Operator solution of the light-front Thirring-Wess model

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We present an operator solution of the Thirring-Wess model, formulated and quantized in terms of light-front (LF) variables. The model describes a system of massless fermions interacting with massive vector bosons in two space-time dimensions. An important ingredient of the solution is a consistent quantization of the two-dimensional massless LF fermion field. The field equations are solved exactly on an operator level and the quantum LF Hamiltonian is derived in terms of independent field variables. The axial anomaly and the interacting correlation functions are computed nonperturbatively from the operator solution. An analogous operator solution in the conventional field theory is briefly described for comparison. While in the LF case the "empty" Fock vacuum is the lowest-energy eigenstate of the full Hamiltonian, the corresponding Hamiltonian in the conventional theory has to be diagonalized in order to find the true physical ground state, which is a dynamical state with a complicated structure. A comment concerning a recently discussed equivalence between the LF and conventional form of field theory concludes the paper.

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I. INTRODUCTION

Quantum field theory (QFT) was originally formulated in the operator "language" using the "natural" parametrization of the space-time in terms of $x^\mu = (t, x, y, z)$. The path integral formalism was later developed based on the canonical formulation. A new branch of the operator (Hamiltonian) version of QFT was founded by Dirac who realized that, in the relativistic theory, there are actually three classes of initial (quantization) hypersurfaces with corresponding choices of the space-time variables [1]. The conventional field theory implies quantization on a spacelike (SL) hypersurface ($t = 0$ being the simplest choice) while the front form implies quantization on a lightlike surface (typically at $x^+ = t + z = 0$). An immediate question of the equivalence between the two versions of QFT emerges. While the notion of equivalence can have a few interpretations, the most relevant meaning appears to us as "predicting the same physical results." This does not imply that the theoretical mechanisms in the two schemes need to be the same. Actually the different mathematical structure of the two forms of the relativistic dynamics suggests different mechanisms. For example, even the structure of field variables [dynamical vs nondynamical (dependent) ones] does not always coincide in the two schemes.

The equivalence issue has been studied since the early days of light-front (LF) quantization. A formal equivalence of the two formulations at the perturbative S-matrix level was established in [2]. A perturbative analysis of the Yukawa model [3] leads to an explicit relation between the physical states in the two forms of the relativistic Hamiltonian dynamics: what appears as an empty Fock vacuum in the interacting LF case is actually a complicated mixture of many-particle states of the SL theory.

The equivalence problem has been analyzed recently in a series of papers by Mannheim, Brodsky, and Lowdon [4–6]. They found a possibility to rewrite quantities like Pauli-Jordan commutator functions (including their equal-time limits) from one form to another, among other things. They argued that because of a general coordinate transformation between the two frameworks they are unitarily equivalent. Their conclusion was that LF and equal-time quantizations are not fundamentally different but essentially represent the same scheme [5].

Some time ago, the area of exactly solvable models was proposed as a suitable testing ground for comparison of the two schemes by the present author [7]. Here we develop this approach further within a concrete dynamics, namely the Thirring-Wess model [8], which describes the interaction of massless fermions with massive vector bosons in $D = 1 + 1$. The model has been proposed independently by Brown [9] and studied also in [10–12]. Thirring and Wess used the method of auxiliary fields ("*Ansätze*") to find the operator solution. Lowenstein and Swieca gave an operator solution of the model [11] as a generalization of their solution of the Schwinger model and computed the corresponding Wightman functions. Brown used a "gauge-invariant" definition of a few operators to arrive at a complicated formulation of the model having no operator solution. The recent study [13] exploits functional and renormalization group methods and correctly predicts the chiral anomaly.

In the present paper, we demonstrate efficiency of the operator methods. We use a Hamiltonian LF formulation

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[14], which is straightforward and transparent. Our approach is based on the original field variables present in the Lagrangian. The key element is an explicit solution of the field equations at the quantum level, which is possible due to the very simple mathematical structure of this two-dimensional dynamics. The axial-vector anomaly and correlation functions of the interacting fields are computed from the operator solution in a straightforward way. A similar operator solution of the model in the conventional SL version of the theory will also be sketched, pointing out a necessity to diagonalize the corresponding SL Hamiltonian, unlike the LF treatment. The full SL treatment of the model is however postponed to a separate paper [15].

II. LF QUANTIZATION OF THE MASSLESS FERMION FIELD AND MASSIVE VECTOR-BOSON FIELD IN TWO DIMENSIONS

First, let us introduce our LF notation. The LF coordinate is $x^\mu = (x^+, x^-) = (x^0 + x^1, x^0 - x^1)$, while $k^\mu = (k^+, k^-)$ (or p^μ) denotes the momentum two-vector. We will use the symbols $\partial_\pm = \frac{\partial}{\partial x^\pm}$, $\hat{k}^- = \frac{\mu^2}{k^+}$, $\hat{k}\cdot x = \frac{1}{2}k^+x^- + \frac{1}{2}\hat{k}^-x^+$. k^+ is the LF momentum and $\hat{k}^-$ the on-shell LF energy. There is no sign ambiguity analogous to $E(k^1) = \pm\sqrt{(k^1)^2 + \mu^2}$ of the SL theory and both k^+, k^- can be taken positive. As for the two-dimensional spinor field $\psi(x)$, we shall work in chiral representation, in which $\gamma^5 = \gamma^0\gamma^1$ is diagonal. The upper component of $\psi(x)$ is then $\psi_1(x)$, and the lower one $\psi_2(x)$. The two-dimensional gamma matrices are $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$, with σ^i being the Pauli matrices.

The two-dimensional massless LF fermion field has the Fock representation [16,17]

$$\psi_2(x^-) = \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi}}\left[b(p^+)e^{-\frac{i}{2}p^+x^-} + d^\dagger(p^+)e^{\frac{i}{2}p^+x^-}\right], \quad (1)$$

$$\{b(p^+), b^\dagger(q^+)\} = \{d(p^+), d^\dagger(q^+)\} = \delta(p^+ - q^+), \quad (2)$$

$$\psi_1(x^+) = \int_0^{+\infty} \frac{dp^-}{\sqrt{4\pi}}\left[\tilde{b}(p^-)e^{-\frac{i}{2}p^-x^+} - \tilde{d}^\dagger(p^-)e^{\frac{i}{2}p^-x^+}\right], \quad (3)$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^-)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^-)\} = \delta(p^- - q^-). \quad (4)$$

These fields satisfy the two-dimensional massless LF Dirac equation $\partial_+\psi_2 = 0$, $\partial_-\psi_1 = 0$ following from the Lagrangian $\mathcal{L} = i/2\bar{\psi}\gamma^\mu\overset{\leftrightarrow}{\partial}_\mu\psi = i\psi_2^\dagger\overset{\leftrightarrow}{\partial}_+\psi_2 + i\psi_1^\dagger\overset{\leftrightarrow}{\partial}_-\psi_1$ and are obtained as massless limits of the corresponding massive fields [16,17]. The two-point functions calculated from $\psi_1(x^+)$ and $\psi_2(x^-)$ coincide with the massless limits of the two-point functions of the massive fields $\psi_1(x^+, x^-)$ and $\psi_2(x^+, x^-)$. This confirms consistency of the proposed quantization scheme.

To derive the representations (1) and (3), one starts from the massive fields

$$\psi_2(x) = \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi}}\left[b(p^+)e^{-\frac{i}{2}p^+x^- - \frac{im^2}{2p^+}x^+} + d^\dagger(p^+)e^{\frac{i}{2}p^+x^- + \frac{im^2}{2p^+}x^+}\right], \quad (5)$$

$$\psi_1(x) = \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi}}\frac{m}{p^+}\left[b(p^+)e^{-\frac{i}{2}p^+x^- - \frac{im^2}{2p^+}x^+} - d^\dagger(p^+)e^{\frac{i}{2}p^+x^- + \frac{im^2}{2p^+}x^+}\right], \quad (6)$$

$$\{b(p^+), b^\dagger(q^+)\} = \{d(p^+), d^\dagger(q^+)\} = \delta(p^+ - q^+), \quad (7)$$

which are the solution of the two-dimensional massive Dirac equations $2i\partial_+\psi_2 = m\psi_1$, $2i\partial_-\psi_1 = m\psi_2$. The $m = 0$ limit of the first, dynamical equation, directly yields (1). To obtain (3), one changes the variables as $p^+ = m^2/p^-$, leading to [18]

$$\psi_1(x) = \int_0^{+\infty} \frac{dp^-}{\sqrt{4\pi}}\frac{m}{p^-}\left[b\left(\frac{m^2}{p^-}\right)e^{-\frac{im^2}{2p^-}x^- - \frac{i}{2}p^-x^+} - d^\dagger\left(\frac{m^2}{p^-}\right)e^{\frac{im^2}{2p^-}x^- + \frac{i}{2}p^-x^+}\right]. \quad (8)$$

The same change of variables in the Fock anticommutation relation (7) gives

$$\left\{b\left(\frac{m^2}{p^-}\right), b^\dagger\left(\frac{m^2}{q^-}\right)\right\} = \delta\left(\frac{m^2}{p^-} - \frac{m^2}{q^-}\right) = \frac{p^-q^-}{m^2}\delta(p^- - q^-). \quad (9)$$

This implies that the operators $\frac{m}{p^-}b(\frac{m^2}{p^-})$ and $\frac{m}{p^-}d(\frac{m^2}{p^-})$ in (8) satisfy the anticommutation relation with mass-independent right-hand side equal to $\delta(p^- - q^-)$. In other words, in analogy to the $\psi_2(x)$ field, we can take the massless limit in these operators with the result

$$\{\tilde{b}(p^-),\tilde{b}^\dagger(q^-)\} = \{\tilde{d}(p^-),\ \tilde{d}^\dagger(q^-)\} = \delta(p^- - q^-),$$
$$\tilde{b}(p^-) \equiv \lim_{m\to 0}\frac{m}{p^-} b\left(\frac{m^2}{p^-}\right),\ \tilde{d}(p^-) \equiv \lim_{m\to 0}\frac{m}{p^-} d\left(\frac{m^2}{p^-}\right). \tag{10}$$

Performing the above change of variables in the free massive LF Hamiltonian

$$P^- = m\int_{-\infty}^{+\infty} dx^-\left[\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2\right],$$
$$P^- = \int_0^{+\infty} dp^+ \frac{m^2}{p^+}\left[b^\dagger(p^+)b(p^+) + d^\dagger(p^+)d(p^+)\right], \tag{11}$$

one finds

$$P^- = \int_0^{+\infty} dp^- p^- \left[\left(\frac{m}{p^-}\right)^2 b^\dagger\left(\frac{m^2}{p^-}\right) b\left(\frac{m^2}{p^-}\right) + \left(\frac{m}{p^-}\right)^2 d^\dagger\left(\frac{m^2}{p^-}\right) d\left(\frac{m^2}{p^-}\right)\right],$$

leading to the massless limit

$$P_0^- = \int_0^{+\infty} dp^- p^- \left[\tilde{b}^\dagger(p^-)\tilde{b}(p^-) + \tilde{d}^\dagger(p^-)\tilde{d}(p^-)\right]. \tag{12}$$

In analogy to the SL massless relation $k^0 = |k^1|$ we have here $k^- = k^+$. Based on the above Fock anticommutation relations, the massless field anticommutators acquire the form

$$\{\psi_1(x^+),\psi_1^\dagger(y^+)\} = \delta(x^+ - y^+), \tag{13}$$
$$\{\psi_2(x^-),\psi_2^\dagger(y^-)\} = \delta(x^- - y^-). \tag{14}$$

The free current $j^\mu(x)$ built from the fields (1) and (3)

$$j^+(x^-) = 2{:}\psi_2{}^\dagger(x^-)\psi_2(x^-){:}, \tag{15}$$
$$j^-(x^+) = 2{:}\psi_1{}^\dagger(x^+)\psi_1(x^+){:} \tag{16}$$

can be bosonized by a Fourier transformation:

$$j^+(x^-) = -\frac{i}{\sqrt{\pi}}\int_0^\infty \frac{dk^+k^+}{\sqrt{4\pi k^+}}[c(k^+)e^{-\frac{i}{2}k^+x^-} - \text{H.c.}], \tag{17}$$

$$[c(k^+), c^\dagger(l^+)] = \delta(k^+ - l^+),$$

$$j^-(x^+) = -\frac{i}{\sqrt{\pi}}\int_0^\infty \frac{dk^-k^-}{\sqrt{4\pi k^-}}[\tilde{c}(k^-)e^{-\frac{i}{2}k^-x^+} - \text{H.c.}], \tag{18}$$

$$[\tilde{c}(k^-), \tilde{c}^\dagger(l^-)] = \delta(k^- - l^-).$$

The boson operators $c(k^+)$, $\tilde{c}(k^-)$ are bilinear in the fermion Fock operators present in (1) and (3) [16,17]:

$$c(k^+) = i\int_0^\infty \frac{dq^+}{2\sqrt{q^+}}\left[b^\dagger(q^+)b(k^+ + q^+) - d^\dagger(q^+)d(k^+ + q^+) + d(q^+)b(k^+ - q^+)\theta(k^+ - q^+)\right], \tag{19}$$

$$\tilde{c}(k^-) = i\int_0^\infty \frac{dq^-}{2\sqrt{q^-}}\left[\tilde{b}^\dagger(q^-)\tilde{b}(k^- + q^-) - \tilde{d}^\dagger(q^-)\tilde{d}(k^- + q^-) + \tilde{d}(q^-)\tilde{b}(k^- - q^-)\theta(k^- - q^-)\right]. \tag{20}$$

The commutators of the current components contain the Schwinger terms,

$$[j^+(x^-), j^+(y^-)] = \frac{2i}{\pi}\partial_{x^-}\delta(x^- - y^-), \qquad [j^-(x^+), j^-(y^+)] = \frac{2i}{\pi}\partial_{x^+}\delta(x^+ - y^+).$$

The free LF vector-meson field is expanded as

$$B^+(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \frac{k^+}{\mu_0} \left[a(k^+)e^{-i\hat{k}\cdot x} + a^\dagger(k^+)e^{i\hat{k}\cdot x}\right], \quad (21)$$

$$B^-(x) = -\int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \frac{\hat{k}^-}{\mu_0} \left[a(k^+)e^{-i\hat{k}\cdot x} + a^\dagger(k^+)e^{i\hat{k}\cdot x}\right].$$

The two components satisfy the condition

$$\partial_\mu B^\mu(x) = \partial_+ B^+(x) + \partial_- B^-(x) = 0 \quad (22)$$

following from the antisymmetry of the $G^{\mu\nu}$ tensor in the free field equation

$$\partial_\mu G^{\mu\nu} + \mu_0^2 B^\nu = 0, \qquad G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu, \quad (23)$$

which follow from the Lagrangian

$$\mathcal{L}_B = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2}\mu_0^2 B_\mu B^\mu, \\ = \frac{1}{2}(\partial_+ B^+ - \partial_- B^-)^2 + \frac{1}{2}\mu_0^2 B^+ B^-. \quad (24)$$

The equal-LF time commutation relations are

$$\left[B^+(x^+, x^-), \Pi^-(x^+, y^-)\right] = ig^{+-}\delta(x^- - y^-), \quad (25)$$

$$\left[B^+(x^+, x^-), 2\partial_+ B^+(x^+, y^-)\right] = i\delta(x^- - y^-), \quad (26)$$

$$\Pi^- = 2(\partial_+ B^+ - \partial_- B^-) = 4\partial_+ B^+, \quad (27)$$

where $g^{\mu\nu}$ is the metric tensor with $g^{+-} = 2$. The expansion (21) leads to the correct form of the energy and momentum operators

$$P_B^- = \int_0^\infty dk^+ \frac{\mu_0^2}{k^+} a^\dagger(k^+)a(k^+), \quad (28)$$

$$P_B^+ = \int_0^\infty dk^+ k^+ a^\dagger(k^+)a(k^+). \quad (29)$$

III. THE LF THIRRING-WESS MODEL

The dynamics of the model is characterized by the covariant-form Lagrangian

$$\mathcal{L} = \frac{i}{2}\bar{\Psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \Psi - \frac{1}{4}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} + \frac{1}{2}\mu^2 \tilde{B}_\mu \tilde{B}^\mu - e\tilde{B}_\mu J^\mu, \quad (30)$$

where

$$\tilde{G}_{\mu\nu} = \partial_\mu \tilde{B}_\nu - \partial_\nu \tilde{B}_\mu, \qquad J^\mu(x) = \bar{\Psi}(x)\gamma^\mu \Psi(x). \quad (31)$$

$\Psi(x)$ and $\tilde{B}^\mu(x)$ are the interacting massless fermion and massive vector fields, respectively. The solvability of the theory means that one can find an operator solution of the coupled system of the Dirac and Proca equations:

$$i\gamma^\mu \partial_\mu \Psi = e\gamma_\mu \tilde{B}^\mu \Psi, \qquad \partial_\mu \tilde{G}^{\mu\nu} + \mu^2 \tilde{B}^\nu = eJ^\nu. \quad (32)$$

For the conserved current $J^\mu(x)$ the vector field satisfies the operator relation

$$\partial_\mu \tilde{B}^\mu(x) = \partial_+ \tilde{B}^+(x^+, x^-) + \partial_- \tilde{B}^-(x^+, x^-) = 0. \quad (33)$$

The LF form of the Lagrangian is

$$\mathcal{L} = i\Psi_2^\dagger \overleftrightarrow{\partial_+} \Psi_2 + i\Psi_1^\dagger \overleftrightarrow{\partial_-} \Psi_1 + \frac{1}{2}(\partial_+ \tilde{B}^+ - \partial_- \tilde{B}^-)^2 \\ + \frac{1}{2}\mu^2 \tilde{B}^+ \tilde{B}^- - \frac{e}{2}\tilde{B}^+ J^- - \frac{e}{2}\tilde{B}^- J^+, \quad (34)$$

with the corresponding coupled LF field equations

$$2i\partial_+ \Psi_2(x) = e\tilde{B}^-(x)\Psi_2(x), \quad (35)$$

$$2i\partial_- \Psi_1(x) = e\tilde{B}^+(x)\Psi_1(x), \quad (36)$$

$$(4\partial_+\partial_- + \mu^2)\tilde{B}^+(x) = eJ^+(x), \quad (37)$$

$$(4\partial_+\partial_- + \mu^2)\tilde{B}^-(x) = eJ^-(x). \quad (38)$$

The solution of the Dirac equation involves the free massless LF fermion field components $\psi_1(x^+)$ and $\psi_2(x^-)$:

$$\Psi_1(x) = e^{-\frac{ie}{2}\int_{-\infty}^{+\infty} dy^- \frac{1}{2}\epsilon(x^- - y^-)\tilde{B}^+(x^+, y^-)} \psi_1(x^+), \quad (39)$$

$$\Psi_2(x) = e^{-\frac{ie}{2}\int_{-\infty}^{+\infty} dy^+ \frac{1}{2}\epsilon(x^+ - y^+)\tilde{B}^-(y^+, x^-)} \psi_2(x^-). \quad (40)$$

$\epsilon(x^\pm)$ is the sign function, $\partial_\pm \epsilon(x^\pm) = 2\delta(x^\pm)$. The nontrivial property of the set of coupled field equations (35)–(38) is that the Proca equation (37) for $\tilde{B}^+$ contains the current J^+ which is constructed from the interacting field Ψ_2 depending on $\tilde{B}^-$, and analogously for the second Proca equation (38). This mixing complicates the solution, which however is still possible due to the relation (33). The interacting currents appearing in the Proca equations can be determined from the solutions (39), (40) (or rather their quantum version, see below) by means of the point-split regularized definition

$$J^+(x) = \lim_{\epsilon \to 0}\left[\Psi_2^\dagger\left(x + \frac{\epsilon}{2}\right)\Psi_2\left(x - \frac{\epsilon}{2}\right) + \text{H.c.}\right], \quad (41)$$

$$J^-(x) = \lim_{\epsilon \to 0}\left[\Psi_1^\dagger\left(x + \frac{\epsilon}{2}\right)\Psi_1\left(x - \frac{\epsilon}{2}\right) + \text{H.c.}\right], \quad (42)$$

Using the free-field relations

$$\psi_2{}^\dagger\left(x^- + \frac{\epsilon^-}{2}\right)\psi_2\left(x^- - \frac{\epsilon^-}{2}\right) = :\psi_2{}^\dagger(x^-)\psi_2(x^-): - V(\epsilon^-),$$
$$\psi_1{}^\dagger\left(x^+ + \frac{\epsilon^+}{2}\right)\psi_1\left(x^+ - \frac{\epsilon^+}{2}\right) = :\psi_1{}^\dagger(x^+)\psi_1(x^+): - V(\epsilon^+),$$
$$V(\epsilon^\pm) = \frac{i}{2\pi}\frac{1}{\epsilon^\pm - i\eta}, \tag{43}$$

one arrives at

$$J^+(x) = j^+(x^-) - \frac{e}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}y^+ \epsilon(x^+ - y^+)\partial_- \tilde{B}^-(y^+, x^-), \tag{44}$$

$$J^-(x) = j^-(x^+) - \frac{e}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}y^- \epsilon(x^- - y^-)\partial_+ \tilde{B}^+(x^+, y^-), \tag{45}$$

where the free current $j^\pm$ is given in (15), (16). Taking into account the relation (33) and performing partial integration, the current $J^\pm$ is seen to be equal to the (normal-ordered) free current plus a quantum correction,

$$J^+(x) = j^+(x^-) - \frac{e}{\pi}\tilde{B}^+(x^+, x^-), \tag{46}$$

$$J^-(x) = j^-(x^+) - \frac{e}{\pi}\tilde{B}^-(x^+, x^-). \tag{47}$$

The remarkable fact is that now the $\tilde{B}^\pm$ part of the current matches the structure of the Proca equations (37), (38), which as a consequence can also be solved exactly (see below). Conservation of the interacting quantum current is tantamount to the condition $\partial_\mu \tilde{B}^\mu = 0$ as one anticipated on the classical level:

$$\partial_+ J^+(x) + \partial_- J^-(x) = -\frac{e}{\pi}(\partial_+ \tilde{B}^+ + \partial_- \tilde{B}^-) = 0, \tag{48}$$

while the divergence of the axial-vector current

$$J_5^\mu(x) = \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) = (J^+(x), -J^-(x)) \tag{49}$$

is nonzero ("anomalous"),

$$\partial_\mu J_5^\mu = -\frac{e}{\pi}(\partial_+ \tilde{B}^+ - \partial_- \tilde{B}^-) = \frac{e}{2\pi}\epsilon_{\mu\nu}\tilde{G}^{\mu\nu}. \tag{50}$$

$\epsilon^{\mu\nu}$ is the antisymmetric tensor. One can see that the axial anomaly is a purely quantum effect, obtained here nonperturbatively from the operator solution of the field equations.

There is a subtlety in the above considerations. Partial integrations in the relations (44) and (45) work only for x^--dependent or x^+-dependent quantities, respectively. In other words, the most general form of $\tilde{B}^+(x)$ can be a combination of $B^+(x^+, x^-)$ and some x^+-independent quantity $b^+(x^-)$, and the most general form of $\tilde{B}^-(x)$ can be a combination of $B^-(x^+, x^-)$ and some x^--independent quantity $b^-(x^+)$. This implies that instead of $\tilde{B}^\pm(x)$ in Eqs. (46) and (47) we have to consider only $B^\pm(x)$, because $b^+(x^-)$ and $b^-(x^+)$ got lost in the partial integration. Therefore, the correct form of the anomaly relation (50) actually reads

$$\partial_\mu J_5^\mu = -\frac{e}{\pi}(\partial_+ B^+ - \partial_- B^-) = \frac{e}{2\pi}\epsilon_{\mu\nu}G^{\mu\nu}. \tag{51}$$

Now, we can insert the currents

$$J^+(x) = j^+(x^-) - \frac{e}{\pi}B^+(x^+, x^-), \tag{52}$$

$$J^-(x) = j^-(x^+) - \frac{e}{\pi}B^-(x^+, x^-) \tag{53}$$

to the Proca equations to obtain

$$(4\partial_+\partial_- + \mu^2)(B^+(x) + b^+(x^-)) = ej^+(x^-) - \frac{e^2}{\pi}B^+(x), \tag{54}$$

$$(4\partial_+\partial_- + \mu^2)(B^-(x) + b^-(x^+)) = ej^-(x^+) - \frac{e^2}{\pi}B^-(x). \tag{55}$$

Collecting the terms, we get

$$(4\partial_+\partial_- + \tilde{\mu}^2)B^\nu(x) = 0, \qquad \mu^2 + \frac{e^2}{\pi} \equiv \tilde{\mu}^2 \tag{56}$$

and

$$\mu^2 b^+(x^-) = ej^+(x^-), \qquad \mu^2 b^-(x^+) = ej^-(x^+). \tag{57}$$

Hence the interacting vector field is given by

$$\tilde{B}^+(x) = B^+(x^+, x^-) + \frac{e}{\mu^2}j^+(x^-), \tag{58}$$

$$\tilde{B}^-(x) = B^-(x^+, x^-) + \frac{e}{\mu^2}j^-(x^+). \tag{59}$$

The vector-boson mass μ got renormalized to the value $\tilde{\mu} = (\mu^2 + e^2/\pi)^{1/2}$ as given in Eq. (56). In this way, we have achieved a complete quantum solution of the coupled field equations of the model. The interacting fields are expressed solely in terms of the free fields $\psi_1(x^+)$, $\psi_2(x^-)$, $B^+(x^+, x^-)$, $B^-(x^+, x^-)$. Because of this, the solutions (39), (40) can be regularized on the quantum level by normal ordering:

$$\Psi_1(x) = e^{-\frac{ie}{2}F_1^{(-)}(x^+,x^-)} e^{-\frac{ie}{2}F_1^{(+)}(x^+,x^-)} \psi_1(x^+), \qquad (60)$$

$$\Psi_2(x) = e^{-\frac{ie}{2}F_2^{(-)}(x^+,x^-)} e^{-\frac{ie}{2}F_2^{(+)}(x^+,x^-)} \psi_2(x^-), \qquad (61)$$

where the fields were decomposed into the positive and negative frequency parts,

$$F_1^{(\pm)}(x^+,x^-) = \int_{-\infty}^{+\infty} \mathrm{d}y^- \frac{1}{2}\epsilon(x^- - y^-)\Big[B^{+(\pm)}(x^+,y^-) + \frac{e}{\mu^2} j^{+(\pm)}(y^-)\Big], \qquad (62)$$

$$F_2^{(\pm)}(x^+,x^-) = \int_{-\infty}^{+\infty} \mathrm{d}y^+ \frac{1}{2}\epsilon(x^+ - y^+)\Big[B^{-(\pm)}(y^+,x^-) + \frac{e}{\mu^2} j^{-(\pm)}(y^+)\Big]. \qquad (63)$$

The fermion field solutions (39), (40) and the vector field solutions (58), (59) can be used to reexpress the starting Lagrangian (34) in terms of the free fields, which are the true field degrees of freedom of the model:

$$\mathcal{L}' = i\psi_2^\dagger \overleftrightarrow{\partial}_+ \psi_2 + i\psi_1^\dagger \overleftrightarrow{\partial}_- \psi_1 + \frac{1}{2}(\partial_+ B^+ - \partial_- B^-)^2 + \frac{1}{2}\tilde{\mu}^2 \left(B^+ + \frac{e}{\mu^2} j^+\right)\left(B^- + \frac{e}{\mu^2} j^-\right). \qquad (64)$$

Calculating the conjugate momenta, one derives the LF Hamiltonian $\hat{P}^- = P^-_{\mathrm{free}} + P^-_{\mathrm{int}}$ as

$$P^-_{\mathrm{free}} = \lim_{m\to 0} \frac{1}{2}\int_{-\infty}^{+\infty} dx^- \Big[m({\psi_2}^\dagger \psi_1 + {\psi_1}^\dagger \psi_2) - \tilde{\mu}^2 B^+ B^-\Big],$$

$$P^-_{\mathrm{int}} = -\frac{e}{2}\int_{-\infty}^{+\infty} dx^- \Big[(B^+ j^- + B^- j^+) + e\frac{\tilde{\mu}^2}{\mu^4} j^+ j^-\Big]. \qquad (65)$$

P^-_{free} represents the free LF energy of massless fermions and massive bosons. Their Fock form is given by Eqs. (12) and (28). The first interacting term vanishes because $j^-(x^+)$ is a constant with respect to the integration variable x^- and the zero mode $B^+_0(x^+)$ (the integral of $B^+(x^+,x^-)$ over the infinite interval x^-) of the massive B^+ field vanishes [19]. The Fock form of P^-_{int} is obtained by inserting the expansions (17), (18) and (21) to the formula (65):

$$\hat{P}^- = \int_0^\infty dp^- p^- (\tilde{b}^\dagger(p^-)\tilde{b}(p^-) + \tilde{d}^\dagger(p^-)\tilde{d}(p^-)) + \int_0^\infty dk^+ \frac{\tilde{\mu}^2}{k^+} a^\dagger(k^+) a(k^+)$$
$$+ \frac{ie\tilde{\mu}}{\sqrt{4\pi}} \int_0^\infty \frac{dk^+}{k^+}\Big[a^\dagger(k^+)\tilde{c}(k^+) - \tilde{c}^\dagger(k^+)a(k^+)\Big] + \frac{i}{2\pi} e^2 \frac{\tilde{\mu}^2}{\mu^4} Q \int_0^\infty dk^+ \sqrt{k^+}\Big[\tilde{c}(k^+) - \tilde{c}^\dagger(k^+)\Big]. \qquad (66)$$

The last term is proportional to Q times $j^-(x^+)$ because only j^+ depends on the integration variable leading to the total charge Q (which commutes with j^-). The $B^- j^+$ term does not contain the $c^\dagger(k^+)a^\dagger(k^+)$ terms due to positivity of k^+ and hence the Fock vacuum is the exact eigenstate of the full LF Hamiltonian.

From the operator solution, the correlation functions can be directly computed. For example,

$$\langle 0|\Psi_2(x){\Psi_2}^\dagger(y)|0\rangle = \langle 0| e^{-\frac{ie}{2}F_2^{(+)}(x)} \psi_2(x^-){\psi_2}^\dagger(y^-) e^{\frac{ie}{2}F_2^{(-)}(y)}|0\rangle$$
$$= \exp\left(\frac{e^2}{\mu^2} D^{(+)}(x-y) - \frac{i}{4\pi}\frac{e^4}{\mu^4}\varepsilon(x^+ - y^+)\right) S^{(+)}_{22}(x^- - y^-). \qquad (67)$$

Here the two-point functions of the free fields are

$$D_1^{(+)}(x-y) = \langle 0|B^+(x)B^+(y)|0\rangle = \int_0^{+\infty} \frac{dk^+ k^+}{4\pi\mu^2} e^{-i\hat{k}\cdot(x-y)} = -\frac{4}{\mu^2}\partial_{x^-}^2 D^{(+)}(x-y), \tag{68}$$

$$S_{22}^{(+)}(x-y) = \langle 0|\psi_2(x^-){\psi_2}^\dagger(y^-)|0\rangle = -\frac{i}{2\pi}\frac{1}{x^- - y^- - i\epsilon}. \tag{69}$$

$D^{(+)}(x)$ is the two-point function of the massive scalar field, equal to $(2\pi)^{-1}K_0(\mu\sqrt{-x^2})$ for $x^2 < 0$, with K_0 being the modified Bessel function. Presence of $D^{(+)}(x)$ in the final result is a consequence of commuting $F_2^{(+)}(x)$ and $F_2^{(-)}(y)$ factors in the exponential, along with the annihilation property $F_2^{(+)}(x)|0\rangle = 0$. Similarly,

$$\langle 0|\Psi_1(x){\Psi_1}^\dagger(y)|0\rangle = \langle 0|e^{-\frac{ie}{2}F_1^{(+)}(x)}\psi_1(x^+){\psi_1}^\dagger(y^+)e^{\frac{ie}{2}F_1^{(-)}(y)}|0\rangle = \exp\left(\frac{e^2}{\mu^2}D^{(+)}(x-y) - \frac{i}{4\pi}\frac{e^4}{\mu^4}\varepsilon(x^- - y^-)\right)S_{11}^{(+)}(x^+ - y^+), \tag{70}$$

where

$$D_2^{(+)}(x-y) = \langle 0|B^-(x)B^-(y)|0\rangle = \frac{\mu^2}{4\pi}\int_{-\infty}^{+\infty}\frac{dk^+}{k^{+3}}e^{-i\hat{k}\cdot(x-y)} = -\frac{4}{\mu^2}\partial_{x^+}^2 D^{(+)}(x-y), \tag{71}$$

$$S_{11}^{(+)}(x-y) = \langle 0|\psi_1(x^+){\psi_1}^\dagger(y^+)|0\rangle = -\frac{i}{2\pi}\frac{1}{x^+ - y^+ - i\epsilon}. \tag{72}$$

It is instructive to briefly compare the LF solution of the Thirring-Wess model with the SL solution obtained in the usual or "instant-form" field theory. The full treatment will be published separately [15]. The SL analysis proceeds in analogy to the LF case. The operator solutions of the coupled field equations are expressed in terms of the SL free fields $B^0(x), B^1(x)$, and $\psi(x)$. The interacting vector current is found to be conserved while the divergence of the interacting axial-vector current is "anomalous" in full agreement with the LF result (51). Expressing the Lagrangian (30) in terms of the true dynamical variables, i.e., the free fields, one arrives at the Hamiltonian

$$H = H_0 + H_{\rm int}, \tag{73}$$

$$H_{\rm int} = -\gamma\mu^2\int dx^1\left[B^0 j^0 - B^1 j^1 + \frac{1}{2}\gamma(j^0 j^0 - j^1 j^1)\right],$$
$$\gamma \equiv \frac{e}{\tilde{\mu}^2}, \qquad \tilde{\mu}^2 \equiv \mu^2 + \frac{e^2}{\pi}. \tag{74}$$

H_0 is the free Hamiltonian of the massless fermion field and the massive vector field. The interacting Hamiltonian $H_{\rm int}$ has a structure similar to the LF Hamiltonian (65), but there is an essential difference: the Hamiltonian (74) of the SL theory contains terms composed solely from creation or annihilation Fock operators of the type $f(k^1)a^\dagger(k^1)c^\dagger(-k^1)$ and $g(k^1)c^\dagger(k^1)c^\dagger(-k^1)$, and thus the Fock vacuum is not its eigenstate. One has to diagonalize $H_{\rm int}$ (74) to find its lowest-energy eigenstate. In the present two-dimensional model with simple dynamics this is possible by a Bogoliubov-type of transformation [15]. However, this also shows explicitly the main difference between the LF and SL forms of quantum field theories that hardly can be considered identical in the interacting case (cf. the conclusion of Ref. [5] that "the LF quantization is the same as the equal-time quantization"). It is true that the equal-t and equal-x^+ commutators can be obtained as different projections from the same (Lorentz-invariant) Pauli-Jordan (or Schwinger) commutator function, but there are other properties that define the "equal-time" quantization scheme on the one hand and the light-front framework on the other one. In particular, status of the vacuum state is fundamentally different in the two schemes. Note also that in the usual canonical treatment the interacting Hamiltonian would simply be $e(j^0B^0 - j^1B^1)$ while our treatment, making use of the knowledge of the operator solution of the Heisenberg field equations, reveals a more complex structure. As in the LF case, this is due to the operator mixing induced by the solution of the coupled field equations, in particular $\tilde{B}^\mu(x) = B^\mu(x) - \frac{e}{\tilde{\mu}^2}j^\mu(x)$.

IV. SUMMARY AND CONCLUSIONS

We have presented an operator solution of the Thirring-Wess model in the light-front form of field theory.

The key ingredient was the possibility to explicitly solve the coupled Dirac and Proca field equations in the two-dimensional space-time, a consistent quantization scheme for two-dimensional massless LF fermion field and a careful definition of the interacting fermionic currents regularized by point splitting. The correct value of the axial-vector "anomaly" as well as the interacting two-point functions were obtained from the operator solution. The LF Hamiltonian was derived and compared to the Hamiltonian of the conventional theory. The main difference between the two forms of the relativistic dynamics is nicely illustrated in the studied model: the Fock vacuum is the physical vacuum in the LF case, while in the SL treatment, the vacuum state has to be found in some nonperturbative calculation.

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[18] The discussion here is a little heuristic. Strictly speaking, these manipulations should better be done on the classical level with $b(p^+)$ and $d(p^+)$ being ordinary functions (Fourier amplitudes), satisfying Poisson brackets, which are replaced by quantum anticommutators after taking the massless limits.
[19] This is explicitly seen in a finite-interval treatment $-L \leq x^- \leq L$ with field periodic in x^-, where the field equation in the zero-mode sector is $\mu^2 B_0^+ = 0$, implying $B_0^+ = 0$.