}{2m_0^2}$

III. LIGHT-CONE REDUCTION AND NONRELATIVISTIC SECOND ORDER FLUID

As we have already discussed, a nonrelativistic fluid can be obtained by LCR of relativistic fluid. The idea follows from the fact that LCR of conformal algebra renders Schrödinger algebra in one lower dimension. We start with relativistic constitutive equations and reduce these equations along a light-cone direction. As a result, the nonrelativistic quantities like energy density, pressure, stress tensor, charge current etc. are given in terms of different components of relativistic stress tensor and $U(1)$ current.

The conservation equations for a $U(1)$ charged relativistic fluid are given by

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu J^\mu = 0. \tag{3.1}$$

We are considering the fluid in a flat background and in absence of any external electric or magnetic fields. The flat metric in $(d+1, 1)$ dimensions is given by

$$ds^2 = -(dx^0)^2 + (dx^{d+1})^2 + \sum_{i=1}^{d} (dx^i)^2. \tag{3.2}$$

In the light-cone frame $x^\pm$, defined by

$$x^+ = \frac{x^0 + x^{d+1}}{\sqrt{2}}, \qquad x^- = \frac{x^0 - x^{d+1}}{\sqrt{2}} \tag{3.3}$$

the metric takes the form

$$ds^2 = -2dx^+dx^- + \sum_{i=1}^{d} (dx^i)^2. \tag{3.4}$$

It turns out that [18,20,30] under LCR the relativistic constitutive equations boil down to the nonrelativistic constitutive equations in one lower dimension. These equations describe a nonrelativistic fluid in one lower dimension.

To obtain the Schrödinger algebra one has to look for the subalgebra of the conformal algebra where all the generators commute with P_- (translation along x^-). Therefore we consider only those solutions to the relativistic equations that do not depend on x^-, i.e. x^- is an isometry direction. To make the reduced theory consistent with discrete light-cone quantization we reduce the theory along x^- direction, and consider x^+ to be the nonrelativistic time.

In the light-cone frame ($x^\mu = \{x^+, x^-, x^i\}$) the nonzero components of metric and partial derivatives are given by

$$g^{+-} = g_{+-} = -1, \qquad g^{ij} = g_{ij} = 1$$
$$\partial_\mu = \{\partial_+, \partial_-, \partial_i\}, \qquad \partial^\mu = \{-\partial_-, -\partial_+, \partial_i\}. \tag{3.5}$$

After reduction, the relativistic constitutive equations are given by

$$\partial_+ T^{+-} + \partial_i T^{i-} = 0, \qquad \partial_+ T^{++} + \partial_i T^{i+} = 0,$$
$$\partial_+ T^{+j} + \partial_i T^{ij} = 0, \qquad \partial_+ J^+ + \partial_i J^i = 0.$$

These equations can be interpreted as dynamical equations governing the motion of nonrelativistic fluid if we identify different components of energy-momentum tensor and charge current with nonrelativistic quantities as follows

$$T^{++} = \rho, \qquad T^{+i} = \rho v^i, \qquad T^{+-} = \epsilon + \frac{1}{2}\rho v^2,$$
$$T^{ij} = t^{ij}, \qquad Q = J^+, \qquad j^i = J^i. \tag{3.6}$$

A relativistic fluid in four space time dimensions is described in terms of its normalized four velocity u^μ, temperature T and charge density q. On the other hand the nonrelativistic fluid in one lower dimension is described in terms of its mass density ρ, pressure p, spatial velocity v^i and charge density Q. Hence the number of fluid variables in both the theories are the same and a mapping between them can be found. Since fluid dynamics admits derivative expansion, the relations between the nonrelativistic and relativistic fluids can be obtained order by order in derivative expansion.

LCR of conformal uncharged and charged fluid up to first order in derivative expansion have been derived in [18,20,27,30]. In a recent paper [31] LCR of conformal uncharged second order fluid has been considered thoroughly. It was shown that the Rivlin-Ericksen fluid [32] is a subclass of such nonrelativistic fluid. In this paper we further extend those calculations. We light-cone reduce a $(3+1)$ dimensional second order Weyl invariant $U(1)$ charged fluid (2.12), (2.13) and obtain the corresponding second order charged nonrelativistic fluid.

To perform the reduction we need to consider two sets of constraints relations. The first set of relations is derived from the conservation laws of the first order corrected stress tensor and charge current. We list the independent first order data in Table II. We have categorized the nonrelativistic fluid data in scalar, vector, and tensor with respect to the $SO(2)$ symmetry group, as the nonrelativistic fluid has an $SO(2)$ symmetry (rotation about z axis).

The second set of constraints can be derived by taking derivatives of the zeroth order conservation laws. The independent second order data are given in Table III. $\tilde{\sigma}_{ij}$ and $\tilde{\omega}_{ij}$ are given by

$$\tilde{\sigma}^{ij} = \partial^i v^j + \partial^j v^i - \delta^{ij}\partial_k v^k, \qquad \tilde{\omega}^{ij} = \partial^i v^j - \partial^j v^i. \tag{3.7}$$

The angle brackets denote symmetric traceless combinations

$$A^{\langle i}B^{j\rangle} = (A^i B^j + A^j B^j - \delta^{ij} A^k B_k). \tag{3.8}$$

TABLE II. Independent nonrelativistic fluid data at first order.

	Data	Constraint	Independent data
Scalar	$\partial_+ T$, $\partial_+\phi$, $\epsilon^{ij}\omega_{ij}$, $\partial_+ u^+$, $\partial_i u^i$	$\partial_\mu T^{\mu+} = 0$, $\partial_\mu T^{\mu-} = 0$, $\partial_\mu J^\mu = 0$	$\partial_i v^i$, $\epsilon^{ij}\tilde{\omega}_{ij}$
Vector	$\partial_i T$, $\partial_i\phi$, $\partial_i u^+$, $\partial_+ u^i$ $\epsilon^{ik}\partial_k T$,$\epsilon^{ik}\partial_k\phi$, $\epsilon^{ik}\partial_k u^+$	$\partial_\mu T^{\mu i} = 0$	$\partial_i\tau$, $\partial_i\nu$, $\partial_i\mu_m$ $\epsilon^{ik}\partial_k\tau$, $\epsilon^{ik}\partial_k\nu$, $\epsilon^{ik}\partial_k\mu_m$
Tensor	σ^{ij}		$\tilde{\sigma}^{ij}$

TABLE III. Independent nonrelativistic fluid data at second order.

	Data	Constraint	Independent data
Scalar	$\partial_+^2 T$, $\partial_+^2\phi$, $\partial_+^2 u^+$, $\partial_i^2 T$, $\partial_i^2\phi$, $\partial_i^2 u^+$, $\partial_+\partial_i u^i$	$\partial_+\partial_\mu T^{\mu+} = 0$, $\partial_+\partial_\mu T^{\mu-} = 0$, $\partial_i\partial_\mu T^{\mu i} = 0$, $\partial_+\partial_\mu J^\mu = 0$	$\partial_i^2\tau$, $\partial_i^2\nu$ $\partial_i^2\mu_m$
Vector	$\partial_j\sigma^{ij}$, $\partial_j\omega^{ij}$, $\partial_+\partial_i u^+$, $\partial_+^2 u^i$ $\partial_+\partial_i T$, $\partial_+\partial_+\phi$ $\epsilon^{il}\partial_k\tilde{\sigma}^{lk}$, $\epsilon^{il}\partial_k\omega^{lk}$	$\partial_+\partial_\mu T^{\mu i} = 0$, $\partial_i\partial_\mu T^{\mu+} = 0$, $\partial_i\partial_\mu T^{\mu-} = 0$ $\partial_i\partial_\mu J^\mu = 0$	$\partial_j\tilde{\sigma}^{ij}$, $\partial_j\tilde{\omega}^{ij}$ $\epsilon^{il}\partial_k\tilde{\sigma}^{lk}$, $\epsilon^{il}\partial_k\tilde{\omega}^{lk}$
Tensor	$\partial^i\partial^j T$, $\partial^i\partial^j\phi$, $\partial^i\partial^j u^+$, $\partial_+\sigma^{ij}$, $\epsilon^{ik}\partial^k\partial^j u^+$, $\epsilon^{ik}\partial^k\partial^j T$ $\epsilon^{ik}\partial^k\partial^j\phi$	$\partial_i\partial_\mu T^{\mu j} = 0$	$\partial^{\langle i}\partial^{j\rangle}\tau$, $\partial^{\langle i}\partial^{j\rangle}\nu$ $\partial^{\langle i}\partial^{j\rangle}\mu_m$, $\epsilon^{\langle ik}\partial^k\partial^{j\rangle}\mu_m$, $\epsilon^{\langle ik}\partial^k\partial^{j\rangle}\tau$, $\epsilon^{\langle ik}\partial^k\partial^{j\rangle}\nu$

The nonrelativistic temperature and $U(1)$ chemical potential are defined as,

$$\tau = \frac{T}{u^+}, \qquad \mu = \frac{\phi}{u^+}. \tag{3.9}$$

The relativistic variables satisfy Euler relation: $E + P - q\phi = TS$. Using the relation between relativistic fluid variables and nonrelativistic fluid variable at leading order [18,20,27]

$$\rho = (E+P)(u^+)^2, \quad p = P, \quad \epsilon = \frac{E-P}{2}, \quad Q = 4\sqrt{3}qu^+ \tag{3.10}$$

we see that the nonrelativistic variables satisfy

$$\epsilon + p - \rho\rho_m - \mu\left(\frac{Q}{4\sqrt{3}}\right) = \tau s \tag{3.11}$$

where ρ_m is given by,

$$\rho_m = -\frac{1}{2(u^+)^2}. \tag{3.12}$$

and $s = u^+ S$ is the entropy density of the nonrelativistic fluid. This equation is interpreted as the Euler relation for the nonrelativistic fluid with an extra fluid variable ρ_m, which can be considered as chemical potential associated with particle number conservation [30]. In [27] a new basis has been introduced in terms of reduced chemical potential ν and redefined mass chemical potential μ_m given by,

$$\nu = \frac{\mu}{\tau}, \qquad \partial_i\mu_m = (u^+)^3\partial_i\rho_m \text{ i.e. } \mu_m = u^+. \tag{3.13}$$

In this paper we use the variables $\{\mu_m, \nu, \tau\}$ to express the nonrelativistic constitutive relations.

In Tables II and III all the nonrelativistic symmetric tensors are traceless. After using the constraint relations we have three independent second order symmetric traceless tensors in the parity even sector and three in the parity odd sector. In addition to these, we have composite symmetric traceless tensors constructed from the independent first order vectors. We have seven such vectors in parity even sector and seven in parity odd sector.[2] Thus, in total we

[2]In higher space dimensions there can be more composite terms like $\sigma^{\langle ik}\sigma^{kj\rangle}$ which identically vanishes in two space dimensions.

have 10 parity even and 10 parity odd second order independent symmetric traceless tensors. The light-cone reduced fluid stress tensor is expected to depend on these independent data.

After a careful computation and using the set of constraints we finally write down the stress tensor in a simplified form given by,

$$t^{ij} = \rho v^i v^j + p\delta^{ij} - n_1\tilde{\sigma}^{ij} + n_{a+1}\partial^{\langle i}\partial^{j\rangle}X_a + c_1\tilde{\sigma}^{\langle ik}\tilde{\omega}^{kj\rangle} + \mathbf{c}_{ab}\partial^{\langle i}X_a\partial^{j\rangle}X_b + g_a\epsilon^{\langle ik}\partial^k\partial^{j\rangle}X_a + \mathbf{g}_{ab}\epsilon^{\langle ik}\partial_k X_a\partial^{j\rangle}X_b + g_{10}\epsilon^{\langle ik}\tilde{\sigma}^{kl}\tilde{\omega}^{lj\rangle}. \quad (3.14)$$

The indices a, b run over 1, 2 and 3, $X_a = \{\mu_m, \nu, \tau\}$ denotes an array of nonrelativistic parameters, $\mathbf{c}_{ab}$ and $\mathbf{g}_{ab}$ are 3×3 transport matrices given by

$$\mathbf{c} = \begin{pmatrix} c_2 & c_3/2 & c_4/2 \\ c_3/2 & c_5 & c_6/2 \\ c_4/2 & c_6/2 & c_7 \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} g_4 & g_5 & g_6 \\ g_5 & g_7 & g_8 \\ g_6 & g_8 & g_9 \end{pmatrix}. \quad (3.15)$$

The other nonrelativistic fluid variables: mass density ρ, pressure p, velocity v^i, energy density ϵ, energy current e_i, and nonrelativistic transports are given in appendix B.

The nonrelativistic charge current which comes with eleven parity even and eleven parity odd transports are given by,

$$\mathcal{J}^i = Qv^i + \mathbf{q}_a\partial^i X_a + \tilde{\mathbf{q}}_a\epsilon^{ik}\partial_k X_a + n_1^{\mathcal{J}}\partial_k\tilde{\sigma}^{ik} + n_2^{\mathcal{J}}\partial_k\tilde{\omega}^{ik} + n_3^{\mathcal{J}}\epsilon^{il}\partial_k\tilde{\sigma}^{lk} + n_4^{\mathcal{J}}\epsilon^{il}\partial_k\tilde{\omega}^{lk} + \mathbf{c}_a^{\mathcal{J}}\tilde{\sigma}^{ik}\partial_k X_a + \tilde{\mathbf{c}}_a^{\mathcal{J}}\tilde{\omega}^{ik}\partial_k X_a + +\bar{\mathbf{c}}_a^{\mathcal{J}}(\partial_k v^k)\partial^i X_a + \mathbf{g}_a^{\mathcal{J}}\epsilon^{il}\tilde{\sigma}^{lk}\partial_k X_a + \tilde{\mathbf{g}}_a^{\mathcal{J}}\epsilon^{il}\tilde{\omega}^{lk}\partial_k X_a + \bar{\mathbf{g}}_a^{\mathcal{J}}(\partial_k v^k)\epsilon^{il}\partial_l X_a \quad (3.16)$$

where

$$\mathbf{q}_a = \{\lambda_q, \sigma_q, \kappa_q\}, \quad \tilde{\mathbf{q}}_a = \{\tilde{\lambda}_q, \tilde{\sigma}_q, \tilde{\kappa}_q\}, \quad \mathbf{c}_a^{\mathcal{J}} = \{c_1^{\mathcal{J}}, c_4^{\mathcal{J}}, c_7^{\mathcal{J}}\}, \quad \tilde{\mathbf{c}}_a^{\mathcal{J}} = \{c_2^{\mathcal{J}}, c_5^{\mathcal{J}}, c_8^{\mathcal{J}}\},$$
$$\bar{\mathbf{c}}_a^{\mathcal{J}} = \{c_3^{\mathcal{J}}, c_6^{\mathcal{J}}, c_9^{\mathcal{J}}\}, \quad \mathbf{g}_a^{\mathcal{J}} = \{g_1^{\mathcal{J}}, g_4^{\mathcal{J}}, g_7^{\mathcal{J}}\}, \quad \tilde{\mathbf{g}}_a^{\mathcal{J}} = \{g_2^{\mathcal{J}}, g_5^{\mathcal{J}}, g_8^{\mathcal{J}}\}, \quad \bar{\mathbf{g}}_a^{\mathcal{J}} = \{g_3^{\mathcal{J}}, g_6^{\mathcal{J}}, g_9^{\mathcal{J}}\} \quad (3.17)$$

are different transport arrays. The nonrelativistic charge density Q and different charge transports are given in appendix B.

As a consistency check we see that in the limit $q \to 0$ all the constitutive relations reduce to those for uncharged fluid found in [31]. In [31] the stress tensor at second order was written in terms of a quantity $\mathcal{B}^{ij} = \partial^i a^j + \partial^j a^i + 2(\partial^i v^k)(\partial^j v_k)$. The acceleration a^i for charged fluid is given by,

$$a^i = -\frac{\partial^i\mu_m}{u^+} - \frac{\partial^i\tau}{\tau} - \frac{\mu q}{(\tau s + \mu q)}\frac{\partial_i\nu}{\nu}. \quad (3.18)$$

In order to match our results with [31] one has to replace $\mathcal{B}^{ij}$ using the above expression of a^i (with $q = 0$) and trade $\partial^i\mu_m$ with a^i using the above relation.

Our next goal is to construct a holographic description of the nonrelativistic fluid studied in this section.

IV. HOLOGRAPHIC DESCRIPTIONOF SECOND ORDER NONRELATIVISTIC FLUID

TsT transformation is a solution generating technique in string theory which has been used to generate a black hole solution with asymptotically Schrödinger isometry [17,22,33–39]. The idea behind is based on the fact that supergravity has more symmetry than the string theory. If we perform TsT transformation on a string theory solution then it generates new solution of string theory, which is guaranteed to be a solution of supergravity theory. To perform the TsT transformation on a five dimensional geometry we first need to uplift the solution to ten dimensions to embed in string theory [17,37,39]. This uplift can be performed such that the 10 dimensional metric is the direct sum of the five dimensional metric (AdS part) and a five sphere which is written as a fibration over CP^2. Such ten dimensional uplift of the five dimensional metric will be a solution of type IIB string theory.

We write the five dimensional metric and the gauge field in the following form,

$$ds^2 = -2u_\mu dx^\mu dr + Su_\mu dx^\mu dr + S_{\mu\nu}dx^\mu dx^\nu$$
$$\mathcal{C} = \mathcal{C}_a dx^a = \mathcal{C}_\mu dx^\mu. \quad (4.1)$$

The exact form of S and $S_{\mu\nu}$ depend on the solution we are considering. Since we are interested in the bulk dual of second order charged fluid (2.7) and (2.8), S and $S_{\mu\nu}$ are given by

$$
\begin{aligned}
S &= 3h(r)\\
S_{\mu\nu} &= -r^2 f(r,m,q)u_\mu u_\nu + r^2 P_{\mu\nu} + \frac{2r^2}{R}F_2\left(\frac{r}{R},\frac{m}{R^4}\right)\sigma_{\mu\nu} + \frac{2}{3}ru_\mu u_\nu(\partial u) - 2ru_\mu(u^\lambda\partial_\lambda u_\nu)\\
&\quad - 2u_\mu\left(\frac{\sqrt{3}\kappa q^3}{mr^4}l_\nu + \frac{6qr^2}{R^7}P^\lambda_\nu\mathcal{D}_\lambda q F_1\left(\frac{r}{R},\frac{m}{R^4}\right)\right) + r^2\alpha_{\mu\nu}(r) + r^2 h(r)P_{\mu\nu} - 12r^2 j_\alpha P^\alpha_\nu u_\mu + \frac{k(r)}{r^2}u_\mu u_\nu\\
\mathcal{C}_\mu &= \frac{\sqrt{3}q}{2r^2}u_\mu + \frac{\sqrt{3}w_2(r)}{2r^2}u_\mu + \frac{3\kappa q^2}{2mr^2}l_\mu - \frac{\sqrt{3}r^5}{2R^8}P^\lambda_\mu\mathcal{D}_\lambda q F_1^{(1,0)}\left(\frac{r}{R},\frac{m}{R^4}\right) + \frac{\sqrt{3}r^5}{2}g_\mu(r).
\end{aligned}
\tag{4.2}
$$

To uplift the five dimensional metric to ten dimensions we introduce the five dimensional Sasaki-Einstein manifold,

$$
ds^2_{SE} = \left(d\psi + \mathcal{P} - \frac{2}{\sqrt{3}}\mathcal{C}\right)^2 + ds^2_{CP^2} \tag{4.3}
$$

where,

$$
\begin{aligned}
\mathcal{P} &= \frac{1}{3}(d\chi_1 + d\chi_2) - \sin^2\alpha(d\chi_2\sin^2\beta + d\chi_1\cos^2\beta)\\
ds^2_{CP^2} &= d\alpha^2 + \sin^2\alpha d\beta^2\\
&\quad + \sin^2\alpha\cos^2\alpha(\cos^2\beta d\chi_1 + \sin^2\beta d\chi_2)^2\\
&\quad + \sin^2\alpha\sin^2\beta\cos^2\beta(d\chi_1 - d\chi_2)^2.
\end{aligned}
\tag{4.4}
$$

We uplift the five dimensional metric (4.1) to ten dimensions as a fibration over CP^2. The ten dimensional metric is given by

$$
\begin{aligned}
ds^2_{10} &= -2u_\mu dx^\mu dr + Su_\mu dx^\mu dr + S_{\mu\nu}dx^\mu dx^\nu\\
&\quad + \left(d\psi + \mathcal{P} - \frac{2}{\sqrt{3}}\mathcal{C}\right)^2 + ds^2_{CP^2}
\end{aligned}
\tag{4.5}
$$

supported by a five form field strength

$$
\begin{aligned}
\mathcal{F}_5 &= 2(1+\star_{10})\left[\left(d\psi + \mathcal{P} - \frac{2}{\sqrt{3}}\mathcal{C}\right)\wedge J_2 - \frac{1}{\sqrt{3}}\star_5\mathcal{G}\right]\wedge J_2,\\
J_2 &= \frac{1}{2}d\mathcal{P},
\end{aligned}
\tag{4.6}
$$

a two form field strength $\mathcal{G} = d\mathcal{C}$ and a dilaton $\Phi = 0$.

To perform the TsT transformation on the ten dimensional solutions (4.5) and (4.6) we need to identify two isometry directions. One such isometry direction is ψ, useful to fabricate a nonrelativistic symmetry from the relativistic one [40]. To identify the other isometry direction we introduce the bulk light-cone coordinates $x^\pm$

$$
\begin{aligned}
x^+ &= (v+z),\\
x^+ &= \frac{1}{2}(v-z)
\end{aligned}
\tag{4.7}
$$

and choose x^- to be the second isometry direction. In the light-cone frame the ten dimensional metric can be written as,

$$
ds^2_{10} = A_1(dx^- + K_1)^2 + ds^2_4 + \left(d\psi + \mathcal{P} - \frac{2}{\sqrt{3}}\mathcal{C}\right)^2 + ds^2_{CP^2} \tag{4.8}
$$

where,

$$
\begin{aligned}
A_1 &= S_{--},\\
K_1 &= \frac{1}{S_{--}}\left(S_{+-}dx^+ - u_- dr + S_{-i}dx^i + \frac{S}{2}u_- dr\right),\\
ds^2_4 &= -2u_{\bar{a}}dx^{\bar{a}}dr + Su_{\bar{a}}dx^{\bar{a}}dr + S_{\bar{a}\bar{b}}dx^{\bar{a}\bar{b}} - A_1K_1^2.
\end{aligned}
\tag{4.9}
$$

Here i, j are running over $\{x, y\}$ and $\bar{a}, \bar{b}$ are running over $\{x^+, x, y\}$. Different components of S: $S_{+-}, S_{-i}, S_{\bar{a}\bar{b}}$ can be computed using (4.2) and the related expressions given in appendix A.

The TsT transformations correspond to performing T-duality along ψ direction, followed by a shift[3] along x^-, i.e., $x^- \to x^- - \psi$. Finally we perform T-duality back along the ψ direction [39,41]. The TsT transformed solutions are given by [39]

$$
\begin{aligned}
&d\hat{s}^2_{10} = \mathcal{M}A_1(dx^- + K_1)^2 + \mathcal{M}\left(d\psi + \mathcal{P} - \frac{2}{\sqrt{3}}\mathcal{C}\right)^2 + ds^2_8, \qquad e^{2\hat{\phi}} = \mathcal{M},\\
&\mathcal{F}_3 = g\wedge d\mathcal{P}, \qquad g = \frac{1}{2}d\mathcal{C}_-, \qquad \hat{B}_2 = \mathcal{A}\wedge\left(d\psi + \mathcal{P} - \frac{2}{\sqrt{3}}\mathcal{C}\right), \qquad \hat{\mathcal{F}}_5 = \mathcal{F}_5 + \hat{B}_2\wedge\mathcal{F}_3
\end{aligned}
\tag{4.10}
$$

[3] Here we set the twist parameter to 1.

where,

$$\mathcal{M} = (1 + A_1)^{-1} \quad \text{and} \quad \mathcal{A} = -\mathcal{M} A_1 (dx^- + K_1). \quad (4.11)$$

The TsT transformed ten dimensional fields can be truncated over S^5 directions [19,33]. After truncation the effective five dimensional solutions consist of a metric, massive vector coming from $\hat{B}_2$,[4] a scalar and a massless vector field. The massless vector field was present in ten dimensions even before the TsT transformation and it remains unaltered after TsT and reduction. We also get an one form in five dimensions but this one form is not an independent excitation. The reduced five dimensional metric in Einstein frame and other fields are given by [17,19],

$$d\hat{s}_E^2 = e^{\frac{4\phi}{3}} A_1 (dx^- + K_1)^2 + e^{\frac{-2\phi}{3}} ds_4^2$$
$$\hat{\mathcal{A}} = -\mathcal{M} A_1 (dx^- + K_1), \qquad g = \frac{1}{2} d\mathcal{C}_-, \qquad e^{2\phi} = \mathcal{M}. \quad (4.12)$$

It can be verified that this reduced 5 dimensional geometry and fields are solution of five dimensional effective action given by [20],

$$S = \frac{1}{16\pi G_5} \int dx^5 \sqrt{-g} \Big(R - \frac{4}{3} (\partial_a \phi)(\partial^a \phi) - \frac{1}{4} e^{\frac{-8\phi}{3}} \mathcal{F}_{ab} \mathcal{F}^{ab} - 4 \mathcal{A}_a \mathcal{A}^a - 4 e^{\frac{2\phi}{3}} (e^{2\phi} - 4) - \frac{1}{3} e^{\frac{4\phi}{3}} \mathcal{G}_{ab} \mathcal{G}^{ab} - 4 e^{2\phi} g_a g^a$$
$$- e^{\frac{-4\phi}{3}} \mathcal{A}_a \mathcal{G}_{bc} \mathcal{A}^a \mathcal{G}^{bc} - \frac{1}{2} e^{\frac{-2\phi}{3}} \Big(-\frac{2}{\sqrt{3}} \mathcal{G}_{ab} - 4 g_a \mathcal{A}_b \Big) \Big(-\frac{2}{\sqrt{3}} \mathcal{G}^{ab} - 4 g^a \mathcal{A}^b \Big) + \frac{4\kappa}{3} \mathcal{G}_{ab} \mathcal{G}_{cd} \mathcal{C}_e \varepsilon^{abcde} \Big). \quad (4.13)$$

To find the boundary stress energy tensor the on-shell variation of the action must be well defined. This can be achieved by adding appropriate boundary terms (Gibbons-Hawking term and counterterms) to the action [17,20]

$$S_{\text{boundary}} = \frac{1}{16\pi G_5} \int d\xi^4 \sqrt{-h} (2K - 6 + \mathcal{A}_a \mathcal{A}^a + 3\phi^2). \quad (4.14)$$

To find the boundary stress energy tensor associated to this five dimensional effective action we cannot directly use Brown-York type analysis [42] as we do not have conformal structure at boundary, because of inhomogeneity in the asymptotic fall off of different metric components. We use the method developed in [26] to obtain the stress tensors and other currents. We summarize the method in the following subsection.

A. Stress energy complex

To determine the boundary stress energy complex for Schrödinger field theory, we consider on-shell variation of action with respect to boundary fields [26,27,43],

$$\delta S = \frac{1}{16\pi G_5} \int d\xi^4 (s_{\alpha\beta} \delta h^{\alpha\beta} + s^\alpha \delta \mathcal{A}_\alpha + s_\phi \delta\phi + \tilde{s}^\alpha \delta \mathcal{C}_\alpha) \quad (4.15)$$

where,

$$s_{\alpha\beta} = \sqrt{-h} \Big(\pi_{\alpha\beta} + 3 h_{\alpha\beta} + \mathcal{A}_\alpha \mathcal{A}_\beta - \frac{1}{2} \mathcal{A}_\gamma \mathcal{A}^\gamma h_{\alpha\beta} - \frac{3}{2} \phi^2 h_{\alpha\beta} \Big)$$
$$s_\alpha = \sqrt{-h} (-n^a \mathcal{F}_{a\alpha} e^{\frac{-8\phi}{3}} + 2\mathcal{A}_\alpha), \qquad s_\phi = -\sqrt{-h} \Big(\frac{8}{3} n^a \partial_a \phi - 3\phi \Big)$$
$$\tilde{s}_\alpha = -\sqrt{-h} \Big(\Big(\frac{4}{3} e^{\frac{4\phi}{3}} + \frac{8}{3} e^{-\frac{2\phi}{3}} + 4 e^{\frac{-4\phi}{3}} \mathcal{A}_b \mathcal{A}^b \Big) n^a \mathcal{G}_{a\alpha} + \frac{8}{\sqrt{3}} e^{-\frac{2\phi}{3}} n^a (g_a \mathcal{A}_\alpha - g_\alpha \mathcal{A}_a) - \frac{16\kappa}{3} n^a \epsilon_{a\alpha bcd} \mathcal{C}^b \mathcal{G}^{cd} \Big) \quad (4.16)$$

here n^a is outward directed unit normal to the boundary, and $\pi_{\alpha\beta} = K_{\alpha\beta} - K h_{\alpha\beta}$ with extrinsic curvature $K_{\alpha\beta} = \nabla_\alpha n_\beta$.

As we have already mentioned, the TsT transformed asymptotic metric is not conformal to flat space time. Therefore we cannot construct the stress tensor by naively varying the action with respect to the boundary data. In [26] a consistent procedure has been developed to find out the stress energy complex when the boundary is degenerate. In this formalism, we first need to write down the boundary fields (metric and other fields) in terms of tangent space indices

[4]To note, $\hat{B}_2$ is not an independent field, as mentioned in (4.10), thus either $\hat{B}_2$ or its field strength does not appear in the final reduced action.

$$\psi^{\alpha\beta\cdots} = e^{\alpha}_{A} e^{\beta}_{B} \cdots \psi^{AB\cdots} \tag{4.17}$$

and consider the variation of the boundary action with respect to independent $\psi^{AB\cdots}$. In our case, the set of boundary fields consists of a metric $h^{\alpha\beta}$, a massive gauge field $\mathcal{A}^{\alpha}$, a massless gauge field $\mathcal{C}^{\alpha}$ and a scalar. We first write these fields in terms of their tangent space indices

$$h^{\alpha\beta} = e^{\alpha}_{A} e^{\beta}_{B} \eta^{AB}, \qquad \mathcal{A}^{\alpha} = e^{\alpha}_{A}\mathcal{A}^{A}, \qquad \mathcal{C}^{\alpha} = e^{\alpha}_{A}\mathcal{C}^{A}, \qquad \phi = \phi \tag{4.18}$$

and consider the variation of the action with respect to e^{A}_{α}, $\mathcal{A}^{A}$, $\mathcal{C}^{a}$, and ϕ

$$\delta S = \frac{1}{16\pi G_5}\int d\xi^4 \Big(T^{\alpha}_{\beta} e^{\beta}_{A} \delta e^{A}_{\alpha} + s_{\alpha} e^{\alpha}_{A} \delta\mathcal{A}^{A} + s_{\phi}\delta\phi + \tilde{s}_{\alpha} e^{\alpha}_{A} \delta\mathcal{C}^{A} \Big) \tag{4.19}$$

where,

$$T^{\alpha}_{\beta} = 2 s^{\alpha}_{\beta} - s^{\alpha}\mathcal{A}_{\beta} - \tilde{s}^{\alpha}\mathcal{C}_{\beta}. \tag{4.20}$$

The components of the boundary stress energy complex has been identified as,

$$\epsilon = T^{+}_{+}, \qquad \epsilon^{i} = T^{i}_{+}, \qquad \rho = T^{+}_{-}, \qquad \rho^{i} = T^{i}_{-},$$
$$t^{i}_{j} = -T^{i}_{j}, \qquad Q = \tilde{s}^{+}, \qquad \mathcal{J}^{i} = \tilde{s}^{i} \tag{4.21}$$

This analysis has been done up to first order in derivative expansion for charged nonrelativistic fluids [20,27]. Our goal is to use this formulation to obtain the second order stress energy complex for charged nonrelativistic fluids.

B. Holographic computation of nonrelativistic constitutive relations

On the gravity side, the independent quantities are mass m, charge q, and four velocity u^{μ} of the boosted black brane. The other quantities temperature T and chemical potential ϕ are related to mass and charge by the following relations [44],

$$m = \frac{\pi^4 T^4}{16}(\gamma+1)^3(3\gamma-1), \qquad q = \frac{\phi}{\sqrt{3}}\frac{\pi^2 T^2}{2}(\gamma+1)^2 \tag{4.22}$$

where,

$$\gamma = \sqrt{1 + \frac{8\phi^2}{3\pi^2 T^2}}. \tag{4.23}$$

The horizon radius is given by,

$$R = \frac{\pi}{2} T(\gamma+1). \tag{4.24}$$

In all holographic calculations the unperturbed horizon radius has been set to 1. So, at the leading (zeroth) order these relations become,

$$m_0 = 1 + q_0^2, \qquad \phi_0 = \frac{\sqrt{3}q_0}{2}, \qquad T_0 = \frac{(2-q_0^2)}{2\pi}, \qquad \gamma_0 = \frac{(2+q_0^2)}{(2-q_0^2)}. \tag{4.25}$$

Following Eqn. (4.21), the zeroth order nonrelativistic quantities are given by,

$$\rho = 4 m_0 (u^{+})^2, \qquad p = m_0, \qquad \epsilon = m_0, \qquad Q = 4\sqrt{3} q_0 u^{+} \tag{4.26}$$

and they satisfy the nonrelativistic Euler's equation $\epsilon + p - \rho\rho_m - \mu\frac{Q}{4\sqrt{3}} = \tau s$ with mass chemical potential ρ_m given by (3.12).

1. Second order calculations

We expand the relativistic mass, charge, and velocities about their values in local rest frame up to second order in derivative expansion

$$\begin{aligned}
m &= m_0 + x^{+}\partial_{+}m + x^{i}\partial_{i}m + \frac{1}{2}((x^{+})^2\partial_{+}^2 m + (x^{i})^2\partial_{i}^2 m + 2x^{+}x^{i}\partial_{+}\partial_{i}m)\\
q &= q_0 + x^{+}\partial_{+}q + x^{i}\partial_{i}q + \frac{1}{2}((x^{+})^2\partial_{+}^2 q + (x^{i})^2\partial_{i}^2 q + 2x^{+}x^{i}\partial_{+}\partial_{i}q)\\
u^{+} &= 1 + x^{+}\partial_{+}u^{+} + x^{i}\partial_{i}u^{+} + \frac{1}{2}((x^{+})^2\partial_{+}^2 u^{+} + (x^{i})^2\partial_{i}^2 u^{+} + 2x^{+}x^{i}\partial_{+}\partial_{i}u^{+})\\
u^{i} &= x^{+}\partial_{+}u^{i} + x^{j}\partial_{j}u^{i} + \frac{1}{2}((x^{+})^2\partial_{+}^2 u^{i} + (x^{j})^2\partial_{j}^2 u^{i} + 2x^{+}x^{j}\partial_{+}\partial_{j}u^{i}).
\end{aligned} \tag{4.27}$$

TABLE IV. Independent first order data from holography.

	Data	Constraint	Independent data
Scalar	$\partial_+ m$, $\partial_+ q$, $\epsilon^{ij}\omega_{ij}$, $\partial_+ u^+$, $\partial_i u^i$	$g^{rb}E_{b+} = 0$, $g^{rb}E_{b-} = 0$, $g^{rb}M_b = 0$	$\partial_i u^i$, $\epsilon^{ij}\omega_{ij}$
Vector	$\partial_i m$, $\partial_i q$, $\partial_i u^+$, $\partial_+ u^i$ $\epsilon^{ik}\partial_k m$,$\epsilon^{ik}\partial_k q$, $\epsilon^{ik}\partial_k u^+$	$g^{rb}E_{bi} = 0$	$\partial_i m$, $\partial_i q$, $\partial_i u^+$ $\epsilon^{ik}\partial_k T$, $\epsilon^{ik}\partial_k q$, $\epsilon^{ik}\partial_k u^+$
Tensor	σ^{ij}		σ^{ij}

Following the holographic procedure described in this subsection and using the dictionary laid down in Eq. (4.21) we calculate the components of stress energy complex corrected up to second order in derivative expansion. In deriving different components of stress energy complex, we use the constraints obtained from Einstein's and Maxwell's equations. In Table IV we list the independent first order holographic fluid data coming form equations of motion. These holographic constraints (Table IV) are similar to the constraints obtained by LCR of relativistic conserved currents (Table II). Using the holographic constraints we write our independent data in terms of derivatives of m, q, and u^+. In addition to these, the holographic fluid also satisfies constraints same as given in Table III, coming from the zeroth order holographic stress tensor and charge current. Using those relations one can obtain the independent second order scalar, vector and tensor data in terms of derivatives of m, q and u^+.

We further use the holographic relations (4.22) to trade derivatives of m and q with the derivatives of nonrelativistic fluid variables: mass chemical potential μ_m, temperature τ and chemical potential ν defined in (3.9) and (3.13). At first order the relations are given by

$$\partial_i m = 4m_0\frac{\partial_i\tau}{\tau} + 6q_0^2\frac{\partial_i\nu}{\nu} + 4m_0(\partial_i\mu_m)$$
$$\partial_i q = 3q_0\frac{\partial_i\tau}{\tau} + \frac{q_0(2+5q_0^2)}{(2+q_0^2)}\frac{\partial_i\nu}{\nu} + 3q_0(\partial_i\mu_m)$$

and at second order they are given by

$$\partial_i\partial_j m = 4m_0(\partial_i\partial_j\mu_m) + 4m_0\frac{\partial_i\partial_j\tau}{\tau} + 6q_0^2\frac{\partial_i\partial_j\nu}{\nu} + \frac{6q_0^2(2+5q_0^2)}{(2+q_0^2)}\frac{\partial_i\nu\partial_j\nu}{\nu^2} + 12m_0\frac{\partial_i\tau\partial_j\tau}{\tau^2} + 12m_0\partial_i\mu_m\partial_j\mu_m + \frac{24q_0^2}{\nu\tau}((\partial_i\nu)(\partial_j\tau) + (\partial_j\nu)(\partial_i\tau)) + \frac{16m_0}{\tau}((\partial_i\mu_m)(\partial_j\tau) + (\partial_j\mu_m)(\partial_i\tau)) + \frac{24q_0^2}{\nu}((\partial_i\mu_m)(\partial_j\nu) + (\partial_j\mu_m)(\partial_i\nu))$$

and

$$\partial_i\partial_j q = 3q_0(\partial_i\partial_j\mu_m) + 3q_0\frac{\partial_i\partial_j\tau}{\tau} + \frac{q_0(2+5q_0^2)}{(2+q_0^2)}\frac{\partial_i\partial_j\nu}{\nu} + \frac{4q_0^3(12+8q_0^2+5q_0^4)}{(2+q_0^2)^3}\frac{\partial_i\nu\partial_j\nu}{\nu^2} + 6q_0\frac{\partial_i\tau\partial_j\tau}{\tau^2} + 6q_0\partial_i\mu_m\partial_j\mu_m + \frac{3q_0(2+5q_0^2)}{(2+q_0^2)\nu\tau}((\partial_i\nu)(\partial_j\tau) + (\partial_j\nu)(\partial_i\tau)) + \frac{9q_0}{\tau}((\partial_i\mu_m)(\partial_j\tau) + (\partial_j\mu_m)(\partial_i\tau)) + \frac{3q_0(2+5q_0^2)}{(2+q_0^2)\nu}((\partial_i\mu_m)(\partial_j\nu) + (\partial_j\mu_m)(\partial_i\nu)).$$

Finally we write the holographic constitutive relations in terms of independent nonrelativistic fluid data give in Table II and III.

After a dedicated computation we obtain the expressions for different holographic constitutive relations and transports. Here we present the expressions for the stress tensor, charge density and charge current. Other nonrelativistic quantities (mass density, energy density, pressure, velocity,energy current) can be found in appendix C. The stress tensor is given by,

$$t^{ij} = \rho v^i v^j + p\delta^{ij} - n_1\tilde{\sigma}^{ij} + n_{a+1}\partial^{\langle i}\partial^{j\rangle}X_a + c_1\tilde{\sigma}^{\langle ik}\tilde{\omega}^{kj\rangle} + \mathbf{c}_{ab}\partial^{\langle i}X_a\partial^{j\rangle}X_b + g_a\epsilon^{\langle ik}\partial^k\partial^{j\rangle}X_a + \mathbf{g}_{ab}\epsilon^{\langle ik}\partial_k X_a\partial^{j\rangle}X_b + g_{10}\epsilon^{\langle ik}\tilde{\sigma}^{kl}\tilde{\omega}^{lj\rangle} \tag{4.28}$$

where $\hat{\mathbf{c}}_{ab}$ and $\hat{\mathbf{g}}_{ab}$ are holographic counterpart of $\mathbf{c}_{ab}$ and $\mathbf{g}_{ab}$ defined in (3.15). Three sets of holographic transport coefficients are given in Tables V, VI, and VII.

TABLE V. List of $\hat{n}_i$.

$\hat{n}_1$	R^3u^+
$\hat{n}_2$	$-\frac{m_0\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{2\sqrt{4m_0-3}}-1$
$\hat{n}_3$	$\frac{-13m_0^3+10m_0^2-3m_0+6}{8\nu m_0^3+8\nu m_0^2}-\frac{3q_0^2\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{4\nu\sqrt{4m_0-3}}$
$\hat{n}_4$	$\frac{1-4m_0}{4m_0\tau}-\frac{m_0\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{2\sqrt{4m_0-3}\tau}$

TABLE VI. List of $\hat{c}_i$.

$\hat{c}_1$	$-\frac{m_0}{4\sqrt{4m_0-3}}\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)$
$\hat{c}_2$	$-\frac{24\kappa^2q_0^4}{m_0}+\frac{3m_0\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{2\sqrt{4m_0-3}}+2m_0+1$
$\hat{c}_3$	$-\frac{36\kappa^2(m_0-1)^3}{\nu m_0^2}+\frac{3(m_0-1)\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{2\nu\sqrt{4m_0-3}}+\frac{(m_0-1)(12m_0^3+19m_0^2-3m_0+6)}{4\nu m_0^2(m_0+1)}$
$\hat{c}_4$	$-\frac{24\kappa^2q_0^4}{m_0\tau}+\frac{m_0\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{\sqrt{4m_0-3}\tau}+\frac{4m_0^2+2m_0-1}{2m_0\tau}$
$\hat{c}_5$	$-\frac{27\kappa^2(m_0-1)^4}{2\nu^2m_0^3}+\frac{3(m_0-1)(m_0+3)(7m_0-9)\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{16\nu^2m_0(m_0+1)\sqrt{4m_0-3}}+\frac{3(m_0-3)^4(m_0-1)\log(2)}{16\nu^2m_0^2(m_0+1)^2}+\frac{(m_0-1)(34m_0^6+210m_0^5+245m_0^4-502m_0^3+390m_0^2-396m_0-333)}{32\nu^2m_0^3(m_0+1)^3}$
$\hat{c}_6$	$-\frac{18\kappa^2(m_0-1)^3}{\nu m_0^2\tau}+\frac{3(m_0-1)\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{4\nu\sqrt{4m_0-3}\tau}-\frac{-3m_0^2-5m_0+8}{2\nu\tau+2\nu m_0\tau}$
$\hat{c}_7$	$-\frac{6\kappa^2q_0^4}{m_0\tau^2}+\frac{3m_0\log\left(\frac{3-\sqrt{4m_0-3}}{\sqrt{4m_0-3}+3}\right)}{4\sqrt{4m_0-3}\tau^2}+\frac{4m_0^2+14m_0-5}{8m_0\tau^2}$

Nonrelativistic charge density Q is given by,

$$Q=4\sqrt{3}qu^+-\frac{6q^2u^+\kappa}{m}\epsilon^{ij}\tilde{\omega}_{ij}-\frac{\sqrt{3}(m_0-2)q_0}{4m_0^2}\tilde{\sigma}_{ij}\tilde{\sigma}^{ij}+\frac{\sqrt{3}q_0^3\kappa^2}{2m_0^2}\tilde{\omega}_{ij}\tilde{\omega}^{ij}+\hat{n}_a^Q(\partial^i\partial_iX_a)+\hat{\lambda}^Q{}_{ab}\partial^iX_a\partial_iX_b+\hat{\hat{\lambda}}^Q_{ab}\epsilon^{ij}\partial_iX_a\partial_jX_b \tag{4.29}$$

where,

$$\hat{\lambda}^Q=\begin{pmatrix}\hat{\lambda}_1 & \hat{\lambda}_4/2 & \hat{\lambda}_6/2\\ \hat{\lambda}_4/2 & \hat{\lambda}_2 & \hat{\lambda}_5/2\\ \hat{\lambda}_6/2 & \hat{\lambda}_5/2 & \hat{\lambda}_3\end{pmatrix},\quad \hat{\hat{\lambda}}^Q=\begin{pmatrix}0 & \hat{\hat{\lambda}}_4/2 & \hat{\hat{\lambda}}_6/2\\ \hat{\hat{\lambda}}_4/2 & 0 & \hat{\hat{\lambda}}_5/2\\ \hat{\hat{\lambda}}_6/2 & \hat{\hat{\lambda}}_5/2 & 0\end{pmatrix}. \tag{4.30}$$

TABLE VII. List of $\hat{g}_i$.

$\hat{g}_1$	$-\frac{2\sqrt{3}\kappa q_0^3}{m_0}$
$\hat{g}_2$	$-\frac{3\sqrt{3}\kappa q_0^5}{2\nu m_0^2}$
$\hat{g}_3$	$-\frac{\sqrt{3}\kappa q_0^3}{m_0\tau}$
$\hat{g}_4$	$\frac{6\sqrt{3}\kappa q_0^3}{m_0}$
$\hat{g}_5$	$\frac{9\sqrt{3}\kappa q_0^5}{2\nu m_0^2}$
$\hat{g}_6$	$\frac{3\sqrt{3}\kappa q_0^3}{m_0\tau}$
$\hat{g}_7$	$\frac{3\sqrt{3}\kappa q_0^5(9q_0^2(q_0^2+2)-m_0(5q_0^2+2))}{2\nu^2m_0^3(q_0^2+2)}$
$\hat{g}_8$	$\frac{3\sqrt{3}\kappa q_0^5}{\nu m_0^2\tau}$
$\hat{g}_9$	$\frac{3\sqrt{3}\kappa q_0^3}{m_0\tau^2}$
$\hat{g}_{10}$	$-\frac{\sqrt{3}\kappa q_0^3}{2m_0}$

TABLE VIII. List of $\hat{n}_i^Q$.

$\hat{n}_1^Q$	$-\frac{2\sqrt{3}\kappa^2q_0^3}{m_0^2}$
$\hat{n}_2^Q$	$-\frac{\sqrt{3}q_0(m_0^3+4m_0^2-15m_0+18)}{8\nu m_0^3}-\frac{3\sqrt{3}\kappa^2q_0^5}{2\nu m_0^3}$
$\hat{n}_3^Q$	$-\frac{3\sqrt{3}q_0^3}{4m_0^2\tau}-\frac{\sqrt{3}\kappa^2q_0^3}{m_0^2\tau}$

TABLE IX. List of $\hat{\lambda}_i$.

$\hat{\lambda}_1$	$\frac{2\sqrt{3}\kappa^2q_0^3}{m_0^2}$
$\hat{\lambda}_2$	$-\frac{3\sqrt{3}q_0^5\kappa^2(5m_0^2-39m_0+60)}{2\nu^2m_0^4(m_0+1)}-\frac{3\sqrt{3}(m_0-3)^2q_0^3\log 2}{4\nu^2m_0^2(m_0+1)}-\frac{\sqrt{3}q_0^3(m_0^4+28m_0^3+24m_0^2+108m_0-369)}{16\nu^2m_0^4(m_0+1)}$
$\hat{\lambda}_3$	$\frac{\sqrt{3}(5-4m_0)q_0}{2m_0^2\tau^2}+\frac{\sqrt{3}\kappa^2(m_0-1)^{3/2}}{m_0^2\tau^2}$
$\hat{\lambda}_4$	$-\frac{\sqrt{3}q_0^3(5m_0-3)}{\nu m_0^2(m_0+1)}-\frac{12\sqrt{3}\kappa^2(m_0-3)^2q_0^3}{\nu m_0^3(m_0+1)}$
$\hat{\lambda}_5$	$-\frac{6\kappa^2\sqrt{3}(m_0-3)^2q_0^3}{\nu m_0^3(m_0+1)\tau}-\frac{\sqrt{3}(m_0-3)^2q_0\log 2}{2\nu m_0(m_0+1)\tau}-\frac{\sqrt{3}q_0(m_0^4+55m_0^3-67m_0^2-15m_0+42)}{8\nu m_0^3(m_0+1)\tau}$
$\hat{\lambda}_6$	$-\frac{3\sqrt{3}q_0^3}{m_0^2\tau}$
$\hat{\hat{\lambda}}_4$	$\frac{3\kappa q_0^2(2m_0^2-9m_0+15)}{\nu m_0^3}$
$\hat{\hat{\lambda}}_5$	$-\frac{3\kappa(m_0-3)q_0^2(2m_0^2-7m_0-3)}{2\nu m_0^3(m_0+1)\tau}$
$\hat{\hat{\lambda}}_6$	$\frac{6\kappa q_0^2}{\tau m_0^2}$

Different $\hat{n}_i^Q$s,and $\hat{\lambda}_i$s are given in Table VIII and Table IX.

Nonrelativistic charge current can be written as,

$$\mathcal{J}^i = Qv^i + \hat{\mathbf{q}}_a\partial^i X_a + \hat{\hat{\mathbf{q}}}_a\epsilon^{ik}\partial_k X_a + \hat{n}_1^{\mathcal{J}}\partial^k\tilde{\sigma}^{ik} + \hat{n}_2^{\mathcal{J}}\partial^k\tilde{\omega}^{ik} + \hat{n}_3^{\mathcal{J}}\epsilon^{ij}\partial^k\tilde{\sigma}^{jk} + \hat{n}_4^{\mathcal{J}}\epsilon^{ij}\partial^k\tilde{\omega}^{jk} + \hat{\mathbf{c}}_a^{\mathcal{J}}\tilde{\sigma}^{ik}\partial_k X_a + \hat{\hat{\mathbf{c}}}_a^{\mathcal{J}}\tilde{\omega}^{ik}\partial_k X_a + \hat{\hat{\hat{\mathbf{c}}}}_a^{\mathcal{J}}(\partial_k v^k)\partial^i X_a + \hat{\mathbf{g}}_a^{\mathcal{J}}\epsilon^{il}\tilde{\sigma}^{lk}\partial_k X_a + \hat{\hat{\mathbf{g}}}_a^{\mathcal{J}}\epsilon^{il}\tilde{\omega}^{lk}\partial_k X_a + \hat{\hat{\hat{\mathbf{g}}}}_a^{\mathcal{J}}(\partial_k v^k)\epsilon^{il}\partial_l X_a. \tag{4.31}$$

Again all the hatted transports are the holographic counterparts of the nonrelativistic transports defined in (3.17). Holographic values of different charge transports are given in Table X, XI, XII, and XIII.

TABLE X. List of $\hat{n}_i^{\mathcal{J}}$.

$\hat{n}_1^{\mathcal{J}}$	$\left(\frac{q_0\sqrt{3}}{2\sqrt{4m_0-3}}\log\left(\frac{3+\sqrt{4m_0-3}}{3-\sqrt{4m_0-3}}\right)+\frac{\sqrt{3}q_0(m_0-2)}{2m_0^2}\right)$
$\hat{n}_2^{\mathcal{J}}$	$\left(-\frac{\sqrt{3}q_0(m_0+2)}{4m_0^2}+\frac{\sqrt{3}\kappa^2q_0^3}{m_0^2}+\frac{q_0\sqrt{3}}{2\sqrt{4m_0-3}}\log\left(\frac{3+\sqrt{4m_0-3}}{3-\sqrt{4m_0-3}}\right)\right)$
$\hat{n}_3^{\mathcal{J}}$	$\frac{6q_0^2\kappa}{m_0^2}$
$\hat{n}_4^{\mathcal{J}}$	$-\frac{3(m_0-2)q_0^2\kappa}{m_0^2}$

TABLE XI. List of $\hat{\mathbf{q}}_a^{\mathcal{J}}$ and $\hat{\hat{\mathbf{q}}}_a^{\mathcal{J}}$.

$\hat{\lambda}_q$	$\frac{\sqrt{3}qR^3}{mu^+}+\frac{2q\tau\sqrt{3}}{mR}\left(\frac{(2R^4+3m)}{T(3\gamma-1)}-\frac{(m+R^4)}{\gamma T}\right)+\frac{q\nu\tau\sqrt{3}}{mR}\left(\frac{\sqrt{3}q(2R^4+3m)}{m}-\frac{(m+R^4)(3\gamma-2)}{\gamma\phi}\right)$
$\hat{\sigma}_q$	$\frac{q\tau u^+\sqrt{3}}{mR}\left(\frac{\sqrt{3}q(2R^4+3m)}{m}-\frac{(m+R^4)(3\gamma-2)}{\gamma\phi}\right)$
κ_q	$\frac{2qu^+\sqrt{3}}{mR}\left(\frac{(2R^4+3m)}{T(3\gamma-1)}-\frac{(m+R^4)}{\gamma T}\right)+\frac{q\nu u^+\sqrt{3}}{mR}\left(\frac{\sqrt{3}q(2R^4+3m)}{m}-\frac{(m+R^4)(3\gamma-2)}{\gamma\phi}\right)$
$\hat{\hat{\lambda}}_q$	$-\frac{12\kappa q^2}{m}\left(1+\frac{2u^+}{3\gamma-1}+\frac{\sqrt{3}q\nu\tau}{m}\right)$
$\hat{\hat{\sigma}}_q$	$-\frac{12\kappa q^2}{m}\left(\tau u^+\frac{\sqrt{3}q}{m}\right)$
$\hat{\hat{\kappa}}_q$	$-\frac{12\kappa q^2}{m}\left(\frac{2}{(3\gamma-1)\tau}+\frac{\sqrt{3}q\nu u^+}{m}\right)$

TABLE XII. List of $\hat{\mathbf{c}}_a^{\mathcal{J}}$, $\hat{\hat{\mathbf{c}}}_a^{\mathcal{J}}$ and $\hat{\hat{\hat{\mathbf{c}}}}_a^{\mathcal{J}}$.

$\hat{c}_1^{\mathcal{J}}$	$\frac{2\sqrt{3}q_0^3}{m_0^2}-\frac{2\sqrt{3}\kappa^2q_0^3}{m_0^2}$
$\hat{c}_2^{\mathcal{J}}$	$\frac{2\sqrt{3}q_0^3}{m_0}-\frac{2\sqrt{3}\kappa^2(12m_0-13)q_0^3}{m_0^2}$
$\hat{c}_3^{\mathcal{J}}$	0
$\hat{c}_4^{\mathcal{J}}$	$\frac{\sqrt{3}(5m_0^2+24m_0-9)q_0^3}{8\nu m_0^3(m_0+1)}+\frac{3\sqrt{3}q_0^3}{4m_0\nu\sqrt{4m_0-3}}\log\left(\frac{3+\sqrt{4m_0-3}}{3-\sqrt{4m_0-3}}\right)-\frac{3\sqrt{3}\kappa^2q_0^5}{2\nu m_0^3}$
$\hat{c}_5^{\mathcal{J}}$	$\frac{\sqrt{3}(11m_0^4-4m_0^3-23m_0^2+12m_0-12)q_0}{8\nu m_0^3(m_0+1)}-\frac{3\sqrt{3}\kappa^2q_0^3(12m_0^3-17m_0^2+12m_0-23)}{2\nu m_0^3(m_0+1)}$
$\hat{c}_6^{\mathcal{J}}$	0
$\hat{c}_7^{\mathcal{J}}$	$\frac{\sqrt{3}(5m_0-6)q_0}{4m_0^2\tau}+\frac{\sqrt{3}q_0}{2\tau\sqrt{4m_0-3}}\log\left(\frac{3+\sqrt{4m_0-3}}{3-\sqrt{4m_0-3}}\right)-\frac{\sqrt{3}\kappa^2q_0^3}{m_0^2\tau}$
$\hat{c}_8^{\mathcal{J}}$	$\frac{\sqrt{3}(4m_0^2-3m_0-2)q_0}{4m_0^2\tau}-\frac{\sqrt{3}\kappa^2q_0^3(12m_0-13)}{m_0^2\tau}$
$\hat{c}_9^{\mathcal{J}}$	0

TABLE XIII. List of $\hat{\mathbf{g}}_a^{\mathcal{J}}$, $\hat{\hat{\mathbf{g}}}_a^{\mathcal{J}}$ and $\hat{\hat{\hat{\mathbf{g}}}}_a^{\mathcal{J}}$.

$\hat{g}_1^{\mathcal{J}}$	0
$\hat{g}_2^{\mathcal{J}}$	0
$\hat{g}_3^{\mathcal{J}}$	0
$\hat{g}_4^{\mathcal{J}}$	$\frac{3q_0^2\kappa}{m_0^2\tau}$
$\hat{g}_5^{\mathcal{J}}$	$\frac{6q_0^2\kappa}{m_0\tau}$
$\hat{g}_6^{\mathcal{J}}$	0
$\hat{g}_7^{\mathcal{J}}$	$\frac{9q_0^6\kappa}{m_0^3(m_0+1)\nu}$
$\hat{g}_8^{\mathcal{J}}$	$\frac{9q_0^4\kappa}{m_0^2\nu}$
$\hat{g}_9^{\mathcal{J}}$	0

V. DISCUSSION: COMPARISON BETWEEN LCR AND TsT

In order to compare the two approaches to obtain the nonrelativistic constitutive relations we first note that the stress tensor (3.14) and charge current (3.16) obtained via LCR have exactly same form as stress tensor (4.28) and charge current (4.31) obtained via TsT transformations. Not only the stress tensor and charge current, one can check that other constitutive relations like mass density, charge density, energy density, energy current etc. have the same structure of terms. The expressions can be found in appendix B and C. The transport coefficients appearing in light-cone reduced stress tensor and charge current depend on relativistic data (i.e., relativistic transports and fluid variables). The same is true for other light-cone reduced constitutive relations also. These relations are very generic and depend on the parent relativistic system (2.12) and (2.13) and do not depend on any holographic model. In appendix B we have listed all these transports. However, if we use the holographic values of the relativistic transports (I) then it turns out that the values of the transports appearing in light-cone reduced constitutive relations exactly match those obtained via TsT. For example let us look at the holographic (TsT) value of $\hat{n}_3$ in table V. The value of n_3 for light-cone reduced fluid depends on the relativistic transports and other fluid variables [see Eqs. (B1) and (B2)]. However, if one uses the holographic values of relativistic transports (I) then it turns out that $\hat{n}_3 = n_3$. The same is true for other transports. Thus we find that the nonrelativistic fluid obtained by LCR or by TsT transformation are identical order by order in derivative expansion. Motivated by this we speculate that the observation is true for any order in derivative expansions.

The matching of TsT and LCR results, we think, is nontrivial for the following reason. It was studied in [22,33] that the TsT transformation of asymptotically AdS space generates a new solution which has Schrödinger isometry at the boundary. On the other hand Schrödinger in the boundary theory can be obtained by LCR of conformal algebra. In this paper we tried to understand this connection in the context of hydrodynamics. LCR reduction of relativistic constitutive equations (in particular conservation equations) renders nonrelativistic constitutive relations, namely continuity equation, Euler equations, energy current equations and charge conservation equation at different orders of derivative expansion. These equations are consistent with Schrödinger isometry. Different nonrelativistic fluid data, i.e., mass density, charge density, fluid velocity, energy current, energy density and different transports can be obtained in terms of relativistic fluid data and transports. Obtaining the nonrelativistic fluid via TsT transformation is rather nontrivial. First of all we need to obtain the correct holographic dual of a relativistic fluid. We then uplift the solution to 10 dimensions and perform the TsT transformation after identifying two isometry directions. The transformation mixes different components of the metric non trivially. In order to find the boundary currents we first reduced the TsT transformed 10 dimensional solution to 5 dimensions and wrote an effective action (following [26]). The boundary currents are obtained from the variation of this effective action following the dictionary [26,27,43]. Since the variation is done with respect to the tangent space variables of the asymptotic fields, there is a further mixing in different components of boundary currents.

Thus after very dedicated calculations we see that the two sides indeed match exactly at every order in derivative expansion which is very nontrivial as well as interesting in its own right. It is worth mentioning that, while performing the TsT transformation, not all of the isometry choices will provide the Schrödinger isometry at the boundary. Here our specific choice of isometry directions is goal driven. As we have mentioned in this paper, the LCR of relativistic conformal algebra boils down to Schrödinger algebra and the TsT transformed black brane solution has Schrödinger isometry at the boundary. Hence, it is not very surprising to expect that such identifications hold between the constitutive relations of a light-cone reduced conformal fluid and a holographic fluid obtained from a TsT transformed local black brane solution. However, this agreement is not very straightforward to see. We had to go through very rigorous calculations at each order in derivative expansion to verify the matching. We believe that it requires further study to understand this agreement at the fundamental level. We keep that for future endeavors.

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APPENDIX A: RELATIVISTIC BULK METRIC

Here we have written down the leading order pieces of the components of bulk charged relativistic metric which appear in the second order correction.

Scalars:

$$
\begin{aligned}
h(r) &= -\frac{1}{12r^2}(2(\partial_i u^+)^2 + 2(\partial_+ u^i)^2 + 2(\partial_x u^y)^2 + 2(\partial_y u^x)^2 + (\partial_x u^x - \partial_y u^y)^2) + \mathcal{O}\left(\frac{1}{r^5}\right)\\
k(r) &= \frac{r^2}{6m_0}(2\partial_+^2 m + 2\partial_i^2 m + 6m_0(\partial_i u^+)^2 + 16m_0(\partial_i u^+)(\partial_+ u^i) - 22m_0(\partial_+ u^i)^2 - 11m_0(\partial_+ u^x)^2\\
&\quad - 3m_0(\partial_x u^x)^2 - 3m_0(\partial_y u^y)^2 + 6m_0(\partial_y u^x)^2 + 16m_0(\partial_y u^x)(\partial_x u^y) - 34m_0(\partial_x u^x)(\partial_y u^y)) + \mathcal{O}\left(\frac{1}{r}\right)\\
w(r) &= \frac{1}{24m_0^2 r}((1+m_0)(-3q_0\partial_+^2 m - 3q_0\partial_i^2 m - +4m_0\partial_+^2 q + 4m_0\partial_i^2 q + 20m_0(\partial_+ q)(\partial_+ u^+)\\
&\quad + 20m_0(\partial_i q)(\partial_+ u^i) + 72m_0 q_0(\partial_+ u^+)^2 + 72m_0 q_0(\partial_+ u^i)^2 + 4q_0 m_0((\partial_i q)^2 + (\partial_+ q)^2 + 9q_0^2(\partial_+ u^+)^2\\
&\quad + 9q_0^2(\partial_+ u^i)^2 + 6q_0(\partial_+ q)(\partial_+ u^+) + 6q_0(\partial_i q)(\partial_+ u^i)) - 32\sqrt{3}q_0 m_0\kappa\epsilon^{ij}((\partial_i q)(\partial_+ u_j)\\
&\quad - (\partial_i q)(\partial_j u^+) - 3q_0(\partial_+ u^+)(\partial_i u_j) + 3q_0(\partial_i u^+)(\partial_j u^+) - (\partial_+ q)(\partial_i u_j))) + \mathcal{O}\left(\frac{1}{r^2}\right)
\end{aligned}
\tag{A1}
$$

Vectors:

$$
\begin{aligned}
j_\mu(r) &= -\frac{1}{12r^2}\left(P^\nu_\mu \mathcal{D}_\lambda \sigma^\lambda_\nu + P^\nu_\mu \mathcal{D}_\lambda \omega^\lambda_\nu\right) + \mathcal{O}\left(\frac{1}{r^5}\right) \\
g_\mu(r) &= -\frac{2}{\sqrt{3}q_0 r^7}\left(\frac{3q^3\kappa}{2m^2}(l^\lambda \sigma_{\lambda\mu}) - \frac{3\sqrt{3}q^2}{8m_0} P^\nu_\mu \mathcal{D}_\lambda \sigma^\lambda_\nu - \frac{\sqrt{3}q^4\kappa^2}{4m^2} P^\nu_\mu \mathcal{D}_\lambda \omega^\lambda_\nu \right. \\
&\quad \left. - \frac{\sqrt{3}q\log 2}{4}\sigma^\lambda_\mu \mathcal{D}_\lambda q + \frac{\sqrt{3}q_0}{16m_0^2}(m_0^2 - 48q_0^2\kappa^2 + 3)\omega^\lambda_\mu \mathcal{D}_\lambda q\right) \qquad \text{(A2)}
\end{aligned}
$$

Tensors:

$$
\begin{aligned}
\alpha_{\mu\nu} &= \frac{1}{r^2}\left(\omega_{\mu\lambda}\sigma^\lambda_\nu + \omega_{\nu\lambda}\sigma^\lambda_\mu + \sigma_{\mu\lambda}\sigma^\lambda_\nu - \frac{P_{\mu\nu}}{3}\sigma^{\alpha\beta}\sigma_{\alpha\beta} - \omega_{\mu\lambda}\omega^\lambda_\nu - \frac{P_{\mu\nu}}{3}\omega^{\alpha\beta}\omega_{\alpha\beta}\right) \\
&\quad + \frac{1}{4r^4}\left(\mathcal{N}_1(u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu}) + \mathcal{N}_2({\omega^\mu}_\lambda \sigma^{\lambda\nu} + {\omega^\nu}_\lambda \sigma^{\lambda\mu}) + \mathcal{N}_3\left({\sigma^\mu}_\lambda \sigma^{\lambda\nu} - \frac{1}{3}P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta}\right)\right. \\
&\quad + \mathcal{N}_4\left({\omega^\mu}_\lambda \omega^{\lambda\nu} + \frac{1}{3}P^{\mu\nu}\omega^{\alpha\beta}\omega_{\alpha\beta}\right) + \mathcal{N}_5(\Pi^{\mu\nu\alpha\beta}\mathcal{D}_\alpha \mathcal{D}_\beta n) + \mathcal{N}_6(\Pi^{\mu\nu\alpha\beta}\mathcal{D}_\alpha n \mathcal{D}_\beta n) \\
&\quad \left. + \mathcal{N}_7\left(\Pi^{\mu\nu\alpha\beta}(\mathcal{D}_\alpha l_\beta + \mathcal{D}_\beta l_\alpha) - \epsilon_{\alpha\beta\gamma\mu}u^\alpha a^\beta(\partial_\nu u^\gamma) - \epsilon_{\alpha\beta\gamma\nu}u^\alpha a^\beta(\partial_\mu u^\gamma) + \frac{2}{3}(l^\alpha a_\alpha)P_{\mu\nu}\right)\right) \qquad \text{(A3)}
\end{aligned}
$$

Here i, j runs over (x, y) and μ, ν over $(+, -, x, y)$.

APPENDIX B: NONRELATIVISTIC CHARGED FLUID FROM LCR

The transport coefficients involved in the stress tensor are given by,

$$
\begin{aligned}
& n_1 = \eta_r u^+, \qquad n_2 = \nu_0\tilde{n}_3\tau_0 + \tilde{n}_4\tau_0 + \tilde{n}_2, \qquad n_3 = \tilde{n}_3\tau_0, \qquad n_4 = \nu_0\tilde{n}_3 + \tilde{n}_4, \\
& c_1 = \frac{\mathcal{N}_2}{2}, \qquad c_2 = \tilde{c}_5\nu_0^2\tau_0^2 + \tilde{c}_6\nu_0\tau_0^2 + \tilde{c}_3\nu_0\tau_0 + \tilde{c}_7\tau_0^2 + \tilde{c}_4\tau_0 + \tilde{c}_2, \\
& c_3 = 2\tilde{c}_5\nu_0\tau_0^2 + \tilde{c}_6\tau_0^2 + \tilde{c}_3\tau_0 + 2\tilde{n}_3\tau_0, \\
& c_4 = 2\tilde{c}_5\nu_0^2\tau_0 + 2\tilde{c}_6\nu_0\tau_0 + \tilde{c}_3\nu_0 + 2\tilde{c}_7\tau_0 + \tilde{c}_4 + 2\nu_0\tilde{n}_3 + 2\tilde{n}_4, \qquad c_5 = \tilde{c}_5\tau_0^2, \\
& c_6 = 2\tilde{c}_5\nu_0\tau_0 + \tilde{c}_6\tau_0 + 2\tilde{n}_3, \qquad c_7 = \tilde{c}_5\nu_0^2 + \tilde{c}_6\nu_0 + \tilde{c}_7 \\
& g_1 = \tau_0(\tilde{g}_2\nu_0 + \tilde{g}_3) + \tilde{g}_1, \qquad g_2 = \tilde{g}_2\tau_0, \qquad g_3 = \tilde{g}_2\nu_0 + \tilde{g}_3, \\
& g_4 = \tau_0(\tilde{g}_7\nu_0^2\tau_0 + 2\tilde{g}_8\nu_0\tau_0 + 2\tilde{g}_5\nu_0 + \tilde{g}_9\tau_0 + 2\tilde{g}_6) + \tilde{g}_4, \qquad g_5 = \tau_0(\tilde{g}_8\tau_0 + \tilde{g}_7\nu\tau + \tilde{g}_2 + \tilde{g}_5), \\
& g_6 = \tilde{g}_7\nu_0^2\tau_0 + 2\tilde{g}_8\nu_0\tau_0 + \tilde{g}_2\nu_0 + \tilde{g}_5\nu_0 + \tilde{g}_9\tau_0 + \tilde{g}_3 + \tilde{g}_6, \qquad g_7 = \tilde{g}_7\tau_0^2, \qquad g_8 = \tilde{g}_8\tau_0 + \tilde{g}_7\tau\nu + \tilde{g}_2, \\
& g_9 = 2\tilde{g}_8\nu_0 + \tilde{g}_7\nu_0^2 + \tilde{g}_9, \qquad g_{10} = -\frac{\mathcal{N}_7}{2} \qquad \text{(B1)}
\end{aligned}
$$

where,

$$
\begin{aligned}
& \tilde{n}_2 = -\frac{{\eta_r}^2}{4p_0}, \qquad \tilde{n}_3 = -\frac{1}{16p_0^2 q_0}\left(2p_0(\mathcal{N}_1 q_0 + 3\mathcal{N}_5 q_0)(\partial_\phi p) - 8\mathcal{N}_5 p_0^2(\partial_\phi q) - q_0(\partial_\phi p){\eta_r}^2\right), \\
& \tilde{n}_4 = -\frac{1}{16p_0^2 q_0}\left(2p_0(\mathcal{N}_1 q_0 + 3\mathcal{N}_5 q_0)(\partial_T p) - 8\mathcal{N}_5 p_0^2(\partial_T q) - q_0(\partial_T p){\eta_r}^2\right) \qquad \text{(B2)}
\end{aligned}
$$

$$\tilde{c}_2 = -\frac{1}{8p_0}((4\mathcal{N}_2 - \mathcal{N}_3 + 4\mathcal{N}_4)p_0 - 3\eta_r{}^2),$$
$$\tilde{c}_3 = \frac{1}{16p_0^2}(p_0((2\mathcal{N}_1 - \mathcal{N}_3 - 4\mathcal{N}_4)(\partial_\phi p) - 4\eta_r(\partial_\phi\eta_r)) + 4(\partial_\phi p)\eta_r{}^2),$$
$$\tilde{c}_4 = \frac{1}{16p_0^2}(p_0((2\mathcal{N}_1 - \mathcal{N}_3 - 4\mathcal{N}_4)(\partial_T p) - 4\eta_r(\partial_T\eta_r)) + 4(\partial_T p)\eta_r{}^2),$$
$$\tilde{c}_5 = -\frac{1}{128p_0^3q_0^2}(16p_0^2(2(4\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_\phi p)(\partial_\phi q) + q_0(\mathcal{N}_1q_0 + 3\mathcal{N}_5q_0)(\partial_\phi^2 p))$$
$$- p_0(((20\mathcal{N}_1 + 4\mathcal{N}_2 + \mathcal{N}_3 - 4\mathcal{N}_4)q_0^2 + 108\mathcal{N}_5q_0^2 + 36\mathcal{N}_6q_0^2)(\partial_\phi p)^2 + 8q_0^2(\partial_\phi p)\eta_r(\partial_\phi\eta_r)$$
$$+ 8q_0^2(\partial_\phi^2 p)\eta_r{}^2) - 64p_0^3(\mathcal{N}_6(\partial_\phi q)^2 + \mathcal{N}_5q_0(\partial_\phi^2 q)) + 17q_0^2(\partial_\phi p)^2\eta_r{}^2),$$
$$\tilde{c}_6 = -\frac{1}{64p_0^3q_0^2}(16p_0^2((4\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_T p)(\partial_\phi q) + (4\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_\phi p)(\partial_T q)$$
$$+ q_0(\mathcal{N}_1q_0 + 3\mathcal{N}_5q_0)(\partial_\phi\partial_T p)) - p_0((\partial_\phi p)(((20\mathcal{N}_1 + 4\mathcal{N}_2 + \mathcal{N}_3 - 4\mathcal{N}_4)q_0^2$$
$$+ 108\mathcal{N}_5q_0^2 + 36\mathcal{N}_6q_0^2)(\partial_T p) + 4q_0^2\eta_r(\partial_T\eta_r)) + 4q_0^2\eta_r((\partial_T p)(\partial_\phi\eta_r) + 2(\partial_\phi\partial_T p)\eta_r))$$
$$- 64p_0^3(\mathcal{N}_6(\partial_\phi q)(\partial_T q) + \mathcal{N}_5q_0(\partial_\phi\partial_T q)) + 17q_0^2(\partial_\phi p)(\partial_T p)\eta_r{}^2),$$
$$\tilde{c}_7 = -\frac{1}{128p_0^3q_0^2}(16p_0^2(2(4\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_T p)(\partial_T q) + q_0(\mathcal{N}_1q_0 + 3\mathcal{N}_5q_0)(\partial_T^2 p))$$
$$- p_0(((20\mathcal{N}_1 + 4\mathcal{N}_2 + \mathcal{N}_3 - 4\mathcal{N}_4)q_0^2 + 108\mathcal{N}_5q_0^2 + 36\mathcal{N}_6q_0^2)(\partial_T p)^2 + 8q_0^2(\partial_T p)\eta_r(\partial_T\eta_r)$$
$$+ 8q_0^2(\partial_T^2 p)\eta_r{}^2) - 64p_0^3(\mathcal{N}_6(\partial_T q)^2 + \mathcal{N}_5q_0(\partial_T^2 q)) + 17q_0^2(\partial_T p)^2\eta_r{}^2)$$
$$\tilde{g}_1 = -\mathcal{N}_7, \qquad \tilde{g}_2 = -\frac{1}{4p_0}(\mathcal{N}_7(\partial_\phi p)), \qquad \tilde{g}_3 = -\frac{1}{4p_0}(\mathcal{N}_7(\partial_T p)) \qquad \tilde{g}_4 = \mathcal{N}_7$$
$$\tilde{g}_5 = \frac{1}{32p_0q_0}((8\mathcal{N}_7q_0 + (6\mathcal{N}_8 - 3\mathcal{N}_9)q_0)(\partial_\phi p) + 4(\mathcal{N}_9 - 2\mathcal{N}_8)p_0(\partial_\phi q))$$
$$\tilde{g}_6 = \frac{1}{32p_0q_0}((8\mathcal{N}_7q_0 + (6\mathcal{N}_8 - 3\mathcal{N}_9)q_0)(\partial_T p) + 4(\mathcal{N}_9 - 2\mathcal{N}_8)p_0(\partial_T q))$$
$$\tilde{g}_7 = -\frac{1}{64p_0^2q_0}(4(2\mathcal{N}_8 + \mathcal{N}_9)p_0(\partial_\phi p)(\partial_\phi q) - 3(8\mathcal{N}_7q_0 + (2\mathcal{N}_8 + \mathcal{N}_9)q_0)(\partial_\phi p)^2 + 16\mathcal{N}_7p_0q_0(\partial_\phi^2 p))$$
$$\tilde{g}_8 = -\frac{1}{64p_0^2q_0}((\partial_\phi p)(2(2\mathcal{N}_8 + \mathcal{N}_9)p_0(\partial_T q) - 3(8\mathcal{N}_7q_0 + (2\mathcal{N}_8 + \mathcal{N}_9)q_0)(\partial_T p))$$
$$+ 2p_0((2\mathcal{N}_8 + \mathcal{N}_9)(\partial_T p)(\partial_\phi q) + 8\mathcal{N}_7q_0(\partial_\phi\partial_T p)))$$
$$\tilde{g}_9 = -\frac{1}{64p_0^2q_0}(4(2\mathcal{N}_8 + \mathcal{N}_9)p_0(\partial_T p)(\partial_T q) - 3(8\mathcal{N}_7q_0 + (2\mathcal{N}_8 + \mathcal{N}_9)q_0)(\partial_T p)^2 + 16\mathcal{N}_7p_0q_0(\partial_T^2 p)). \tag{B3}$$

The transports appearing in the charge current are given by,

$$\lambda_q = \eta_r\left(\frac{\sqrt{3}q}{Pu^+} - \frac{\sqrt{3}qTu^+(\partial_T p)}{4P^2} - \frac{3q^2(\partial_\phi p)}{8u^+P^2}\right) + \lambda_r\left(\frac{\sqrt{3}(3q\partial_T p - 4P\partial_T q)Tu^+}{P} + \frac{3q(3q\partial_\phi p - 4P\partial_\phi q)}{2u^+P}\right)$$
$$\kappa_q = \frac{\eta_r}{\tau}\left(-\frac{\sqrt{3}qT(u^+)^2(\partial_T p)}{4P^2} - \frac{3q^2(\partial_\phi p)}{8u^+P^2}\right) + \frac{\lambda_r}{\tau}\left(\frac{\sqrt{3}(3q\partial_T p - 4P\partial_T q)T(u^+)^2}{P} + \frac{3q(3q\partial_\phi p - 4P\partial_\phi q)}{2u^+P}\right)$$
$$\sigma_q = \eta_r\left(-\frac{\sqrt{3}q(\partial_\phi p)T}{4P^2}\right) + \lambda_r\left(\frac{\sqrt{3}(3q\partial_\phi p - 4P\partial_\phi q)T}{P}\right),$$

$$\tilde{\lambda}_q = -\xi_r\left(\frac{1}{u^+} + \frac{(\partial_T p)Tu^+}{4P} + \frac{(\sqrt{3}q\partial_\phi p)}{8u^+P}\right)$$
$$\tilde{\kappa}_q = -\frac{\xi_r}{\tau}\left(\frac{(\partial_T p)T(u^+)^2}{4P} + \frac{\sqrt{3}q(\partial_\phi p)}{8u^+P}\right), \qquad \tilde{\sigma}_q = -\xi_r\left(\frac{(\partial_\phi p)T}{4P}\right) \tag{B4}$$

$$c_1^{\mathcal{J}} = \tilde{c}_1^{\mathcal{J}} + u^+\tau\nu\tilde{c}_4^{\mathcal{J}} + u^+\tau\tilde{c}_7^{\mathcal{J}}, \qquad c_2^{\mathcal{J}} = \tilde{c}_2^{\mathcal{J}} + u^+\tau\nu\tilde{c}_5^{\mathcal{J}} + u^+\tau\tilde{c}_8^{\mathcal{J}}, \qquad c_3^{\mathcal{J}} = \tilde{c}_3^{\mathcal{J}} + u^+\tau\nu\tilde{c}_6^{\mathcal{J}} + u^+\tau\tilde{c}_9^{\mathcal{J}}$$
$$c_7^{\mathcal{J}} = \nu u^+\tilde{c}_4^{\mathcal{J}} + \tilde{c}_7^{\mathcal{J}}, \qquad c_8^{\mathcal{J}} = \nu u^+\tilde{c}_5^{\mathcal{J}} + \tilde{c}_8^{\mathcal{J}}, \qquad c_9^{\mathcal{J}} = \nu u^+\tilde{c}_6^{\mathcal{J}} + \tilde{c}_9^{\mathcal{J}}$$
$$c_4^{\mathcal{J}} = u^+\tau\tilde{c}_4^{\mathcal{J}}, \qquad c_5^{\mathcal{J}} = u^+\tau\tilde{c}_5^{\mathcal{J}}, \qquad c_6^{\mathcal{J}} = u^+\tau\tilde{c}_6^{\mathcal{J}}$$
$$g_1^{\mathcal{J}} = \tilde{g}_1^{\mathcal{J}} + u^+\tau\nu\tilde{g}_4^{\mathcal{J}} + u^+\tau\tilde{g}_7^{\mathcal{J}}, \qquad g_2^{\mathcal{J}} = \tilde{g}_2^{\mathcal{J}} + u^+\tau\nu\tilde{g}_5^{\mathcal{J}} + u^+\tau\tilde{g}_8^{\mathcal{J}}, \qquad g_3^{\mathcal{J}} = \tilde{g}_3^{\mathcal{J}} + u^+\tau\nu\tilde{g}_6^{\mathcal{J}} + u^+\tau\tilde{g}_9^{\mathcal{J}}$$
$$g_7^{\mathcal{J}} = \nu u^+\tilde{g}_4^{\mathcal{J}} + \tilde{g}_7^{\mathcal{J}}, \qquad g_8^{\mathcal{J}} = \nu u^+\tilde{g}_5^{\mathcal{J}} + \tilde{g}_8^{\mathcal{J}}, \qquad g_9^{\mathcal{J}} = \nu u^+\tilde{g}_6^{\mathcal{J}} + \tilde{g}_9^{\mathcal{J}}$$
$$g_4^{\mathcal{J}} = u^+\tau\tilde{g}_4^{\mathcal{J}}, \qquad g_5^{\mathcal{J}} = u^+\tau\tilde{g}_5^{\mathcal{J}}, \qquad g_6^{\mathcal{J}} = u^+\tau\tilde{g}_6^{\mathcal{J}} \tag{B5}$$

where,

$$n_1^{\mathcal{J}} = \frac{-\sqrt{3}p_0q_0(\mathcal{N}_1 + 12\eta_r\lambda_r) + 2\gamma_1 p_0^2 + \sqrt{3}q_0\eta_r^2}{2p_0^2}$$
$$n_2^{\mathcal{J}} = \frac{-2\sqrt{3}p_0q_0(\mathcal{N}_1 + 6\eta_r\lambda_r) + 2(\gamma_1+\gamma_2)p_0^2 + \sqrt{3}q_0\eta_r^2}{4p_0^2}$$
$$n_3^{\mathcal{J}} = \frac{\eta_r\xi_r}{2p_0}, \qquad n_4^{\mathcal{J}} = \frac{\eta_r\xi_r - 4\sqrt{3}q\mathcal{N}_7}{4p_0}$$
$$\tilde{c}_1^{\mathcal{J}} = \frac{\sqrt{3}p_0q_0(\mathcal{N}_1 - 2\mathcal{N}_2 - \mathcal{N}_3 - 12\eta_r\lambda_r) + 2(\gamma_1-\gamma_2)p_0^2 + \sqrt{3}q_0\eta_r^2}{4p_0^2}$$
$$\tilde{c}_2^{\mathcal{J}} = \frac{\sqrt{3}p_0q_0(\mathcal{N}_1 + 2\mathcal{N}_2 - 4\mathcal{N}_4 + 12\eta_r\lambda_r) - 2(\gamma_1-\gamma_2)p_0^2 - \sqrt{3}q_0\eta_r^2}{4p_0^2}$$
$$\tilde{c}_3^{\mathcal{J}} = \frac{1}{8p_0^2((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}\Big(\sqrt{3}q_0(\eta_r - 12p_0\lambda_r)(3\eta_r((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))$$
$$+ q_0(3(\partial_\phi p)(\partial_T\eta_r) - 3(\partial_T p)(\partial_\phi\eta_r)) + 4p_0((\partial_T q)(\partial_\phi\eta_r) - (\partial_\phi q)(\partial_T\eta_r)))\Big)$$
$$\tilde{c}_4^{\mathcal{J}} = \frac{1}{16p_0^2q_0}\Big(p_0(8\sqrt{3}q_0(\mathcal{N}_5(\partial_\phi q) - 6q_0\lambda_r(\partial_\phi\eta_r)) + (6\gamma_1q_0 - 2\gamma_2q_0 - 6\gamma_4q_0)(\partial_\phi p))$$
$$+ \sqrt{3}q_0(((-3\mathcal{N}_1 - 2\mathcal{N}_2 + \mathcal{N}_3)q_0 - 6\mathcal{N}_5q_0)(\partial_\phi p) + 4q_0\eta_r(\partial_\phi\eta_r)) + 8\gamma_4p_0^2(\partial_\phi q)\Big)$$
$$\tilde{c}_5^{\mathcal{J}} = \frac{1}{16p_0^3q_0}\Big(2p_0^2(4\sqrt{3}\mathcal{N}_5q_0(\partial_\phi q) + ((\gamma_1+\gamma_2)q_0 - 3\gamma_5q_0)(\partial_\phi p))$$
$$- \sqrt{3}p_0q_0(\partial_\phi p)(q_0(\mathcal{N}_1 + 2\mathcal{N}_2 + 4\mathcal{N}_4 + 12\eta_r\lambda_r) + 6\mathcal{N}_5q_0) + \sqrt{3}q_0^2\eta_r^2(\partial_\phi p) + 8\gamma_5p_0^3(\partial_\phi q)\Big)$$
$$\tilde{c}_6^{\mathcal{J}} = -\frac{1}{32p_0^3((\partial_T p)(\partial_\phi q) - (\partial_\phi p(\partial_T q))}\Big(\sqrt{3}q_0(\partial_\phi p)(\eta_r - 12p_0\lambda_r)(3\eta_r((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))$$
$$+ q_0(3(\partial_\phi p)(\partial_T\eta_r) - 3(\partial_T p)(\partial_\phi\eta_r)) + 4p_0((\partial_T q)(\partial_\phi\eta_r) - (\partial_\phi q)(\partial_T\eta_r)))\Big)$$

$$\tilde{c}_7^{\mathcal{J}} = \frac{1}{16p_0^2q_0}\Big(p_0(8\sqrt{3}q_0(\mathcal{N}_5(\partial_T q) - 6q_0\lambda_r(\partial_T\eta_r)) + (6\gamma_1 q_0 - 2\gamma_2 q_0 - 6\gamma_4 q_0)(\partial_T p)) + \sqrt{3}q_0(((-3\mathcal{N}_1 - 2\mathcal{N}_2 + \mathcal{N}_3)q_0 - 6\mathcal{N}_5 q_0)(\partial_T p) + q_0\eta_r(\partial_T\eta_r)) + 8\gamma_4 p_0^2(\partial_T q)\Big)$$

$$\tilde{c}_8^{\mathcal{J}} = \frac{1}{16p_0^3q_0}\Big(2p_0^2(4\sqrt{3}\mathcal{N}_5 q_0(\partial_T q) + ((\gamma_1+\gamma_2)q_0 - 3\gamma_5 q_0)(\partial_T p)) - \sqrt{3}p_0q_0(\partial_T p)(q_0(\mathcal{N}_1 + 2\mathcal{N}_2 + 4\mathcal{N}_4 + 12\eta_r\lambda_r) + 6\mathcal{N}_5 q_0) + \sqrt{3}q_0^2\eta_r^2(\partial_T p) + 8\gamma_5 p_0^3(\partial_T q)\Big)$$

$$\tilde{c}_9^{\mathcal{J}} = \frac{1}{32p_0^3((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}\big(\sqrt{3}q_0(\partial_T p)(\eta_r - 12p_0\lambda_r)(\eta_r(3(\partial_\phi p)(\partial_T q) - 3(\partial_T p)(\partial_\phi q)) + 3q_0((\partial_T p)(\partial_\phi\eta_r) - (\partial_\phi p)(\partial_T\eta_r)) + 4p_0((\partial_\phi q)(\partial_T\eta_r) - (\partial_T q)(\partial_\phi\eta_r)))\Big)$$

$$\tilde{g}_1^{\mathcal{J}} = \frac{1}{4}\left(2\gamma_3 + \frac{\eta_r\xi_r}{p_0}\right), \qquad \tilde{g}_2^{\mathcal{J}} = \frac{\gamma_3}{2} - \frac{2\sqrt{3}\mathcal{N}_7 q_0}{p_0}$$

$$\tilde{g}_3^{\mathcal{J}} = \frac{1}{8p_0((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}(\xi_r(3\eta_r((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q)) + q_0(3(\partial_\phi p)(\partial_T\eta_r) - 3(\partial_T p)(\partial_\phi\eta_r)) + 4p_0((\partial_T q)(\partial_\phi\eta_r) - (\partial_\phi q)(\partial_T\eta_r))))$$

$$\tilde{g}_4^{\mathcal{J}} = \frac{1}{16p_0^2q_0}\Big(2p_0(-2\sqrt{3}\mathcal{N}_9 q_0(\partial_\phi q) + \gamma_3 q_0(\partial_\phi p) + 2q_0\xi_r(\partial_\phi\eta_r)) + 3\sqrt{3}\mathcal{N}_9 q_0^2(\partial_\phi p)\Big)$$

$$\tilde{g}_5^{\mathcal{J}} = \frac{1}{8p_0^2q_0}\Big(\sqrt{3}q_0(4\mathcal{N}_7 q_0 + 3\mathcal{N}_8 q_0)(\partial_\phi p) - p_0(4\sqrt{3}\mathcal{N}_8 q_0(\partial_\phi q) + \gamma_3 q_0(\partial_\phi p))\Big)$$

$$\tilde{g}_6^{\mathcal{J}} = \frac{1}{32p_0^2((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}\Big(\xi_r(\partial_\phi p)(\eta_r(3(\partial_\phi p)(\partial_T q) - 3(\partial_T p)(\partial_\phi q)) + 3q_0((\partial_T p)(\partial_\phi\eta_r) - (\partial_\phi p)(\partial_T\eta_r)) + 4p_0((\partial_\phi q)(\partial_T\eta_r) - (\partial_T q)(\partial_\phi\eta_r)))\Big)$$

$$\tilde{g}_7^{\mathcal{J}} = \frac{1}{16p_0^2q_0}\Big(2p_0(-2\sqrt{3}\mathcal{N}_9 q_0(\partial_T q) + \gamma_3 q_0(\partial_T p) + 2q_0\xi_r(\partial_T\eta_r)) + 3\sqrt{3}\mathcal{N}_9 q_0^2(\partial_T p)\Big)$$

$$\tilde{g}_8^{\mathcal{J}} = \frac{1}{8p_0^2q_0}\Big(\sqrt{3}q_0(4\mathcal{N}_7 q_0 + 3\mathcal{N}_8 q_0)(\partial_T p) - p_0(4\sqrt{3}\mathcal{N}_8 q_0(\partial_T q) + \gamma_3 q_0(\partial_T p))\Big)$$

$$\tilde{g}_9^{\mathcal{J}} = \frac{1}{32p_0^2((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}\Big(\xi_r(\partial_T p)(\eta_r(3(\partial_\phi p)(\partial_T q) - 3(\partial_T p)(\partial_\phi q)) + 3q_0((\partial_T p)(\partial_\phi\eta_r) - (\partial_\phi p)(\partial_T\eta_r)) + 4p_0((\partial_\phi q)(\partial_T\eta_r) - (\partial_T q)(\partial_\phi\eta_r)))\Big). \tag{B6}$$

Density,

$$\rho = (E+P)(u^+)^2 + \tilde{n}_1^{(\rho)}(\partial_k\partial_k\mu_m) + \tilde{n}_2^{(\rho)}(\partial_k\partial_k\phi) + \tilde{n}_1^{(\rho)}(\partial_k\partial_k T) + \tilde{c}_1^{(\rho)}(\partial_k\mu_m)(\partial^k\mu_m) + \tilde{c}_2^{(\rho)}(\partial_k\phi)(\partial^k\phi) + \tilde{c}_3^{(\rho)}(\partial_k T)(\partial^k T) + \tilde{c}_4^{(\rho)}(\partial_k\mu_m)(\partial^k\phi) + \tilde{c}_5^{(\rho)}(\partial_k T)(\partial^k\phi) + \tilde{c}_6^{(\rho)}(\partial_k\mu_m)(\partial^k T) + \tilde{c}_7^{(\rho)}\sigma^{ij}\sigma_{ij} + \tilde{c}_8^{(\rho)}\omega^{ij}\omega_{ij} + \tilde{g}_1^{(\rho)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j\phi) + \tilde{g}_2^{(\rho)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j T) + \tilde{g}_3^{(\rho)}\epsilon^{ij}(\partial_i T)(\partial_j\phi) \tag{B7}$$

where,

$$
\begin{aligned}
\tilde{n}_1^{(\rho)} &= -\frac{\eta_r^2}{2p_0}, \qquad \tilde{n}_2^{(\rho)} = \frac{1}{24p_0^2 q_0}((6\mathcal{N}_5 p_0 q_0(\partial_\phi p) + 3q_0\eta_r^2(\partial_\phi p) - 8\mathcal{N}_5 p_0^2(\partial_\phi q)))\\
\tilde{n}_3^{(\rho)} &= \frac{1}{24p_0^2 q_0}((6\mathcal{N}_5 p_0 q_0(\partial_T p) + 3q_0\eta_r^2(\partial_T p) - 8\mathcal{N}_5 p_0^2(\partial_T q)))\\
\tilde{c}_1^{(\rho)} &= \frac{1}{12p_0}(((12\mathcal{N}_2 + \mathcal{N}_3 - 4\mathcal{N}_4)p_0 + 8\eta_r^2))\\
\tilde{c}_2^{(\rho)} &= -\frac{1}{192p_0^3 q_0^2}((16q_0^2\eta_r^2(\partial_\phi p)^2 - 16p_0^2(2(5\mathcal{N}_5 q_0 + 3\mathcal{N}_6 q_0)(\partial_\phi p)(\partial_\phi q) + 3\mathcal{N}_5 q_0^2(\partial_\phi^2 p))\\
&\quad + p_0(((12\mathcal{N}_1 + 12\mathcal{N}_2 - \mathcal{N}_3 + 4\mathcal{N}_4)q_0^2 + 132\mathcal{N}_5 q_0^2 + 36\mathcal{N}_6 q_0^2)(\partial_\phi p)^2\\
&\quad - 24q_0^2\eta_r(\partial_\phi p)(\partial_\phi \eta_r) - 24q_0^2\eta_r^2(\partial_\phi^2 p)) + 64p_0^3(\mathcal{N}_6(\partial_\phi q)^2 + \mathcal{N}_5 q_0(\partial_\phi^2 q))))\\
\tilde{c}_3^{(\rho)} &= -\frac{1}{192p_0^3 q_0^2}((16q_0^2\eta_r^2(\partial_T p)^2 - 16p_0^2(2(5\mathcal{N}_5 q_0 + 3\mathcal{N}_6 q_0)(\partial_T p)(\partial_T q) + 3\mathcal{N}_5 q_0^2(\partial_T^2 p))\\
&\quad + p_0(((12\mathcal{N}_1 + 12\mathcal{N}_2 - \mathcal{N}_3 + 4\mathcal{N}_4)q_0^2 + 132\mathcal{N}_5 q_0^2 + 36\mathcal{N}_6 q_0^2)(\partial_T p)^2\\
&\quad - 24q_0^2\eta_r(\partial_T p)(\partial_T \eta_r) - 24q_0^2\eta_r^2(\partial_T^2 p)) + 64p_0^3(\mathcal{N}_6(\partial_T q)^2 + \mathcal{N}_5 q_0(\partial_T^2 q))))\\
\tilde{c}_4^{(\rho)} &= -\frac{1}{24p_0^2}((5\eta_r^2(\partial_\phi p) + p_0((-6\mathcal{N}_1 + \mathcal{N}_3 + 4\mathcal{N}_4)(\partial_\phi p) + 12\eta_r(\partial_\phi \eta_r))))\\
\tilde{c}_5^{(\rho)} &= \frac{1}{96p_0^3 q_0^2}((-16q_0^2\eta_r^2(\partial_\phi p)(\partial_T p) + 16p_0^2((5\mathcal{N}_5 q_0 + 3\mathcal{N}_6 q_0)(\partial_\phi q)(\partial_T p)\\
&\quad + (5\mathcal{N}_5 q_0 + 3\mathcal{N}_6 q_0)(\partial_\phi p)(\partial_T q) + 3\mathcal{N}_5 q_0^2(\partial_\phi \partial_T p))\\
&\quad + p_0(-(\partial_\phi p)(((12\mathcal{N}_1 + 12\mathcal{N}_2 - \mathcal{N}_3 + 4\mathcal{N}_4)q_0^2 + 132\mathcal{N}_5 q_0^2 + 36\mathcal{N}_6 q_0^2)(\partial_T p)\\
&\quad - 12q_0^2\eta_r(\partial_T \eta_r)) + 12q_0^2\eta_r((\partial_\phi \eta_r)(\partial_T p) + 2\eta_r(\partial_T \partial_\phi p))) - 64p_0^3(\mathcal{N}_6(\partial_\phi q)(\partial_T q) + \mathcal{N}_5 q_0(\partial_\phi \partial_T q))))\\
\tilde{c}_6^{(\rho)} &= -\frac{1}{24p_0^2}((5\eta_r^2(\partial_T p) + p_0((-6\mathcal{N}_1 + \mathcal{N}_3 + 4\mathcal{N}_4)(\partial_T p) + 12\eta_r(\partial_T \eta_r))))\\
\tilde{c}_7^{(\rho)} &= -\frac{p_0\mathcal{N}_3 - 4\eta_r^2}{12p_0}, \qquad \tilde{c}_8^{(\rho)} = \frac{\mathcal{N}_4}{3}\\
\tilde{g}_1^{(\rho)} &= \frac{1}{24p_0 q_0}((2\mathcal{N}_8 + 3\mathcal{N}_9)(3q_0(\partial_\phi p) - 4p_0(\partial_\phi q)))\\
\tilde{g}_2^{(\rho)} &= \frac{1}{24p_0 q_0}((2\mathcal{N}_8 + 3\mathcal{N}_9)(3q_0(\partial_T p) - 4p_0(\partial_T q)))\\
\tilde{g}_3^{(\rho)} &= -\frac{1}{24p_0 q_0}((2\mathcal{N}_8 - 3\mathcal{N}_9)((\partial_\phi q)(\partial_T p) - (\partial_\phi p)(\partial_T q))).
\end{aligned} \tag{B8}
$$

Velocity,

$$
\begin{aligned}
v^i &= \frac{u^i}{u^+} + \frac{\eta_r}{16P^2u^+}\left(\partial_i P - 4P\frac{\partial_i u^+}{u^+}\right) + \tilde{n}_1^{(v)}\partial_k\sigma^{ik} + \tilde{n}_2^{(v)}\partial_k\omega^{ik} + \tilde{n}_3^{(v)}\epsilon^{il}\partial_k\sigma^{lk} + \tilde{n}_4^{(v)}\epsilon^{il}\partial_k\omega^{lk}\\
&\quad + \tilde{c}_1^{(v)}\sigma^{ik}(\partial_k\mu_m) + \tilde{c}_2^{(v)}\omega^{ik}(\partial_k\mu_m) + \tilde{c}_3^{(v)}(\partial_k v^k)(\partial_i\mu_m) + \tilde{c}_4^{(v)}\sigma^{ik}(\partial_k\phi) + \tilde{c}_5^{(v)}\omega^{ik}(\partial_k\phi)\\
&\quad + \tilde{c}_6^{(v)}(\partial_k v^k)(\partial_i\phi) + \tilde{c}_7^{(v)}\sigma^{ik}(\partial_k T) + \tilde{c}_8^{(v)}\omega^{ik}(\partial_k T) + \tilde{c}_9^{(v)}(\partial_k v^k)(\partial^i T) + \tilde{g}_1^{(v)}\epsilon^{il}\sigma^{lk}(\partial_k\mu_m)\\
&\quad + \tilde{g}_2^{(v)}\epsilon^{il}\omega^{lk}(\partial_k\mu_m) + \tilde{g}_3^{(v)}(\partial_k v^k)\epsilon^{il}(\partial_l\mu_m) + \tilde{g}_4^{(v)}\epsilon^{il}\sigma^{lk}(\partial_k\phi) + \tilde{g}_5^{(v)}\epsilon^{il}\omega^{lk}(\partial_k\phi)\\
&\quad + \tilde{g}_6^{(v)}(\partial_k v^k)\epsilon^{il}(\partial_l\phi) + \tilde{g}_7^{(v)}\epsilon^{il}\sigma^{lk}(\partial_k T) + \tilde{g}_8^{(v)}\epsilon^{il}\omega^{lk}(\partial_k T) + \tilde{g}_9^{(v)}(\partial_k v^k)\epsilon^{il}(\partial^l T)
\end{aligned} \tag{B9}
$$

where,

$$\tilde{n}_1^{(v)} = \frac{\mathcal{N}_1 p_0 - \eta_r^2}{8p_0^2}, \qquad \tilde{n}_2^{(v)} = \frac{2\mathcal{N}_1 p_0 - \eta_r^2}{16p_0^2}, \qquad \tilde{n}_3^{(v)} = 0, \qquad \tilde{n}_4^{(v)} = \frac{\mathcal{N}_7}{4p_0},$$
$$\tilde{c}_1^{(v)} = \frac{(-\mathcal{N}_1 + 2\mathcal{N}_2 + \mathcal{N}_3)p_0 - \eta_r^2}{16p_0^2}, \qquad \tilde{c}_2^{(v)} = \frac{(\mathcal{N}_1 + 2\mathcal{N}_2 - 4\mathcal{N}_4)p_0 - \eta_r^2}{16p_0^2}$$
$$\tilde{c}_3^{(v)} = \frac{1}{32p_0^2((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}(\eta_r(\eta_r(3(\partial_\phi p)(\partial_T q) - 3(\partial_T p)(\partial_\phi q)) + (\partial_\phi \eta_r)(3q_0(\partial_T p) - 4p_0(\partial_T q)) + (\partial_T \eta_r)(4p_0(\partial_\phi q) - 3q_0(\partial_\phi p))))$$
$$\tilde{c}_4^{(v)} = -\frac{1}{64p_0^2 q_0}(-3\mathcal{N}_1 q_0(\partial_\phi p) - 2\mathcal{N}_2 q_0(\partial_\phi p) + \mathcal{N}_3 q_0(\partial_\phi p) - 6\mathcal{N}_5 q_0(\partial_\phi p) + 8\mathcal{N}_5 p_0(\partial_\phi q) + 4q_0\eta_r(\partial_\phi \eta_r))$$
$$\tilde{c}_5^{(v)} = -\frac{1}{64p_0^3 q_0}(\mathcal{N}_1 p_0 q_0(\partial_\phi p) + 2\mathcal{N}_2 p_0 q_0(\partial_\phi p) + 4\mathcal{N}_4 p_0 q_0(\partial_\phi p) + 6\mathcal{N}_5 p_0 q_0(\partial_\phi p) - 8\mathcal{N}_5 p_0^2(\partial_\phi q) - q_0\eta_r^2(\partial_\phi p))$$
$$\tilde{c}_6^{(v)} = \frac{1}{128p_0^3((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}(4p_0\eta_r(\partial_\phi p)(\partial_T q)(\partial_\phi \eta_r) - 4p_0\eta_r(\partial_\phi p)(\partial_\phi q)(\partial_T \eta_r) + 3\eta_r^2(\partial_\phi p)(\partial_T p)(\partial_\phi q) - 3\eta_r^2(\partial_\phi p)^2(\partial_T q) - 3q_0\eta_r(\partial_\phi p)(\partial_T p)(\partial_\phi \eta_r) + 3q_0\eta_r(\partial_\phi p)^2(\partial_T \eta_r))$$
$$\tilde{c}_7^{(v)} = \frac{1}{64p_0^2 q_0}(((3\mathcal{N}_1 + 2\mathcal{N}_2 - \mathcal{N}_3)q_0 + 6\mathcal{N}_5 q_0)(\partial_T p) - 4(2\mathcal{N}_5 p_0(\partial_T q) + q_0\eta_r(\partial_T \eta_r)))$$
$$\tilde{c}_8^{(v)} = \frac{1}{64p_0^3 q_0}(-p_0((\mathcal{N}_1 + 2\mathcal{N}_2 + 4\mathcal{N}_4)q_0 + 6\mathcal{N}_5 q_0)(\partial_T p) + 8\mathcal{N}_5 p_0^2(\partial_T q) + q_0\eta_r^2(\partial_T p))$$
$$\tilde{c}_9^{(v)} = \frac{1}{128p_0^3((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}(\eta_r(\partial_T p)(3\eta_r((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q)) + q_0(3(\partial_\phi p)(\partial_T \eta_r) - 3(\partial_T p)(\partial_\phi \eta_r)) + 4p_0((\partial_T q)(\partial_\phi \eta_r) - (\partial_\phi q)(\partial_T \eta_r))))$$
$$\tilde{g}_1^{(v)} = 0, \qquad \tilde{g}_2^{(v)} = -\frac{\mathcal{N}_7}{2p_0}, \qquad \tilde{g}_3^{(v)} = 0, \qquad \tilde{g}_4^{(v)} = \frac{1}{64p_0^2 q_0}(\mathcal{N}_9(4p_0(\partial_\phi q) - 3q_0(\partial_\phi p))),$$
$$\tilde{g}_5^{(v)} = -\frac{1}{32p_0^2 q_0}(-4\mathcal{N}_7 q_0(\partial_\phi p) - 3\mathcal{N}_8 q_0(\partial_\phi p) + 4\mathcal{N}_8 p_0(\partial_\phi q)), \qquad \tilde{g}_6^{(v)} = 0,$$
$$\tilde{g}_7^{(v)} = \frac{1}{64p_0^2 q_0}(\mathcal{N}_9(4p_0(\partial_T q) - 3q_0(\partial_T p))),$$
$$\tilde{g}_8^{(v)} = \frac{1}{32p_0^2 q_0}((4\mathcal{N}_7 q_0 + 3\mathcal{N}_8 q_0)(\partial_T p) - 4\mathcal{N}_8 p_0(\partial_T q)), \qquad \tilde{g}_9^{(v)} = 0. \tag{B10}$$

Pressure,

$$p = P + \tilde{n}_1^{(p)}(\partial_k\partial_k\mu_m) + \tilde{n}_2^{(p)}(\partial_k\partial_k\phi) + \tilde{n}_1^{(p)}(\partial_k\partial_k T) + \tilde{c}_1^{(p)}(\partial_k\mu_m)(\partial^k\mu_m) + \tilde{c}_2^{(p)}(\partial_k\phi)(\partial^k\phi) + \tilde{c}_3^{(p)}(\partial_k T)(\partial^k T) + \tilde{c}_4^{(p)}(\partial_k\mu_m)(\partial^k\phi) + \tilde{c}_5^{(p)}(\partial_k T)(\partial^k\phi) + \tilde{c}_6^{(p)}(\partial_k\mu_m)(\partial^k T) + \tilde{c}_7^{(p)}\sigma^{ij}\sigma_{ij} + \tilde{c}_8^{(p)}\omega^{ij}\omega_{ij} + \tilde{g}_1^{(p)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j\phi) + \tilde{g}_2^{(p)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j T) + \tilde{g}_3^{(p)}\epsilon^{ij}(\partial_i T)(\partial_j\phi) \tag{B11}$$

where,

$$\tilde{n}_1^{(p)} = \frac{1}{4p_0}({\eta_r}^2), \qquad \tilde{n}_2^{(p)} = \frac{1}{48p_0^2q_0}(-6N_5p_0q_0(\partial_\phi p) + 8N_5p_0^2(\partial_\phi q) - 3q_0(\partial_\phi p){\eta_r}^2)$$
$$\tilde{n}_3^{(p)} = \frac{1}{48p_0^2q_0}(-6N_5p_0q_0(\partial_T p) + 8N_5p_0^2(\partial_T q) - 3q_0(\partial_T p){\eta_r}^2)$$
$$\tilde{c}_1^{(p)} = \frac{1}{24p_0}(-11{\eta_r}^2 - 12N_2p_0 - N_3p_0 + 4N_4p_0)$$
$$\begin{aligned}\tilde{c}_2^{(p)} = \frac{1}{384p_0^3q_0^2}(&-160N_5p_0^2q_0(\partial_\phi p)(\partial_\phi q) - 96N_6p_0^2q_0(\partial_\phi p)(\partial_\phi q) - 48N_5p_0^2q_0^2(\partial_\phi^2 p)\\ &+ 12N_1p_0q_0^2(\partial_\phi p)^2 + 12N_2p_0q_0^2(\partial_\phi p)^2 - N_3p_0q_0^2(\partial_\phi p)^2 + 4N_4p_0q_0^2(\partial_\phi p)^2\\ &+ 132N_5p_0q_0^2(\partial_\phi p)^2 + 36N_6p_0q_0^2(\partial_\phi p)^2 + 64N_6p_0^3(\partial_\phi q)^2 + 64N_5p_0^3q_0(\partial_\phi^2 q)\\ &- 24p_0q_0^2(\partial_\phi p)\eta_r(\partial_\phi \eta_r) - 24p_0q_0^2(\partial_\phi^2 p){\eta_r}^2 + 13q_0^2(\partial_\phi p)^2{\eta_r}^2)\end{aligned}$$
$$\begin{aligned}\tilde{c}_3^{(p)} = \frac{1}{384p_0^3q_0^2}(&-160N_5p_0^2q_0(\partial_T p)(\partial_T q) - 96N_6p_0^2q_0(\partial_T p)(\partial_T q) - 48N_5p_0^2q_0^2(\partial_T^2 p)\\ &+ 12N_1p_0q_0^2(\partial_T p)^2 + 12N_2p_0q_0^2(\partial_T p)^2 - N_3p_0q_0^2(\partial_T p)^2 + 4N_4p_0q_0^2(\partial_T p)^2\\ &+ 132N_5p_0q_0^2(\partial_T p)^2 + 36N_6p_0q_0^2(\partial_T p)^2 + 64N_6p_0^3(\partial_T q)^2 + 64N_5p_0^3q_0(\partial_T^2 q)\\ &- 24p_0q_0^2(\partial_T p)\eta_r(\partial_T \eta_r) - 24p_0q_0^2(\partial_T^2 p){\eta_r}^2 + 13q_0^2(\partial_T p)^2{\eta_r}^2)\end{aligned}$$
$$\tilde{c}_4^{(p)} = \frac{1}{48p_0^2}(-6N_1p_0(\partial_\phi p) + N_3p_0(\partial_\phi p) + 4N_4p_0(\partial_\phi p) + 8(\partial_\phi p){\eta_r}^2 + 12p_0(\partial_\phi \eta_r)\eta_r)$$
$$\tilde{c}_5^{(p)} = \frac{1}{48p_0^2}(-6N_1p_0(\partial_T p) + N_3p_0(\partial_T p) + 4N_4p_0(\partial_T p) + 8(\partial_T p){\eta_r}^2 + 12p_0(\partial_T \eta_r)\eta_r)$$
$$\begin{aligned}\tilde{c}_6^{(p)} = \frac{1}{384p_0^3q_0^2}(&-32p_0^2(5N_5q_0(\partial_T p)(\partial_\phi q)\\ &+ 3N_6q_0(\partial_T p)(\partial_\phi q) + 5N_5q_0(\partial_\phi p)(\partial_T q) + 3N_6q_0(\partial_\phi p)(\partial_T q) + 3N_5q_0^2(\partial_\phi\partial_T p))\\ &+ 2p_0(12N_1q_0^2(\partial_\phi p)(\partial_T p) + 12N_2q_0^2(\partial_\phi p)(\partial_T p) - N_3q_0^2(\partial_\phi p)(\partial_T p) + 4N_4q_0^2(\partial_\phi p)(\partial_T p)\\ &+ 132N_5q_0^2(\partial_\phi p)(\partial_T p) + 36N_6q_0^2(\partial_\phi p)(\partial_T p) - 12q_0^2(\partial_T p)\eta_r(\partial_\phi \eta_r) - 12q_0^2(\partial_\phi p)\eta_r(\partial_T \eta_r)\\ &- 24q_0^2(\partial_\phi\partial_T p){\eta_r}^2) + 128p_0^3(N_6(\partial_\phi q)(\partial_T q) + N_5q_0(\partial_\phi\partial_T q)) + 26q_0^2(\partial_\phi p)(\partial_T p){\eta_r}^2)\end{aligned}$$
$$\tilde{c}_7^{(p)} = \frac{1}{24p_0}(N_3p_0 - 4{\eta_r}^2), \qquad \tilde{c}_8^{(p)} = -\frac{N_4}{6},$$
$$\tilde{g}_1^{(p)} = \frac{1}{48p_0q_0}(-6N_8q_0(\partial_\phi p) - 9N_9q_0(\partial_\phi p) + 8N_8p_0(\partial_\phi q) + 12N_9p_0(\partial_\phi q)),$$
$$\tilde{g}_2^{(p)} = \frac{1}{48p_0q_0}(-6N_8q_0(\partial_T p) - 9N_9q_0(\partial_T p) + 8N_8p_0(\partial_T q) + 12N_9p_0(\partial_T q))$$
$$\tilde{g}_3^{(p)} = -\frac{1}{48p_0q_0}(-2N_8(\partial_T p)(\partial_\phi q) + 3N_9(\partial_T p)(\partial_\phi q) + 2N_8(\partial_\phi p)(\partial_T q) - 3N_9(\partial_\phi p)(\partial_T q)). \tag{B12}$$

Energy,

$$\begin{aligned}\epsilon = \frac{E-P}{2} + \frac{1}{2}\rho v^2 &+ \tilde{n}_1^{(\epsilon)}(\partial_k\partial_k\mu_m) + \tilde{n}_2^{(\epsilon)}(\partial_k\partial_k\phi) + \tilde{n}_1^{(\epsilon)}(\partial_k\partial_k T) + \tilde{c}_1^{(\epsilon)}(\partial_k\mu_m)(\partial^k\mu_m)\\ &+ \tilde{c}_2^{(\epsilon)}(\partial_k\phi)(\partial^k\phi) + \tilde{c}_3^{(\epsilon)}(\partial_k T)(\partial^k T) + \tilde{c}_4^{(\epsilon)}(\partial_k\mu_m)(\partial^k\phi) + \tilde{c}_5^{(\epsilon)}(\partial_k T)(\partial^k\phi)\\ &+ \tilde{c}_6^{(\epsilon)}(\partial_k\mu_m)(\partial^k T) + \tilde{c}_7^{(\epsilon)}\sigma^{ij}\sigma_{ij} + \tilde{c}_8^{(\epsilon)}\omega^{ij}\omega_{ij} + \tilde{g}_1^{(\epsilon)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j\phi)\\ &+ \tilde{g}_2^{(\epsilon)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j T) + \tilde{g}_3^{(\epsilon)}\epsilon^{ij}(\partial_i T)(\partial_j\phi)\end{aligned} \tag{B13}$$

where,

$$\tilde{n}_1^{(\epsilon)} = \frac{\eta_r^2}{4p_0}, \qquad \tilde{n}_2^{(\epsilon)} = -\frac{1}{48p_0^2q_0}(6\mathcal{N}_5 p_0 q_0(\partial_\phi p) - 8\mathcal{N}_5 p_0^2(\partial_\phi q) + 3q_0\eta_r^2(\partial_\phi p))$$
$$\tilde{n}_3^{(\epsilon)} = -\frac{1}{48p_0^2q_0}(6\mathcal{N}_5 p_0 q_0(\partial_T p) - 8\mathcal{N}_5 p_0^2(\partial_T q) + 3q_0\eta_r^2(\partial_T p)),$$
$$\tilde{c}_1^{(\epsilon)} = -\frac{1}{24p_0}((12\mathcal{N}_2 + \mathcal{N}_3 - 4\mathcal{N}_4)p_0 + 11\eta_r^2)$$
$$\begin{aligned}\tilde{c}_2^{(\epsilon)} = \frac{1}{384p_0^3q_0^2}&(-16p_0^2(2(5\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_\phi p)(\partial_\phi q) + 3\mathcal{N}_5q_0^2(\partial_\phi^2 p)) \\ &+ p_0(((12\mathcal{N}_1 + 12\mathcal{N}_2 - \mathcal{N}_3 + 4\mathcal{N}_4)q_0^2 + 132\mathcal{N}_5q_0^2 + 36\mathcal{N}_6q_0^2)(\partial_\phi p)^2 \\ &- 24q_0^2\eta_r(\partial_\phi p)(\partial_\phi\eta_r) - 24q_0^2\eta_r^2(\partial_\phi^2 p)) + 64p_0^3(\mathcal{N}_6(\partial_\phi q)^2 + \mathcal{N}_5q_0(\partial_\phi^2 q)) + 13q_0^2\eta_r^2(\partial_\phi p)^2)\end{aligned}$$
$$\begin{aligned}\tilde{c}_3^{(\epsilon)} = \frac{1}{384p_0^3q_0^2}&(-16p_0^2(2(5\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_T p)(\partial_T q) + 3\mathcal{N}_5q_0^2(\partial_T^2 p)) \\ &+ p_0(((12\mathcal{N}_1 + 12\mathcal{N}_2 - \mathcal{N}_3 + 4\mathcal{N}_4)q_0^2 + 132\mathcal{N}_5q_0^2 + 36\mathcal{N}_6q_0^2)(\partial_T p)^2 \\ &- 24q_0^2\eta_r(\partial_T p)(\partial_T\eta_r) - 24q_0^2\eta_r^2(\partial_T^2 p)) + 64p_0^3(\mathcal{N}_6(\partial_T q)^2 + \mathcal{N}_5q_0(\partial_T^2 q)) + 13q_0^2\eta_r^2(\partial_T p)^2)\end{aligned}$$
$$\tilde{c}_4^{(\epsilon)} = \frac{1}{48p_0^2}(p_0((-6\mathcal{N}_1 + \mathcal{N}_3 + 4\mathcal{N}_4)(\partial_\phi p) + 12(\partial_\phi\eta_r)) + 8\eta_r^2(\partial_\phi p))$$
$$\tilde{c}_5^{(\epsilon)} = \frac{1}{48p_0^2}(p_0((-6\mathcal{N}_1 + \mathcal{N}_3 + 4\mathcal{N}_4)(\partial_T p) + 12(\partial_T\eta_r)) + 8\eta_r^2(\partial_T p))$$
$$\begin{aligned}\tilde{c}_6^{(\epsilon)} = \frac{1}{192p_0^3q_0^2}&(-16p_0^2((5\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_T p)(\partial_\phi q) + (5\mathcal{N}_5q_0 + 3\mathcal{N}_6q_0)(\partial_\phi p)(\partial_T q) \\ &+ 3\mathcal{N}_5q_0^2(\partial_T\partial_\phi p)) + p_0((\partial_\phi p)(((12\mathcal{N}_1 + 12\mathcal{N}_2 - \mathcal{N}_3 + 4\mathcal{N}_4)q_0^2 + 132\mathcal{N}_5q_0^2 \\ &+ 36\mathcal{N}_6q_0^2)(\partial_T p) - 12q_0^2(\partial_T\eta_r)) - 12q_0^2\eta_r(2\eta_r(\partial_T\partial_\phi p) + (\partial_T p)(\partial_\phi\eta_r))) + 64p_0^3(\mathcal{N}_6(\partial_\phi q)(\partial_T q) \\ &+ \mathcal{N}_5q_0(\partial_T\partial_\phi q)) + 13q_0^2\eta_r^2(\partial_\phi p)(\partial_T p))\end{aligned}$$
$$\tilde{c}_7^{(\epsilon)} = \frac{1}{24}\left(\mathcal{N}_3 - \frac{4\eta_r^2}{p_0}\right), \qquad \tilde{c}_8^{(\epsilon)} = -\frac{\mathcal{N}_4}{6}$$
$$\tilde{g}_1^{(\epsilon)} = \frac{1}{48p_0q_0}((2\mathcal{N}_8 + 3\mathcal{N}_9)(4p_0(\partial_\phi q) - 3q_0(\partial_\phi p))), \qquad \tilde{g}_2^{(\epsilon)} = \frac{1}{48p_0q_0}((2\mathcal{N}_8 + 3\mathcal{N}_9)(4p_0(\partial_T q) - 3q_0(\partial_T p)))$$
$$\tilde{g}_3^{(\epsilon)} = -\frac{1}{48p_0q_0}((2\mathcal{N}_8 - 3\mathcal{N}_9)((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))). \tag{B14}$$

Energy current,

$$\begin{aligned}e^i = &\left(e + P + \frac{1}{2}\rho v^2\right)v^i + \eta_r\left(\frac{\partial_i u^+}{(u^+)^2} - \frac{\partial_i P}{4Pu^+}\right) - \eta_r\sigma_{ij}v^j + \tilde{n}_1^{(ec)}\partial_k\sigma^{ik} + \tilde{n}_2^{(ec)}\partial_k\omega^{ik} + \tilde{n}_3^{(ec)}\epsilon^{il}\partial_k\sigma^{lk} \\ &+ \tilde{n}_4^{(ec)}\epsilon^{il}\partial_k\omega^{lk} + \tilde{c}_1^{(ec)}\sigma^{ik}(\partial_k\mu_m) + \tilde{c}_2^{(ec)}\omega^{ik}(\partial_k\mu_m) + \tilde{c}_3^{(ec)}(\partial_k v^k)(\partial_i\mu_m) + \tilde{c}_4^{(ec)}\sigma^{ik}(\partial_k\phi) \\ &+ \tilde{c}_5^{(ec)}\omega^{ik}(\partial_k\phi) + \tilde{c}_6^{(ec)}(\partial_k v^k)(\partial_i\phi) + \tilde{c}_7^{(ec)}\sigma^{ik}(\partial_k T) + \tilde{c}_8^{(ec)}\omega^{ik}(\partial_k T) + \tilde{c}_9^{(ec)}(\partial_k v^k)(\partial^i T) \\ &+ \tilde{g}_1^{(ec)}\epsilon^{il}\sigma^{lk}(\partial_k\mu_m) + \tilde{g}_2^{(ec)}\epsilon^{il}\omega^{lk}(\partial_k\mu_m) + \tilde{g}_3^{(ec)}(\partial_k v^k)\epsilon^{il}(\partial_l\mu_m) + \tilde{g}_4^{(ec)}\epsilon^{il}\sigma^{lk}(\partial_k\phi) + \tilde{g}_5^{(ec)}\epsilon^{il}\omega^{lk}(\partial_k\phi) \\ &+ \tilde{g}_6^{(ec)}(\partial_k v^k)\epsilon^{il}(\partial_l\phi) + \tilde{g}_7^{(ec)}\epsilon^{il}\sigma^{lk}(\partial_k T) + \tilde{g}_8^{(ec)}\epsilon^{il}\omega^{lk}(\partial_k T) + \tilde{g}_9^{(ec)}(\partial_k v^k)\epsilon^{il}(\partial^l T)\end{aligned} \tag{B15}$$

where,

$$
\begin{aligned}
\tilde{n}_1^{(ec)} &= \frac{1}{2}\left(\frac{\eta_r^2}{p_0} - \mathcal{N}_1\right), \qquad \tilde{n}_2^{(ec)} = \frac{\eta_r^2 - 2\mathcal{N}_1 p_0}{4p_0}, \qquad \tilde{n}_3^{(ec)} = 0, \qquad \tilde{n}_4^{(ec)} = -\mathcal{N}_7,\\
\tilde{c}_1^{(ec)} &= \frac{1}{4}(\mathcal{N}_1 - 2\mathcal{N}_2 - \mathcal{N}_3), \qquad \tilde{c}_2^{(ec)} = \frac{1}{4}\left(\mathcal{N}_1 + 2\mathcal{N}_2 - 4\mathcal{N}_4 - \frac{\eta_r^2}{p_0}\right)\\
\tilde{c}_3^{(ec)} &= \frac{1}{8p_0((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}(\eta_r(3\eta_r((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q)) + 4p_0((\partial_T q)(\partial_\phi \eta_r)\\
&\quad - (\partial_\phi q)(\partial_T \eta_r)) + q_0(3(\partial_\phi p)(\partial_T \eta_r) - 3(\partial_T p)(\partial_\phi \eta_r))))\\
\tilde{c}_4^{(ec)} &= \frac{1}{16p_0^2 q_0}(p_0(((-3\mathcal{N}_1 - 2\mathcal{N}_2 + \mathcal{N}_3)q_0 - 6\mathcal{N}_5 q_0)(\partial_\phi p) + 4q_0\eta_r(\partial_\phi \eta_r)) + 8\mathcal{N}_5 p_0^2(\partial_\phi q) + q_0\eta_r^2(\partial_\phi p))\\
\tilde{c}_5^{(ec)} &= \frac{1}{16p_0^2 q_0}(-p_0((\mathcal{N}_1 + 2\mathcal{N}_2 + 4\mathcal{N}_4)q_0 + 6\mathcal{N}_5 q_0)(\partial_\phi p) + 8\mathcal{N}_5 p_0^2(\partial_\phi q) + q_0\eta_r^2(\partial_\phi p))\\
\tilde{c}_6^{(ec)} &= \frac{1}{32p_0^2((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}(\eta_r(\partial_\phi p)(\eta_r(3(\partial_\phi p)(\partial_T q) - 3(\partial_T p)(\partial_\phi q)) + 3q_0((\partial_T p)(\partial_\phi \eta_r)\\
&\quad - (\partial_\phi p)(\partial_T \eta_r)) + 4p_0((\partial_\phi q)(\partial_T \eta_r) - (\partial_T q)(\partial_\phi \eta_r))))\\
\tilde{c}_7^{(ec)} &= \frac{1}{16p_0^2 q_0}(p_0(((-3\mathcal{N}_1 - 2\mathcal{N}_2 + \mathcal{N}_3)q_0 - 6\mathcal{N}_5 q_0)(\partial_T p) + 4q_0\eta_r(\partial_T \eta_r)) + 8\mathcal{N}_5 p_0^2(\partial_T q) + q_0\eta_r^2(\partial_T p))\\
\tilde{c}_8^{(ec)} &= \frac{1}{16p_0^2 q_0}(-p_0((\mathcal{N}_1 + 2\mathcal{N}_2 + 4\mathcal{N}_4)q_0 + 6\mathcal{N}_5 q_0)(\partial_T p) + 8\mathcal{N}_5 p_0^2(\partial_T q) + q_0\eta_r^2(\partial_T p))\\
\tilde{c}_9^{(ec)} &= \frac{1}{32p_0^2((\partial_T p)(\partial_\phi q) - (\partial_\phi p)(\partial_T q))}(\eta_r(\partial_T p)(\eta_r(3(\partial_\phi p)(\partial_T q) - 3(\partial_T p)(\partial_\phi q)) + 3q_0((\partial_T p)(\partial_\phi \eta_r)\\
&\quad - (\partial_\phi p)(\partial_T \eta_r)) + 4p_0((\partial_\phi q)(\partial_T \eta_r) - (\partial_T q)(\partial_\phi \eta_r))))\\
\tilde{g}_1^{(ec)} &= 0, \qquad \tilde{g}_2^{(ec)} = -2\mathcal{N}_7, \qquad \tilde{g}_3^{(ec)} = 0, \qquad \tilde{g}_4^{(ec)} = \frac{1}{16p_0 q_0}(\mathcal{N}_9(3q_0(\partial_\phi p) - 4p_0(\partial_\phi q)))\\
\tilde{g}_5^{(ec)} &= \frac{1}{8p_0 q_0}((4\mathcal{N}_7 q_0 + 3\mathcal{N}_8 q_0)(\partial_\phi p) - 4\mathcal{N}_8 p_0(\partial_\phi q)), \qquad \tilde{g}_6^{(ec)} = 0,\\
\tilde{g}_7^{(ec)} &= \frac{1}{16p_0 q_0}(\mathcal{N}_9(3q_0(\partial_T p) - 4p_0(\partial_T q)))\\
\tilde{g}_8^{(ec)} &= \frac{1}{8p_0 q_0}((4\mathcal{N}_7 q_0 + 3\mathcal{N}_8 q_0)(\partial_T p) - 4\mathcal{N}_8 p_0(\partial_T q)), \qquad \tilde{g}_9^{(ec)} = 0.
\end{aligned} \tag{B16}
$$

Charge,

$$
\begin{aligned}
Q &= 4\sqrt{3}qu^+ - \xi_r(u^+)^2\epsilon^{ij}(\partial_i v_j) + \tilde{n}_1^{(q)}(\partial_k\partial_k\mu_m) + \tilde{n}_2^{(q)}(\partial_k\partial_k\phi) + \tilde{n}_1^{(q)}(\partial_k\partial_k T) + \tilde{c}_1^{(q)}(\partial_k\mu_m)(\partial^k\mu_m)\\
&\quad + \tilde{c}_2^{(q)}(\partial_k\phi)(\partial^k\phi) + \tilde{c}_3^{(q)}(\partial_k T)(\partial^k T) + \tilde{c}_4^{(q)}(\partial_k\mu_m)(\partial^k\phi) + \tilde{c}_5^{(q)}(\partial_k T)(\partial^k\phi) + \tilde{c}_6^{(q)}(\partial_k\mu_m)(\partial^k T)\\
&\quad + \tilde{c}_7^{(q)}\sigma^{ij}\sigma_{ij} + \tilde{c}_8^{(q)}\omega^{ij}\omega_{ij} + \tilde{g}_1^{(q)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j\phi) + \tilde{g}_2^{(q)}\epsilon^{ij}(\partial_i\mu_m)(\partial_j T) + \tilde{g}_3^{(q)}\epsilon^{ij}(\partial_i T)(\partial_j\phi)
\end{aligned} \tag{B17}
$$

$$\tilde{n}_1^{(q)} = \frac{1}{2p_0}\Big((\gamma_1-\gamma_2)p_0 - 6\sqrt{3}q_0\eta_r\lambda_r\Big)$$
$$\tilde{n}_2^{(q)} = -\frac{1}{8p_0^2}\Big(p_0(\partial_\phi p)(\gamma_1+\gamma_2-24\sqrt{3}q_0\lambda_r^2) - 6\sqrt{3}q_0\eta_r\lambda_r(\partial_\phi p) + 32\sqrt{3}p_0^2\lambda_r^2(\partial_\phi q)\Big)$$
$$\tilde{n}_3^{(q)} = -\frac{1}{8p_0^2}\Big(p_0(\partial_T p)(\gamma_1+\gamma_2-24\sqrt{3}q_0\lambda_r^2) - 6\sqrt{3}q_0\eta_r\lambda_r(\partial_T p) + 32\sqrt{3}p_0^2\lambda_r^2(\partial_T q)\Big)$$
$$\tilde{c}_1^{(q)} = \frac{1}{2p_0}\Big((\gamma_2-\gamma_1)p_0 + 8\sqrt{3}q_0\eta_r\lambda_r\Big)$$
$$\begin{aligned}\tilde{c}_2^{(q)} = \frac{1}{32p_0^3q_0}\Big(&4p_0q_0(-p_0(\partial_\phi^2 p)(\gamma_1+\gamma_2-24\sqrt{3}q_0\lambda_r^2) + 6\sqrt{3}q_0\eta_r\lambda_r(\partial_\phi^2 p) - 32\sqrt{3}p_0^2\lambda_r(\lambda_r(\partial_\phi^2 q)\\ &+ (\partial_\phi q)(\partial_\phi\lambda_r))) + 4p_0(\partial_\phi p)(6\sqrt{3}q_0^2\lambda_r(4p_0(\partial_\phi\lambda_r) + (\partial_\phi\eta_r)) - p_0(\partial_\phi q)(\gamma_4+\gamma_5-24\sqrt{3}q_0\lambda_r^2))\\ &+ (\partial_\phi p)^2(4\gamma_2p_0q_0 + q_0(3(\gamma_4+\gamma_5)p_0 - 16\sqrt{3}q_0\lambda_r(6p_0\lambda_r+\eta_r)))\Big)\end{aligned}$$
$$\begin{aligned}\tilde{c}_3^{(q)} = \frac{1}{32p_0^3q_0}\Big(&4p_0q_0(-p_0(\partial_T^2 p)(\gamma_1+\gamma_2-24\sqrt{3}q_0\lambda_r^2) + 6\sqrt{3}q_0\eta_r\lambda_r(\partial_T^2 p) - 32\sqrt{3}p_0^2\lambda_r(\lambda_r(\partial_T^2 q)\\ &+ (\partial_T q)(\partial_T\lambda_r))) + 4p_0(\partial_T p)(6\sqrt{3}q_0^2\lambda_r(4p_0(\partial_T\lambda_r) + (\partial_T\eta_r)) - p_0(\partial_T q)(\gamma_4+\gamma_5-24\sqrt{3}q_0\lambda_r^2))\\ &+ (\partial_T p)^2(4\gamma_2p_0q_0 + q_0(3(\gamma_4+\gamma_5)p_0 - 16\sqrt{3}q_0\lambda_r(6p_0\lambda_r+\eta_r)))\Big)\end{aligned}$$
$$\tilde{c}_4^{(q)} = \frac{1}{8p_0^2q_0}\Big((\partial_\phi p)(p_0(4\gamma_1q_0 + 3(\gamma_5-\gamma_4)q_0) - 10\sqrt{3}q_0^2\eta_r\lambda_r) + 4p_0((\gamma_4-\gamma_5)p_0(\partial_\phi q) - 6\sqrt{3}q_0^2\lambda_r(\partial_\phi\eta_r))\Big)$$
$$\tilde{c}_5^{(q)} = \frac{1}{8p_0^2q_0}\Big((\partial_T p)(p_0(4\gamma_1q_0 + 3(\gamma_5-\gamma_4)q_0) - 10\sqrt{3}q_0^2\eta_r\lambda_r) + 4p_0((\gamma_4-\gamma_5)p_0(\partial_T q) - 6\sqrt{3}q_0^2\lambda_r(\partial_T\eta_r))\Big)$$
$$\begin{aligned}\tilde{c}_6^{(q)} = \frac{1}{16p_0^3q_0}\Big(&(\partial_\phi p)((\partial_T p)(4\gamma_2p_0q_0 + q_0(3p_0(\gamma_4+\gamma_5) - 16\sqrt{3}q_0\lambda_r(6p_0\lambda_r+\eta_r)))\\ &+ 2p_0(6\sqrt{3}q_0^2\lambda_r(4p_0(\partial_T\lambda_r) + (\partial_T\eta_r)) - p_0(\partial_T q)(\gamma_4+\gamma_5-24\sqrt{3}q_0\lambda_r^2)))\\ &+ 2p_0(2q_0(p_0((-\gamma_1-\gamma_2)(\partial_T\partial_\phi p) - 16\sqrt{3}p_0\lambda_r(2\lambda_r(\partial_T\partial_\phi q) + (\partial_T q)(\partial_\phi\lambda_r)))\\ &+ 3\sqrt{3}q_0\lambda_r(2(\partial_T\partial_\phi p)(4p_0\lambda_r+\eta_r) + (\partial_T p)(4p_0(\partial_\phi\lambda_r) + (\partial_\phi\eta_r))))\\ &+ p_0(\partial_\phi q)((\partial_T p)(-\gamma_4-\gamma_5+24\sqrt{3}q_0\lambda_r^2) - 32\sqrt{3}p_0q_0\lambda_r(\partial_T\lambda_r)))\Big)\end{aligned}$$
$$\tilde{c}_7^{(q)} = -\frac{1}{4p_0}\Big(\gamma_1p_0 - 8\sqrt{3}q_0\eta_r\lambda_r\Big), \qquad \tilde{c}_8^{(q)} = \frac{\gamma_2}{4}$$
$$\tilde{g}_1^{(q)} = \frac{1}{16p_0^2}\Big(4p_0((\partial_\phi p)(-\gamma_3-2\lambda_r\xi_r) + \xi_r(\partial_\phi\eta_r)) - 5\eta_r\xi_r(\partial_\phi p) + 16p_0^2\lambda_r(\partial_\phi\xi)\Big)$$
$$\tilde{g}_2^{(q)} = \frac{1}{16p_0^2}\Big(4p_0((\partial_T p)(-\gamma_3-2\lambda_r\xi_r) + \xi_r(\partial_T\eta_r)) - 5\eta_r\xi_r(\partial_T p) + 16p_0^2\lambda_r(\partial_T\xi)\Big)$$
$$\tilde{g}_3^{(q)} = \frac{1}{16p_0^2}\Big(\xi_r((\partial_T p)(\partial_\phi\eta_r) - (\partial_\phi p)(\partial_T\eta_r)) + 4p_0\lambda_r((\partial_\phi p)(\partial_T\xi) - (\partial_T p)(\partial_\phi\xi))\Big). \tag{B18}$$

APPENDIX C: HOLOGRAPHIC NONRELATIVISTIC CHARGED FLUID

Here we are presenting all the quantities (not presented in the main text) describing nonrelativistic charged holographic fluid, in terms of mass, charge, and local velocities of black brane.

The pressure of the holographic fluid is given by,

$$\begin{aligned} p &= m + \frac{1}{4m_0}(\partial_i\partial^i u^+) + \frac{(m_0-2)}{16m_0^2}(\partial_i\partial^i m) - \frac{q_0}{12m_0}(\partial_i\partial^i q) \\ &+ \frac{m_0}{2\sqrt{4m_0-3}}\log\left(\frac{3+\sqrt{4m_0-3}}{3-\sqrt{4m_0-3}}\right)\left(\frac{1}{16m_0^2}(\partial^i m)(\partial_i m) - \frac{1}{2}(\partial_i m)(\partial^i u^+) + (\partial_i u^+)(\partial^i u^+)\right) \\ &- \frac{\log(2)}{4}\left((\partial_i q)(\partial^i q) + \frac{9q_0^2}{16m_0^2}(\partial_i m)(\partial^i m) - \frac{3q_0}{2m_0}(\partial_i q)(\partial^i m)\right) - \frac{2q_0^4\kappa^2}{m_0}\left(\frac{\partial_i m}{4m_0} + \partial_i u^+\right)^2 \\ &+ \frac{(22m_0^3 - 44m_0^2 - 181m_0 + 237)}{24(m_0-3)m_0(16m_0^2)}(\partial^i m)(\partial_i m) - \frac{(6m_0^2 - 8m_0 - 39)q_0}{48(m_0-3)m_0^2}(\partial^i m)(\partial_i q) \\ &- \frac{(2m_0^3 - 3m_0^2 - 4m_0 + 12)}{24(m_0-3)m_0^2}(\partial^i m)(\partial_i u^+) - \frac{1}{12}(\partial^i q)(\partial_i q) + \frac{3q_0}{2(m_0-3)m_0}(\partial^i q)(\partial_i u^+) \\ &- \frac{(4m_0^2 - 2m_0 + 11)}{24m_0}(\partial^i u^+)(\partial_i u^+) - \frac{(2-m_0)}{12m_0}\sigma^{ij}\sigma_{ij} + \left(\frac{2q_0^4\kappa^2}{m_0} - \frac{q_0^2}{6}\right)\omega^{ij}\omega_{ij}. \end{aligned} \tag{C1}$$

Mass density is given by,

$$\begin{aligned} \rho &= 4m(u^+)^2 + \frac{2-m_0}{6m_0}\sigma^{ij}\sigma_{ij} + \frac{q_0^2}{3}\omega^{ij}\omega_{ij} + \frac{4q_0^4\kappa^2}{m_0}(l^i l_i - \omega^{ij}\omega_{ij}) + \frac{q_0}{6m_0}\partial_i\partial^i q - \frac{1}{2m_0}\partial_i\partial^i u^+ \\ &+ \frac{(2-m_0)}{8m_0^2}\partial_i\partial^i m - \frac{\log 2}{2}\partial_i q\partial_i q + \frac{3q_0\log 2}{4m_0}\partial_i q\partial_i m - \frac{9q_0^2\log 2}{32m_0^2}\partial_i m\partial_i m \\ &- \frac{m_0}{2\sqrt{4m_0-3}}\log\left(\frac{3-\sqrt{4m_0-3}}{3+\sqrt{4m_0-3}}\right)\left(\frac{1}{4m_0}(\partial_i m) - (\partial_i u^+)\right)^2 - \frac{3q_0}{2(m_0-3)m_0}\partial_i q\partial_i u^+ \\ &+ \frac{q_0}{4m_0}\frac{(-39-8m_0+6m_0^2)}{6m_0(m_0-3)}\partial_i q\partial_i m + \frac{1}{6}\partial_i q\partial_i q + \frac{(4-m_0+2m_0^2)}{6m_0}(\partial_i u^+)(\partial_i u^+) \\ &+ \frac{(123-92m_0-22m_0^2+11m_0^3)}{96m_0^3(m_0-3)}\partial_i m\partial_i m + \frac{(15-5m_0-6m_0^2+4m_0^3)}{24m_0^2(m_0-3)}\partial_i u^+\partial_i m \end{aligned} \tag{C2}$$

Nonrelativistic velocity has the following expression,

$$\begin{aligned} v_i &= \frac{u_i}{u^+} - \frac{1}{\rho}\left(\frac{3u^+(q^2-m) + (2m_0-3)u^+}{2(m_0-3)}\right)\left(\frac{\partial_i u^+}{u^+} - \frac{1}{4m}\partial_i m\right) + \frac{\sqrt{3}q_0^3\kappa}{4m_0^2}\epsilon_{ij}\partial_k\omega_{jk} \\ &+ \frac{\sqrt{3}q_0^3\kappa}{8m_0^3}\epsilon_{ij}(\partial_k m)\omega_{jk} + \frac{\sqrt{3}q_0^3\kappa}{2m_0^2}\epsilon_{ij}(\partial_k u^+)\omega_{jk} + \frac{(4m_0-1)}{16m_0^2}\partial_k\omega_{ik} + \frac{(2m_0-1)}{8m_0^2}\partial_k\sigma_{ik} \\ &- \frac{1}{8\sqrt{4m_0-3}}\log\left(\frac{3+\sqrt{4m_0-3}}{3-\sqrt{4m_0-3}}\right)\left(\partial_k\sigma_{ik} + \partial_k\omega_{ik} - (\partial_k u^+)\sigma_{ik} + \frac{1}{4m_0}(\partial_k m)\sigma_{ik}\right) \\ &+ \frac{1}{16m_0^2}\sigma_{ik}\left(\frac{q_0(m_0-6)}{(m_0-3)}(\partial_k q) - (\partial_k u^+) + \frac{(m_0^2+6m_0-9)}{4m_0(m_0-3)}(\partial_k m)\right) \\ &- \frac{1}{16m_0^2}\omega_{ik}\left(q_0(\partial_k q) + (1+2m_0-4m_0^2)(\partial_k u^+) - \frac{(4m_0^2-3m_0-2)}{4m_0}(\partial_k m)\right). \end{aligned} \tag{C3}$$

Energy density is given by

$$
\begin{aligned}
e = {} & m + \frac{1}{2}\rho v^2 - \frac{2q_0^4\kappa^2}{m_0}\omega^{ij}\omega_{ij} + \log(2)\left(\frac{1}{4}\partial_i q\partial_i q - \frac{3q_0}{8m_0}\partial_i q\partial_i m + \frac{9q_0^2}{64m_0^2}\partial_i m\partial_i m\right) \\
& - \frac{m_0}{4\sqrt{4m_0-3}}\log\left(\frac{3+\sqrt{4m_0-3}}{3-\sqrt{4m_0-3}}\right)(\partial_i u^+ - \frac{\partial_i m}{4m_0})^2 - \frac{q_0}{12m_0}\partial_i\partial_i q + \frac{1}{4m_0}\partial_i\partial_i u^+ \\
& - \frac{1}{12}\partial_i q\partial_i q + \frac{3q_0}{4m_0(m_0-3)}\partial_i q\partial_i u^+ + \frac{q_0(6m_0^2-8m_0-39)}{48m_0^2(m_0-3)}\partial_i q\partial_i m \\
& - \frac{(4m_0^2-2m_0+11)}{24m_0}\partial_i u^+\partial_i u^+ + \frac{q_0^4\kappa^2}{8m_0^3}\partial_i m\partial_i m + \frac{q_0^4\kappa^2}{m_0^2}\partial_i m\partial_i u^+ \\
& + \frac{(-12+4m_0+3m_0^2-2m_0^3)}{(24(-3+m_0)m_0^2)}\partial_i m\partial_i u^+ + \frac{(m_0-2)}{12m_0}\sigma_{ij}\sigma^{ij} + \frac{q_0^2}{6}\omega_{ij}\omega^{ij} \\
& + \frac{(-237+181m_0+44m_0^2-22m_0^3)}{(384(-3+m_0)m_0^3)}\partial_i m\partial_i m + \frac{2q_0^4\kappa^2}{m_0}(\partial_i u^+)(\partial^i u^+).
\end{aligned}
\tag{C4}
$$

Finally, the energy current is given by,

$$
\begin{aligned}
e^i = {} & \left(e + p + \frac{1}{2}\rho v^2\right)v^i + \frac{R^3}{u^+}\left(\frac{\partial_i u^+}{u^+} - \frac{\partial_i m}{4m}\right) - R^3\sigma^{ik}v^k - \frac{\sqrt{3}q_0^3\kappa}{m_0}\epsilon^{ik}\partial^j\omega^{kj} \\
& + \frac{q_0^3\kappa\sqrt{3}}{2m_0^2}\epsilon^{ij}\omega^{jk}\partial^k m - \frac{12q_0^4\kappa^2}{m_0}\omega^{ij}\left(\partial^j u^+ + \frac{1}{4m_0}\partial^j m\right) + \frac{(1-2m_0)}{2m_0}\partial^k\sigma^{ik} \\
& + \frac{m_0}{2\sqrt{4m_0-3}}\log\left(\frac{3-\sqrt{4m_0-3}}{3+\sqrt{4m_0-3}}\right)\left(\sigma^{ik}(\partial^k u^+) - \partial^k\sigma^{ik} - \partial^k\omega^{ik} - \frac{1}{4m_0}\sigma^{ik}(\partial^k m)\right) \\
& + \frac{(1-4m_0)}{4m-0}\partial^k\omega^{ik} - \frac{(m_0-6)q_0}{4m_0(m_0-3)}\sigma^{ik}(\partial^k q) - \frac{q_0}{4m_0}\omega^{ik}(\partial^k q) - \frac{2\sqrt{3}q_0^3\kappa}{m_0}\epsilon^{ij}\omega^{jk}\partial^k u^+ \\
& + \frac{(4m_0^2-2m_0-1)}{4m_0}\sigma^{ik}(\partial^k u^+) - \left(\frac{(m_0^2+5m_0-6)}{16m_0^2(m_0-3)}\sigma^{ik}(\partial^k m) - \frac{(4m_0^2-2-3m_0)}{16m_0^2}\omega^{ik}(\partial^k m)\right)
\end{aligned}
\tag{C5}
$$

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