

Probing strong field $f(R)$ gravity and ultradense matter with the structure and thermal evolution of neutron stars

Martin Nava-Callejas^{*} and Dany Page

*Instituto de Astronomía, Universidad Nacional Autónoma de México,
Ciudad de México, CDMX 04510, Mexico*

Mikhail V. Beznogov

*National Institute for Physics and Nuclear Engineering (IFIN-HH), RO-077125 Bucharest, Romania
and Instituto de Astronomía, Universidad Nacional Autónoma de México,
Ciudad de México, CDMX 04510, Mexico*



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Thermal evolution of neutron stars is studied in the $f(R) = R + \alpha R^2$ theory of gravity. We first review the equations of stellar structure and evolution for a spherically symmetric spacetime plus a perfect fluid at rest. We then present numerical results for the structure of neutron stars using four nucleonic dense matter equations of state and a series of gravity theories for α ranging from zero, i.e., general relativity, up to $\alpha \approx 10^{16} \text{ cm}^2$. We emphasize properties of these neutron star models that are of relevance for their thermal evolution as the threshold masses for enhanced neutrino emission by the direct Urca process, the proper volume of the stellar cores where this neutrino emission is allowed, the crust thickness, and the surface gravitational acceleration that directly impact the observable effective temperature. Finally, we numerically solve the equations of thermal evolution and explicitly analyze the effects of altering gravity. We find that uncertainties in the dense matter microphysics, such as the core chemical composition and superfluidity/superconductivity properties, as well as the astrophysical uncertainties on the chemical composition of the surface layers, have a much stronger impact than possible modifications of gravity within the studied family of $f(R)$ theories. We conclude that within this family of gravity theories, conclusions from previous studies of neutron star thermal evolution are not significantly altered by modification of gravity theory. Conversely, this implies that neutron star cooling modeling may not be a useful tool to constrain deviations of gravity from Einstein theory unless these are much more radical than in the $f(R) = R + \alpha R^2$ framework.

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I. INTRODUCTION

Neutron stars contain one of the densest forms of matter and the strongest gravitational fields in the observable Universe. Possibly higher densities have been reached in relativistic heavy ion collisions but only in a short-lived unstable form and with extremely high energy densities [1], while the densest form of matter in the interior of black holes is not observable. Neutron stars can also be copious neutrino emitters [2,3] and some exhibit the strongest known magnetic fields [4]. While the strongest gravitational potentials are found near the horizon of black holes, higher curvatures are present within neutron stars due to their smaller radii [5]. As such, they are prime candidates to study the interplay between the four fundamental forces of Nature.

Immediately after A. Einstein presented his final formulation of the field equations of general relativity (GR) in

1915 [6,7], D. Hilbert showed that these can be very elegantly deduced from a minimal action principle assuming a Lagrangian density given by the Ricci scalar R [8]. Such formulation would pave the way for exploring alternative theories of gravitation to GR following a simple, albeit powerful, formalism. For instance, the idea of replacing R with a more general function $f(R)$ can be traced back to H. Weyl who already in 1918 proposed the addition of an R^2 term to the Lagrangian density [9], while keeping $g_{\mu\nu}$ as the sole dynamic field generating gravity. Removing this latter assumption proved to be a theoretically stimulating alternative to GR, possibly the most famous example being that of Jordan-Brans-Dicke scalar-tensor theory [10] (for a more complete story, see Ref. [11]). From a purely theoretical point of view, exploring alternate theories of gravitation emerge from the attempts of merging gravity and quantum field theory. A hint suggesting going beyond GR is necessary for this task can be found, for instance, in the relative success of incorporating higher-order powers of the scalar curvature R

^{*}mnava@astro.unam.mx

in the Lagrangian density [12,13], although this approach is still not exempt of limitations such as renormalization or the physical scales of importance for these higher-order terms [14–16]. From an experimental/observational point of view probing gravity can be ordered from the so-called *strong gravitational regime*, characterized by large values of $G\langle M\rangle/c^2\langle r\rangle$ (e.g. ~ 0.17 for neutron stars) to systems exhibiting a low value belonging to the *weak gravitational regime* (e.g. $\sim 10^{-6}$ for the Sun, $\sim 10^{-7}$ for a typical disk galaxy). Since within GR there exist marked differences between both regimes, it is suspected the former one might be the best arena to test the validity of GR by contrasting its predictions with those from alternative theories of gravity. It is in the not so weak field of the solar vicinity of the Mercury orbit that deviations of gravitation from Newton description were first evidenced and it is reasonable to expect that inadequacies of Einstein gravity may be first found in the strong field environment of neutron stars or stellar mass black holes [5].

It is, however, in the galaxy level ultra weak field regime that hints of the need of modification to the Newton/Einstein description were found by the invocation of the presence of *dark matter* in galaxy clusters [17] and within galaxies through the anomalies in their rotation curve [18,19]. In spite of extensive searches, dark matter has so far eluded detection [20,21]. It was proposed that a modification of the dynamics (“MOND”) could be an alternative to the presence of this unidentified dark matter [22] (see, e.g., for a review [23]) and these ideas have been extended within the schemes of alternate theories of gravitation [24–26]. On a wider cosmological scale alternate theories of gravitation have been extensively invoked [27,28] to provide a different explanation to the problems of the *late accelerated expansion* and the *early inflationary phase* of the Universe as, within GR, the existence of the exotic *dark energy* (occupying nearly $\sim 70\%$ of the total energy-matter budget while remaining invisible) is necessary to explain these phenomena. For example, $f(R)$ extensions of GR gained significant interest as a natural explanation for either an early Universe inflationary phase (as, e.g., $f(R) = R + \alpha R^2$ [29]) or late Universe accelerated expansion (as, e.g., $f(R) = R - \mu^4/R$ [30] or $f(R) = R - \beta R_*[1 - \exp(-R/R_*)]$ [31]), both considering $g_{\mu\nu}$ as the only dynamical field generating gravity throughout its interplay with ordinary baryonic matter. Consequently, among the zoo of possible extensions of gravity theory beyond GR [32], $f(R)$ theories have remained popular as they provide a conceptually simple scheme with minimal departure from the very successful Einstein theory [33,34].

Going beyond $f(R) = R$, however, has the enormous cost that the field equations are not anymore of second order. When dealing with the structure of a spherical star, the result is that the Tolman-Oppenheimer-Volkoff equations of hydrostatic equilibrium, which are a simple set of 3 first order ordinary differential equations in GR

(see, e.g., [35]) that are straightforward to integrate, are now replaced by much more complicated equations. Newtonian stars have been amply studied in the context of the weak field limits solar system constraint (see, e.g., the reviews [27,28,32,33]). It was, however, initially suggested [36] that in the case of relativistic stars, i.e., stars in which the pressure P is of comparable magnitude to the energy density ϵ , a curvature singularity may develop, unless the theory’s parameters are very finely tuned, and casted doubts on the overall viability of $f(R)$ gravity theories. This was apparently confirmed in Ref. [37] who studied the textbook model of a uniform density compact star and found that it could not sustain massive enough such objects to describe a neutron star. However, simple neutron star models built on a polytropic equation of state were soon presented where no singularity appeared [38–45], and even uniform density neutron star models could actually be developed [46]. The possibility of the existence of neutron star models in $f(R)$ gravity was definitely confirmed in works that employed realistic equations of states [39,43] and presented the first $M - R$ diagrams: it was then demonstrated that equations of state that may be ruled out within GR because they fail the test of being able to sustain a $2M_\odot$ neutron star (the then highest measured pulsar mass [47]) may pass this test within an $f(R) = R + \alpha R^2$ gravity theory [43]. Many further works confirmed these first results as, e.g., [48–53]. In particular, these results showed that “exotic” models of dense matter as, e.g., matter containing hyperons that soften the equation of state, which may not be able to sustain a $2M_\odot$ within GR may be able to do it when gravity is described by an $f(R)$ theory [54] or variants of it as Eddington inspired Born-Infeld gravity [55]. Subsequently, models of rotating neutron stars were presented (see, e.g., [56,57]) as well as neutron star in the presence of ultrastrong magnetic fields (see, e.g., [58,59]). More recently, attention has been drawn toward gravitational wave emission during neutron star-neutron star mergers (see, e.g., [60–62]) and the possible existence of supermassive neutron stars in the light of the GW190814 event [63].

More intimate properties of a system, beyond its ground state properties, are revealed by its excitation levels that manifest themselves at finite temperatures. Studies of the thermal evolution of neutron stars open a window on such finite temperature properties as transport coefficients, neutrino emission, and specific heat [64–66]. For example, the occurrence of fast neutrino emission by processes such as the direct Urca one, in which neutrinos are emitted when baryons undergo circular beta and inverse beta transformations, provides us with information about the details of the chemical composition of the matter such as the proton abundance [67,68] or the presence of constituents beyond nucleons, such as meson condensates [69–71], hyperons [72], or deconfined quark matter [73]. Moreover, the single particle excitation spectrum of a baryonic constituent can

be strongly modified by pairing, as, e.g., in neutron superfluidity or proton superconductivity, with dramatic consequences on the thermal evolution of the star (see, e.g., [74]).

Our focus in the present work will precisely be the interplay between the structure and evolution of neutron stars and strong field gravity. We see it as a two way issue: how can we use what we know and understand, mitigated by what we do not know and/or do not understand, about neutron stars to constrain strong field gravity and, reciprocally, how do uncertainties in our description of strong field gravity limit our possibilities to constrain the properties of dense matter by studying neutron stars. We will restrict ourselves to modified (compared to GR) gravity theories with $f(R) = R + \alpha R^2$. However, we consider that the function $f(R)$ does not necessarily have a single analytical expression representing it in the whole possible range of R values. We thus take the position that the simple analytical expression $f(R) = R + \alpha R^2$ represents $f(R)$ in the extreme environment of a compact star while a different expression could describe it in the context of the Solar system and on Earth, or at galactic and cosmological scales. As such, constraints from cosmology, the Solar system or the Eöt-Wash experiment [75], all in the context of $R \rightarrow 0$, are not expected to constrain the α of our $f(R)$.

As is well known [34,37,46,76], it is possible in principle to rewrite the $f(R)$ Lagrangian density as a scalar field theory one by defining an auxiliary field $\Phi_{\text{aux}} = df/dR$ and an associated potential $V(\Phi_{\text{aux}})$. At that point, the usage of conformal transformations becomes frequent as it allows to (a) recast the Lagrangian density as $\mathcal{L} \sim R^* + \mathcal{L}(\Phi_{\text{aux}}^*)$, i.e., GR with an appropriate scalar curvature R^* and conformally transformed scalar-field Φ_{aux}^* , (b) deduce equations of motion which are, in some sense, easier to handle numerically [32,52,57,77]. The turn-back of this approach is the necessity of clarifying what we must understand as *physical equivalence* of the field equations and/or the observable quantities. Thus, we opt for analyzing the original Lagrangian density, its equations of motion and physical predictions without redefining fields or using conformal transformations.

The structure of this paper is as follows: in Sec. II the field equations of the αR^2 theory are reviewed, considering a perfect fluid and a static and spherically symmetric spacetime. In Sec. III, the essential aspects of the energy balance and heat transport equations are discussed. The numerical implementation of the equations and the results are given in Secs. IV and V. We discuss our results in Sec. VI and briefly conclude in Sec. VII.

II. THE $f(R)$ GRAVITY THEORY AND THE STELLAR STRUCTURE EQUATIONS

To avoid ambiguities we first summarize our geometric conventions. We adopt the metric signature $(-, +, +, +)$ and take the Riemann tensor as

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\beta\mu} \quad (1)$$

where ∂_μ denotes partial derivation with respect to the coordinate x^μ and $\Gamma^\alpha_{\mu\nu}$ is the Levi-Civita connection

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\epsilon} (\partial_\mu g_{\nu\epsilon} + \partial_\nu g_{\mu\epsilon} - \partial_\epsilon g_{\mu\nu}), \quad (2)$$

which serves to define the covariant derivative ∇_μ for this work as, e.g., for a rank two tensor T^α_β ,

$$\nabla_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\sigma\mu} T^\sigma_\beta - \Gamma^\sigma_{\mu\beta} T^\alpha_\sigma. \quad (3)$$

The contraction $R^\alpha_{\beta\alpha\nu} = R_{\beta\nu}$ is the Ricci tensor, its trace $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature and the combination

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (4)$$

is the Einstein tensor. Regarding the D'Alembertian operator applied to a scalar field ϕ , it is defined by

$$\square\phi = \nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}} \partial_\nu [\sqrt{-g} g^{\mu\nu} \partial_\mu \phi], \quad (5)$$

with g the determinant of the metric tensor. Finally, for single-variable functions we employ either $df(x)/dx$ or f' to denote derivation with respect to the argument x .

Regarded as an effective field theory for gravity, the motivation for this theory consists in the replacement of the standard Hilbert-Einstein Lagrangian density $\mathcal{L} \propto R$ with a function of the scalar curvature $f(R)$, which in order to be ghost-free must satisfy [52]:

$$\frac{df}{dR} \geq 0, \quad \frac{d^2 f}{dR^2} \geq 0. \quad (6)$$

Being an extension of GR, gravity is exclusively described by the metric tensor, whose dynamics is enclosed in the action functional which, for the particular choice of $f(R) = R + \alpha R^2$, is

$$S = \int \sqrt{-g} d^4x \frac{1}{c} \left[\frac{c^4}{16\pi G} (R + \alpha R^2) + \mathcal{L}_M \right] \quad (7)$$

where c is the speed of light in vacuum, G is Newton's constant, α a non-negative real number with dimensions of length squared, and $\mathcal{L}_M$ is the matter Lagrangian. By performing a null variation with respect to $g^{\mu\nu}$, i.e. $\delta S = 0$, one arrives to the field equations [32,34,44,78]:

$$(1 + 2\alpha R) G_{\mu\nu} + \frac{\alpha R^2}{2} g_{\mu\nu} = 2\alpha [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] R + \frac{8\pi G}{c^4} T_{\mu\nu} \quad (8)$$

where $T_{\mu\nu}$ is the energy-momentum tensor [79]

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \quad (9)$$

which, for this theory of gravity, still satisfies the conservation law

$$\nabla_\mu T^{\mu\nu} = 0. \quad (10)$$

In contrast to GR, the trace $T = g^{\mu\nu}T_{\mu\nu}$ and the scalar curvature R are related by a differential expression,

$$(6\alpha\Box - 1)R = \frac{8\pi G}{c^4}T, \quad (11)$$

which constitutes an additional constraint for the variables of the model.

To study nonrotating neutron stars, we model the metric as static and spherically symmetric (SSS):

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + e^{2\Lambda(r)}dr^2 + r^2d\Omega^2, \quad (12)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The coordinates in this metric have exactly the same interpretation as in GR. The 2D metric on a surface (sphere) with r and t constant is identical to the one in Euclidean space, in particular the area of this sphere is $S(r) = 4\pi r^2$ and r can be called the “areal radius.” The proper radial length, dl , measured by a local observer at rest on the shell is related to dr by $dl = \exp[\Lambda(r)]dr$ while their proper time interval $d\tau$ is given by $d\tau = \exp[\Phi(r)]dt$. We also impose the condition that the metric tends to *Schwarzschild's*

$$\begin{aligned} \Lambda(r) + \Phi(r) &= 0, \\ e^{2\Phi(r)} &= \left[1 - \frac{2GM}{c^2r}\right]^{-1}, \end{aligned} \quad (13)$$

as $r \rightarrow \infty$. Hence, t can be interpreted as the proper time of an observer at rest at infinity and M the (total) gravitational mass of the star.

For neutron stars, we are working under the assumption that they are adequately described by a perfect fluid defined by its pressure P and energy density ε providing a simple energy momentum tensor

$$T_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu + Pg_{\mu\nu}, \quad (14)$$

with u^μ the fluid rest-frame unitary 4-velocity, $u^\mu u_\mu = -1$. By inserting this tensor and the SSS metric in the field equations, we obtain the following set of second-order nonlinear differential equations [78,80,81] as a generalized set of TOV equations for hydrostatic equilibrium for a spherically symmetric star

$$\frac{d\Phi}{dr} = \frac{1}{A_1} \left\{ \frac{4\pi G P r e^{2\Lambda}}{c^4} - \frac{[A_2 e^{2\Lambda} + 2A_3]}{4r} \right\}, \quad (15)$$

$$\frac{d\Lambda}{dr} = \frac{1}{A_1} \left\{ \frac{4\pi G \varepsilon r e^{2\Lambda}}{c^4} + \frac{[A_2 e^{2\Lambda} + 2A_3]}{4r} + ar \frac{d^2 R}{dr^2} \right\}, \quad (16)$$

$$\frac{d^2 R}{dr^2} = \frac{1}{A_6} \left\{ \frac{A_1 A_4 e^{2\Lambda}}{6\alpha} - \frac{1}{r} \left[A_5 - \frac{A_2 e^{2\Lambda}}{2} \right] \frac{dR}{dr} \right\}, \quad (17)$$

where, for simplicity of notation, the following functions have been defined:

$$A_1 = 1 + 2\alpha R + ar \frac{dR}{dr}, \quad (18)$$

$$A_2 = \alpha r^2 R^2 - 4\alpha R - 2, \quad (19)$$

$$A_3 = 1 + 2\alpha R + 4ar \frac{dR}{dr}, \quad (20)$$

$$A_4 = \frac{8\pi G}{c^4} (3P - \varepsilon) + R, \quad (21)$$

$$A_5 = \frac{4\pi G}{c^4} r^2 e^{2\Lambda} (P - \varepsilon) + 1 + 2\alpha R - 2ar \frac{dR}{dr}, \quad (22)$$

$$A_6 = 1 + 2\alpha R. \quad (23)$$

From the r -component of the energy-momentum tensor conservation law, Eq. (10), we deduce the differential equation for the pressure,

$$\frac{dP}{dr} = -(P + \varepsilon) \frac{d\Phi}{dr}. \quad (24)$$

We thus have five structural functions, $\Phi(r)$, $\Lambda(r)$, $R(r)$, $P(r)$, and $\varepsilon(r)$, and four equations, Eqs. (15), (16), (17), and (24), which need to be completed by an EOS giving a relationship of the form $P = P(\varepsilon)$ [82]. Instead of ε it is also customary to employ the mass density ρ such that $\varepsilon = \rho c^2$. Microscopic calculations of the EOS and astrophysical processes in general are commonly phrased in terms of baryons through their correspondent density n , such that $P = P(n)$ and $\varepsilon = \varepsilon(n)$. For our study the temperature dependence of the EOS can be neglected and we do not need further equations to determine the structure of the star, except for the outer layers of the envelope as discussed below in Sec. VA.

For compact objects, the exterior is defined as the region where both pressure and energy density are identically zero. From the differential equations, it is clear that these conditions do not imply a vanishing scalar curvature outside the objects, but it is nevertheless possible to demand it as an asymptotic limit, recovering Schwarzschild's metric in the process, Eq. (13). Indeed, from the differential equations for Λ and Φ we have

$$\frac{d\Phi}{dr} + \frac{d\Lambda}{dr} = \frac{\alpha r}{A_1} \frac{d^2 R}{dr^2}. \quad (25)$$

For the left hand side to vanish, $d^2 R/dr^2 = 0$, which implies that $R=0$ is a possible asymptotic solution. Aside from this fact, let us stress that the treatment of R as an independent variable has the advantage of keeping a second order system of equations instead of a fourth-order one: applying the geometric definition $R = g^{\mu\nu} R_{\mu\nu}$ to the SSS metric leads to $R \propto \Phi''$, hence $R'' \propto \Phi^{(4)}$. Notice that in GR the scalar curvature is given by

$$R^{(\text{GR})} = \frac{8\pi G}{c^4} (\varepsilon - 3P) \quad (26)$$

and naturally vanishes outside the star. In the center of a neutron star $R^{(\text{GR})}$ can be positive or negative and values of $|R^{(\text{GR})}|$ of the order of 10^{-12} cm^{-2} are found.

We will describe the boundary conditions needed to solve these structure equations in details in Sec. IV A.

A. Terminology and notations

We will simply refer to this theory of gravity with $f(R) = R + \alpha R^2$ as the “ αR^2 theory” or as “ αR^2 gravity.” For the parameter α , we adopt the following notation:

$$\alpha_X = \left(\frac{GM_\odot}{c^2} \right)^2 \times 10^{X-10} = 2.18 \times 10^X \text{ cm}^2, \quad (27)$$

considering for this work the cases $X = 9, 10, 11, 12, 14$, and 16.

III. THE STELLAR EVOLUTION EQUATIONS

The equations describing the thermal evolution of a spherically symmetric star in the metric (12) depend only on the form of this SSS metric and not on the particular theory of gravity, within certain conditions (see last paragraphs of this section). They are derived in all details, within the GR framework, in Ref. [83,84] and summarized in [85] and we here sketch their derivation to convince the reader that the very same equations are still valid as long as the SSS metric of Eq. (12) is employed.

Let $T(r)$ be the temperature on the shell of areal radius r and $L(r)$ the diffusive luminosity at the same position. Both T and L refer to the physical quantities measured by a local observer at rest on the said shell [86]. In flat space-time L can be assumed to be related to the gradient of T through the Fourier law

$$L = -4\pi r^2 K \frac{dT}{dr} \quad (28)$$

where K is the thermal conductivity in the shell (as it would be measured by a local observer at rest). Still in

Minkowski space-time, energy conservation in this shell is expressed by

$$\frac{dL}{dr} = 4\pi r^2 \left[q_h - q_\nu - n \frac{d(\varepsilon/n)}{dt} + \frac{P}{n} \frac{dn}{dt} \right] \quad (29)$$

where q_ν is the neutrino emissivity, i.e., the energy loss per unit time and unit volume due to neutrino emission, and similarly q_h represents heat injection by such processes as nuclear reactions, magnetic field decay, friction due to differential rotation, etc. When transcribing these two equations to curved space-time, one has to consider that dr and dt in Eqs. (28) and (29) are proper quantities, i.e., they are dl and $d\tau$, respectively, and the derivative must thus be replaced as

$$\frac{d}{dt} \rightarrow \frac{d}{d\tau} = e^{-\Phi(r)} \frac{d}{dt} \quad (30)$$

and

$$\frac{d}{dr} \rightarrow \frac{d}{dl} = e^{-\Lambda(r)} \frac{d}{dr} \quad (31)$$

while the factors $4\pi r^2$ remain without change since r is an areal coordinate. A crucial point is that in the presence of gravity conservation of energy must include the gravitational energy and hence, in the space-time with metric (12) the conserved energy is the *redshifted* energy. If $E(r)$ is some energy measured at r by some local observer at rest, the corresponding redshifted energy is $\hat{E}(r) \equiv \exp[\Phi(r)]E(r)$. A radial derivative dE/dr involves comparing the energies $E(r + \delta r)$ and $E(r)$: $E(r + \delta r)$ must be moved from $r + \delta r$ to r and become $\hat{E} = \exp[\Phi(r + \delta r) - \Phi(r)]E(r + \delta r)$ so that the radial derivative becomes

$$\frac{dE}{dr} \rightarrow \lim_{\delta r \rightarrow 0} \frac{\hat{E} - E(r)}{\delta r} = e^{-\Lambda} \frac{d(e^\Phi E)}{dr}. \quad (32)$$

This same rule applies to T in Eq. (28) since it represents the energy of the diffusing particles that generate L . For the derivative of L in Eq. (29), however, two redshift factors are necessary because L is energy per unit time and time is itself blue-shifted. With these consideration it is easy to verify that the stellar evolution equations in the curved space-time described by the SSS metric (12) are

$$\frac{d(e^\Phi T)}{dr} = -\frac{1}{K} \frac{e^{\Phi+\Lambda} L}{4\pi r^2} \quad (33)$$

and

$$\begin{aligned} & \frac{d(e^{2\Phi} L)}{dr} \\ &= 4\pi r^2 e^\Lambda \left[e^{2\Phi} q_h - e^{2\Phi} q_\nu - e^\Phi n \frac{d(\varepsilon/n)}{dt} + \frac{e^\Phi P}{n} \frac{dn}{dt} \right] \end{aligned} \quad (34)$$

TABLE I. Examples of neutrino emitting processes in neutron star cores^a.

Name	Process	Emissivity ^b q_ν (erg cm ⁻³ s ⁻¹)	
Modified Urca (neutron branch)	$n + n \rightarrow n + p + e^- + \bar{\nu}_e$ $n + p + e^- \rightarrow n + n + \nu_e$	$\sim 2 \times 10^{21} \mathcal{R} T_9^8$	Slow
Modified Urca (proton branch)	$p + n \rightarrow p + p + e^- + \bar{\nu}_e$ $p + p + e^- \rightarrow p + n + \nu_e$	$\sim 10^{21} \mathcal{R} T_9^8$	Slow
Bremsstrahlung	$n + n \rightarrow n + n + \nu \bar{\nu}$ $n + p \rightarrow n + p + \nu \bar{\nu}$ $p + p \rightarrow p + p + \nu \bar{\nu}$	$\sim 10^{19} \mathcal{R} T_9^8$	Slow
Cooper pair breaking and formation ^c	$n + n \rightarrow [nn] + \nu \bar{\nu}$ $p + p \rightarrow [pp] + \nu \bar{\nu}$	$\sim 5 \times 10^{21} \mathcal{R} T_9^7$ $\sim 5 \times 10^{19} \mathcal{R} T_9^7$	Medium
Direct Urca	$n \rightarrow p + e^- + \bar{\nu}_e$ $p + e^- \rightarrow n + \nu_e$	$\sim 10^{27} \mathcal{R} T_9^6$	Fast

^aTable from [65].^b T_9 denotes T in units of 10^9 K and for each process the “control coefficient” $\mathcal{R} = \mathcal{R}(T/T_c)$ is introduced to take into account the extra temperature dependence due to pairing. Each process has its own corresponding $\mathcal{R}$.^c $[nn]$ and $[pp]$ denote neutron and proton Cooper pairs.

which have exactly the same form as in the well known GR formulation. We will describe the boundary conditions needed to solve these evolution equations in details in Sec. VA.

Within Newtonian thermodynamics a system is isothermal when T is uniform within it which implies, by Eq. (28), that $L = 0$. In curved space-time we invert the argument and call “isothermal” a star in which $L = 0$ which then implies that $\tilde{T}(r) \equiv e^{\Phi(r)} T(r)$ is uniform, i.e., independent of r . $\tilde{T}(r)$ is commonly called the redshifted temperature.

Apparently, the energy balance and heat transport equations are insensible to the gravity theory chosen. This holds only if certain conditions are met: first, an equivalence principle must exist in the gravity theory under consideration. Second, if additional fields (either scalar, vector or tensor ones) are included in the Lagrangian density (which is not the case for the present work), it must be verified that no thermodynamical consequences arise from their introduction. Finally, if conformal transformations are employed (either to simplify the task of solving the field equations or to invoke the analogy with a scalar field), both structure and thermal differential equations must be solved in the same frame. In the present work, all the equations and analysis are carried out without invoking conformal transformations due to the intricacy of constructing physical frames and the interpretation of the emerging results from the variables involved [33,34,87–89].

A. Physical ingredients

In the interior where matter is degenerate we can simplify the term $nd(\epsilon/n)/dt$ in Eq. (34) as

$$n \frac{d(\epsilon/n)}{dt} = c_v \frac{dT}{dt} \quad (35)$$

with

$$c_v = n \frac{d(\epsilon/n)}{dT} \quad (36)$$

being the specific heat, per unit volume, at constant volume. Notice that in degenerate matter there is no need to distinguish c_v from the specific heat at constant pressure c_p [90]. The microphysics ingredients, specific heat c_v , thermal conductivity K and neutrino emissivity q_ν , we employ are standard and described in [91,92].

For completeness we briefly present in Table I the dominant neutrino emission processes occurring in the core that we consider with schematic expressions for their emissivities and mark them as either “slow,” “medium,” or “fast” to briefly qualify their efficiency. The direct Urca (DUrca) process, being a highly efficient mechanism of neutrino emission, plays a fundamental role in the cooling of neutron stars [67,68,93]. This process is only possible when the proton fraction is high enough, a criterium that can simply be stated as

$$n_p^{1/3} + n_e^{1/3} \geq n_n^{1/3} \quad (37)$$

where n_n , n_p , and n_e are the (number) densities of neutrons, protons, and electrons, respectively. This criterium is very restrictive and is only satisfied at high densities above a threshold density ρ_{DU} which is dependent on the EOS.

Of outmost importance in the cooling of neutron stars is the occurrence of pairing, i.e., superfluidity/superconductivity, of baryons which results from the instability of the Fermi surface under the formation of Cooper pairs in the presence of an attractive interaction [94,95]. Various type of pairing are possible depending on the spin and angular momentum of the pairs, the most relevant ones being 1S_0 (in spectroscopic notation, i.e., corresponding to spin-singlet

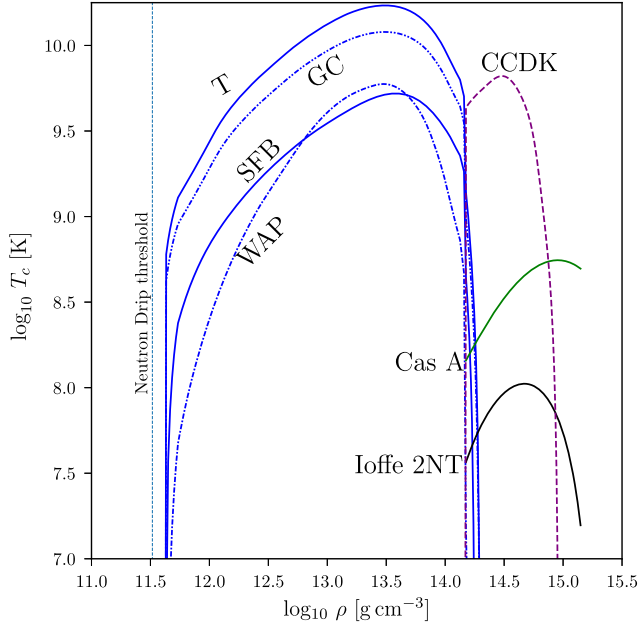


FIG. 1. Superfluidity/superconductivity critical temperature T_c vs mass density ρ for the nucleon pairing gaps considered in this work: SFB [98], WAP [99], GC [100], and T [101] for 1S_0 neutrons; Min a [91], Cas A [102] and Ioffe 2NT [103] for 3P_2 neutrons and CCDK [104] for 1S_0 for protons.

with orbital angular momentum $L = 0$ and total momentum $J = 0$ and 3P_2 (corresponding to spin-triplet with $L = 1$ and $J = 2$). We refer the reader to the reviews [96,97] for extensive presentations of the topic. The first important effect of pairing for our purpose is the appearance of a gap at the Fermi surface in the single particle excitation spectrum, once the temperature is below the phase transition critical temperature T_c , that results in a strong suppression, often as a Boltzmann-like exponential, of all processes involving the paired component. This possibly affects all microphysics ingredients as c_v , K , and q_v . The second important effect is the opening of a new neutrino emission channel in which the energy released in the formation of a Cooper pair is emitted as a $\nu\bar{\nu}$ pair. The pairing phase transition, being of second order, implies the continuous formation and breaking of Cooper pairs as T drops below T_c which renders this process, dubbed the “Cooper pair breaking and formation,” or PBF process, very efficient. For the pairing critical temperature, among the many published models, we choose a few representative ones that we plot in Fig. 1. In our models, we include the following channels: $X = ^1S_0$ and $Y = ^3P_2$ for neutrons, and $Z = ^1S_0$ for protons. For clarity in the discussion, we adopt the notation (X, Y, Z) for labeling the models.

IV. STELLAR STRUCTURE

We first here describe our numerical method to solve the stellar structure equations and then examine several aspects

of this structure that have a potential direct impact on the stellar thermal evolution.

A. Numerical implementation

The structure equations, Eqs. (15), (16), (17), and (24), were solved employing a 4th order Runge-Kutta method of adaptative stepsize, implemented in FORTRAN77 language [105]. The radial interval was split into $[0, r_*]$ and $[r_*, r_g]$, where r_* is the radius of the star and r_g is referred as the gravitational radius. The first is formally defined as the point where $P(r_*) = 0$. Numerically, it translates into $P(r_*) \sim \mathcal{O}(10^{-27}P_0)$, with P_0 being the central pressure. This implies that for $r \in [r_*, r_g]$ the system is solved by employing $\varepsilon = P = 0$. Regarding r_g , it is defined as the coordinate where $R(r_g) \sim \mathcal{O}(10^{-30}R_0)$ beyond which the Schwarzschild metric can be applied.

By demanding regularity of the equations at the origin of coordinates, one obtains the boundary conditions [44,60,80,106]

$$\Lambda(0) = 0, \left. \frac{d\Lambda}{dr} \right|_{r=0} = \left. \frac{d\Phi}{dr} \right|_{r=0} = \left. \frac{dR}{dr} \right|_{r=0} = 0. \quad (38)$$

Since $\Phi(r)$ is absent from the right-hand side of the equations, a finite $\Phi(0)$ can be chosen at the beginning of the calculations, and at the end of them perform a re-scaling $\Phi(r) \rightarrow \Phi(r) - \Lambda(r_g) - \Phi(r_g)$ in order to recover Schwarzschild’s solution for $r \geq r_g$.

There is no restriction on the initial condition on the pressure, P_0 , which can be chosen arbitrarily, thus generating a family of solutions. The corresponding energy density, ε_0 , or mass density, ρ_0 , are then obtained directly from the EOS. In this paper we consider four core EOSs: APR [107], arising from a nonrelativistic model with relativistic corrections, and three EOSs, MS-A1, MS-B1, MS-C1 which are built from relativistic mean field calculations [108,109]. These are completed by a low density EOS for the inner crust from [110] and for the outer crust from [111]. Our motivation for using the three EOSs MS-A1, MS-B1, and MS-C1, instead of some more popular EOSs as has been done in other works on neutron star structure, is that these EOSs were specifically constructed to comply not only with recent restrictions on the neutron star radius deduced from both NICER and LIGO/Virgo observations but, moreover, to display a broad range of critical masses for the onset of the direct Urca process, a critical issue for neutron star evolution not considered in previous works restricted to the study of just neutron star structure, going from 1.5 up to $1.9M_\odot$ (and reaching almost $2M_\odot$ once the APR EOS is included). We refer the reader to Appendix A of Ref. [109] for details.

Given a central density ρ_0 , or central pressure P_0 , to find R_0 a combination of shooting and bisection methods was implemented. Two examples of such integrations are shown

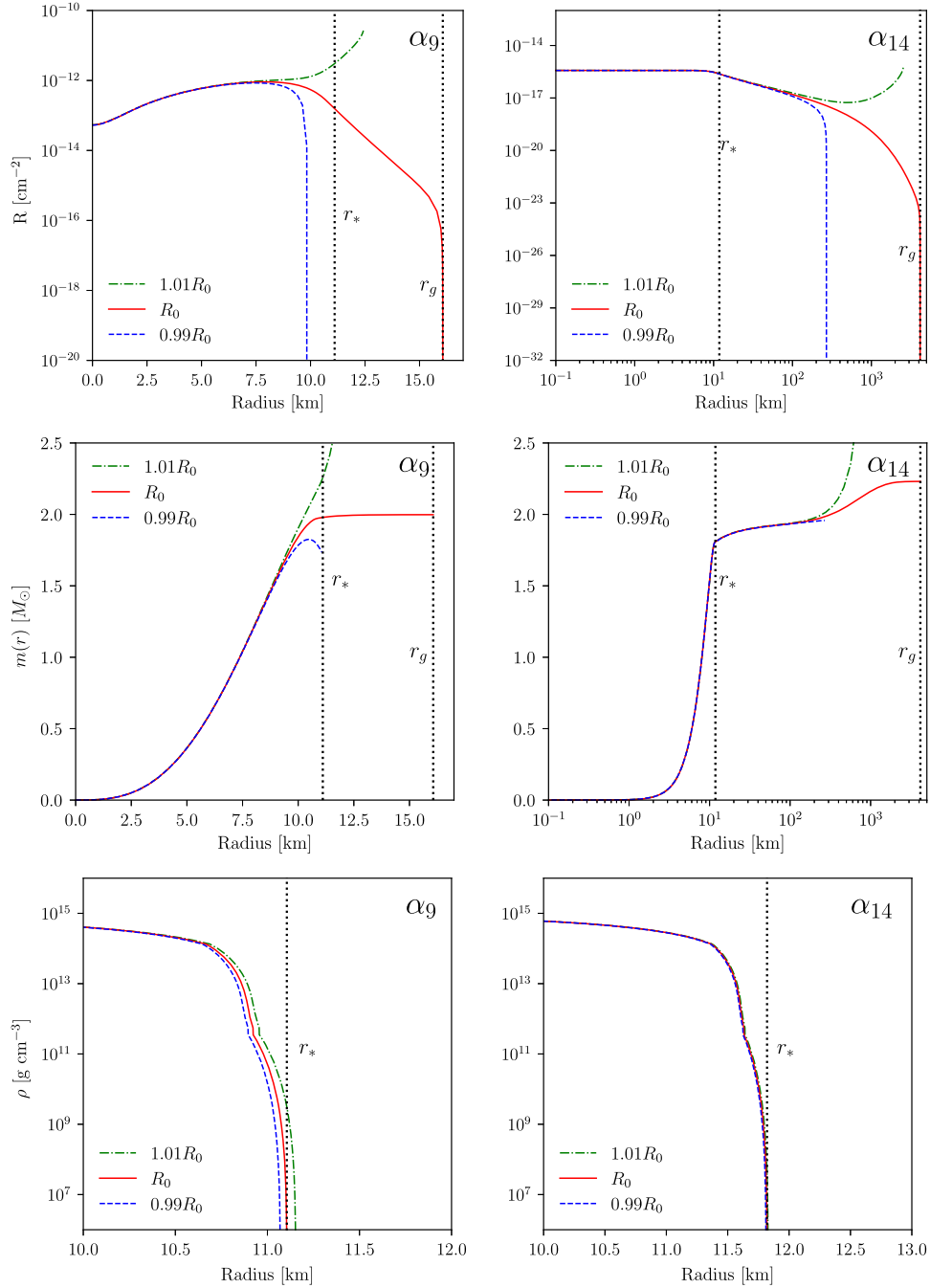


FIG. 2. Scalar curvature R (upper panels), mass function m (middle panels) and density ρ in the outer layers (lower panels) for a neutron star model built with the APR EOS and central density $\rho_0 = 1.55 \times 10^{15} \text{ g cm}^{-3}$ in $\alpha_9 R^2$ (left panels) and $\alpha_{14} R^2$ (right panels) gravity with the respective locations of their stellar, r_* , and gravitational, r_g , radii. In each panel the three curves show the values corresponding to the central value of R , R_0 , that leads to the Schwarzschild metric beyond r_g and slightly modified values, $0.99R_0$ and $1.01R_0$, that exhibit the divergent behavior of the solutions.

in Fig. 2 that illustrate the cause of the numerical subtleties needed to find the correct solution: the solution that leads to the Schwarzschild metric at infinity is a bifurcation point (see, e.g., [46,80,112]). This figure also shows that for small values of α such as α_9 the Schwarzschild limit is reached already a few kilometres above the stellar surface

while for extreme cases such as α_{14} it is only at distances of several thousands of kilometres that this limit is numerically obtained. Obtaining a value of the central R_0 as close as possible to the exact one is important to describe the metric outside the star, however, as the lower panels of this figure illustrate, the ρ profile is much more stable than R

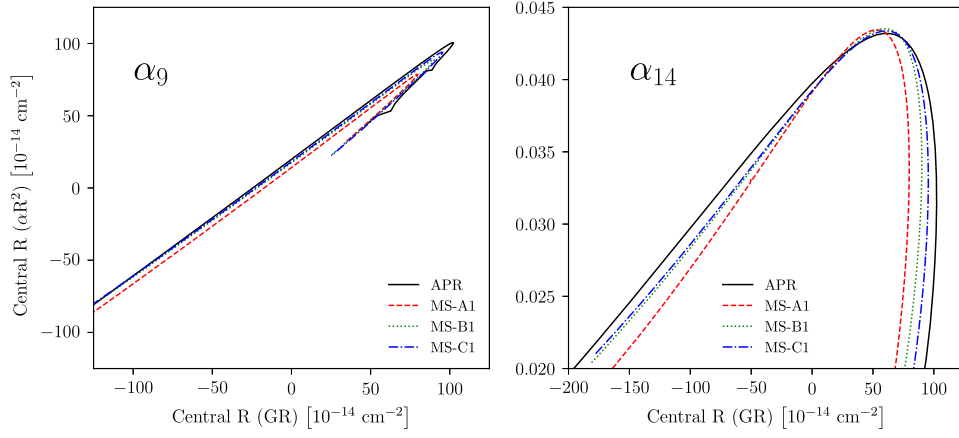


FIG. 3. Scalar curvature at the star's center for $\alpha_9 R^2$ (left panel) and $\alpha_{14} R^2$ (right panel) gravity, as a function of its GR counterpart, using four EOSs to build the neutron star models.

one. In order to overcome these issues, we demanded the bisection method to stop when the length of the interval for seeking R_0 was inferior to 10^{-27} cm^{-2} . This stiff condition allowed us to obtain variations below 0.05% for both the final masses $m(r_g)$ and radii r_* (see Sec. IV B for the precise meaning of masses), although the stability of the ρ profile guarantees even smaller variations of density in this stiff scenario. Taking advantage of an EOS-independent, parabolic-shape relation between $R(0)$ and $R^{\text{GR}}(0)$, as illustrated on Fig. 3, it becomes easier to generate families of models when varying ρ_0 once a first solution has been found.

Finally, it is important to notice that given the values of R found after integrations, for the whole range of α we have explored and as illustrated in Fig. 3, αR is always smaller than 10^{-2} , i.e., the term αR^2 always results to be a small correction to R in the Lagrangian density $\mathcal{L}$. Increasing values of α actually force smaller values of R when the structure equations are solved: while R^{GR} is always $\sim 10^{-12} \text{ cm}^{-2}$ in the center of a neutron star, in αR^2 gravity R_0 is significantly reduced. A consequence of this, as we will show below, is that when values of α beyond 10^{14} cm^2 are considered the resulting structure of a neutron star remains almost identical to the α_{14} case, i.e., there is a clear saturation of the effects of αR^2 gravity at large α . However, when considering the shape of the metric outside the star, increasing values of α beyond α_{14} still imply increasing values of r_g .

B. Gravitational and baryonic masses

Three separate gravitational mass concepts appear in αR^2 modified gravity: the surface and the total ones [60,106]. In contradistinction, in GR the surface and total gravitational masses coincide.

In analogy to the Schwarzschild metric, we can introduce a mass function, $m(r)$, by writing

$$\exp(2\Lambda) = \frac{1}{1 - 2Gm(r)/rc^2}. \quad (39)$$

Under the condition that we recover the Schwarzschild metric in the limit $r \rightarrow \infty$ we obtain the *total* gravitational mass of the star as

$$M \equiv M_{\text{tot}} := \lim_{r \rightarrow \infty} m(r). \quad (40)$$

We will also consider a *surface* gravitational mass as $M_{\text{surf}} = m(r = r_*)$ that will be of importance when studying the surface gravitational acceleration in Sec. IV F. In practice we use $M_{\text{tot}} = m(r = r_g)$ and will often omit the “tot” subindex, so that M will always refer to the total gravitational mass. Another mass concept frequently encountered is the *baryonic* mass, M_{bar} , defined through the total baryon number of the star, N_{bar} , as

$$M_{\text{bar}} \equiv m_B N_{\text{bar}} = m_B \int_0^{r_*} 4\pi r^2 n(r) e^{\Lambda(r)} dr, \quad (41)$$

where the $e^{\Lambda(r)}$ factor corrects for the proper radial length from the areal radius r , and m_B is the “baryon mass” taken as the nucleon mass. These concepts are necessary to understand both the cooling setup and results: the surface gravitational mass is the value we shall employ to define the surface gravity (see Eq. (48) below), while the total gravitational mass is the value measured in the case of binary systems. The baryonic mass is just a reference mass and is not a measurable quantity but is a useful conserved value in some contexts.

For any given EOS, and a chosen gravity theory, we can calculate a family of neutron star models by varying the central density ρ_0 . In the left panel of Fig. 4 we compare the gravitational mass in GR and the total and surface gravitational masses in αR^2 gravity for various values of α as function of the baryonic mass when using the APR EOS,

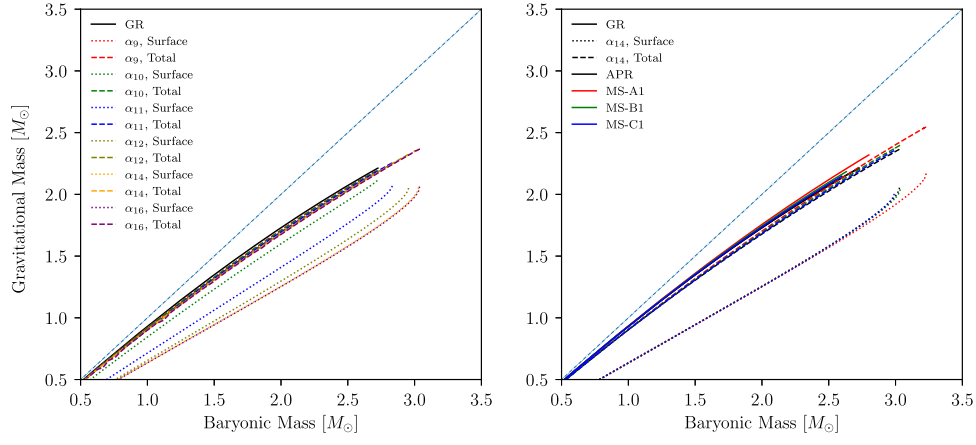


FIG. 4. Total and surface gravitational masses versus baryonic mass, in GR and various αR^2 gravity. In the left panel we consider a range of α_X 's and a fixed EOS, the APR one. In the right panel we explore our four EOSs for the extreme α_{14} .

while in the right panel we consider our four EOSs and the GR and $\alpha_{14} R^2$ gravity cases. We first notice that for all four EOSs and all α considered, M_{tot} in αR^2 gravity is close but always smaller than the corresponding GR M for a given M_{bar} : in spite of a 7 orders of magnitude variation in α we find that M_{tot} is at most 5% smaller than the corresponding GR M . However, when considering M_{surf} , it can be significantly smaller than M_{tot} , and the GR M , the difference being a strong function of α . Most interesting here is to notice that the difference between M_{tot} and M_{surf} is almost constant, i.e., independent of M_{bar} , in the range of observed values of M_{tot} , i.e., between 1.2 and $2M_{\odot}$.

With the same procedure of varying ρ_0 we can generate $M - r_*$ and $M - \rho_0$ curves which we present in Figs. 5 and 6. In agreement with previous works on the subject [52,53,78,81,106], we find that M_{tot} can reach larger values in the αR^2 theory than in GR. [113] The left panel of Fig. 5 shows that there is always a crossing of the $M - r_*$ curves

of GR and any αR^2 theory and that at masses below the crossing point αR^2 neutron star models have smaller radii while above this point they have larger radii than GR models of the same total gravitational mass. (For small values of α , as α_9 and α_{10} , there appears a second crossing point a M close to the maximum mass.) On the right panel of Fig. 5 we also find a crossing point on the $M - \rho_0$ curves: significant differences in the predicted total gravitational masses, for a given ρ_0 , between GR and αR^2 theories with large α appear above this crossing point while at lower central densities the differences are much smaller. We can also notice that the central density for the maximum mass is progressively lower as α grows in the case of the APR EOS.

These considerations are also valid in the case of the other three EOSs we consider as shown in Fig. 6. Finally, we see that α_{14} and α_{16} results are almost identical and illustrate the saturation of the αR^2 effects of which we will find more examples below.

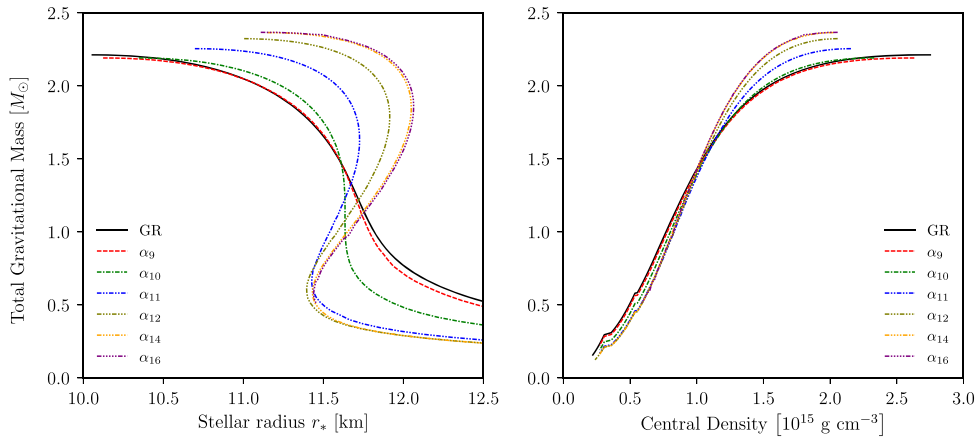


FIG. 5. Total gravitational mass versus: (left panel) stellar radius r_* ; (right panel) central density, considering the APR EOS and both GR and αR^2 theories and various α_X 's as labeled.

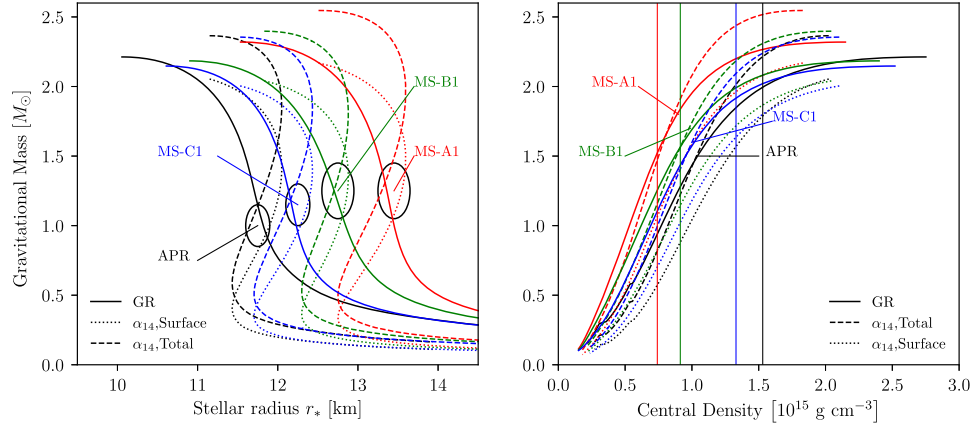


FIG. 6. Gravitational mass for the labeled EOSs, in GR and $\alpha_{14}R^2$ gravity, against: (left) stellar radius r_* ; (right) central density. The four vertical lines in the right panel mark the DUrca critical densities, ρ_{DU} , for MS-A1, MS-B1, MS-C1 and APR, by order of increasing density.

C. Stellar proper volume and heat capacity

The proper volume of a star is naturally defined as

$$V = \int_0^{r_*} 4\pi r^2 e^{\Lambda(r)} dr, \quad (42)$$

where the $e^{\Lambda(r)}$ factor corrects for the proper radial length from the areal radius r . On Fig. 7, the stellar proper volume of the objects is plotted against the total gravitational mass. There is a clear EOS-independent trend that at high masses, higher than about $1.4 - 1.5M_\odot$ depending on the EOS, the proper volume of the star in αR^2 gravity is larger than in GR

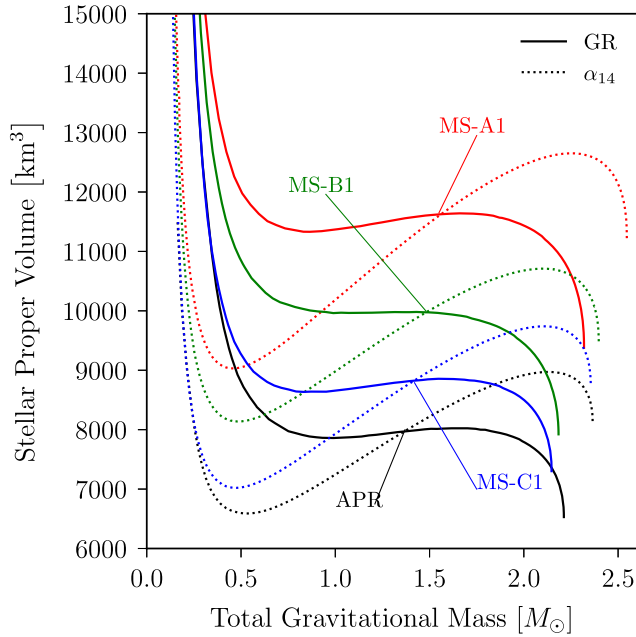


FIG. 7. Proper stellar volume, considering our four EOSs, in GR versus $\alpha_{14}R^2$ gravity as a function of the total gravitational mass of the star.

for α_{14} while at lower masses the effect is inverted. This effect parallels the relative variation in radii exemplified in the left panel of Fig. 6.

The stellar volume is clearly not directly measurable. However, the total stellar heat capacity, which is proportional to the stellar volume, is potentially measurable through the overall neutron star temperature change under heating from a strong outburst of accretion in a binary system [114]. Recent results in the case of the neutron star in the HETE J1900.1-2455 system [115] have shown that, after a ten year long accretion outburst, constraints on its heat capacity may be getting close to the theoretically predicted value.

In Fig. 8 we explore the effect of gravity theory on the expected values of the total stellar core heat capacity C_{core} taken at a fiducial uniform redshifted temperature $\tilde{T} = 10^8$ K. In the left panel of this figure one can see an increase in C_{core} with increasing values of α which, for the range of plotted masses, parallels the increase in stellar volume. However, in the right panel we show that this small increase of C_{core} , even in the extreme case of $\alpha_{14}R^2$ gravity, is smaller than the variability resulting from uncertainties on the EOS.

D. The DUrca critical mass and volume

The allowance of the DUrca process has a dramatic effect on the cooling of a neutron star [93] and while it is clear that ρ_{DU} is independent of the gravity theory, depending only on the EOS, the critical stellar mass M_{DU} at which the DUrca process becomes allowed does depend on the gravity theory. Specifically, the values of ρ_{DU} are 7.41×10^{14} , 9.13×10^{14} , 1.33×10^{15} and 1.53×10^{15} g cm $^{-3}$ for the MS-A1, MS-B1, MS-C1 and APR EOS, respectively [109]. Similarly, the proper volume of the inner core where the DUrca process is allowed is of importance. This quantity is defined as

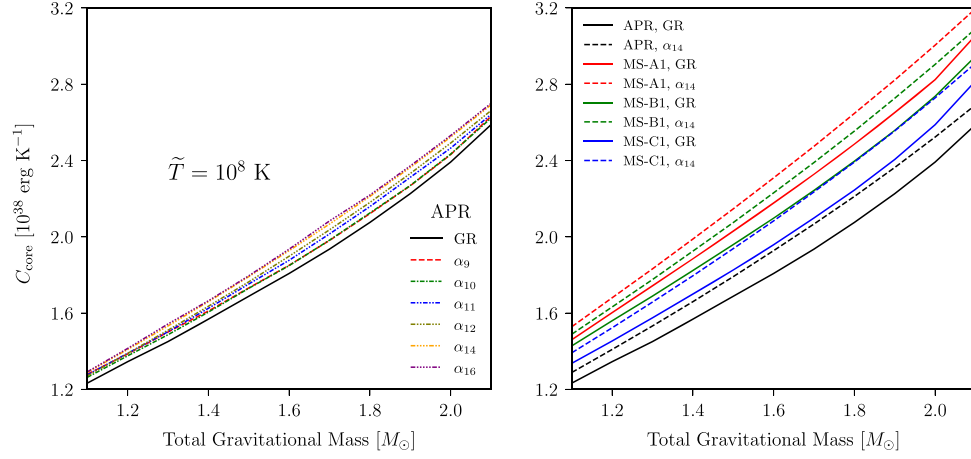


FIG. 8. Total heat capacity of an isothermal core, at a uniform redshifted temperature $\tilde{T} = 10^8$ K, as a function of the stellar mass in GR and various αR^2 gravity. Left panel: APR EOS in GR and six α_X 's. Right panel: our four chosen EOSs in GR and α_{14} gravity.

$$V_{\text{DU}} = \int_0^{r_{\text{DU}}} 4\pi r^2 e^{\Lambda(r)} dr, \quad (43)$$

with r_{DU} being the last radial coordinate where the criterium of Eq. (37) is satisfied, i.e., where ρ_{DU} is reached.

In the left panel of Fig. 9 we show how V_{DU} grows with M for the APR EOS in GR and in $6\alpha R^2$ theories. One sees that the threshold mass M_{DU} increases significantly as α is increasing while the maximum V_{DU} , reached at the maximum mass, is smaller in αR^2 gravity than in GR and has a nonmonotonic behavior with respect to α . However, the details of this behavior are EOS dependent. When considering other EOSs, in the right panel of Fig. 9, the same behavior, increase of M_{DU} in α_{14} gravity compared to GR, is seen for the EOS MS-C1 while for MS-B1 there is little change in M_{DU} and for MS-A1 the contrary happens, M_{DU} is lower in α_{14} gravity than in GR. The reason for this difference is simple when one considers the crossing point of the

$M - \rho_0$ curves seen in the right panels of Fig. 5 and 6: ρ_{DU} for MS-A1 and MS-B1 is close to the crossing point, i.e., in the density regime where the $M - \rho_0$ curve is little affected by α while for the EOSs MS-C1 and APR, ρ_{DU} is much higher than the crossing point and changes in α have a large impact.

E. Crust thickness

Again in the case of transiently accreting neutron stars in binary systems, the thermal response of the crust has now being monitored in several cases (see, e.g., [116] for a brief review). The basic physical property setting the conditions for this response is the thickness of the crust, thermal timescales being proportional to the square of the length-scale of the evolving region.

Let r_{cc} be the radial coordinate of the crust-core transition, assumed in the present work to occur at a density $\rho_{cc} = 1.4 \times 10^{14} \text{ g cm}^{-3}$ [117]. The crust thickness is simply calculated as

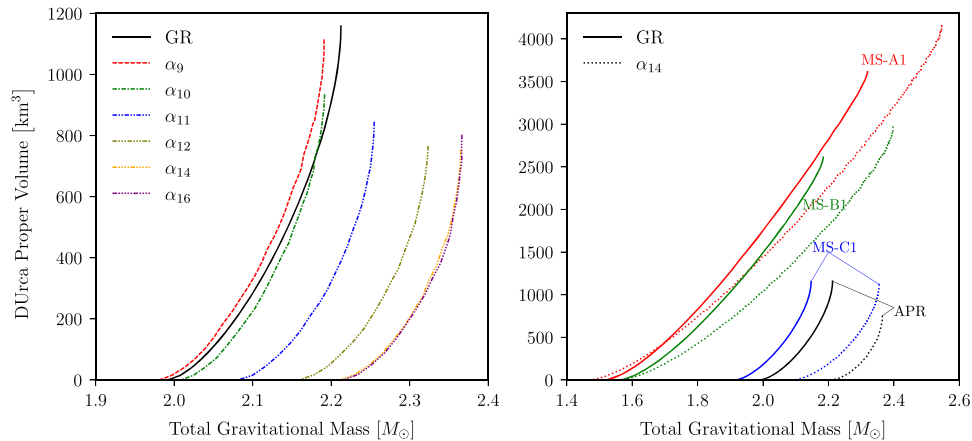


FIG. 9. Proper volume of DURCA process against total gravitational mass in GR versus αR^2 gravity. Left panel: APR EOS in GR and six α_X 's. Right panel: our four chosen EOSs in GR and α_{14} gravity.

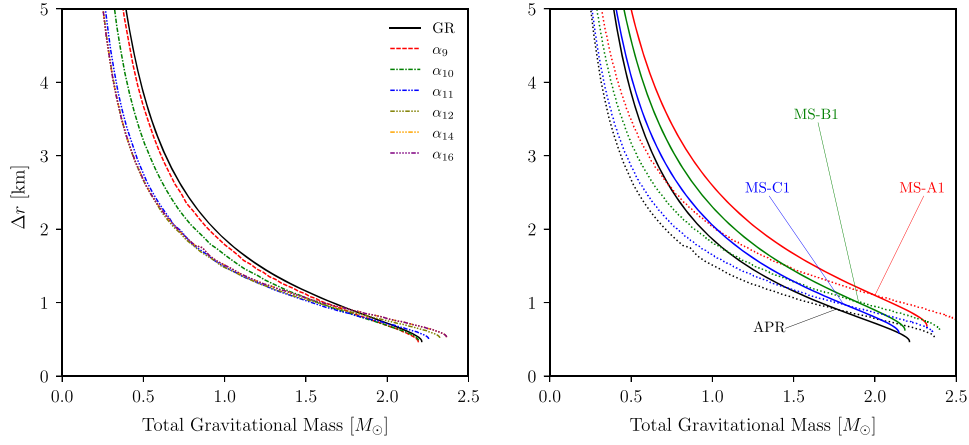


FIG. 10. Crust thickness Δr as a function of the total gravitational mass, in GR and various αR^2 gravity. Left panel: APR EOS in GR and six α_χ 's. Right panel: our four chosen EOSs in GR and α_{14} gravity.

$$\Delta r = \int_{r_{cc}}^{r_s} e^{\Lambda(r)} dr \quad (44)$$

with again an $e^{\Lambda(r)}$ factor to correct from the areal radius to the proper radius. On the left panel of Fig. 10 we can notice that at masses below $1.8M_\odot$ the crust thickness gradually diminishes as α increases while it slightly increases at $M \gtrsim 1.8M_\odot$. This reduction in Δr is very significant at low masses while in the relevant mass range $1.2 - 2.2M_\odot$ changes in Δr are effectively minor. It is interesting to notice that at $0.5M_\odot$ the radius of the star shrinks by 1 km when comparing GR and α_{14} models (see Fig. 5) which appears to be almost totally due to the shrinking of the crust that is also of the order of 1 km. In contradistinction, at larger masses the change in r_* is much larger than the change in Δr , i.e., most of the changes are occurring in the core. The right panel of Fig. 10 shows that these effects are quite independent of the employed EOS.

F. Surface gravity

The last structural characteristic with an impact on thermal evolution that we consider is the surface gravitational acceleration g_s . This quantity affects several important observable properties as, e.g., the shape of the thermal spectrum observable in the soft x-ray band (see, e.g., for [118–120]) and the thickness of the envelope that relates the surface temperature to the interior one (see, e.g., [121] and Sec. VA).

In general relativity, near the surface of the star the hydrostatic equilibrium equation reduces to

$$\frac{dP}{dr} = -g_s \rho, \quad (45)$$

where g_s is the *local surface gravity* defined by

$$g_s := g(r)|_{r=r_*} \quad (46)$$

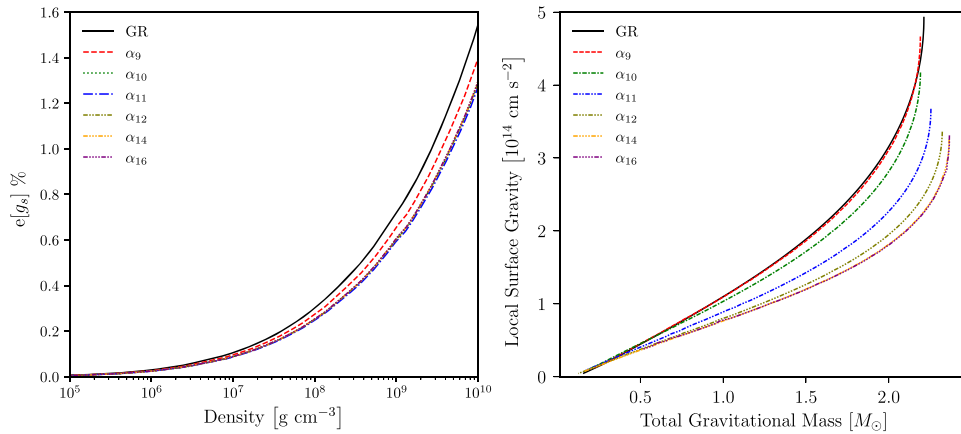


FIG. 11. Gravitational acceleration at the surface of neutron star models built with GR versus αR^2 gravity. Left panel: percent difference between g_s and $g(r)$ at densities close to the surface of the star, considering a model using the APR EOS with $\rho_0 = 1.5 \times 10^{15} \text{ g cm}^{-3}$ in both GR and six α_χ 's. Right panel: local surface gravity for the APR EOS and the indicated α_χ 's as functions of the total gravitational mass.

with

$$g(r) = c^2 e^{-\Lambda(r)} \frac{d\Phi}{dr}. \quad (47)$$

We, thus, obtain a simple expression for the surface gravity in GR as

$$g_s^{(\text{GR})} = \frac{GM}{r_*^2} \left[1 - \frac{2GM}{c^2 r_*} \right]^{-1/2}. \quad (48)$$

with M being the total gravitational mass. Within the αR^2 theory it is trivial to verify that Eqs. (45), (46), and (47), also hold. To verify that Eq. (48) holds let us define

$$e[g_s]\% = 100 \left| \frac{g_s - g(r)}{g_s} \right| \quad (49)$$

where g_s is calculated from Eq. (48) with $M \rightarrow M_{\text{surf}}$ and $g(r)$ from Eq. (47). This quantity is plotted in the left panel of Fig. 11 which proves that employing Eq. (48) for g_s is accurate to within 1% within αR^2 models as well as in GR. It is, hence, acceptable to keep g_s as defined above but with $M = M_{\text{surf}}$:

$$g_s^{(\alpha R^2)} = \frac{GM_{\text{surf}}}{r_*^2} \left[1 - \frac{2GM_{\text{surf}}}{c^2 r_*} \right]^{-1/2}. \quad (50)$$

On the right panel of Fig. 11 it is clear that αR^2 -stars have lower values of g_s in comparison with their GR counterparts for the same (total) gravitational mass, a direct effect of reduction of M_{surf} with respect to M_{tot} .

V. THERMAL EVOLUTION

A. Numerical implementation and envelope models

For simulating the thermal evolution of neutron stars, the fully relativistic 1-D code NSCOOL was employed [122]. As is traditional in neutron star cooling modeling, the star is split in two parts: the interior where the EOS is treated as temperature independent, and a low density outer region, the envelope, where the EOS is T -dependent. In the interior the Eqs. (33) and (34) are solved with a fully implicit Heyney scheme with the inner boundary condition

$$L(r=0) = 0 \quad (51)$$

and an outer boundary condition, at a density ρ_b , provided by an envelope model. In such a scheme the hydrostatic structure of the interior does not change with time and is only calculated once at the beginning of the simulation. For the core EOS we consider the same 4 models as in Sec. IV.

In an envelope model for the outer boundary condition it is assumed that (1) there are no significant energy sources or sinks within it and (2) the thermal timescale of this thin layer is much shorter than the one of the interior. Thus, the

time derivatives in Eq. (34) are neglected and, as a result, in the envelope the luminosity is uniform, i.e., $L(r) = L_*$ where L_* is the surface luminosity, and only the heat transport equation (33) needs to be solved. The integration starts with choosing a surface or *effective* temperature, T_e , defining the luminosity as

$$L_* = 4\pi r_*^2 \sigma_{\text{SB}} T_e^4 \quad (52)$$

in analogy with the blackbody flux $F = \sigma_{\text{SB}} T^4$, σ_{SB} being the Stefan-Boltzmann constant. Using a photospheric boundary condition (typically the Eddington condition, see, e.g., [123]) the density ρ_e and pressure P_e corresponding to T_e are obtained and from this Eq. (33), and the simplified version of the hydrostatic equilibrium (45) adequate for a thin envelope, are integrated inward until ρ reaches ρ_b with a temperature T_b . Notice that, once taken into account the trivial adjustment of g_s from Eq. (48) to Eq. (50), the equations to be solved, Eq. (33) and (45), are the same as in GR and thus all envelope models calculated within GR automatically apply within αR^2 gravity. For a given g_s , by varying T_e one gets a family of models generating what is commonly referred to as a “ $T_e - T_b$ relationship,” $T_e = T_e(T_b, g_s)$, which we take as our outer boundary condition: at ρ_b the temperature T_b and luminosity L_b must be such that

$$L_b = L_* = 4\pi r_*^2 \sigma_{\text{SB}} T_e(T_b, g_s)^4. \quad (53)$$

It turns out there is a simple scaling of T_e in terms of g_s [124] as $T_e(T_b, g_s) = g_{s14}^{1/4} T_e(T_b, g_{s14} = 1)$ where $g_{s14} = g_s / 10^{14} \text{ cm s}^{-2}$. This envelope acts as a thermal insulator between the hot interior and the surface and a lower g_s allows for a thicker envelope, hence a more efficient insulation, resulting in a lower T_e , for a given T_b , as described by this scaling. One frequently encounters the redshifted effective temperature and we will follow standard notations and write it as $T_e^\infty \equiv e^{\Phi(r_*)} T_e$.

Various types of envelope models have been presented considering variations in the strength and geometry of the magnetic field [125] and the chemical composition [126]. It turns out that, excluding the case of magnetar size magnetic fields, the effects of the chemical composition are much more significant than the magnetic ones [127] and we will here neglect the latter. Variation in the chemical composition range from the presence of low- Z elements such as ^4He and ^{12}C up to iron-like ones. Low- Z material having higher thermal conductivity than high- Z ones, their presence in the envelope will result in an increase of T_e , for a given T_b , compared to the case of an envelope made of the latter. This effect is accounted in the $T_e - T_b$ relationship through the adimensional parameter

$$\eta = g_{s,14}^2 \frac{\Delta M_L}{M_{\text{surf}}} = \left(\frac{P_L}{1.193 \times 10^{34} \text{ dyn cm}^{-2}} \right), \quad (54)$$

where ΔM_L is the mass of this light element layer, M_{surf} is the surface gravitational mass of the star, and P_L is the maximum pressure at which low- Z nuclei exist. When the outer layers of a newborn neutron star are formed at very high temperatures it is natural to expect that matter will reach its ground state, commonly called “catalyzed matter,” that consist of iron-peak nuclei (see, e.g., Sec. 3.1 in [128]). However late fall-back after the supernova explosion or posterior accretion may, or may not, deposit light elements at the surface and this results in a large uncertainty in η that is an inherent aspect of the cooling theory. There is to date, unfortunately, no known way of determining the thickness of this light element layer.

Some insight on the basic issues of neutron star cooling can be gained by considering simplified models that allow for illustrative analytical solutions as presented, e.g., in the reviews of Refs. [74,91,129].

After briefly reviewing observational data in the next subsection VB we will present our results by first considering, in Sec. VC, simple cooling models that emphasize the difference between M_{tot} and M_{surf} and in Sec. VD consider more realistic models that include pairing and uncertainties in the envelope composition. Finally we focus on two particular stars, “Cas A” in Sec. VE and “NS 1987A” in Sec. VF, that are the two youngest known neutron stars and present interesting features.

B. Observational data on isolated cooling neutron stars

One of the Holy Grails of early x-ray astronomy was direct observation of the surface of a neutron star, i.e., detection of its thermal emission. Observations by the *ROSAT* satellite in the late 1980s finally reached this goal in the case of three nearby pulsars (see, e.g., the argumentation presented by H. Ögelman in [130]). Three decades later, data on the surface temperature of isolated cooling neutron stars are available for about fifty objects and we display in Fig. 12 the latest compilation from [131]. We will not here get into the tricky description and discussion of the numerous source of error and uncertainty in these data but simply refer to [131] (earlier similar content can be found, e.g., in [91,92,132]) and will only mention a few key points (and a few key bibliographic references). The first choice to be made when analyzing soft x-ray data likely to be thermal photons from the neutron star surface is the model spectrum to be applied: is a simple blackbody adequate or is a detailed atmosphere model needed [133]? This choice is related to the total absence of direct information, e.g., from spectral lines, about the chemical composition of the stellar atmosphere: is it made of iron-like nuclei as in a textbook neutron star [35], in which case a blackbody spectrum is a reasonable first approximation, or have light elements been deposited at the surface after the birth of the neutron star? Fitting with light element atmosphere models commonly lead to inferred surface temperatures that are about twice lower than the one inferred from a blackbody fit [134].

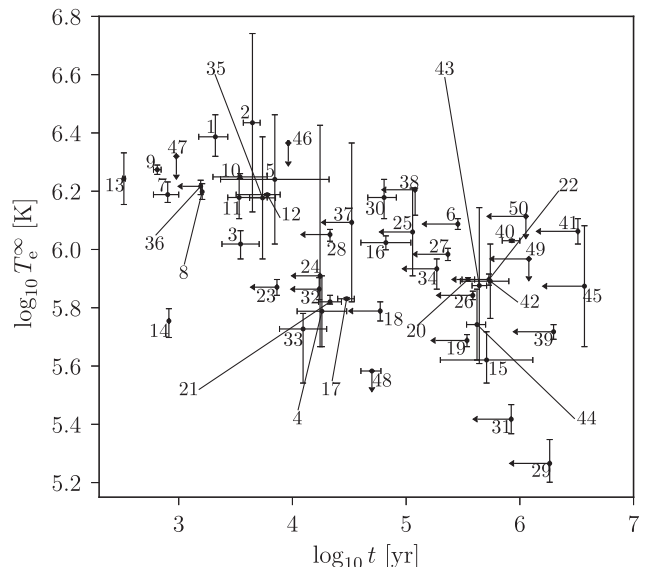


FIG. 12. The sample of observational data considered for this work, taken from [131]. Error bars correspond to 1σ . Plotted objects are: I weakly magnetized thermal emitters (1) 1E 0102.2 – 7219, (2) RX J0822.0 – 4300 (in Puppis A), (3) CXOU J085201.4 – 461753 (in Vela Jr.), (4) 2XMM J104608.7 – 594306 (in Homunculus), (5) 1E 1207.4 – 5209, (6) 1RXS J141256.0 + 792204, a.k.a. “Calvera”, (7) CXOU J160103.1 – 513353, (8) 1WGA J1713.4 – 3949, (9) XMMU J172054.5 – 372652, (10) XMMU J173203.3 – 344518, (11) CXOU J181852.0 – 150213, (12) CXOU J185238.6 + 004020 (in Kes 79), (13) CXOU J232327.8 + 584842 (in Cas A), II ordinary pulsars (14) PSR J0205 + 6449 (in 3C 58), (15) PSR J0357 + 3205, a.k.a. “Morla”, (16) PSR J0538 + 2817, (17) CXOU J061705.3 + 222127, (18) PSR J0633 + 0632, (19) PSR J0633 + 1746, a.k.a. “Geminga”, (20) PSR B0656 + 14, (21) PSR B0833 – 45 (in Vela), (22) PSR B1055 – 52, (23) PSR J1357 – 6429, (24) PSR B1706 – 44, (25) PSR J1740 + 1000, (26) PSR J1741 – 2054, (27) PSR B1822 – 09, (28) PSR B1823 – 13, (29) PSR J1836 + 5925, (30) PSR B1951 + 32 (in CTB 80), (31) PSR J1957 + 5033, (32) PSR J2021 + 3651, (33) PSR B2334 + 61, III high-field pulsars (34) PSR J0726 – 2612, (35) PSR J1119 – 6127, (36) PSR B1509 – 58, (37) PSR J1718 – 3718, (38) PSR J1819 – 1458, IV “The Magnificent Seven” (39) RX J0420.0 – 5022, (40) RX J0720.4 – 3125, (41) RX J0806.4 – 4123, (42) RX J1308.6 + 2127, (43) RX J1605.3 + 3249, (44) RX J1856.5 – 3754, (45) RX J2143.0 + 0654, V neutron stars with only upper limits on thermal emission (46) PSR J0007 + 7303 (in CTA 1), (47) PSR B0531 + 21 (in Crab), (48) PSR B1727 – 47 (in RCW 114), (49) PSR J2043 + 2740, (50) PSR B2224 + 65 (in Guitar nebula). To help the reader we also added, for several famous stars, their nickname or, in parenthesis, the associated supernova remnant.

In the presence of a strong magnetic field atomic structure is dramatically altered resulting in significant changes in the prediction of the spectral shape [135]. In that case, the strength of the dipolar component of the magnetic field at the surface of the star can be roughly estimated from the pulsar spin-down, but the presence of strong multipolar

components cannot be excluded [125]. Moreover, the presence of a strong magnetic field results in a nonuniform surface temperature distribution [136,137] obscuring the relationship between a single value of surface temperature obtained from the spectral fit and the effective temperature. Almost complete lack of information about the mass M and radius r_* of the observed neutron stars imply that their values usually have to be guessed. M and r_* determine the surface gravity, that controls the atmosphere thickness, and the redshift. The distance D and age t of the stars have observational estimates but often with large uncertainties. The soft x-ray spectrum is often contaminated by a harder component whose origin is not always clear and, moreover, it can be reprocessed by resonant scattering when the photons have to pass through in the magnetosphere when escaping from the surface [138]. Finally, a significant fraction of the soft photons are absorbed by the interstellar matter and their contribution to the original spectrum has to be reconstructed from the distribution of the detected ones. The reader can find complementary discussions in the reviews [119,120,139,140].

This plethora of uncertainties means that the inferred effective temperature, and age, of any isolated neutron star has to be taken with caution. However, having now at hand some 50 such values one can hope that they present a credible overall picture of isolated neutron star cooling, Fig. 12, and that the spread, at a given age, in deduced temperatures is a real feature implying a broad range of evolutions, likely related to a mass effect, and not a cumulative result of the above described uncertainties.

C. Simple cooling models and impact of local surface gravity

As a first step in our exploration we consider a neutron star model with a total gravitational mass of $1.4M_\odot$ built with the APR EOS, without superfluidity/superconductivity

effects and employing an envelope containing only heavy elements. The cooling of such a star is shown in the left panel of Fig. 13 comparing the GR case with five modified gravity cases. We see a very rough agreement between the predicted T_e^∞ and the observational data. Better agreement with the data can be reached with more sophisticated models that will be presented below.

In standard GR cooling, one distinguishes two different eras, named according to the particle species that dominate energy losses. Initially we have the *neutrino cooling era* lasting till an age of $\sim 10^{5-6}$ yrs at which time the slope of the $T_e^\infty - t$ curve significantly increases, indicating the transition to the *photon cooling era*. These epochs are clearly visible in the αR^2 models as well. During the first era we see that increasing α results in models with lower T_e^∞ while during the second era this behavior slightly inverts, yielding less steep slopes.

Notice that during the first few decades, the “plateau,” very little evolution is seen as surface temperature is completely controlled by the slow evolution of the crust, region which is not yet sensitive to the temperature of the deep interior. At ages $\sim 10^2$ yrs, heat has had enough time to flow from the crust into the core, thus provoking a rapid surface temperature drop after which crust and core have reached an isothermal state and, consequently, the surface temperature is a direct reflection of core’s temperature. These different evolutionary phases are described in more details in, e.g., Ref. [65,92,129].

To understand the difference in cooling behavior appearing in the left panel of Fig. 13 when gravity is modified, we notice that it is unlikely due to a change in the deepest part of the star since for the APR EOS and a $1.4M_\odot$ star the central density practically does not change with gravity, as seen in the right panel of Fig. 5. However, the left panel of the same figure shows that the radius of the star significantly increases with α which itself implies a decrease of

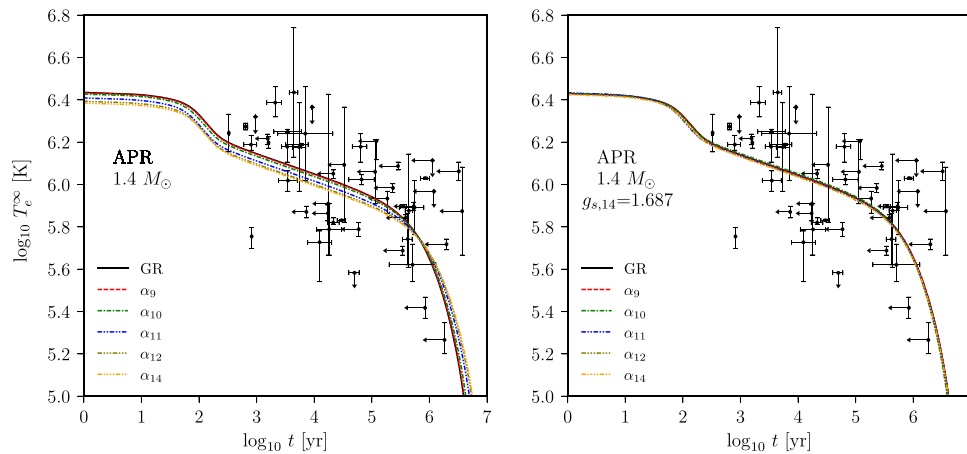


FIG. 13. Left panel: cooling curves of a simple $1.4M_\odot$ APR-neutron star model, within GR and αR^2 gravity for five α_X 's. Right panel: same set of models but with the surface gravity g_s fixed to the GR model's value, $1.687 \times 10^{14} \text{ cm s}^{-2}$. In both panels superfluidity/superconductivity is absent and the envelope is made of heavy elements.

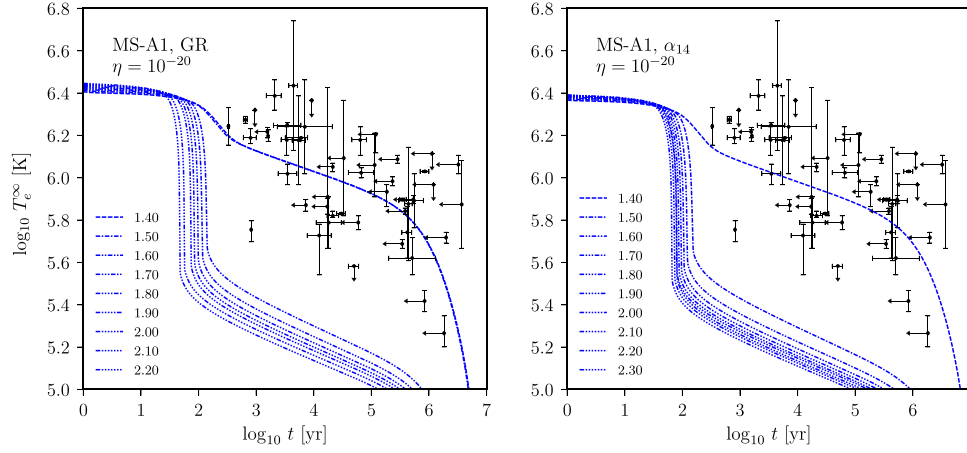


FIG. 14. Cooling curves for the MS-A1 EOS, considering GR in the left panel and $\alpha_{14}R^2$ gravity in the right panel. Superfluidity is absent in both panels, and η is fixed to 10^{-20} (i.e., heavy elements envelope).

the surface gravity, exhibited in the right panel of Fig. 11. This simple change of g_s has an immediate effect on T_e through the thickening of the envelope [see paragraph below Eq. (53)]. To illustrate this, in the right panel of Fig. 13 we present the same models but where the value of g_s , as it enters the “ $T_e - T_b$ relationship,” has been artificially fixed to its GR value, demonstrating that change in g_s explains most of the effect of modified gravity in these simple models.

Having explained the similarities and differences that both gravity theories exhibit in simple cooling curves, in Fig. 14 we compare a larger set of neutron star mass-models for the MS-A1 EOS. Among the covered masses, in GR the 1.4 and 1.5 $M_\odot$ models are below the critical mass M_{DU} and the dramatic effect of the DUrca process is clearly seen in all more massive stars. During the early plateau T_e^∞ is controlled by the crust and is not yet sensitive to the cooling of the core and the effect of the DUrca process only appears at the end of this phase. In the case of the α_{14} models, their behaviors are very similar to the GR case, with the significant difference that the critical mass M_{DU} is lower than in GR resulting in the 1.5 $M_\odot$ model already exhibiting fast DUrca cooling.

D. Realistic cooling models: Superfluidity/superconductivity and light element envelopes

After the presentation of very simple models in the previous section we are now in position to consider a case study of realistic scenarios in which two additional essential ingredients are present, namely pairing which can strongly suppress neutrino emission and the presence of light elements in the envelope which significantly alters the observable T_e^∞ . We will compare models in GR and in α_{14} built with the MS-A1 EOS, considering masses between 1.1 and 1.7 $M_\odot$ in steps of 0.1 $M_\odot$. In the GR case models up to 1.5 $M_\odot$ have slow neutrino emission and only the 1.6 and 1.7 $M_\odot$ models have the DUrca process allowed, while in

the α_{14} case the 1.5, 1.6, and 1.7 $M_\odot$ models have the DUrca process acting. Models with higher mass result in even lower temperatures than the 1.7 $M_\odot$ ones and we do not include them to avoid saturating the figures.

We present in Fig. 15 the cooling of these more complex models where several distinct effects can be noticed. A first look shows that the predicted range of T_e^∞ is broader than those of the previous section (see Fig. 14), effectively covering the inferred values of most of the 50 neutron stars of our adopted dataset. This is in line with similar results known for a long time [91,93]. For models with masses below M_{DU} , there is very little mass dependence and they form bands in the $\log_{10} T_e^\infty - \log_{10} t$ plots, different bands corresponding to different assumed depths of the light element layer in the envelope [parametrized by η , see Eq. (54)]. For models with masses above M_{DU} there is a clear mass dependence determined by the size of the inner core region where the DUrca process is acting and, for a given mass, we again have a broad range of predicted $\log_{10} T_e^\infty$ depending on the assumed value of η .

The vertical spread of the cooling curves with models of mass below M_{DU} , which arises from the varying amount of light elements in their envelopes, allows to fit the hottest stars on the sample. On the other hand, models with $M \geq M_{\text{DU}}$ in which the fast neutrino emission of the DUrca process has been controlled by pairing can explain the colder objects.

Let us consider in more detail the effects of increasing the amount of light elements in the envelope, parametrized by η in Eq. (54), by considering the neutrino cooling era of stars with masses below M_{DU} . We have five bands of cooling curves for the adopted five values of η , each band encompassing masses from 1.1 to 1.5 $M_\odot$ in GR and 1.1 to 1.4 $M_\odot$ in α_{14} : varying the mass has little effect, more massive stars being only slightly cooler than less massive ones, while increasing η significantly raises T_e^∞ because of the higher thermal conductivity of light elements compared

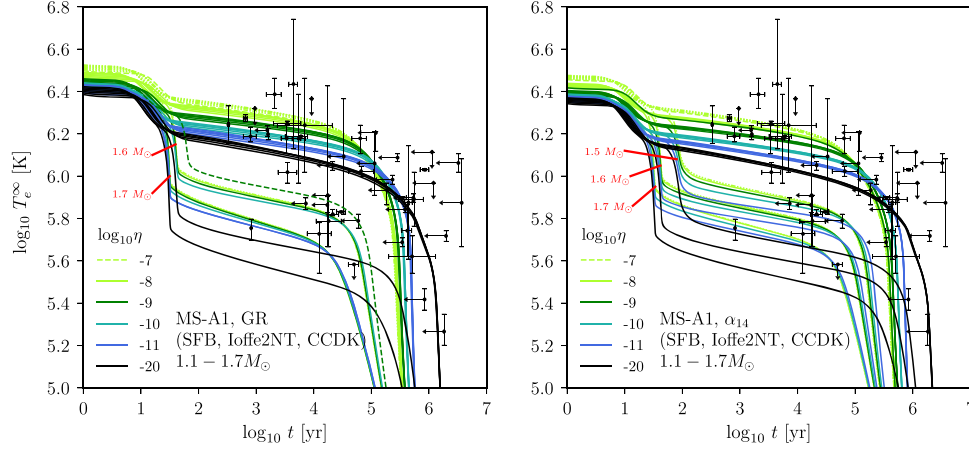


FIG. 15. Cooling curves for the MS-A1 EOS, considering a range of stellar masses and several values of η and the indicated superfluidity gaps. Left panel: GR. Right panel: $\alpha_{14}R^2$ gravity. Models with masses above M_{DU} are explicitly labeled. Dashed curve in the left panel is a $1.55M_{\odot}$ model. See text for description of the various cooling curves.

to iron-peak ones. Within such a type of scenario the spread of T_e^{∞} in hot young neutron stars is a result of their having accreted different amounts of matter in their past, either when they were formed in a supernova explosion or during their later evolution. Notice that during the neutrino cooling era models with different values of η have identical internal temperatures, since neutrino losses control this temperature, and the ones with larger η just appears warmer at the surface. When entering the photon cooling era the situation is reversed: models with larger η will emit more photons that result in a faster cooling. This inversion is clearly seen in both in the GR and the α_{14} cases and one can also notice that models with larger η and larger photon luminosity naturally enter the photon cooling era earlier than models with lower η .

The spectacular effect of suppression of the fast DURca neutrino emission is clearly seen by comparing Fig. 14 with Fig. 15: in the first case, all models with the DURca process acting have T_e^{∞} well below all observed values while once the neutrino emission is controlled by pairing, T_e^{∞} comparable to observed ones are easily obtained. Together with the effects of the presence of light elements in the envelope, we are able to generate families of models that easily cover the range of T_e^{∞} for the coldest observed young neutron stars.

We emphasize that the neutron star mass range that describes stars with ages below 10^5 years, depends on the assumed EOS. In the case of the APR EOS, with $M_{\text{DU}} = 1.97M_{\odot}$, models with masses above $2M_{\odot}$ are needed to produce cold stars in that age range.

For curiosity, there is an interesting effect that is clearly visible in the $1.5M_{\odot}$ models within $\alpha_{14}R^2$ gravity. After the first temperature drop due to the thermal relaxation of the crust at ages just above 10^1 yrs, there is a second, deeper, temperature drop at ages just below 10^2 yrs: this second temperature drop results from the thermal relaxation of the

core whose innermost part cools very rapidly from the DURca neutrino losses while the outer core's thermal energy flows into the inner core on a finite timescale which turns out to be of the order of 10^2 years. This can be visualized as a “cold front” propagating outward which results in a drop in T_e^{∞} when it reaches the surface of the star. Similar behaviors have been described previously in Refs. [141,142] This long timescale is due to the fact that in this particular model the DURca inner core is very small as the central density is just barely higher than ρ_{DU} . The $1.6M_{\odot}$ and $1.7M_{\odot}$ models, in both GR and $\alpha_{14}R^2$ gravity, also exhibit the same effect but on a shorter timescale, because they have a much larger inner core and thus thinner outer core, which by superficial examination may be confused with an extended crust relaxation.

Finally, comparing the two panels of Fig. 15, it is clear that the small differences in cooling behaviors when altering the gravity theory from GR to αR^2 are much smaller than the effects of either microphysical (as, e.g., pairing) or astrophysical (as mass or envelope composition) ingredients.

E. The cooling of the Cassiopeia A neutron star

The neutron star in the Cassiopeia A supernova remnant was discovered in the Chandra first light observation [144]. It is the youngest known neutron star, its supernova remnant having a kinematic age of ~ 340 yrs [145] that is supported by a possible historical observation of the supernova by J. Flamsteed on August 16th, 1684 [146]. Its soft x-ray thermal spectrum is well described by a non-magnetized carbon atmosphere model [147] implying a surface temperature $\sim 2 \times 10^6$ K. Analysis of 10 years of Chandra observations of the supernova remnant pointed to a rapid cooling of the neutron star [148], its effective temperature having apparently decreased by $\sim 4\%$ during

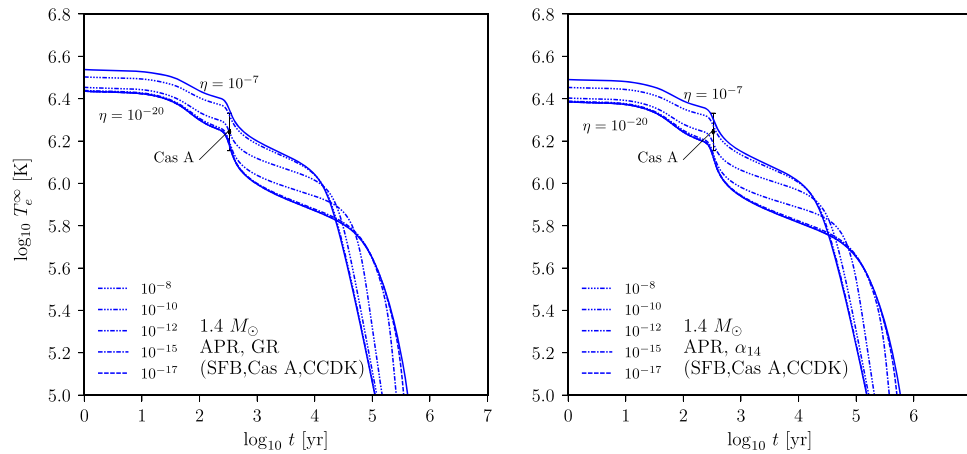


FIG. 16. Comparison of a $1.4M_{\odot}$ APR-neutron star model with several values of η in either GR (left panel) and $\alpha_{14}R^2$ gravity (right panel). The superfluidity/superconductivity gaps employed are indicated on each panel and were chosen to induce a rapid cooling of the star at the age of the Cas A neutron star.

that time span. Analyses of another decade of observations reached a similar conclusion [149,150] while a different approach [151–153] found a much smaller decrease in T_e^{∞} , if any at all.

We here accept the possibility of rapid cooling of this neutron star and the interpretation of this evolution being due to a recent sudden increase in the neutrino luminosity resulting from the triggering of the PBF process at the initiation of the neutron 3P_2 superfluidity phase transition [102,154]. In this scenario, protons became superconducting at an early time, suppressing neutrino emission from all processes involving protons (modified Urca, n-p and p-p bremsstrahlung, see Table I), thus slowing the early cooling of the star. Due to this slow initial cooling, the core of the neutron star reached the neutron 3P_2 superfluidity critical temperature T_c only very recently, from which it was inferred that T_c must be of the order of $\sim 5 \times 10^8$ K [102,154].

We display in Fig. 16 this model in GR and in α_{14} modified gravity considering a star with a total gravitational mass of $1.4M_{\odot}$ and pairing gaps chosen to produce the desired cooling behavior. The model in the left panel, reproducing the results of [102], displays a rapid cooling at ages around 300 yrs and an effective temperature $T_e^{\infty} \simeq 1.55 \times 10^6$ K at the present age of the neutron star. As seen previously, α_{14} models are slightly colder than GR ones but a simple adjustment in the depth of the light element layer in the envelope can counteract this effect: with $\eta \simeq 10^{-10}$ (see Eq. (54), compared to $\eta \simeq 10^{-12}$ in the GR case) we again can reproduce the observed T_e^{∞} and have a rapid cooling.

F. The cooling of NS 1987A

As a last case study we consider the possible recent discovery of the neutron star NS 1987A produced by the

supernova SN 1987A, whose putative existence was revealed by the presence of a blob of warm dust close to center of explosion [109,155]. From the luminosity of the blob a luminosity $\sim 1-3 \times 10^{35}$ erg s $^{-1}$ is inferred for the neutron star. With an age of 28 yrs at the time of the discovery observation, in 2015 by the ALMA observatory [155], this very young object allows us to probe the thermal evolution of a neutron star during the early plateau. This stage of evolution is quite insensitive to the physics of the core and the dominant characteristics that influence it are: in the first place, the thickness of the crust which geometrically determines the cooling timescale; secondly, the extent of 1S_0 neutron superfluidity in the inner crust which can strongly reduce the specific heat and fasten the cooling, and, finally, the thickness of the light elements layer in the envelope which controls the emerging photon luminosity.

Within GR, this scenario was studied in detail in [109] and we reproduce in Fig. 17 one set of models from that work and compare them with their $\alpha_{14}R^2$ gravity counterparts. This early evolution being dominated by crust physics and since, as we showed in Fig. 10, the size of the crust happens to be little affected by the change in the gravity models for stars of mass $\sim 1.4M_{\odot}$, the only gravity effect of relevance is the reduction of the surface mass M_{surf} which controls the thickness of the envelope and the light element layer: comparing the two panels of Fig. 17 one sees that the only significant difference is an almost uniform reduction of the luminosity in the $\alpha_{14}R^2$ models compared to their GR counterparts, the same effect we have identified as a reduction of T_e^{∞} in § V C. As a result, the interpretation of the warm dust blob as being a sign of the presence of NS 1987A is still sustainable within the αR^2 theory. However, an extreme modification as in the α_{14} case requires a significantly thicker light element surface layer: in GR gravity a light element layer reaching a density of $\rho_L \sim 10^9$ g cm $^{-3}$ results in luminosities comparable to the lower

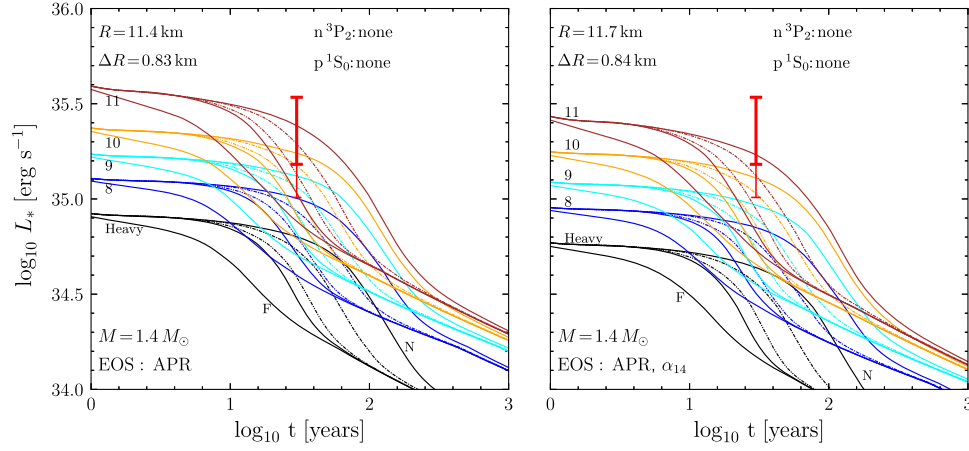


FIG. 17. Cooling curves for a $1.4M_{\odot}$ APR neutron star model. Left panel: GR. Right panel: $\alpha_{14}R^2$ gravity. In each panel the (red) vertical bar marks the 1σ range of the inferred surface thermal luminosity of the putative neutron star NS 1987A (the lower extension of the bar extends the range when taking into account preheating of the dust by ^{44}Ti decay). In each case five families of cooling models are presented that have different amount of light elements in their envelopes, ranging from no light elements and marked as “heavy” and light elements present up to a density of $\rho_L = 10^L \text{ g cm}^{-3}$ with $L = 8, 9, 10$, and 11 , as labeled. Within each family, the linestyle corresponds to a different assumed neutron $^1\text{S}_0$ superfluid gap: continuous lines correspond to assuming inner crust neutrons are either fully superfluid, labeled as “F” in the heavy element envelope case, or not superfluid at all, labeled as “N”, while the three other lines have linestyle as in Fig. 1.

requirement of $L_* \sim 10^{35} \text{ erg s}^{-1}$ while within the extreme case of $\alpha_{14}R^2$ gravity $\rho_L \sim 10^{10} \text{ g cm}^{-3}$ is implied. However, the analysis of Ref. [109] showed that, even if light elements such as O may avoid destruction by nuclear burning to such high densities as $\rho_L \sim 10^{10} \text{ g cm}^{-3}$ in the hot environment of a young neutron star envelope (see Fig. 3 of that paper), how such a thick layer of light elements may have been deposited is unclear. Even accretion at the Eddington rate (i.e., at about $10^5 \text{ g cm}^{-2} \text{ s}^{-1}$) would have taken about 20 yrs and the resulting luminosity, $L_{\text{Edd}} \sim 10^{38} \text{ erg s}^{-1}$, would have exceeded observational upper limits on the luminosity of that object [156]. Thus, extreme change in gravity within the αR^2 gravity scheme may present a challenge for this interpretation of the ALMA observation revealing the presence of the neutron star NS 1987A.

VI. DISCUSSION

In this work we have studied in detail how a shift in the description of gravity from general relativity to an $f(R)$ type of theory, in the particular case of $f(R) = R + \alpha R^2$, modifies our description of the structure and thermal evolution of neutron stars. Conversely, we also examined the potentialities for constraining such modification of gravity theory using neutron star observables.

With regard to the structure of dense matter above $10^{13} \text{ g cm}^{-3}$ [157], due to the uncertainties on the nature of strong interactions at high densities and the limitations of many-body theory in this regime, we employed four different EOSs that cover the present range of expectations in the case this matter consists only of nucleons and leptons [109,158]. Specifically, the EOSs MS-A1, MS-B1, and

MS-C1, were designed so that, together with the APR EOS, the deduced radius of a $1.4M_{\odot}$ neutron star is between 11.6 and 13.2 km, as the latest observational constraints indicate is the most likely range [159], and all four have a maximum mass above $2.1M_{\odot}$ in agreement with the most massive pulsar mass measurements, $1.97 \pm 0.04M_{\odot}$ for PSR J1614 – 2230 [47], $2.01 \pm 0.04M_{\odot}$ for PSR J0348 + 0432 [160], and $2.17 \pm 0.11M_{\odot}$ for PSR J0740 + 6620 [161]. We do not expect the consideration of EOSs with more degrees of freedom, as, e.g., the inclusion of hyperons or deconfined quarks, to have any significant impact on our conclusions as we discuss below.

As a first step we studied global properties of neutron stars, their mass and radius. Three different mass concepts appear, the baryonic mass, M_{bar} , and the total and surface, M_{tot} and M_{surf} , gravitational masses. The latter two are obtained from the metric, Eq. (39), and are identical in GR while $M_{\text{surf}} < M_{\text{tot}}$ in αR^2 theories. Neutron star mass measurements are typically obtained from binary orbital motion, i.e. in the regime $r \gg r_*$, and are thus providing us with M_{tot} . For this reason we label all our neutron star models with the total gravitational mass of the model. The baryonic mass is not directly measurable but is an intrinsic property of the neutron star that is independent of gravity, being just its baryon number multiplied by the nucleon mass. The surface gravitational mass is potentially measurable as it should strongly affect the bending of photon trajectories nearby the stellar surface [137,162,163] and light curve modeling may be able to find differences between M_{surf} and the binary motion inferred M_{tot} in the case of very accurate modeling as possible with NICER data (see, e.g., Ref. [164]). We found, see Fig. 4, that

the difference between M_{tot} and M_{surf} depends only on α and almost nothing else for nonrotating stars, at least within the range of measured gravitational masses and the presently restricted range of EOSs. This difference can reach half a solar mass in the extreme case of $\alpha_{14}R^2$ gravity opening a promising road to confirm or refute such modifications of gravity.

With respect to the effects of modified gravity on the mass-radius and mass-central density relationships our results, presented in the Figs. 5 and 6, confirmed results of previous authors, e.g., [52,53,78,106].

The main effect is that, independently of the employed EOS, larger gravitational masses are reachable within αR^2 gravity theories. Stellar radii are larger, except for very low mass stars, and central densities are lower, for high mass stars, in αR^2 gravity stellar models than in the corresponding GR models with the same M_{tot} , but what is precisely meant by “very low mass” and “high mass” is dependent on the EOS.

We considered finer details of the internal structure, such as the total proper physical volume of the star, Fig. 7, the threshold mass for the direct Urca neutrino emission process, Fig. 9, the thickness of the crust, Fig. 10, and the surface gravitational acceleration g_s , Fig. 11. In the cases of the volume and crust thickness, we found no difference between GR and αR^2 gravity that we could expect to lead to some clear observable signal. A reduction of g_s by up to a factor of two is possible and its consequences are discussed below in the context of thermal evolution. Regarding the direct Urca process, its occurrence is determined by the matter density being above some EOS dependent critical density ρ_{DU} and any star whose central density ρ_0 is above ρ_{DU} can undergo very fast neutrino cooling. A possible observable, or observationally deducible, quantity would be the critical total gravitational mass M_{DU} at which ρ_0 reaches ρ_{DU} . In the case ρ_{DU} is around or below $10^{15} \text{ g cm}^{-3}$, the difference in M_{tot} between GR and αR^2 gravity is quite small and thus M_{DU} is not significantly altered by this class of modified gravity. In contradistinction, in the case of an EOS for which ρ_{DU} is well above $10^{15} \text{ g cm}^{-3}$, as in our two chosen EOSs APR and MS-C1, M_{DU} is increased by up to $0.2M_{\odot}$ in $\alpha_{14}R^2$ gravity compared to GR: at present time the actual value of ρ_{DU} is uncertain, as exemplified by the range it covers in our four chosen EOSs, but, being optimistic, future progress in both dense matter theory to constrain ρ_{DU} and mass measurements of some neutron stars undergoing fast neutrino cooling could provide us with a handle to constraint modifications to gravity theory in the case ρ_{DU} turns out to be large. Fast neutrino emission through some form of direct Urca process can also be the result of the presence of deconfined quark matter, hyperons, or charged meson condensates (see, e.g., [82] for a brief description) and in all these cases there is also a corresponding critical density so that the same considerations

apply. Evidence for fast neutrino cooling can be inferred from neutron stars in transiently accreting binary systems [132,165] for which mass measurements, or at least mass estimates, are possible. At present time a few such systems have been identified, as SAX J1808.4-3658 [166,167], 1H 1905 + 000 [168], MXB 1659-29 [169], and HETE J1900.1-2455 [115], but none of them has mass measurement.

Turning now to the impact of modified gravity on the cooling of neutron stars we first showed in Sec. III that the thermal evolution equations retain the same form in αR^2 gravity as they have in GR. This has the convenient implication that a neutron star cooling code such as NSCOOL [122] can be immediately used without any change. In a first, very simple scenario, we exhibited in Fig. 13 a trivial effect of αR^2 gravity in that the reduction in g_s results in slightly lower surface temperatures during the neutrino cooling era. However, even small changes in the envelope chemical composition, as displayed, e.g., in Fig. 15, can induce a rise in the surface temperature during the neutrino cooling era that can be much more important than the small decrease due to the αR^2 gravity reduction in g_s . This effect is thus completely hidden by the “noise” from the uncertainty in the envelope chemical composition for which we usually have almost no constraint. Considering a set of stellar models covering a range of masses we showed Fig. 14 the immediate effect that a model, the $1.5M_{\odot}$ one in this particular example, that undergoes slow neutrino cooling in GR gravity because its mass is below $M_{\text{DU}} = 1.509M_{\odot}$, will experience fast neutrino cooling in $\alpha_{14}R^2$ gravity because its mass is now above new value of M_{DU} which has decreased to $1.474M_{\odot}$. As long as no mass measurements are available for isolated neutron stars such an effect is unfortunately only of academic interest and, moreover, a small change in ρ_{DU} due to a small change in the EOS properties can have the same effect of modifying M_{DU} , without any change in gravity modeling.

After these first results, we considered three case studies for a better evaluation of the magnitude of the impact of modifying the gravity theory on our ability to infer properties of dense matter from such studies. In the first case we revisited the scenario [93,170] in which fast neutrino emission, which results in very cold neutron stars if uninhibited, is controlled by pairing. This scenario is well known [93,170] (see, also e.g., the reviews [64,65] and the recent works [132,171]) to result in a wide range of predicted effective temperatures that allow to interpret all present data on isolated cooling neutron stars and we showed in Fig. 15 that the same conclusion holds even within the most extreme case of $\alpha_{14}R^2$ gravity theory. Since increasing α has no further noticeable effect compared to α_{14} while at smaller α the modifications are even smaller, we can conclude that the validity of this family of cooling scenarios, fast neutrino emission controlled by pairing, is not altered by modification of the gravity theory within the αR^2 scheme.

In the second case we revisited the intriguing possible observation of the cooling of an isolated neutron star in real time, the neutron star in the supernova remnant Cassiopeia A. We considered the specific scenario of sudden increase of the star's neutrino luminosity due to the very recent onset of neutron 3P_2 pairing in its core [102,154] and found, as shown in Fig. 16, that a small increase in the assumed thickness of the light element layer in its envelope, an essentially unknown quantity, is sufficient to counterbalance the effect of the change in gravity theory as already mentioned in the above described simple models. As a result, the underlying physical interpretation of the observation as due to the appearance of superfluidity is not affected at all.

Our last case study is the analysis of the recent identification of a likely first sign of the neutron star, dubbed NS 1987A, created in the SN 1987A supernova event [109,155]. A key point of the argumentation is that the expected thermal luminosity of a ~ 30 years old neutron star is sufficient to energize the dust blob suspected to conceal NS 1987A which needs an energy input of at least 10^{35} erg s $^{-1}$. Producing such a high thermal luminosity was shown to be possible, within GR gravity, in the case the outer layer of the neutron star contain a significant amount of light elements, as C or O [109]. In the case of αR^2 gravity, as shown in Sec. VC, slightly lower thermal luminosities are obtained with the consequences for NS 1987A, as shown in Fig. 17, that a thicker layer of light elements is needed compared to the requirements of analogous models in GR gravity. In the case of $\alpha_{14} R^2$ gravity, light elements would need to be present up to a density $\rho_L \sim 10^{10}$ g cm $^{-3}$ that may be difficult to reach as we discussed in Sec. VF. *If the existence of NS 1987A, and with such a high thermal luminosity, is confirmed, it may offer us an interesting object to test gravity theory.*

When finishing the present work we became aware of a similar work by Dohi *et al.* [77] in the context of the very similar theory $f(R) = R + R^{4/3}$. The difference between both theories appears to be small and the differences in the results are very small. In the case where cooling curves in GR and $\alpha R^{4/3}$ gravity are compared, as in their Figs. 11, 13, and 15, their results are in perfect agreement with ours. These authors label their neutron star models using M_{surf} instead of M_{tot} : as a result they find that the threshold mass M_{DU} can be much smaller in $\alpha R^{4/3}$ gravity than in GR. However, once employing the total mass M_{tot} as we do, considering this to be the measured one when a measurement is available, the difference is actually much smaller. In their conclusion, these authors mention that the crust relaxation time, which they refer to as the “knee,” can be affected by gravity theory and be used as a probe for it, a claim we strongly disagree with: we found that the crust thickness, a major ingredient in the crust relaxation time, is little affected by gravity changes, at least with the scheme we studied here. Moreover, there are many physical

ingredients in the crust that are still poorly determined and have a much larger effect than changing the underlying theory of gravity, as we have shown in our Figs. 13–17 and has been described with much detail within GR in, e.g. [92].

VII. CONCLUSIONS

We have presented an extensive study of the structure and evolution of isolated neutron stars within the $f(R) = R + \alpha R^2$ extension of gravity theory beyond the GR case of $f(R) = R$. We found, as was already hinted by previous authors who only studied the structure, that the deviations from GR saturate in both the structure and the evolution when α grows and theories with $\alpha \geq 10^{14}$ cm $^{-2}$ result in essentially identical predictions. This result has the important consequence that the whole range of predictions from this family of gravity theories can thus be covered by actually scanning only a finite range of values of α .

For a given high density EOS, predicted values for the maximum mass and the radii of stars with masses above about $1M_{\odot}$ are larger within αR^2 gravity theories than within GR gravity. However, the changes when going from GR to αR^2 gravity are comparable in size to changes that are obtained within GR simply by considering different EOSs. Thus, using neutron star mass and radius measurements to constrain theories of gravity do not seem to be a promising approach, as long as present uncertainties on the high density EOS remain. Nevertheless we have been able to identify one property, the surface gravitational mass M_{surf} , which is potentially measurable through light curve modeling, and differs in αR^2 gravity from the total gravitational mass M_{tot} that is measured from binary orbit, while both masses are identical in GR gravity. Most importantly, we have shown, see Fig. 4, that the difference $M_{\text{tot}} - M_{\text{surf}}$ depends only on α and turns out to be almost independent of M_{tot} and of the high density EOS. Measuring this difference would be a direct measurement of α , within the αR^2 gravity scheme.

Regarding the thermal evolution of neutron stars, many physical and astrophysical ingredients beyond just the EOS are needed for a realistic description. We have shown that uncertainties in these many ingredients have stronger impact in the models than the change in structure resulting from a change in gravity theory, at least with the αR^2 gravity scheme. By studying a few well-known cooling scenarios we have shown that modified gravity within the αR^2 scheme alters neither their validity nor the conclusions that may be derived from them regarding properties of dense matter. We nevertheless found one case, the neutron star NS 1987A in the remnant of SN 1987A, whose deduced high thermal luminosity requires the presence of a very thick layer of light elements at its surface: within αR^2 gravity with large α the presence of an unrealistically thick layer of light elements may be required resulting in a tension between such gravity theories and the interpretation

of the ALMA detected warm blob of dust luminosity being due to the heating from the young neutron star [109].

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