Minimally modified gravity with auxiliary constraints formalism

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We investigate the possibility of reducing the number of degrees of freedom (d.o.f.) starting from generic metric theories of gravity by introducing multiple auxiliary constraints (ACs), under the restriction of retaining spatial covariance as a gauge symmetry. Arbitrary numbers of scalar-, vector- and tensor-type ACs are considered *a priori*, yet we find that no vector- and tensor-type constraints should be introduced, and that scalar-type ACs should be no more than four for the purpose of constructing minimally modified gravity (MMG) theories which propagate only two tensorial d.o.f., like general relativity (GR). Through a detailed Hamiltonian analysis, we exhaust all the possible classifications of ACs and find out the corresponding minimalizing and symmetrizing conditions for obtaining the MMG theories. In particular, no condition is required in the case of four ACs, hence in this case the theory can couple with matter consistently and naturally. To illustrate our formalism, we build a concrete model for this specific case by using the Cayley-Hamilton theorem and derive the dispersion relation of the gravitational waves, which is subject to constraints from the observations.

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I. INTRODUCTION

After the first binary black hole coalescence event, GW150914, detected by LIGO in 2015 [1], there have been more than fifty compact binary merger events reported by the LIGO-Virgo Scientific Collaborations (LVSC) [2,3] heralding the era of gravitational wave (GW) for astrophysics and cosmology which provides the first-ever window to explore the nature of gravity in the strong-field regime. So far, GR stands strongly against tests such as consistency checks [4], merger remnants [5] and the properties of the generation and propagation (e.g. the propagating speed, dispersion relations and polarization states) of GWs [6–13] (see Ref. [14] for reviews).

In particular, due to the spacetime diffeomorphism symmetry, GR predicts that only two transverse-traceless tensor modes of GWs are propagating. This is a significant feature to distinguish GR from usual modified gravity theories [15,16], in which additionally nontensorial polarization mode(s) [17–19] could be propagating as well. For instance, there is a scalar mode beyond the two tensor modes in the $f(R)$ and generic scalar-tensor (ST) theories [20–25] (see also [26–28]), the vector modes appear in the vector-tensor theories (e.g. the Einstein-Æther theory [29]) while both scalar and tensor modes could be excited simultaneously in addition to the tensor modes in scalar-vector-tensor theories, such as the TeVeS theory [30]. Therefore, people have put a lot of effort into extracting information on the polarization states from GWs signals [31–35] for the purpose of falsifying gravity theories via polarization states of GWs. However, limited by the orientation of the detectors arranged at LVSC, so far the best we have known about the polarization states from the GWs signals is that the purely tensor polarization is strongly favored over the purely scalar or vector polarizations [10]. In light of the establishment of the network of ground-based detectors including advanced LIGO, Virgo [36–38], KAGRA [39,40] and LIGO India, more information will be available and we may then have the probing ability for separating polarization modes in the near future.

Naturally, in this context there comes a crucial question, that is, whether GR is the unique theory with two tensorial degrees of freedom (TTDOF) of gravity. According to the Lovelock's theorem [41,42], GR is the unique theory in which the metric is the only field in the gravity sector and obeys second order equations of motion with general covariance and locality in the four dimensional spacetime and therefore GR is indeed the unique TTDOF gravity

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theory when the assumptions of Lovelock's theorem are preserved.

However, this uniqueness does not hold anymore when these assumptions are relaxed to some extent.[1] In fact, an alternative gravity theory with TTDOF was proposed and dubbed the Cuscuton in 2007 [48] by introducing an incompressible (i.e. propagating at infinite sound speed) scalar field $\phi(t,\vec{x})$ with a nonvanishing and timelike vacuum expectation value of its first derivative. The scalar d.o.f. with infinite sound speed in the Cuscuton manifests itself as an instantaneous mode. As discussed in [49–51] for a generalized instantaneous mode called a shadowy mode in the context of U-DHOST and VCDM respectively, such a mode with infinite sound speed satisfies an elliptic equation on each constant-ϕ hypersurface. In the so-called unitary gauge, i.e. with $\phi = \phi(t)$ [52], these hypersurfaces on which the elliptic equation is defined agree with time slices and thus the equation of motion for the instantaneous/shadowy mode does not include time derivatives, meaning that the Cuscuton only propagates the TTDOF [53]. In other choices of time slicing, the equation of motion for the instantaneous/shadowy mode includes time derivatives but is still elliptic. Therefore by imposing a proper boundary condition the instantaneous/shadowy mode is uniquely determined by other degrees of freedom. As a result the Cuscuton propagates TTDOF only, irrespective of whether the unitary gauge is adopted or not (see Refs. [49–51] for corresponding discussions in U-DHOST and VCDM).

Generally, in the unitary gauge, general covariance is broken into the spatial covariance therefore the Cuscuton (or a generic ST theory) can be classified as a spatially covariant (SC) framework [54] and inversely the ST theory can be also recovered from a SC theory by introducing a Stueckelberg field [55–57]. Due to the reduction of symmetry, in addition to the two tensorial d.o.f., the SC gravity theory with a nondynamical lapse function propagates one scalar d.o.f. [58]. Nevertheless, a class of TTDOF gravity theories was proposed within a special SC framework where the lapse function enters the Lagrangian linearly. The resulting theory was dubbed the MMG theory [59], which indicates that GR is modified without changing its d.o.f. Similarly, a class of TTDOF theories was identified within the more general ST theory with higher order derivatives under the unitary gauge [60]. In the case of a nondynamical lapse function, the general conditions to eliminate the scalar d.o.f. were found in [61,62]. This idea was also applied to the more general SC framework with a dynamical lapse function that enters the Lagrangian nonlinearly [63]. The conditions to have TTDOF in the presence of a dynamical lapse function were analyzed in [64].

According to the above works, within the general SC framework, MMG theories exist as long as some additional conditions are satisfied, which we dub the TTDOF conditions. The example of the first TTDOF condition arose in [60], which we dubbed the degeneracy condition to indicate that the sector of the lapse function and extrinsic curvature is degenerate. The self-consistency condition identified in [59], which we dub the second TTDOF condition, is used to prevent the number of phase space dimensions from being odd.

The TTDOF conditions are nonlinear functional differential equations of the Lagrangian, which are difficult to be solved in general. We are only able to solve them with some particular ansatz of actions, such as, the square root gravity in [59], the extended Cuscuton in [60] and the quadratic extrinsic curvature (QEC) gravity in [61] (see the cosmological constraints to this model in [65,66]). Another difficulty of MMG theories is the problem of coupling with matter consistently, which happens when there are extra first-class constraint(s) (other than the original six first-class constraints associated with the spatial diffeomorphism) appearing in the MMG theories [67]. More precisely, the extra first-class constraint(s) would be downgraded to be second-class when the theory is naively coupled with Lorentz covariant matter, at which point the suppressed scalar d.o.f. arises again.

The resolution of these difficulties arising for MMG theories become more transparent in the phase space than in the configuration space. This is because counting the number of d.o.f. is transparent by means of an explicit constraint analysis in the phase space. For this reason, a number of works have utilized the Hamiltonian approach. For instance, by imposing the linearity of the lapse function in the Hamiltonian [68] instead of the Lagrangian [59], the self-consistency condition was reformulated in a much simpler expression by solving the simplified condition. In these works, the $f(\mathcal{H})$ theory which is a particular MMG theory constructed in [67] was rediscovered and the "kink" model based on the $f(\mathcal{H})$ theory was shown later to fit the Planck data better than the ΛCDM model [69]. The matter-coupling problem can also be addressed in the phase space. As another instance, in [70–72], the authors introduce the so-called gauge fixing condition to the Hamiltonian of the MMG theory obtained by performing a canonical transformation of GR. As a result, the first-class constraint

[1] Recently, the authors in [43] attempted to circumvent the Lovelock's theorem to construct an alternative gravity theory with TTDOF by rescaling the coupling constant of the Gauss-Bonnet term and therefore nontrivially modifying the Einstein's field equation in four dimensional spacetime. However, the Lovelock's theorem is a statement about the equations of motion (but not just about the action) and therefore directly excludes such a possibility. Indeed, it was soon realized that taking the limit of Gauss-Bonnet term from higher dimension to 4 dimension is path-dependent which should be regularized properly and, as a consequence, additional d.o.f. appears in the regularized 4D Einstein-Gauss-Bonnet theories (see Ref. [44] for reviews). Alternatively, by breaking the 4D diffeomorphism invariance down to the 3D spatial diffeomorphism invariance, one can construct a consistent 4D Einstein-Gauss-Bonnet theory with TTDOF [45–47]. We will briefly introduce this theory in Appendix B 3.

associated with temporal diffeomorphism is split into a pair of second-class constraints, which allow the theory to couple with the Lorentz covariant matter consistently. Another proposal for addressing the same problem is adopted in [67,73,74], where the additional first-class constraint is maintained by modifying the Hamiltonian constraint of the matter sector such that the constraints algebra is kept closed. However, as a price, matter no longer behaves in the usual Lorentz covariant manner.

Inspired by the above works, a more straightforward approach to constructing the MMG theories was proposed in [75], where the so-called (scalar-type) auxiliary constraint (AC) is introduced to a general total Hamiltonian respecting the spatial diffeomorphism with a nondynamical lapse function. The Hamiltonian carries two tensorial and one scalar d.o.f. at the beginning. The AC is used to constrain the trajectories of canonical variables such that the unwanted scalar d.o.f. is suppressed. It is therefore introduced to assist in locating the MMG theories in the space of theories and is eventually part of the definition of the theory. This AC can be also considered as the generalization of the gauge-fixing condition that addresses the matter-coupling problem mentioned above. Nevertheless, the phase space constrained by the AC is still insufficient to ensure a MMG theory because the AC is introduced via a generic function of the canonical variables. Generally, additional TTDOF conditions are still needed, which are renamed as the "minimalizing conditions" in [75], to underline that they are the conditions that "minimalize" the space of more general theories into the MMG theory space. Even though the AC is initially introduced by hand, we emphasize that it is actually nothing but one possible kind of constraint structure for the MMG theories and just part of the definition of the theory. In principle there could be more than one AC, which can be thought of as yet unrevealed territory among the constraint structures of the MMG theories.

In this work, we are going to complete the constraint analysis for the MMG theories with multiple ACs. In principle, not only the scalar-type but also the vector- and the tensor-type ACs might possibly appear. Hence we will first investigate the possibilities of minimizing the number of d.o.f. by introducing an arbitrary number of any of the above types of ACs to a general total Hamiltonian, which still preserves spatial diffeomorphism invariance. Throughout, it is also important to limit the number of ACs, otherwise the system will be overconstrained thus become physically inconsistent. In order to determine the maximum number of each type of ACs, we will first assume that they are all classified as second-class then count the number of d.o.f. with respect to an arbitrary background via the Hamiltonian analysis. By requiring that the number of each type of d.o.f. be non-negative, we will find limits on the number of each type of ACs. In fact, as we will see in the next section, the number of scalar-type ACs should not exceed four and no vector- and tensor-type ACs are allowed, because the phase space in the vector sector is already sufficiently constrained by the spatial diffeomorphism constraints, while the tensor sector should not be constrained in order to have the correct number of d.o.f.

Following this, we will construct a consistent SC framework with multiple (scalar-type) ACs as our starting point for searching the MMG theories. According to the number of introduced ACs, we will divide the theories into four cases and for the purpose of obtaining the MMG theories, the scalar d.o.f. should be completely eliminated by the primary and secondary constraints, which will divide the ACs into different classifications for each case. To further contrive the classification, the ACs and the canonical Hamiltonian should satisfy not only the corresponding minimalizing conditions mentioned above but also symmetrizing conditions which are the sufficient but not necessary conditions to end up with a MMG theory and are required to enhance the gauge symmetries of the theory. As a results, we will exhaust all the possible constraint structures for the MMG theories with multiple ACs, thus leading to a complete classification.

To illustrate this formalism, we will construct a concrete model with four ACs by using the Cayley-Hamilton theorem. This theory can couple with matter consistently without further conditions. We will investigate the dispersion relation of tensor perturbations around a flat FLRW background, and show how some coefficients of the theory can be constrained by the observations.

The rest of the current paper is organized as follows. In Sec. II, we investigate the possibilities of reducing the number of d.o.f. by introducing different types of ACs and determine a consistent general framework with multiple ACs as our starting point for searching the MMG theories. In Sec. III, we find the minimalizing and symmetrizing conditions for each class of MMG theories, each of which is described in Secs. III A–III D. As an illustrative example, we construct a concrete model with four ACs and study the modified dispersion relation of the GWs by performing a tensor perturbation in Sec. IV. Finally, we conclude this work in Sec. V.

II. A CONSISTENT FRAMEWORK WITH AUXILIARY CONSTRAINTS

In this section, we will investigate the possibilities of reducing the number of d.o.f. by introducing multiple primary ACs and construct a consistent framework as our starting point for searching for the MMG theories. We will adopt the Arnowitt-Deser-Misner (ADM) formalism in which the lapse function, shift vector, induced metric and their conjugate momenta are denoted by $\{N, N^i, h_{ij}; \pi, \pi_i, \pi^{ij}\}$ respectively and ∇_i is the spatially covariant derivative compatible with h_{ij}. Without loss of generality, we start with the following general total Hamiltonian

$$H_{\rm T} = \int {\rm d}^3x[\mathcal{H}(N,\pi,h_{ij},\pi^{ij};\nabla_i) + N^i\mathcal{H}_i + \lambda^i\pi_i + \mu_{\rm n}\mathcal{S}^{\rm n} + \nu^i_{\rm m}\mathcal{V}^{\rm m}_i + \rho^{ij}_{\rm r}\mathcal{T}^{\rm r}_{ij}], \tag{1}$$

where $\mathcal{H}$ is a generic function of $(N,\pi,h_{ij},\pi^{ij};\nabla_i)$ which, with the second term, corresponds the usual canonical Hamiltonian and N^i, λ^i, $\mu_{\rm n}$, $\nu^i_{\rm m}$ and $\rho^{ij}_{\rm r}$ play the role of the Lagrange multipliers corresponding to the following constraints,

$$\mathcal{H}_i \approx 0_i, \qquad \pi_i \approx 0_i, \tag{2}$$

which are associated with the spatial diffeomorphism with $\mathcal{H}_i$ the momentum constraints, and

$$\mathcal{S}^{\rm n} \approx 0^{\rm n}, \qquad \mathcal{V}^{\rm m}_i \approx 0^{\rm m}_i, \qquad \mathcal{T}^{\rm r}_{ij} \approx 0^{\rm r}_{ij}, \tag{3}$$

which denote the introduced scalar-, vector- and (symmetric rank-2) tensor-type ACs with n, m and r the corresponding indices, respectively. Here, scalar-, vector- and tensor-types refer to the transformation properties under the spatial diffeomorphism generated by $\mathcal{H}_i$.[2] Throughout this work, "≈" represents "weak equality" that holds only on the constrained subspace $\Gamma_{\rm C}$ of the phase space.

The terminology "primary constraint" is usually referred to constraints due to a singular Lagrangian, from which we cannot solve all the conjugate momenta. In particular, in the case of GR or general SC theories, in (2) $\pi_i \approx 0_i$ are the primary constraints due to the absence of the velocity of the shift vector N^i in the Lagrangian, while $\mathcal{H}_i \approx 0_i$ are the so-called secondary constraints, which arise after making use of the equations of motion. In this work, since we start from the Hamiltonian in the phase space from the beginning, a "primary constraint" is merely referred to a constraint that is introduced by hand when defining the total Hamiltonian. In this sense, both $\pi_i \approx 0_i$ and $\mathcal{H}_i \approx 0_i$, as well as constraints in (3), are treated as primary constraints in this work.

We now make some comments on the above introduced constraints. First, as explained in Sec. I, in light of the restriction by the Lovelock's theorem, in order to enlarge the space of theories such that there is space for searching for the MMG theories, we reduce the symmetries of the theory from the general covariance to the spatial covariance. This requires that the spatial diffeomorphism constraints (2) must be present in the total Hamiltonian (1). For convenience, we will adopt the extended definition for the momentum constraints [58,63,76,77]

$$\mathcal{H}_i \equiv -2\sqrt{h}\nabla_j\frac{\pi^j_i}{\sqrt{h}} + \pi\nabla_iN + \pi_j\nabla_iN^j + \sqrt{h}\nabla_j\frac{\pi_iN^j}{\sqrt{h}} \approx 0_i, \tag{4}$$

which satisfy the following property [63]

$$[\mathcal{H}_i(\vec{x}), Q(\vec{y})] \approx 0_i, \quad \forall\ Q\approx 0, \tag{5}$$

with Q an arbitrary quantity weakly vanishing on the constrained phase space $\Gamma_{\rm C}$. The Poisson bracket $[\cdot,\cdot]$ is defined by

$$[\mathcal{F},\mathcal{G}] \equiv \int {\rm d}^3z\sum_I\left(\frac{\delta\mathcal{F}}{\delta\Phi_I(\vec{z})}\frac{\delta\mathcal{G}}{\delta\Pi^I(\vec{z})} - \frac{\delta\mathcal{F}}{\delta\Pi^I(\vec{z})}\frac{\delta\mathcal{G}}{\delta\Phi_I(\vec{z})}\right), \tag{6}$$

where we formally denote the ADM variables with Φ_I and their conjugate momenta with Π^I. By using the property in (5), it is easy to show that the constraints in (2), i.e. the spatial diffeomorphism generators, are explicitly classified as the first-class in terms of Dirac's terminology. They eliminate 12 dimensions from the (in total) 20-dimensional phase space, leaving us with 8 dimensions at each point of the spacetime, i.e. four local d.o.f.

Second, for the purpose of obtaining the MMG theories, one or several additional constraints are needed in order to reduce the number of d.o.f. from four to two. In this work, we perform this by introducing multiple ACs (3) to the total Hamiltonian (1) as part of the primary constraints of the theory. Our previous work [75], in which we introduced only one scalar-type AC with the assumption of a nondynamical lapse function, can be considered a preliminary work in that regard. In principle, there could be more than one AC appearing in the Hamiltonian, including *a priori* all types of constraints, i.e. scalar $\mathcal{S}^{\rm n}$, vector $\mathcal{V}^{\rm m}_i$ and tensor $\mathcal{T}^{\rm r}_{ij}$, which are generic functions of $(N,\pi,h_{ij},\pi^{ij};\nabla_i)$ with the indices n, m and r labeling the number of each type of AC respectively. Note that it is fair to assume that the ACs are linearly independent from each other in order to avoid unnecessary complexity. Obviously, there must be an upper limit of the number of ACs, otherwise the system will become overconstrained and inconsistent physically even without coupling to any external fields. The maximum number of each type of ACs can be obtained in a scenario

[2]By using the spatial metric h_{ij}, its inverse h^{ij} and the spatial covariant derivative ∇_i, one could decompose a spatial vector into the transverse and longitudinal parts, a spatial (symmetric rank-2) tensor into the transverse-traceless, traceless-longitudinal and trace parts, as far as the inverse of the Laplace operator is unique on the spatial manifold. In particular, the decomposition of a spatial (symmetric rank-2) tensor into the traceless and trace parts does not introduce any nonlocality and thus is easy to adopt. Nonetheless, we shall not employ such decompositions since the main conclusion of the present paper does not change. Hereafter, we thus simply classify ACs under the transformation properties under the spatial diffeomorphism.

where all of the ACs (3) are assumed to be of the second-class. In this case, we count the number of d.o.f. as

$$\begin{aligned}\#_{\rm dof} &= \frac{1}{2}(\#_{\rm var}\times 2 - \#_{\rm 1st}\times 2 - \#_{\rm 2nd}) \\ &= \frac{1}{2}[(4_{\rm s}+4_{\rm v}+2_{\rm t})\times 2 - (1_{\rm s}+2_{\rm v})\times 2\times 2 \\ &\quad - 1_{\rm s}\times\mathcal{N} - (1_{\rm s}+2_{\rm v})\times\mathcal{M} - (2_{\rm s}+2_{\rm v}+2_{\rm t})\times\mathcal{R}] \\ &= (2_{\rm t}-\mathcal{R}_{\rm t}) - (\mathcal{M}_{\rm v}+\mathcal{R}_{\rm v}) \\ &\quad + \frac{1}{2}(4_{\rm s}-\mathcal{N}_{\rm s}-\mathcal{M}_{\rm s}-2\times\mathcal{R}_{\rm s}), \end{aligned} \tag{7}$$

where $\mathcal{N}$, $\mathcal{M}$, $\mathcal{R}$ are the numbers of constraints in each type, and we use the subscripts s, v and t to denote the scalar, vector and tensor d.o.f. respectively. The classification of d.o.f. into various types are to be understood in the sense of spatial diffeomorphism with respect to an arbitrary background. Since the phase space is spanned by N, N^i, h_{ij} and their conjugate momenta, N will contribute one scalar d.o.f., i.e. $1_{\rm s}$ in (7), N^i will contribute one scalar and two vectorial d.o.f. accounting for $1_{\rm s}+2_{\rm v}$ and h_{ij} will contribute two scalar, two vectorial and two tensorial d.o.f., accounting for $2_{\rm s}+2_{\rm v}+2_{\rm t}$. Together with their conjugate momenta, the dimension of the phase space is therefore $(4_{\rm s}+4_{\rm v}+2_{\rm t})\times 2$. The number of d.o.f. removed by the ACs (3) are counted in the similar way in (7). Note that the tensor-type ACs, $\mathcal{T}^{\rm r}_{ij}\approx 0^{\rm r}_{ij}$, are symmetric with respect to the subscripts therefore $\mathcal{T}^{\rm r}_{ij}\approx 0^{\rm r}_{ij}$ account for $-(2_{\rm s}+2_{\rm v}+2_{\rm t})\times\mathcal{R}$ in (7). The number of d.o.f. in each type should not be negative in the absence of external fields otherwise the theory is physically inconsistent. Therefore, from the last line of (7), we require

$$2-\mathcal{R}\geq 0,\qquad \mathcal{M}+\mathcal{R}\leq 0, \tag{8}$$

and

$$4-\mathcal{N}-\mathcal{M}-2\mathcal{R}\geq 0, \tag{9}$$

which gives

$$\mathcal{R}=0,\qquad \mathcal{M}=0,\qquad \mathcal{N}\leq 4. \tag{10}$$

We conclude that none of the tensor- and vector-type ACs are allowed, and no more than four scalar-type ACs should be introduced. In the case with four (necessarily second-class) independent scalar-type ACs, from (7) we see that (1) turns out to be an MMG theory automatically, i.e. without requiring any further condition. In the next section, we will confirm this result again, via a more detailed Hamiltonian analysis accounting for all the possible classifications of the ACs.

Before getting into the next section, we justify the introduction of (scalar-type) ACs as follows. Even though the ACs are introduced by hand in this work, we emphasize that they are nothing but part of the definition of the MMG theories. In other words, it is not possible to construct an MMG theory without introducing ACs because without any ACs, the number of d.o.f. in the theory (1) is four. Therefore additional constraints are necessary to reduce the number of d.o.f. One may consider the possibility that extra constraint may come from the particular choice of the free function $\mathcal{H}(N,\pi,h_{ij},\pi^{ij};\nabla_i)$ in (1), for example by requiring that the lapse N plays the role of a Lagrange multiplier. However, this also yields the constraint of $\pi\approx 0$, which is actually a typical choice of the scalar-type AC and has been adopted in the previous related works [59–61,68,75] based on the assumption of a nondynamical lapse. In GR, $\pi\approx 0$ is one of the constraints naturally required by the 4-dimensional spacetime diffeomorphism. In the more general framework with only spatial covariance, the lapse function could be dynamical in principle [63,64,78], therefore the conjugate momentum π does not correspond to a constraint in general. From the viewpoint of the formalism presented here, the constraint $\pi\approx 0$ is noting but a specific scalar-type AC, which generates the constraint originally imposed on $\mathcal{H}$. Another example of the AC in the literature is the gauge fixing term introduced in [70–72]. By fixing the gauge condition, which by itself is of the second class, the first-class constraint becomes a second class constraint. As a result, the theory is able to couple with matter consistently [67,73,74] (see also [79–81]).

Based on the above discussions, we construct a consistent framework for searching for the MMG theories in the vacuum by introducing the ACs as follows:

$$H_{\rm T}=\int d^3x(\mathcal{H}+\mu_{\rm n}\mathcal{S}^{\rm n}+N^i\mathcal{H}_i+\lambda^i\pi_i), \tag{11}$$

where $\mathcal{H}$ and $\mathcal{S}^{\rm n}$ with $\rm n=1,\ldots,\mathcal{N}(\mathcal{N}\leq 4)$ are generic functions of $(N,\pi,h_{ij},\pi^{ij};\nabla_i)$. Before we start to search for the MMG theories based on (11), it is convenient to split the total Hamiltonian (11) into two parts

$$H_{\rm T}=H_{\rm D}+H_{\rm P}, \tag{12}$$

where $H_{\rm D}$ denotes the part of the Hamiltonian corresponding to the spatial diffeomorphism

$$H_{\rm D}\equiv\int d^3x(N^i\mathcal{H}_i+\lambda^i\pi_i), \tag{13}$$

and the rest in (11) is denoted by

$$H_{\rm P}\equiv\int d^3x(\mathcal{H}+\mu_{\rm n}\mathcal{S}^{\rm n}), \tag{14}$$

which we dub the "partial" Hamiltonian [75]. Clearly, H_{D} is fixed in all the SC gravity theories, and thus H_{P} plays the central role in the following discussions, which has nothing to do with N^i and π_i. Indeed, we are allowed to deduct the $\{N^i, \pi_i\}$-sector from the system in the first place since the spatial diffeomorphism constraints (2) are retained and considered as the first-class. Therefore, the specificities of the theory such as the number of d.o.f. will be completely encoded in H_{P} with an $20 - 6 \times 2 = 8$ dimensional phase space. One can however restore the neglected part H_{D} without any difficulty in the following discussions. With the partial Hamiltonian H_{P} (14), the number of d.o.f. of the theory is formally counted as

$$\begin{aligned}\#_{\mathrm{dof}} &= \frac{1}{2}[(2_{\mathrm{t}} + 2_{\mathrm{s}}) \times 2 - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}}] \\ &= 2_{\mathrm{t}} + \frac{1}{2}(4_{\mathrm{s}} - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}}), \end{aligned} \tag{15}$$

where $\#^{\mathrm{s}}_{\mathrm{1st}}$ and $\#^{\mathrm{s}}_{\mathrm{2nd}}$ are the numbers of the first- and the second-class constraints, which include the primary ACs and the possible secondary constraints generated from the ACs. Clearly, $\#^{\mathrm{s}}_{\mathrm{1st}}$ and $\#^{\mathrm{s}}_{\mathrm{2nd}}$ should satisfy

$$\mathcal{N} \leq \#^{\mathrm{s}}_{\mathrm{1st}} + \#^{\mathrm{s}}_{\mathrm{2nd}} \leq 4, \tag{16}$$

and

$$4 - \#^{\mathrm{s}}_{\mathrm{1st}} \times 2 - \#^{\mathrm{s}}_{\mathrm{2nd}} = 0, \tag{17}$$

since, for the sake of having an MMG theory, the scalar d.o.f. should be completely eliminated. Combining (16) with (17), we are able to exhaust all the possible constraint structures for the MMG theories case by case, which we will do in the next section.

To conclude this section, by performing a Legendre transformation of the Hamiltonian (11), we get the corresponding formal form of the action as follows

$$S = \int \mathrm{d}t\mathrm{d}^3x[N(\pi F + 2\pi^{ij}K_{ij}) - \mathcal{H} - \mu_{\mathrm{n}}\mathcal{S}^{\mathrm{n}}], \tag{18}$$

in which π and π^{ij} should be understood as the solutions of the following canonical equations

$$NF = \frac{\delta H_{\mathrm{P}}}{\delta \pi} \quad \text{and} \quad 2NK_{ij} = \frac{\delta H_{\mathrm{P}}}{\delta \pi^{ij}}. \tag{19}$$

Here, H_{P} is the partial Hamiltonian defined in (14) and we denote

$$F \equiv \frac{1}{N}(\dot{N} - N^i\nabla_i N), \qquad K_{ij} \equiv \frac{1}{2N}(\dot{h}_{ij} - 2\nabla_{(i}N_{j)}), \tag{20}$$

which play the roles of velocities of the lapse and spatial metric in the action and the latter is nothing but the extrinsic curvature. Here and throughout the paper, the overdot "." denotes a time derivative. The Lagrange multipliers μ_{n} in (18) are determined as follows. According to the classification of the ACs, the corresponding μ_{n} will be fixed by the consistency conditions of the ACs or kept as some general functions. From (18) with (19), we see that the ACs not only appear in the action directly but also become part of the canonical equations, hence will influence how the velocities enter the action. Once we solve for the momenta π and π^{ij} as functions of F and K_{ij} and substitute them into (18), in principle, we will find the action corresponding to the Hamiltonian (11).

III. MINIMALLY MODIFIED GRAVITY WITH AUXILIARY CONSTRAINT(S)

In this section, we are going to classify the ACs and their secondary constraints by solving Eqs. (17) with (16), and then find out the corresponding conditions needed in order to fully satisfy the classifications so that we are able to exhaust all the possible constraint structures for the MMG theories. As mentioned in the last section, one requires no condition in the case with four second-class (scalar-type) ACs, so we start the discussion from the case with four ACs, with more details than the previous section.

A. Case with four auxiliary constraints

Let us start with the case with four ACs, i.e.

$$H_{\mathrm{P}} \equiv \int \mathrm{d}^3x(\mathcal{H} + \mu_{\mathrm{n}}\mathcal{S}^{\mathrm{n}}) \quad \text{with} \quad \mathrm{n} = 1, 2, 3, 4. \tag{21}$$

By solving Eq. (16) with (17), we find the unique class

$$\#^{\mathrm{s}}_{\mathrm{1st}} = 0, \qquad \#^{\mathrm{s}}_{\mathrm{2nd}} = 4, \quad (\text{ID key: IV-0-4}) \tag{22}$$

which implies that the four ACs, $\mathcal{S}^{\mathrm{n}} \approx 0$, must be all of the second-class for the sake of having an MMG theory. (For convenience, we will give each type of classification an identification key. We label this case IV-0-4.) This is also consistent with the discussions in the last section. As a consequence, there are two important properties to be stressed for this case.

First, as an MMG theory, $\mathcal{H}$ and $\mathcal{S}^{\mathrm{n}}$ in (21) can be arbitrarily chosen as the independent functions of $(N, \pi, h_{ij}, \pi^{ij}; \nabla_i)$, since no condition is required to (21). In the cases with less number of ACs, however, this is not true in general, some conditions on $\mathcal{H}$ and $\mathcal{S}^{\mathrm{n}}$ are needed to obtain the MMG theories. This complexity is fortunately evaded in this case because we have introduced a sufficient number of additional constraints.

Second, the MMG theory (21) with (13) can be coupled with matter consistently. A common problem for the MMG theories that include the first-class constraint(s) [in addition to the spatial diffeomorphism constraints (2)] is how to couple with matter consistently. Naive coupling with matter may change the constraint structure of the theory and thus make the additional first-class constraint(s) become the second-class, thus reintroduce the scalar mode(s) suppressed before. Some strategies for dealing with this problem have been adopted, for example in [67,70,71,73,74], by making the gauge symmetries of the gravity- and matter-sector match each other. Owing to the absence of additional first-class constraint in (21), the gauge symmetry of this theory, (21) with (13), is exactly the spatial diffeomorphism. Thus the coupling problem is automatically solved as long as the matter field preserves the spatial covariance as well. For instance, the total Hamiltonian consistently coupled with a scalar field ϕ with its conjugate momentum p can be written as

$$H_{\mathrm{T}} \equiv \int \mathrm{d}^3x(\hat{\mathcal{H}} + \mu_{\mathrm{n}}\mathcal{S}^{\mathrm{n}} + N^i\hat{\mathcal{H}}_i + \lambda^i\pi_i) \tag{23}$$

where the momentum constraints are extended to

$$\hat{\mathcal{H}}_i \equiv \mathcal{H}_i + p\nabla_i\phi \approx 0_i, \tag{24}$$

with $\mathcal{H}_i$ defined in (4) and the matter coupled with gravity through the following generic function

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}(N, \pi, h_{ij}, \pi^{ij}, \phi, p; \nabla_i). \tag{25}$$

Note that, at the classical level, the Lorentz covariance of the scalar matter could be enhanced by particular choices of the generic function $\hat{\mathcal{H}}$ although, at the quantum level, it may be violated via the loops induced effect from the Lorentz-violating gravitons [79].

To conclude, we show that (21) is a partial Hamiltonian of an MMG theory without requiring any condition. It can be used to couple with the matter consistently therefore (23) provides an extensive yet simple framework for investigating the cosmological properties of the MMG theories. As an illustrating example, we will construct a concrete model (by making some choices for the free functions) in Sec. IV.

B. Case with three auxiliary constraints

In the case of three ACs,

$$H_{\mathrm{P}} \equiv \int \mathrm{d}^3x(\mathcal{H} + \mu_{\mathrm{n}}\mathcal{S}^{\mathrm{n}}) \quad \text{with} \quad \mathrm{n} = 1, 2, 3, \tag{26}$$

there are two possible classifications according to Eq. (16) with (17)

$$\#^{\mathrm{s}}_{\mathrm{1st}} = 1, \qquad \#^{\mathrm{s}}_{\mathrm{2nd}} = 2, \quad (\text{ID key: III-1-2}) \tag{27}$$

or

$$\#^{\mathrm{s}}_{\mathrm{1st}} = 0, \qquad \#^{\mathrm{s}}_{\mathrm{2nd}} = 4. \quad (\text{ID key: III-0-4}) \tag{28}$$

The first case (34), i.e. of type III-1-2, means that one of the ACs in (26) is first-class and the other two are second-class. The existence of a first-class constraint implies that the "partial" Dirac matrix

$$[\mathcal{S}^{\mathrm{n}}(\vec{x}), \mathcal{S}^{\mathrm{n}'}(\vec{y})], \tag{29}$$

is degenerate in one dimension and the corresponding linear combination of $\{\mathcal{S}^{\mathrm{n}}\}$ has vanishing Poisson bracket with $\mathcal{H}$, which will yield some conditions on the generic functions $\mathcal{H}$ and $\mathcal{S}^{\mathrm{n}}$. Hence, different from the case with four ACs, these functions cannot be arbitrarily chosen anymore.

As a simple case in which the matrix (29) is degenerate in one dimension, let us suppose

$$[\mathcal{S}^1(\vec{x}), \mathcal{S}^{\mathrm{n}}(\vec{y})] \approx 0^{\mathrm{n}}. \tag{30}$$

We are always able to make linear combinations among the ACs and redefine the Lagrange multipliers. So, without loss of generality, we will continue the discussion with the pattern in (30) for its simplicity. The pattern (30) actually defines the conditions to be imposed on the ACs in order to have an MMG theory. We dub this kind of conditions the "minimalizing conditions" [75], since this kind of condition helps to narrow the space of theories down to the MMG subspace. In other words, the scalar d.o.f. are completely eliminated by introducing the ACs satisfying the minimalizing conditions. Once the minimalizing conditions (30) are satisfied, we should check the consistency condition of $\mathcal{S}^1 \approx 0$, which is

$$\dot{\mathcal{S}}^1(\vec{x}) = [\mathcal{S}^1(\vec{x}), H_{\mathrm{P}}] = \int \mathrm{d}^3y[\mathcal{S}^1(\vec{x}), \mathcal{H}(\vec{y})] \approx 0. \tag{31}$$

According to the classification (27), there should be no secondary constraint generated from (31), which implies that

$$[\mathcal{S}^1(\vec{x}), \mathcal{H}(\vec{y})] \approx 0, \tag{32}$$

must be satisfied. This is however not a necessary condition for obtaining an MMG theory. In fact, if Eq. (32) is not satisfied, then a secondary constraint is generated in (31) and the system is classified into the class (28), i.e. of type III-0-4. So no matter whether the condition (32) is satisfied or not, we will always have an MMG theory as long as the minimalizing condition (30) is satisfied. The condition (32) is just used to enhance the gauge symmetry of the theory without altering the number of d.o.f. We therefore dub this kind of condition the "symmetrizing condition."

We conclude that the partial Hamiltonian (26) describes an MMG theory as long as the minimalizing condition (30) is satisfied and the symmetries of the theory will be enhanced when the symmetrizing condition (32) is satisfied. In this case, the enhanced gauge symmetry helps to suppress the unwanted scalar d.o.f., which is exactly what happens in GR where the would-be scalar d.o.f. is completely suppressed by the spacetime diffeomorphism. One may ask what kind of gauge symmetry will be enhanced by adopting the symmetrizing conditions [e.g. (32)] beyond the spatial diffeomorphism. Some interesting opinions have been discussed in [74,82]. A detailed analysis is however beyond the scope of the current work.

C. Case with two auxiliary constraints

In the case with two scalar-type ACs, i.e.

$$H_{\mathrm{P}} \equiv \int \mathrm{d}^3x(\mathcal{H} + \mu_{\mathrm{n}}\mathcal{S}^{\mathrm{n}}) \quad \text{with} \quad \mathrm{n} = 1, 2, \tag{33}$$

we find the possible classes from Eqs. (16) with (17) as follows

$$\#^{\mathrm{s}}_{1\mathrm{st}} = 2, \qquad \#^{\mathrm{s}}_{2\mathrm{nd}} = 0, \quad (\text{ID key: II-2-0}) \tag{34}$$

$$\#^{\mathrm{s}}_{1\mathrm{st}} = 1, \qquad \#^{\mathrm{s}}_{2\mathrm{nd}} = 2, \quad (\text{ID key: II-1-2}) \tag{35}$$

and

$$\#^{\mathrm{s}}_{1\mathrm{st}} = 0, \qquad \#^{\mathrm{s}}_{2\mathrm{nd}} = 4. \quad (\text{ID key: II-0-4}) \tag{36}$$

Let us explain the implications of these three classes one by one:

(1) Obviously, the first class (34), i.e. type II-2-0, covers the case in which both of the ACs are first-class without generating secondary constraints. By performing a similar analysis as the one conducted in the last case (27), it is easy to find the following minimalizing conditions for this class

$$[\mathcal{S}^1(\vec{x}), \mathcal{S}^{\mathrm{n}}(\vec{y})] \approx 0^{\mathrm{n}}, \tag{37}$$

and

$$[\mathcal{S}^2(\vec{x}), \mathcal{S}^2(\vec{y})] \approx 0, \tag{38}$$

with the symmetrizing conditions as

$$[\mathcal{S}^{\mathrm{n}}(\vec{x}), \mathcal{H}(\vec{y})] \approx 0^{\mathrm{n}}, \tag{39}$$

which prevent the secondary constraints from being generated.

(2) Similarly, the second class (35), i.e. of type II-1-2, implies that there is one secondary constraint generated from the two primary constraints and one of these three constraints is first-class. However, it is easy to show that the secondary constraint could not be first-class, because then the corresponding primary constraint would become first-class simultaneously. As a result there would be two first- and one second-class constraints, which lead to a negative number of scalar d.o.f. Therefore, without loss of generality, we set the secondary constraint as being generated by the time evolution of $\mathcal{S}^1$, i.e., $\dot{\mathcal{S}}^1 \approx 0$. According to whether the first-class constraint is $\mathcal{S}^1 \approx 0$ or $\mathcal{S}^2 \approx 0$, we divide the class (35) into two parallel patterns:

(a) The first-class constraint is $\mathcal{S}^1 \approx 0$ which yields the same minimalizing condition as in (37) but with

$$[\mathcal{S}^1(\vec{x}), \dot{\mathcal{S}}^1(\vec{y})] \approx 0. \tag{40}$$

Once (40) is satisfied, generally, the consistency condition of $\dot{\mathcal{S}}^1 \approx 0$ will generate a tertiary constraint as

$$\ddot{\mathcal{S}}^1(\vec{x}) = \int \mathrm{d}^3y\{[\dot{\mathcal{S}}^1(\vec{x}), \mathcal{H}(\vec{y})] + \mu_2(\vec{y})[\dot{\mathcal{S}}^1(\vec{x}), \mathcal{S}^2(\vec{y})]\} \approx 0, \tag{41}$$

where the Lagrange multiplier μ_2 has been fixed by the consistency condition of $\mathcal{S}^2 \approx 0$. However by requiring (35), $\ddot{\mathcal{S}}^1 \approx 0$ should be prevented from being a nontrivial constraint therefore (41) is forced to be a symmetrizing condition. We label this case with the identification key II-1-2a to distinguish it from the next case.

(b) The first-class constraint is $\mathcal{S}^2 \approx 0$, which gives the same minimalizing conditions (37) and (38). The symmetrizing conditions (39) should be replaced by

$$[\mathcal{S}^2(\vec{x}), \dot{\mathcal{S}}^1(\vec{y})] \approx 0, \tag{42}$$

and

$$[\mathcal{S}^2(\vec{x}), \mathcal{H}(\vec{y})] \approx 0, \tag{43}$$

to ensure that $\mathcal{S}^2 \approx 0$ is first-class and generates no secondary constraint. Correspondingly, this case is labeled II-1-2b.

(3) The last class (36), i.e. of type II-0-4, can be simply achieved by giving up the symmetrizing conditions (41), which leads to two primary, one secondary, and one tertiary constraints, or (42) with (43), which leads to two primary and two secondary constraints. All the resulting constraints are the second-class and both choices require the same minimalizing conditions (37), with (40) or (38), respectively.

We summarize for this case that we find two kinds of minimalizing conditions, i.e. (37), with (38) or (40) respectively, for the partial Hamiltonian (33) with two ACs. For the former, i.e. (37) with (38), one should complement the symmetrizing conditions (39), or (42) with (43). In the latter case, i.e., (37) with (40), one should impose the symmetrizing condition (41). In the previous work, we have studied a special case of theories in the class (33) in [75] where $\pi \approx 0$ is specifically chosen as one of the ACs and the particular minimalizing conditions of (37) and (40) for the other AC were also discovered.

D. Case with one auxiliary constraint

As the last case, we study the theory with only one AC, i.e.

$$H_{\mathrm{P}} \equiv \int \mathrm{d}^3 x(\mathcal{H} + \mu_1 \mathcal{S}^1), \tag{44}$$

which may develop into the same classification as in (34)–(36), i.e.

$$\#^{\mathrm{s}}_{1\mathrm{st}} = 2, \qquad \#^{\mathrm{s}}_{2\mathrm{nd}} = 0, \quad (\mathrm{ID\,key:\ I\text{-}2\text{-}0}) \tag{45}$$

$$\#^{\mathrm{s}}_{1\mathrm{st}} = 1, \qquad \#^{\mathrm{s}}_{2\mathrm{nd}} = 2, \quad (\mathrm{ID\,key:\ I\text{-}1\text{-}2}) \tag{46}$$

and

$$\#^{\mathrm{s}}_{1\mathrm{st}} = 0, \qquad \#^{\mathrm{s}}_{2\mathrm{nd}} = 4. \quad (\mathrm{ID\,key:\ I\text{-}0\text{-}4}) \tag{47}$$

It is obvious in this case that the total number of constraints is equal to the level of the secondary constraints since $\mathcal{S}^1 \approx 0$ is the only primary constraint and the secondary constraints must be generated from it step by step.

(1) In the first class (45), i.e. of type I-2-0, the primary constraint $\mathcal{S}^1 \approx 0$ and the secondary constraint $\dot{\mathcal{S}}^1 \approx 0$ are both first-class constraints and it is easy to find the minimalizing conditions as

$$[\mathcal{S}^1(\vec{x}), \mathcal{S}^1(\vec{y})] \approx 0, \tag{48}$$

$$[\mathcal{S}^1(\vec{x}), \dot{\mathcal{S}}^1(\vec{y})] \approx 0, \tag{49}$$

and

$$[\dot{\mathcal{S}}^1(\vec{x}), \dot{\mathcal{S}}^1(\vec{y})] \approx 0, \tag{50}$$

with the symmetrizing condition

$$[\dot{\mathcal{S}}^1(\vec{x}), \mathcal{H}(\vec{y})] \approx 0, \tag{51}$$

which prevents further secondary constraints from being generated.

(2) The second class (46), i.e. of type I-1-2, implies that only one of the three constraints, i.e., the primary constraint $\mathcal{S}^1 \approx 0$, the secondary constraint $\dot{\mathcal{S}}^1 \approx 0$ and the tertiary constraint $\ddot{\mathcal{S}}^1 \approx 0$, is first-class. Similarly to what happens to the class of (35), in this case, the tertiary constraint $\ddot{\mathcal{S}}^1 \approx 0$ is not allowed to be first-class. Therefore, according to whether the first-class is taken by $\mathcal{S}^1 \approx 0$ or by $\dot{\mathcal{S}}^1 \approx 0$, we have the following two scenarios for (46):

(a) If the first-class constraint is $\mathcal{S}^1 \approx 0$, one requires the same minimalizing conditions (48) and (49) but with

$$[\mathcal{S}^1(\vec{x}), \ddot{\mathcal{S}}^1(\vec{y})] \approx 0, \tag{52}$$

instead of (50). In order to prevent the generation of a quaternary constraint we should require the following symmetrizing condition

$$[\ddot{\mathcal{S}}^1(\vec{x}), \mathcal{H}(\vec{y})] \approx 0. \tag{53}$$

We label this case with the identification key I-1-2a to distinguish it from the next case.

(b) On the other hand, choosing $\dot{\mathcal{S}}^1 \approx 0$ as the first-class constraint leads exactly to the same minimalizing conditions as in (48)–(50) but with a different symmetrizing condition

$$[\dot{\mathcal{S}}^1(\vec{x}), \ddot{\mathcal{S}}^1(\vec{y})] \approx 0. \tag{54}$$

This case is labeled with the identification key I-1-2b correspondingly.

(3) The last class (47), i.e. of type I-0-4, requires that the primary, secondary, tertiary and quaternary constraints are all second-class, which yields the same minimalizing conditions as in case a) i.e., (48), (49) and (52), without requiring the symmetrizing condition.

In summary, we find two possible kinds of minimalizing conditions for the partial Hamiltonian (44) with only one AC, both of which consist in (48) and (49), with (50) or (52) respectively. We can choose the symmetrizing conditions as either (51) or (54) for the former, and (53) for the latter. We also point out that these two kinds of minimalizing conditions become equivalent, i.e.

$$[\dot{\mathcal{S}}^1(\vec{x}), \dot{\mathcal{S}}^1(\vec{y})] = [\mathcal{S}^1(\vec{x}), \ddot{\mathcal{S}}^1(\vec{y})] \approx 0, \tag{55}$$

when (49) is satisfied strongly

$$[\mathcal{S}^1(\vec{x}), \dot{\mathcal{S}}^1(\vec{y})] = 0. \tag{56}$$

This is just what happened in [68] where $\mathcal{S}^1 \approx 0$ and $\mathcal{H}$ are respectively specified as $\pi \approx 0$ and

$$\mathcal{H} = \mathcal{V} + N\mathcal{H}_0, \tag{57}$$

where $\mathcal{V}$ and $\mathcal{H}_0$ are two general functions of $(h_{ij}, \pi^{ij}; \nabla_i)$ so that both of the minimalizing conditions (48) and (49) are automatically satisfied strongly and (55) reduces to

TABLE I. The minimalizing and symmetrizing conditions.

# ACs	Minimalizing conditions	Symmetrizing conditions	Classifications	Identification key	Examples
$\#^{s}=4$	None	None	$\#^{s}_{1st}=0$, $\#^{s}_{2nd}=4$	IV-0-4	Mixed traces
$\#^{s}=3$	$[\mathcal{S}^1,\mathcal{S}^n]$	$[\mathcal{S}^1,\mathcal{H}]$	$\#^{s}_{1st}=1$, $\#^{s}_{2nd}=2$	III-1-2	Unknown
		None	$\#^{s}_{1st}=0$, $\#^{s}_{2nd}=4$	III-0-4	Unknown
$\#^{s}=2$	$[\mathcal{S}^1,\mathcal{S}^n]$ & $[\mathcal{S}^2,\mathcal{S}^2]$	$[\mathcal{S}^1,\mathcal{H}]$ & $[\mathcal{S}^2,\mathcal{H}]$	$\#^{s}_{1st}=2$, $\#^{s}_{2nd}=0$	II-2-0	Unknown
		$[\mathcal{S}^2,\dot{\mathcal{S}}^1]$ & $[\mathcal{S}^2,\mathcal{H}]$	$\#^{s}_{1st}=1$, $\#^{s}_{2nd}=2$	II-1-2b	Unknown
		None	$\#^{s}_{1st}=0$, $\#^{s}_{2nd}=4$	II-0-4b	Linear AC
	$[\mathcal{S}^1,\mathcal{S}^n]$ & $[\mathcal{S}^1,\dot{\mathcal{S}}^1]$	$[\dot{\mathcal{S}}^1,H_P]$	$\#^{s}_{1st}=1$, $\#^{s}_{2nd}=2$	II-1-2a	4dEGB
		None	$\#^{s}_{1st}=0$, $\#^{s}_{2nd}=4$	II-0-4a	Unknown
$\#^{s}=1$	$[\mathcal{S}^1,\mathcal{S}^1]$, $[\mathcal{S}^1,\dot{\mathcal{S}}^1]$ & $[\dot{\mathcal{S}}^1,\dot{\mathcal{S}}^1]$	$[\dot{\mathcal{S}}^1,\mathcal{H}]$	$\#^{s}_{1st}=2$, $\#^{s}_{2nd}=0$	I-2-0	GR & $f(\mathcal{H})$
		$[\dot{\mathcal{S}}^1,\ddot{\mathcal{S}}^1]$	$\#^{s}_{1st}=1$, $\#^{s}_{2nd}=2$	I-1-2b	Unknown
	$[\mathcal{S}^1,\mathcal{S}^1]$, $[\mathcal{S}^1,\dot{\mathcal{S}}^1]$ & $[\mathcal{S}^1,\ddot{\mathcal{S}}^1]$	$[\ddot{\mathcal{S}}^1,\mathcal{H}]$	$\#^{s}_{1st}=1$, $\#^{s}_{2nd}=2$	I-1-2a	Cuscuton & QEC
		None	$\#^{s}_{1st}=0$, $\#^{s}_{2nd}=4$	I-0-4	Unknown

Note that we simply denote the condition $[\cdot(\vec{x}),\cdot(\vec{y})]\approx 0$ by $[\cdot,\cdot]$ in the table.

$$[\mathcal{H}_0(\vec{x}),\mathcal{H}_0(\vec{y})]\approx 0. \tag{58}$$

With all the cases studied above, we have exhausted all of the possible constraint structures for the MMG theories with AC(s) and found the corresponding minimalizing and symmetrizing conditions for each class. For convenience, we summarize the results of Secs. III A to III D in the table in Appendix A. In the next section, as an illustrating example for our formalism, we will show how to construct a concrete MMG model corresponding to the case with four ACs by using the generalized Cayley-Hamilton theorem. We also describe some known MMG theories in Appendix B as some concrete examples for the different classifications listed in Table I to better illustrate our formalism.

IV. THE CAYLEY-HAMILTON CONSTRUCTION WITH MIXED TRACES CONSTRAINTS

According to the discussions in Sec. III A, the total Hamiltonian with four ACs describes a broad consistent framework to construct MMG theories, which reads

$$H_{\mathrm{T}}=\int \mathrm{d}^3x(\mathcal{H}+\mu_{\mathrm{n}}\mathcal{S}^{\mathrm{n}}+N^i\mathcal{H}_i+\lambda^i\pi_i), \tag{59}$$

where $\mathcal{H}$ and $\mathcal{S}^{\mathrm{n}}$ $(\mathrm{n}=1,2,3,4)$ are generic functions of $(N,\pi,h_{ij},\pi^{ij};\nabla_i)$, of which the forms can be taken arbitrarily. This theory is able to couple with matter consistently and in a general manner, for example in (25). In order to pick out a concrete MMG model from this class of theories, we will choose some restrictions for the generic functions $\mathcal{H}$ and $\mathcal{S}^{\mathrm{n}}$.

First, for simplicity, we assume that the lapse N is nondynamical, as being considered in [59–61,68,75], which means its conjugate momentum π plays the role of an AC. We take

$$\mathcal{S}^4=\pi\approx 0, \tag{60}$$

therefore the remaining functions $\mathcal{H}$ and $\mathcal{S}^{\mathrm{I}}$ $(\mathrm{I}=1,2,3)$ can be rewritten as generic functions of $(N,h_{ij},\pi^{ij};\nabla_i)$ on the constrained hypersurface or by redefinition of μ_4.

Next we adopt the same restriction as what was imposed in [75], i.e. that the spatial derivative ∇_i appears in the theory in terms of the Ricci tensor R_{ij} only. Thus we consider the $\mathcal{H}$ and $\mathcal{S}^{\mathrm{I}}$ to be generic functions of (N,R_{ij},π^{ij}) only. A motivation for picking this restriction is that according to the generalized Cayley-Hamilton theorem [83] (see also the Appendix in [75]), $\mathcal{H}$ and $\mathcal{S}^{\mathrm{I}}$ can then be equivalently recast into generic functions of the traces $(N,\mathcal{R}^{\mathrm{I}},\Pi^{\mathrm{I}},\mathcal{Q}^{\mathrm{I}})$ constructed from R_{ij} and π^{ij}, in which we denote the traces of R_{ij} by

$$\mathcal{R}^{\mathrm{I}}\equiv\{R^i_i,R^i_jR^j_i,R^i_jR^j_kR^k_i\}, \tag{61}$$

the traces of π_{ij} by

$$\Pi^{\mathrm{I}}\equiv\{\pi^i_i,\pi^i_j\pi^j_i,\pi^i_j\pi^j_k\pi^k_i\}, \tag{62}$$

and the mixed traces by

$$\mathcal{Q}^{\mathrm{I}}\equiv\{R^i_j\pi^j_i,R^i_j\pi^j_k\pi^k_i,R^i_jR^j_k\pi^k_i\}. \tag{63}$$

In the model constructed in [75], the mixed traces terms (63) were dropped for simplicity. Instead, in the current work, we use them to construct ACs by choosing

$$\mathcal{S}^{\mathrm{I}}=\mathcal{Q}^{\mathrm{I}}-\mathcal{P}^{\mathrm{I}}(N)\approx 0, \tag{64}$$

where $\mathcal{P}^{\mathrm{I}}$ are generic functions of N. Please keep in mind that we have complete freedom to determine the form for the

ACs in this theory and each choice may define a different MMG theory. As a result, if we put together the above choices, we have picked a concrete MMG model from the class of theories (59) as

$$H_{\mathrm{T}}^{(\mathrm{C.H.})} = \int \mathrm{d}^3x[\mathcal{H}^{(\mathrm{C.H.})} + N^i\mathcal{H}_i + \lambda^i\pi_i + \lambda\pi + \mu_{\mathrm{I}}(\mathcal{Q}^{\mathrm{I}} - \mathcal{P}^{\mathrm{I}})], \quad (65)$$

where the fourth Lagrange multiplier is written as $\mu_4 \equiv \lambda$ and $\mathcal{H}^{(\mathrm{C.H.})}$ is a free function of $(N, \mathcal{R}^{\mathrm{I}}, \Pi^{\mathrm{I}})$ on the constrained hypersurface for this MMG model (65). We will dub this model the Cayley-Hamilton construction with mixed traces constraints.

A. The dispersion relation

In order to investigate the properties of the model (65) in a cosmological setting, we will derive the dispersion relation of gravitational waves as tensor perturbations on a cosmological background within this model. First, according to (18) and (19), we can easily obtain the corresponding action of the Hamiltonian (65) as follows

$$S^{(\mathrm{C.H.})} = \int \mathrm{d}t\mathrm{d}^3x[2NK_{ij}\pi^{ij} - \mathcal{H}^{(\mathrm{C.H.})} - \mu_{\mathrm{I}}(\mathcal{Q}^{\mathrm{I}} - \mathcal{P}^{\mathrm{I}})], \quad (66)$$

where π^{ij} should be understood as the solution of

$$2NK_{ij} = \frac{\partial\mathcal{H}^{(\mathrm{C.H.})}}{\partial\pi^{ij}} + \mu_{\mathrm{I}}\frac{\partial\mathcal{Q}^{\mathrm{I}}}{\partial\pi^{ij}}, \quad (67)$$

and will rely on the concrete choice of $\mathcal{H}^{(\mathrm{C.H.})}$. The Lagrange multipliers μ_{I} have been fixed by the consistency condition of $\pi \approx 0$ as

$$\mu_{\mathrm{I}} = \left(\frac{\partial\mathcal{P}^{\mathrm{I}}}{\partial N}\right)^{-1}\frac{\partial\mathcal{H}^{(\mathrm{C.H.})}}{\partial N}. \quad (68)$$

The tensor perturbations of the action (66) are defined around a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background with

$$\mu_{\mathrm{I}} = \bar{\mu}_{\mathrm{I}}(t), \qquad N = 1, \qquad N^i = 0, \qquad h_{ij} = a(t)^2\mathfrak{g}_{ij}, \quad (69)$$

where (and also throughout the rest of this paper) a bar“ - ” represents the background values and $a(t)$ is the scale factor of the background FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \quad (70)$$

We expand

$$\mathfrak{g}_{ij} \equiv \delta_{ij} + \gamma_{ij} + \frac{1}{2!}\gamma_{ik}\gamma^k{}_j + \frac{1}{3!}\gamma_{ik}\gamma^k{}_l\gamma^l{}_j + \cdots, \quad (71)$$

with the tensor perturbation $\gamma^i{}_j$ satisfying the transverse and traceless conditions

$$\partial_i\gamma^i{}_j = 0, \qquad \gamma^i_i = 0. \quad (72)$$

Note that we have turned off the scalar- and vector-type perturbations and in this subsection, spatial indices are raised and lowered by δ^{ij} and δ_{ij}.

For generality, we keep $\mathcal{H}^{(\mathrm{C.H.})}$ as a general function. However this also means that we are only able to solve (67) order by order for π^{ij}. By substituting this solution back into the action (66), we can nonetheless find the following quadratic action

$$S_2^{(\mathrm{C.H.})} = \int \mathrm{d}t\mathrm{d}^3x\frac{1}{4}\left(\mathcal{G}_0(t)\dot{\gamma}_{ij}\dot{\gamma}^{ij} + \mathcal{W}_0(t)\gamma_{ij}\frac{\Delta}{a^2}\gamma^{ij} - \mathcal{W}_2(t)\gamma_{ij}\frac{\Delta^2}{a^4}\gamma^{ij}\right), \quad (73)$$

where

$$\mathcal{G}_0(t) \equiv \left[\left(\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^2}\right)^2 - 3\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^1} - 2H\right)\right]^{-1/2}, \quad (74)$$

$$\mathcal{W}_0(t) \equiv -\frac{\partial\bar{\mathcal{H}}}{\partial\mathcal{R}^1} + \varpi_0(t), \quad (75)$$

and

$$\mathcal{W}_2(t) \equiv \frac{\partial\bar{\mathcal{H}}}{\partial\mathcal{R}^2} + \varpi_2(t), \quad (76)$$

with

$$\begin{aligned}\varpi_0(t) \equiv & -\frac{1}{2}\mathcal{G}_0\dot{\bar{\mu}}_1 + \left(3\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\right)^{-1}\left(\mathcal{G}_0\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^2} - 1\right)\dot{\bar{\mu}}_2 \\ & + \left[\left(3\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\right)^{-1}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^2} - \mathcal{G}_0^{-1}\right) - \frac{\dot{\mathcal{G}}_0}{2} + \mathcal{G}_0H\right]\bar{\mu}_1 \\ & + \left(3\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\mathcal{G}_0\right)^{-2}\left[-1 + \mathcal{G}_0\left(3\mathcal{G}_0\left(\mathcal{G}_0\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\frac{\partial\dot{\bar{\mathcal{H}}}}{\partial\Pi^2} + \frac{\partial\dot{\bar{\mathcal{H}}}}{\partial\Pi^3} + 2\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}H\right) + \frac{\partial\bar{\mathcal{H}}}{\partial\Pi^2}\left(2 - 3\mathcal{G}_0\left(\mathcal{G}_0\frac{\partial\dot{\bar{\mathcal{H}}}}{\partial\Pi^3} - \frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\dot{\mathcal{G}}_0 + 2\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\mathcal{G}_0H\right)\right) - \frac{\partial\bar{\mathcal{H}}}{\partial\Pi^2}^2\mathcal{G}_0\right)\right]\bar{\mu}_2,\end{aligned} \quad (77)$$

and

$$\varpi_2(t) \equiv \left(6\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\right)^{-2}\left[12\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\left(\mathcal{G}_0^{-1} - \frac{\partial\bar{\mathcal{H}}}{\partial\Pi^2}\right)\bar{\mu}_3 - \mathcal{G}_0\left(3\frac{\partial\bar{\mathcal{H}}}{\partial\Pi^3}\bar{\mu}_1 + 2\left(\mathcal{G}_0^{-1} - \frac{\partial\bar{\mathcal{H}}}{\partial\Pi^2}\right)\bar{\mu}_2\right)^2\right]. \quad (78)$$

Here, $\mathcal{H}^{(\mathrm{C.H.})}$ is simply denoted by $\mathcal{H}$ for short and $H \equiv \dot{a}/a$ is the Hubble parameter. Note that the background values of the Lagrange multipliers $\bar{\mu}_{\mathrm{I}}$ in (77) and (78) have been fixed by (68).

In order to prevent tensor perturbations from being ghosts, i.e. from acquiring a negative kinetic term, only the positive branch of $\mathcal{G}_0(t)$ is taken in (74) which holds for $\frac{\partial \bar{\mathcal{H}}}{\partial \Pi^2} > 0$. The dispersion relation can be immediately read from (73) as [84]

$$\begin{aligned}\omega_{\mathrm{T}}^2 &= \frac{\mathcal{W}_0(\tau)}{\mathcal{G}_0(\tau)}\frac{k^2}{a^2} + \frac{\mathcal{W}_2(\tau)}{\mathcal{G}_0(\tau)}\frac{k^4}{a^4} \\ &= \frac{k^2}{a^2}\mathcal{G}_0^{-1}\left[\varpi_0 - \frac{\partial \bar{\mathcal{H}}}{\partial \mathcal{R}^1} + \left(\varpi_2 + \frac{\partial \bar{\mathcal{H}}}{\partial \mathcal{R}^2}\right)\frac{k^2}{a^2}\right]. \end{aligned} \quad (79)$$

On large scales, the speed of gravitational waves $c_{\mathrm{T}} = \omega_{\mathrm{T}}/k = 1$ when

$$\frac{\partial \bar{\mathcal{H}}}{\partial \mathcal{R}^1} = \varpi_0 - \mathcal{G}_0. \quad (80)$$

According to the observation of the speed of gravitational waves [11,13] and the modified dispersion relation [12], we should impose the following constraints to (79):

$$-3 \times 10^{-15} < \frac{\mathcal{W}_0}{\mathcal{G}_0} - 1 < 7 \times 10^{-16}, \quad (81)$$

and

$$\left|\frac{\mathcal{W}_2}{\mathcal{G}_0}\right| < 10^{-19}\ \mathrm{peV}^{-2}, \quad (82)$$

where 1 peV $\simeq h \times 250$ Hz with h the Planck constant.

V. CONCLUSIONS

In this work, we have searched for all the possible Hamiltonian structures for minimally modified gravity (MMG) theories with multiple auxiliary constraints (ACs) in the phase space. To do this, we have first investigated the possibilities of reducing the number of degree(s) of freedom (d.o.f.) by introducing ACs to the total Hamiltonian (1) while respecting spatial diffeomorphism. An arbitrary number of scalar-, vector- and tensor-type ACs have been considered *a priori*, and in order to extract the maximum number for each type of ACs, they were first assumed to be linearly independent second-class constraints following Dirac's terminology. By counting the number of each type of d.o.f. with respect to an arbitrary background in (7), and requiring that this number be non-negative, we have found that no vector- and no tensor-type should be introduced, and that there should be no more than four scalar-type of ACs in the absence of external fields.[3] In fact, the vectorial d.o.f. have been completely eliminated by the spatial diffeomorphism constraints (2) and the existence of vector- or tensor-type ACs would lead to a negative result, which is physically inconsistent in the vacuum. Hence, on the premise of retaining spatial covariance, we have determined a consistent framework (11) with no more than four (scalar-type) ACs, $\mathcal{S}^{\mathrm{n}} \approx 0^{\mathrm{n}}$ $[\mathrm{n} = 1, \ldots, \mathcal{N}(\mathcal{N} \leq 4)]$, as our starting point for searching for MMG theories which propagate only two tensorial d.o.f. By this request, the residual scalar d.o.f. should all be completely suppressed by the ACs and the possible secondary constraints being generated from the consistency conditions of the ACs. According to the number of introduced primary ACs, we have exhausted all possible classes of primary and secondary constraints and have found out the corresponding conditions for their consistency.

In Sec. III A with four ACs (21), we have confirmed that all of the ACs must be classified as second-class, and that one requires no extra condition to obtain a MMG theory. Therefore, (21) together with (13) provide an extensive framework for us to investigate several MMG theories with four ACs $\mathcal{S}^{\mathrm{n}} \approx 0^{\mathrm{n}}$, thus leaving in addition to the free function $\mathcal{H}$, four other arbitrary choices of independent functions of $(N, \pi, h_{ij}, \pi^{ij}; \nabla_i)$, i.e. a lot of freedom in terms of model building. More importantly, this type of structure can be used to couple with matter consistently in a very general manner in (25). In Sec. IV, as an illustrating example of this type of structure, we have constructed a practical MMG model by starting from (59). For simplicity, we have restricted ourselves to the special case of a nondynamical lapse which yields $\pi \approx 0$ as one of the four ACs. We have further chosen that the spatial derivative ∇_i should only appear in the theory in the form of the Ricci tensor R_{ij} and, according to the generalized Cayley-Hamilton theorem, we were therefore able to recast $\mathcal{H}$ and $\mathcal{S}^{\mathrm{I}}$ into generic functions of $(N, \mathcal{R}^{\mathrm{I}}, \Pi^{\mathrm{I}}, \mathcal{Q}^{\mathrm{I}})$. By choosing the mixed traces $\mathcal{Q}^{\mathrm{I}}$ as the remaining ACs (64), we were able to pick out an interesting MMG model (65) which we dub the Cayley-Hamilton construction with mixed traces constraints. In order to investigate the properties of this model in the cosmological setting, we have studied the tensor perturbations of the corresponding action (66) on an FLRW background up to the quadratic order and thereby have derived the modified dispersion relation (79) for the gravitational waves (GW) from which we have found that the speed of GW is unity on large scales when (80) is satisfied. Besides, in order to prevent the tensor perturbations from being ghosts, the free function $\mathcal{H}^{(\mathrm{C.H.})}(N, \mathcal{R}^{\mathrm{I}}, \Pi^{\mathrm{I}})$ in the model must satisfy $\frac{\partial \bar{\mathcal{H}}}{\partial \Pi^2} > 0$ and the constraints (81) and (82) from observations.

[3]This conclusion holds as far as we seek theories with only two tensorial d.o.f., even if we allow the traceless or transverse-traceless parts of tensor-type ACs (or the transverse parts of vector-type ACs) to be imposed separately.

In Secs. III B to III D, we have determined all possible classes of primary and secondary constraints for the MMG theories with three, two and one AC(s) respectively. Different from the case with four ACs, in these cases with less AC(s), minimalizing conditions are required. These are the sufficient conditions to suppress the scalar d.o.f. completely, e.g., (30) for the case with three ACs. In particular, with a nondynamical lapse, the specific minimalizing conditions, (37) with (38) or (40), for the case with two ACs had already been discovered in [75]; similarly, the minimalizing conditions (48) and (49) with (50) or (52) had already been discovered [68] respectively. On the other hand, with fewer ACs, there is room for the appearance of extra gauge symmetries other than the spatial diffeomorphism retained from the beginning of the construction. In order to allow for the extra gauge symmetries, so-called symmetrizing conditions have been imposed, e.g. (32) for the case with three ACs. The symmetrizing conditions of course help to suppress the scalar d.o.f., however, they are neither necessary nor sufficient conditions for obtaining a MMG theory because we are always able to fix the gauge symmetries by simply giving up these symmetrizing conditions. Nevertheless they are important for investigating the gauge symmetries of MMG theories [74,82] and it would be interesting to clarify these properties in a future work. We have summarized the minimalizing and symmetrizing conditions for each class in Table I. With this, we have exhausted all the possible constraint structures of the MMG theories with multiple ACs. To better illustrate our formalism, we have collected some MMG theories in Appendix B as concrete examples for some of the classifications listed in Table I.

We finish this paper with the following comments. First, as mentioned previously, we should clarify the possible gauge symmetries for the MMG theories and examine the possible deviations from the spacetime diffeomorphism. Second, we have not constructed a concrete model for the case with three ACs which is also an interesting case of MMG theory and should be studied in the future. Third, the cosmological behavior and evolution with matter of the Cayley-Hamilton construction with mixed traces constraints (65) should be investigated and tested against the observations. It would also be interesting to see whether it can be used to address current issues within cosmology, e.g. the Hubble tension or the dark energy [85,86]. And more importantly, we should find the features of this theory with respect to GWs and see whether it can be practically distinguished from GR. Lastly, we comment on the symplectic structure modified by the ACs. The effects of ACs on the symplectic structure are essentially the same as what usual constraints do. If all the ACs (and induced secondary constraints) are classified into the second-class, the induced two-form has a maximum rank and the second-class ACs can be taken into account by the use of appropriate Dirac brackets. On the other hand, in the case with the first-class ACs, the induced two-form is maximally degenerate and the first-class ACs need to be imposed on the quantum states. We will come back to these questions in the near future.

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APPENDIX A: SUMMARY OF THE MINIMALIZING AND SYMMETRIZING CONDITIONS

We summarize the minimalizing and symmetrizing conditions for each class discussed in Sec. III A to III D in Table I.

APPENDIX B: SOME KNOWN EXAMPLES OF MMG THEORIES

In this appendix, we collect some known MMG theories as concrete examples for some of the classifications in Table I. Especially, the lapse functions of the MMG theories collected in this appendix are all considered to be nondynamical that, which in our terminology, implies a specific AC, i.e. $\mathcal{S}^1 = \pi \approx 0$, in the Hamiltonian of the theories.

1. The QEC gravity

A model of SCG theory with TTDOF was proposed in the [61] whose action is quadratic in the extrinsic curvature (QEC) and by performing a Legendre transformation we can obtain the total Hamiltonian of the QEC gravity as

$$H_{\mathrm{T}}^{(\mathrm{QEC})} = \int \mathrm{d}^3x (\mathcal{V}^{(\mathrm{QEC})} + N\mathcal{H}_0^{(\mathrm{QEC})} + N^i\mathcal{H}_i + \lambda^i\pi_i + \lambda\pi), \tag{B1}$$

in which the Hamiltonian constraint of the QEC gravity can be written as

$$\mathcal{H}_0^{(\mathrm{QEC})} \equiv \frac{1}{2\sqrt{h}}\mathcal{G}_{ij,kl}^{(\mathrm{W.D.})}\pi^{ij}\pi^{kl} - \sqrt{h}[\rho_1(t) + \rho_2(t)R], \tag{B2}$$

with the Wheeler-DeWitt metric [87]

$$\mathcal{G}_{kl,mn}^{(\mathrm{W.D.})} \equiv 2h_{k(m}h_{n)l} - h_{kl}h_{mn}, \tag{B3}$$

and the part with no lapse is

$$\mathcal{V}^{(\mathrm{QEC})} \equiv \frac{1}{2\sqrt{h}}\mathcal{V}^{(\mathrm{QEC})}_{ij,kl}\pi^{ij}\pi^{kl} - \sqrt{h}[\rho_3(t)+\rho_4(t)R], \tag{B4}$$

with

$$\mathcal{V}^{(\mathrm{QEC})}_{kl,mn} \equiv 2\beta_2 h_{k(m}h_{n)l} - \frac{1}{3}(\beta_1+2\beta_2)h_{kl}h_{mn}. \tag{B5}$$

The coefficients β_1, β_2 and $\rho_1 \sim \rho_4$ are all general functions of time.

From the viewpoint of our formalism, given the total Hamiltonian of QEC gravity (B1), we have

$$\mathcal{H} = \mathcal{V}^{(\mathrm{QEC})} + N\mathcal{H}_0^{(\mathrm{QEC})}, \qquad \mathcal{S}^1 = \pi \approx 0, \tag{B6}$$

and one can check that the minimalizing conditions (48), (49) and (52) and the symmetrizing condition (53) are satisfied, therefore the QEC gravity generally belongs to the I-1-2a type of MMG theory.

A special case of the QEC gravity is the Cuscuton theory, which corresponds to

$$\mathcal{V}^{(\mathrm{Cus})} = -\sqrt{h}\mu^2(t). \tag{B7}$$

In particular, if we set $\mathcal{V}^{(\mathrm{QEC})} = 0$, then the symmetrizing condition (53) is trivially satisfied, which turns the QEC gravity into the I-2-0 type of MMG theory. Especially, if we further set the coefficient in front of the Ricci scalar to unity, i.e. $\rho_2 = 1$, we recover general relativity.

2. The $f(\mathcal{H})$ theory

A more general I-2-0 type MMG theory was proposed in [67,68] whose total Hamiltonian can be written as

$$H_{\mathrm{T}}^{(\mathrm{fH})} = \int \mathrm{d}^3x[Nf(\mathcal{H}_{\mathrm{gr}}) + N^i\mathcal{H}_i + \lambda^i\pi_i + \lambda\pi], \tag{B8}$$

where the Hamiltonian constraint $\mathcal{H}_0$ is chosen as a function of the Hamiltonian constraint in GR, i.e.,

$$\mathcal{H}_0^{(\mathrm{fH})} \equiv f(\mathcal{H}_{\mathrm{gr}}), \quad \text{with} \quad \mathcal{H}_{\mathrm{gr}} \equiv \frac{1}{2\sqrt{h}}\mathcal{G}^{(\mathrm{W.D.})}_{ij,kl}\pi^{ij}\pi^{kl} - R, \tag{B9}$$

the free function f being the reason for it to be dubbed the $f(\mathcal{H})$ theory. It is easy to check that the $f(\mathcal{H})$ theory (B8) satisfies the minimalizing conditions (48), (49) and (50) and the symmetrizing condition (51), which points it to the I-2-0 type of MMG theory.[4]

[4]As other examples of the I-2-0 type of MMG theories, two different MMG theories with the square root form of the Hamiltonians were constructed earlier and dubbed the "intrinsic time gravity" in [88] and "square root gravity" in [59] respectively.

3. The consistent $d \to 4$ EGB gravity

A concrete example with an additional AC other than $\pi \approx 0$ is the consistent $d \to 4$ Einstein-Gauss-Bonnet (4dEGB) gravity [79] in which a gauge fixing condition is introduced in order to cure the inconsistency of the initial theory [43]. As a result, the total Hamiltonian of the consistent 4dEGB gravity can be expressed as follows

$$H_{\mathrm{T}}^{(\mathrm{4dEGB})} = \int \mathrm{d}^3x(N\mathcal{H}_0^{(\mathrm{4dEGB})} + N^i\mathcal{H}_i + \lambda^i\pi_i + \lambda\pi + \mu\, {}^3\mathcal{G}), \tag{B10}$$

and the Hamiltonian constraint is determined by

$$\mathcal{H}_0^{(\mathrm{4dEGB})} \equiv \frac{\sqrt{h}}{2\kappa^2}\left[2\Lambda - \mathcal{M} + \tilde{\alpha}\left(4\mathcal{M}_{ij}\mathcal{M}^{ij} - \frac{3}{2}\mathcal{M}^2\right)\right], \tag{B11}$$

where κ is the gravitational coupling constant and $\tilde{\alpha}$ is the Gauss-Bonnet coupling with

$$\mathcal{M}_{ij} \equiv R_{ij} + \mathcal{K}^k_k\mathcal{K}_{ij} - \mathcal{K}_{ik}\mathcal{K}^k_j. \tag{B12}$$

The $\mathcal{K}_{ij}$ in (B12) should be understood as the solution of

$$\pi^i_j = \frac{\sqrt{h}}{2\kappa^2}\Bigg[\mathcal{K}^i_j - \mathcal{K}\delta^i_j - \frac{8}{3}\tilde{\alpha}\delta^{ikl}_{jrs}\mathcal{K}^r_k \times \left(R^s_l - \frac{1}{4}\delta^s_l R + \frac{1}{2}\left(\mathcal{M}^s_l - \frac{1}{4}\delta^s_l\mathcal{M}\right)\right)\Bigg], \tag{B13}$$

with $\delta^{ikl}_{jrs} \equiv 3!\delta^{[i}_r\delta^j_s\delta^{k]}_t$. The gauge fixing condition ${}^3\mathcal{G}$ is introduced as a general function of $(h_{ij}, \pi^{ij}; \nabla_i)$ and to match 4dEGB gravity (B10) with our framework, we take

$$\mathcal{H} = N\mathcal{H}_0^{(\mathrm{4dEGB})}, \qquad \mathcal{S}^1 = \pi \approx 0, \qquad \mathcal{S}^2 = {}^3\mathcal{G} \approx 0. \tag{B14}$$

One can check that the minimalizing conditions (37) and (40) as well as the symmetrizing condition (41) are satisfied, which implies that the consistent 4dEGB gravity (B10) belongs to the II-1-2a type of MMG theory.

A general framework for the II-1-2a type of MMG theory was proposed in [75]

$$H_{\mathrm{T}} = \int \mathrm{d}^3x(\mathcal{V} + N\mathcal{H}_0 + N^i\mathcal{H}_i + \lambda^i\pi_i + \lambda\pi + \nu\varphi_0), \tag{B15}$$

where the free function $\mathcal{V}$, the Hamiltonian constraint $\mathcal{H}_0$ and the AC φ_0 are arbitrary functions of $(h_{ij}, \pi^{ij}; \nabla_i)$.

4. The Cayley-Hamilton construction with a linear AC

In our previous work [75], we constructed a concrete MMG theory with a linear AC by applying the generalized Cayley-Hamilton theorem as

$$H_{\mathrm{T}}^{(\mathrm{LAC})} = \int \mathrm{d}^3x(\mathcal{H}^{(\mathrm{C.H.})} + N^i\mathcal{H}_i + \lambda^i\pi_i + \lambda\pi + \nu\hat{\varphi}), \quad \text{(B16)}$$

where the $\mathcal{H}^{(\mathrm{C.H.})}$ is identical to the free function in (65) and $\hat{\varphi}$ is called the linear AC with the following form

$$\hat{\varphi} \equiv c_1(t)\pi^i_i + c_2(t)\sqrt{h}R^i_i + c_3(t)\sqrt{h}\nabla^2\frac{\pi^i_i}{\sqrt{h}}. \quad \text{(B17)}$$

We have demonstrated that the total Hamiltonian (B16) belongs to the II-0-4b type of MMG theory in [75] in which we also proposed a general framework for this type MMG theory as

$$H_{\mathrm{T}} = \int \mathrm{d}^3x(\mathcal{H} + N^i\mathcal{H}_i + \lambda^i\pi_i + \lambda\pi + \nu\tilde{\varphi}), \quad \text{(B18)}$$

where the free function $\mathcal{H}$ and the AC $\tilde{\varphi}$ are arbitrary functions of $(N, h_{ij}, \pi^{ij}; \nabla_i)$ and (h_{ij}, π^{ij}) respectively.

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