Black hole perturbations in Maxwell-Horndeski theories

Ryotaro Kase[1] and Shinji Tsujikawa[2]
[1]*Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*
[2]*Department of Physics, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*

(Received 29 January 2023; accepted 1 May 2023; published 19 May 2023)

We study the linear stability of black holes in Maxwell-Horndeski theories where a $U(1)$ gauge-invariant vector field is coupled to a scalar field with the Lagrangian of full Horndeski theories. The perturbations on a static and spherically symmetric background can be decomposed into odd- and even-parity modes under the expansion of spherical harmonics with multipoles l. For $l \geq 2$, the odd-parity sector contains two propagating degrees of freedom associated with the gravitational and vector field perturbations. In the even-parity sector, there are three dynamical perturbations arising from the scalar field besides the gravitational and vector field perturbations. For these five propagating degrees of freedom, we derive conditions for the absence of ghost/Laplacian stabilities along the radial and angular directions. We also discuss the stability of black holes for $l = 0$ and $l = 1$, in which case no additional conditions are imposed to those obtained for $l \geq 2$. We apply our general results to Einstein-Maxwell-dilaton-Gauss-Bonnet theory and Einstein-Born-Infeld-dilaton gravity and show that hairy black hole solutions present in these theories can be consistent with all the linear stability conditions. In regularized four-dimensional Einstein-Gauss-Bonnet gravity with a Maxwell field, however, exact charged black hole solutions known in the literature are prone to instabilities of even-parity perturbations besides a strong coupling problem with a vanishing kinetic term of the radion mode.

DOI: 10.1103/PhysRevD.107.104045

I. INTRODUCTION

Black holes (BHs) are the fundamental objects arising as a solution to the Einstein field equation in General Relativity (GR). On a static and spherically symmetric background, the Schwarzschild geometry characterized by a single mass parameter M is a unique asymptotically flat solution in GR without matter. In the presence of a Maxwell field, there is a static BH with an electric charge q known as a Reissner-Nordström (RN) solution. Allowing the rotation of BHs leads to a Kerr solution containing an angular momentum J. In Einstein-Maxwell theory without additional matter, there is a uniqueness theorem stating that stationary and asymptotically flat BHs are characterized only by three parameters, i.e., M, q, and J [1–4].

If we take an extra degree of freedom into account, it is possible to have additional BH "hairs" to those present in GR without matter. For a minimally coupled canonical scalar field ϕ [4,5] and k-essence [6] as well as for a nonminimally coupled scalar field with the Ricci scalar R of the form $G_4(\phi)R$ [7–10], it is known that BHs do not have scalar hairs. If the scalar field is coupled to a Gauss-Bonnet (GB) term $R_{\rm GB}^2$ of the form $\xi(\phi)R_{\rm GB}^2$ [11–13], where $\xi(\phi)$ is a function of ϕ, the existence of asymptotically flat hairy BH solutions was shown for the dilatonic coupling $\xi(\phi) \propto e^{-\lambda\phi}$ [14–24] and the linear coupling $\xi(\phi) \propto \phi$ [25,26]. It was also found that, for the scalar-GB coupling $\xi(\phi)$ with even power-law functions of ϕ, a phenomenon called spontaneous scalarization of BHs can occur [27–34], analogous to spontaneous scalarization of neutron stars induced by a nonminimal coupling with the Ricci scalar [35].

The scalar-GB coupling mentioned above belongs to a subclass of Horndeski theories with second-order Euler equations of motion [36,37]. If we consider a time-independent scalar field on the static and spherically symmetric background, there are some other subclasses of Horndeski theories in which hairy BH solutions are present. One example is a scalar nonminimal derivative coupling $\phi G_{\mu\nu}\nabla^\mu\nabla^\nu\phi$ to the Einstein tensor $G_{\mu\nu}$, in which case nonasymptotically flat BH solutions are present [38–43]. However, it was recently recognized that these solutions are unstable against linear perturbations around the BH horizon [44]. In so-called regularized four-dimensional Einstein-Gauss-Bonnet (4DEGB) theory [45] where the GB coupling $\hat{\alpha}_{\rm GB}R_{\rm GB}^2$ in a D-dimensional spacetime is rescaled as $\hat{\alpha}_{\rm GB} \to \alpha_{\rm GB}/(D-4)$ on a $(D-4)$-dimensional maximally symmetric flat space [46,47], there exists an exact hairy BH solution respecting the asymptotic flatness. The 4DEGB gravity also belongs to a subclass of Horndeski theories with the scalar field playing the role of a radion [48], so the linear stability conditions derived in

Refs. [49–51] for full Horndeski theories can be applied to this case as well. The recent study [52] showed that the exact BH solution present in 4DEGB gravity is not only unstable but also plagued by a strong coupling problem.

The instabilities of BHs found in nonminimal derivative coupling and 4DEGB theories are related to a finite scalar field kinetic term $X = -(1/2)\nabla^\mu\phi\nabla_\mu\phi$ on the horizon [44]. If we try to search for asymptotically flat hairy BHs with a static scalar field in full Horndeski theories, models with regular coupling functions $G_{2,3,4,5}$ of ϕ and X generally result in no-hair Schwarzschild solutions [53]. The exceptional case is the scalar-GB coupling $\xi(\phi)R^2_{\rm GB}$ mentioned above, in which case the corresponding hairy BHs can be consistent with all the linear stability conditions in a small GB coupling regime. For a scalar field having the dependence of time t in the form $\phi = q_c t + \Psi(r)$, where q_c is a constant and $\Psi(r)$ is a function of the radial coordinate, it is known that a stealth Schwarzschild solution is also present [54,55]. It is still fair to say that the construction of asymptotically flat hairy BHs free from instabilities is limited in the framework of Horndeski theories, especially for a time-independent scalar field.

If we consider an electromagnetic tensor $F_{\mu\nu}$ coupled to the scalar field ϕ, there are more possibilities for realizing hairy BHs. From the theoretical perspective, heterotic string theory gives rise to a coupling between the dilaton field ϕ and Maxwell field strength $F = -F_{\mu\nu}F^{\mu\nu}/4$. In Einstein-Maxwell-dilaton theory given by the Lagrangian $\mathcal{L} = R + 4X + 4e^{-2\phi}F$, Gibbons and Maeda (GM) [56] and Garfinkle, Horowitz, and Strominger (GHS) [57] found charged hairy BH solutions with a nonvanishing dilaton. The dilatonic hair appears as a result of the coupling with the electromagnetic field. We note that, for scalar-vector couplings $\xi(\phi)F$ with even power-law functions of ϕ in ξ, the RN BH can trigger tachyonic instability to evolve into a scalarized charged BH [58–62]. The low energy effective action in string theory also contains a coupling between the dilaton and the GB term as a next-to-leading order term of the inverse string tension α'. In the presence of the dilatonic coupling with both Maxwell and GB terms, Mignemi and Stewart [63] showed the existence of hairy BH solutions by using an expansion in terms of the small coupling α' (see Refs. [64,65] for related works).

In 4DEGB gravity, the hairy BH said before corresponds to an exact solution without a Maxwell field. Analogous to the case of string theory, one can incorporate an electromagnetic field in the four-dimensional effective action. Indeed, there exists an exact charged BH solution in 4DEGB gravity [66], which is analogous to those derived in higher-dimensional setups [67–69]. It is not yet clarified whether this charged BH has the problems of instability and strong coupling mentioned above.

In open string theory, there are possible corrections to the Maxwell action arising from couplings of the Abelian gauge field to bosonic strings [70–72]. The tree-level effective electromagnetic action coincides with a nonlinear action of Born and Infeld (BI) given by the Lagrangian $\mathcal{L} = (4/b^2)(1-\sqrt{1-2b^2F})$ [73]. At leading order in the expansion with respect to a small coupling constant b, the BI action recovers the Maxwell Lagrangian $\mathcal{L} = 4F$. In the four-dimensional Einstein-BI gravity, there is an exact BH solution whose metric differs from the RN solution [74–76]. One can deal with such a nonlinear electromagnetism by considering a general function of $G_2(F)$ in the Lagrangian. If there is a scalar field ϕ coupled with the Maxwell field, the Lagrangian can be further extended to the form $G_2(\phi, X, F)$. Indeed, in Einstein-Born-Infeld-dilaton gravity where the dilaton is coupled to the BI field, the existence of hairy BH solutions is also known [77–81].

In this paper, we study the stability of static and spherically symmetric BH solutions in four-dimensional Maxwell-Horndeski theories where the scalar field ϕ with the Horndeski Lagrangian is coupled to a $U(1)$ gauge-invariant vector field through the coupling $G_2(\phi, X, F)$. A similar study was performed in Ref. [82] for the Lagrangian $\mathcal{L} = G_2(\phi, X, F) + G_4(\phi)R$, but our analysis is more general in that the scalar field sector is described by the full Horndeski Lagrangian. By doing this, we can accommodate the stabilities of hairy BH solutions present in all the theories mentioned above, especially those containing the GB term.

We decompose the types of perturbations into the odd- and even-parity sectors and derive all the linear stability conditions of five dynamical perturbations. In particular we will derive the propagation speeds of even-parity perturbations along the angular direction in the limit of large multipoles l, which are missing in most of the papers about BH perturbations in the literature [50,82]. We note that, in full Horndeski theories with a perfect fluid, all the linear stability conditions including the angular propagation speeds were derived in Ref. [51], which can be applied to the BH case as well (see also Ref. [83]). Indeed, the angular Laplacian stability is important to exclude hairy BHs arising in nonminimal derivative coupling theories [44,44] and in 4DEGB gravity [52]. Neutron stars with scalar hairs present in the same theories are also prone to similar instability problems [51,84,85].

After deriving all the linear stability conditions of odd- and even-parity perturbations in Maxwell-Horndeski theories, we will apply them to concrete hairy BH solutions present in Einstein-Maxwell-dilaton theory, Einstein-BI-dilaton gravity, Einstein-Maxwell-dilaton-GB theory, and 4DEGB gravity. While the first three theories allow the existence of charged BHs consistent with all the linear stability conditions, the exact charged BH solution present in 4DEGB gravity suffers from Laplacian instability of even-parity perturbations as well as the strong coupling problem. The nature of instabilities is similar to what was found for the uncharged exact BH solution in 4DEGB gravity [52]. Thus, our stability criteria in Maxwell-Horndeski theories are useful to exclude some BH solutions or constrain allowed parameter spaces. The second-order actions of odd- and even-parity perturbations and resulting field equations of motion can be also applied to the computation of BH quasinormal modes.

This paper is organized as follows. In Sec. II, we derive the field equations of motion in Maxwell-Horndeski theories on the static and spherically symmetric background. In Sec. III, we obtain conditions for the absence of ghost/Laplacian instabilities in the odd-parity sector and show that the propagation of vector field perturbation is luminal with the other stability conditions similar to those in Horndeski theories. In Sec. IV, we derive the second-order action of even-parity perturbations and clarify how the vector field perturbation affects the linear stability conditions. Since the number of dynamical degrees of freedom (DOFs) depends on the multipole l in the expansion of spherical harmonics, we discuss the cases $l \geq 2$, $l = 0$, and $l = 1$, in turn. In Sec. V, we apply our general results to the stability of hairy BHs present in several classes of theories mentioned above. Section VI is devoted to conclusions.

II. MAXWELL-HORNDESKI THEORIES

We consider a scalar field ϕ in the framework of Horndeski theories with second-order Euler equations of motion [36]. We also incorporate a $U(1)$ gauge-invariant vector field A_μ with the field strength tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, where ∇_μ is a covariant-derivative operator. The vector field Lagrangian depends on a scalar quantity

$$F \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{2.1}$$

We allow the existence of couplings between the scalar and vector fields of the form $G_2(\phi, X, F)$, where G_2 is a function of ϕ, $X = -(1/2)\nabla^\mu \phi \nabla_\mu \phi$, and F. The action of Maxwell-Horndeski theories is given by

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\Big\{G_2(\phi,X,F) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R + G_{4,X}(\phi,X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)] + G_5(\phi,X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}(\phi,X)[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)]\Big\}, \tag{2.2}$$

where g is a determinant of the metric tensor $g_{\mu\nu}$, and G_3, G_4, G_5 are functions of ϕ and X. We use the notations $\Box\phi \equiv \nabla^\mu\nabla_\mu\phi$ and $G_{j,\phi} \equiv \partial G_j/\partial\phi$, $G_{j,X} \equiv \partial G_j/\partial X$, $G_{j,\phi X} \equiv \partial^2 G_j/(\partial X \partial\phi)$ $(j = 2,3,4,5)$, and so on. The action (2.2) is invariant under the shift $A_\mu \to A_\mu + \partial_\mu \chi$, so the theory respects a $U(1)$ gauge symmetry. Introducing the gauge-invariant vector field A_μ to the Horndeski action gives rise to two additional dynamical DOFs to those in Horndeski theories (one scalar and two tensor modes). Hence the total propagating DOFs are five in Maxwell-Horndeski theories.[1]

In this section, we derive the background equations of motion on a static and spherically symmetric spacetime given by the line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + h^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.3}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$, and t, r, and (θ, φ) represent the time, radial, and angular coordinates, respectively, and f and h are functions of r. Since we are interested in the stability of BHs outside the horizon, we will consider positive values of f and h. On the background (2.3), we consider a time-independent scalar field with the radial dependence

$$\phi = \phi(r). \tag{2.4}$$

As we mentioned in Introduction, Maxwell-Horndeski theories given by the action (2.2) can accommodate a variety of hairy BH solutions known in the literature. For the vector field, we consider the following configuration [98]:

$$A_\mu = [A_0(r), A_1(r), 0, 0]. \tag{2.5}$$

In the $U(1)$ gauge-invariant theory under consideration now, the longitudinal mode $A_1(r)$ does not contribute to the background equations. The scalar quantities X and F reduce, respectively, to

$$X = -\frac{1}{2}h\phi'^2, \qquad F = \frac{h}{2f}A_0'^2, \tag{2.6}$$

where a prime represents the derivative with respect to r.

Varying the action (2.2) with respect to $g_{\mu\nu}$, the (00), (11), (22) components of gravitational field equations of motion are given, respectively, by

[1]If we consider generalized Proca (GP) theories [86–88] with a $U(1)$-symmetry breaking gauge field, there is an additional longitudinal propagation of the vector field. It is known that there are hairy BH solutions in GP theories [89–93], but our analysis in this paper does not accommodate such cases. Readers may refer to Refs. [94–97] for BH perturbations (mostly in the odd-parity sector) in GP theories and its extensions.

$$\mathcal{E}_{00} \equiv \left(C_1 + \frac{C_2}{r} + \frac{C_3}{r^2}\right)\phi'' + \left(\frac{\phi'}{2h}C_1 + \frac{C_4}{r} + \frac{C_5}{r^2}\right)h' + C_6 + \frac{C_7}{r} + \frac{C_8}{r^2} - \frac{h}{f}G_{2,F}A_0'^2 = 0, \tag{2.7}$$

$$\mathcal{E}_{11} \equiv -\left(\frac{\phi'}{2h}C_1 + \frac{C_4}{r} + \frac{C_5}{r^2}\right)\frac{hf'}{f} + C_9 - \frac{2\phi'}{r}C_1 - \frac{1}{r^2}\left[\frac{\phi'}{2h}C_2 + (h-1)C_4\right] + \frac{h}{f}G_{2,F}A_0'^2 = 0, \tag{2.8}$$

$$\mathcal{E}_{22} \equiv \left[\left\{C_2 + \frac{(2h-1)\phi' C_3 + 2hC_5}{h\phi' r}\right\}\frac{f'}{4f} + C_1 + \frac{C_2}{2r}\right]\phi'' + \frac{1}{4f}\left(2hC_4 - \phi' C_2 + \frac{2hC_5 - \phi' C_3}{r}\right)\left(f'' - \frac{f'^2}{2f}\right) + \left[C_4 + \frac{2h(2h+1)C_5 - \phi' C_3}{2h^2 r}\right]\frac{f'h'}{4f} + \left(\frac{C_7}{4} + \frac{C_{10}}{r}\right)\frac{f'}{f} + \left(\frac{\phi'}{h}C_1 + \frac{C_4}{r}\right)\frac{h'}{2} + C_6 + \frac{C_7}{2r} = 0, \tag{2.9}$$

where a prime represents the derivative with respect to r, and the coefficients are given by

$$\begin{aligned}
C_1 &= -h^2(G_{3,X} - 2G_{4,\phi X})\phi'^2 - 2G_{4,\phi}h,\\
C_2 &= 2h^3(2G_{4,XX} - G_{5,\phi X})\phi'^3 - 4h^2(G_{4,X} - G_{5,\phi})\phi',\\
C_3 &= -h^4 G_{5,XX}\phi'^4 + h^2 G_{5,X}(3h-1)\phi'^2,\\
C_4 &= h^2(2G_{4,XX} - G_{5,\phi X})\phi'^4 + h(3G_{5,\phi} - 4G_{4,X})\phi'^2 - 2G_4,\\
C_5 &= -\frac{1}{2}[G_{5,XX}h^3\phi'^5 - hG_{5,X}(5h-1)\phi'^3],\\
C_6 &= h(G_{3,\phi} - 2G_{4,\phi\phi})\phi'^2 + G_2,\\
C_7 &= -2h^2(2G_{4,\phi X} - G_{5,\phi\phi})\phi'^3 - 4G_{4,\phi}h\phi',\\
C_8 &= G_{5,\phi X}h^3\phi'^4 - h(2G_{4,X}h - G_{5,\phi}h - G_{5,\phi})\phi'^2 - 2G_4(h-1),\\
C_9 &= -h(G_{2,X} - G_{3,\phi})\phi'^2 - G_2,\\
C_{10} &= \frac{1}{2}G_{5,\phi X}h^3\phi'^4 - \frac{1}{2}h^2(2G_{4,X} - G_{5,\phi})\phi'^2 - G_4 h.
\end{aligned} \tag{2.10}$$

The scalar field equation of motion following from the variation of (2.2) with respect to ϕ gives

$$\frac{1}{r^2}\sqrt{\frac{h}{f}}\left(r^2\sqrt{\frac{f}{h}}J^r\right)' + \frac{\partial \mathcal{E}}{\partial \phi} = 0, \tag{2.11}$$

where

$$J^r = \left(C_1 + \frac{C_2}{r} + \frac{C_3}{r^2}\right)\frac{f'}{2f} - \frac{C_6 + C_9}{\phi'} + \frac{2}{r}C_1 + \frac{1}{r^2}\left(\frac{1+h}{2h}C_2 - \frac{C_4 + C_8 - 2C_{10}}{\phi'}\right), \tag{2.12}$$

$$\mathcal{E} = \left[C_1 + \frac{1}{r^2}\left(\frac{C_3}{2h} - \frac{C_5}{\phi'}\right)\right]\left(\phi'' + \frac{\phi' h'}{2h}\right) + \left[\frac{\phi'}{2}C_2 - hC_4 + \frac{1}{2r}\left(\frac{\phi'}{2}C_3 - hC_5\right)\right]\frac{f'}{rf} + C_6 + \frac{1}{r^2}\left(\frac{\phi'}{2}C_2 - hC_4 + C_8 - 2C_{10}\right). \tag{2.13}$$

Varying the action (2.2) with respect to A_0, it follows that

$$\mathcal{E}_{A_0} \equiv \left(G_{2,F}\sqrt{\frac{h}{f}}r^2 A_0'\right)' = 0, \tag{2.14}$$

whose integrated solution is given by

$$A_0' = \frac{1}{G_{2,F}}\sqrt{\frac{f}{h}}\frac{q_0}{r^2}, \tag{2.15}$$

where q_0 is constant. We will only focus on the case of an electric charge q_0 without considering the magnetic charge.

On using Eqs. (2.7)–(2.9) and (2.14), the scalar field Eq. (2.11) can be expressed as

$$\mathcal{E}_\phi \equiv -\frac{2}{\phi'}\left[\frac{f'}{2f}\mathcal{E}_{00} + \mathcal{E}_{11}' + \left(\frac{f'}{2f} + \frac{2}{r}\right)\mathcal{E}_{11} + \frac{2}{r}\mathcal{E}_{22} - \frac{A_0'}{r^2}\sqrt{\frac{h}{f}}\mathcal{E}_{A_0}\right] = 0. \tag{2.16}$$

We note that some of the coefficients appearing in the second-order action of even-parity perturbations derived later can be expressed in terms of the partial ϕ derivatives of $\mathcal{E}_\phi$ and $\mathcal{E}_{11}$.

III. ODD-PARITY PERTURBATIONS

On top of the static and spherically symmetric background (2.3), we decompose the metric tensor into the background and perturbed parts as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where a bar represents the background quantity. Under the rotation in the (θ, φ) plane, the metric perturbations $h_{\mu\nu}$ can be separated into odd- and even-parity modes [99,100]. Expanding $h_{\mu\nu}$ in terms of the spherical harmonics $Y_{lm}(\theta, \varphi)$, the odd- and even-modes of perturbations have parities $(-1)^{l+1}$ and $(-1)^l$, respectively. In the odd-parity sector, the components of $h_{\mu\nu}$ are given by

$$
\begin{aligned}
&h_{tt} = h_{tr} = h_{rr} = 0,\\
&h_{ta} = \sum_{l,m} Q(t,r) E_{ab} \nabla^b Y_{lm}(\theta,\varphi), \qquad h_{ra} = \sum_{l,m} W(t,r) E_{ab} \nabla^b Y_{lm}(\theta,\varphi),\\
&h_{ab} = \frac{1}{2}\sum_{l,m} U(t,r)[E_a{}^c \nabla_c \nabla_b Y_{lm}(\theta,\varphi) + E_b{}^c \nabla_c \nabla_a Y_{lm}(\theta,\varphi)],
\end{aligned} \tag{3.1}
$$

where Q, W, and U are functions of t and r, and the subscripts a and b denote either θ or φ [49,101–103]. In a formal sense, we should write subscripts l and m for the variables Q, W, and U, but we omit them for brevity. We note that E_{ab} is an antisymmetric tensor with nonvanishing components $E_{\theta\varphi} = -E_{\varphi\theta} = \sin\theta$. The scalar field ϕ does not have an odd-parity perturbation, so it is equivalent to the background value $\phi(r)$. The vector field A_μ in the odd-parity sector has the following perturbed components:

$$
\delta A_t = \delta A_r = 0, \qquad \delta A_a = \sum_{l,m} \delta A(t,r) E_{ab} \nabla^b Y_{lm}(\theta,\varphi), \tag{3.2}
$$

where δA depends on t and r.

Under a gauge transformation $x_\mu \to x_\mu + \xi_\mu$, where $\xi_t = 0$, $\xi_r = 0$, and $\xi_a = \sum_{l,m} \Lambda(t,r) E_{ab} \nabla^b Y_{lm}(\theta,\varphi)$, the metric perturbations transform as $Q \to Q + \dot{\Lambda}$, $W \to W + \Lambda' - 2\Lambda/r$, and $U \to U + 2\Lambda$, where a dot represents the derivative with respect to t. In the following, we choose the gauge

$$
U = 0, \tag{3.3}
$$

which fixes the scalar Λ in ξ_a.

We expand the action (2.2) up to quadratic order in odd-parity perturbations. For this purpose it is sufficient to focus on the axisymmetric modes of perturbations characterized by $m = 0$, since the nonaxisymmetric modes with $m \neq 0$ can be restored under the suitable rotation by virtue of the spherical symmetry on the background [104]. We perform the integrals with respect to θ and φ by using the following properties:

$$
\begin{aligned}
&\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta Y_{l0,\theta}^2 \sin\theta = L,\\
&\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \left(\frac{Y_{l0,\theta}^2}{\sin\theta} + Y_{l0,\theta\theta}^2 \sin\theta \right) = L^2,
\end{aligned} \tag{3.4}
$$

where

$$
L \equiv l(l+1). \tag{3.5}
$$

We also exploit the background Eqs. (2.7), (2.8), and (2.14) to eliminate the terms G_2, $G_{2,X}$, and A_0''. After integrating the action $\mathcal{S}$ with respect to t and r, the second-order action can be expressed in the form

$$
\mathcal{S}_{\rm odd} = \sum_l L \int \mathrm{d}t\mathrm{d}r \mathcal{L}_{\rm odd}, \tag{3.6}
$$

where

$$
\begin{aligned}
\mathcal{L}_{\rm odd} = {}& \frac{\sqrt{h}}{4\sqrt{f}} \mathcal{H} \left(\dot{W} - Q' + \frac{2Q}{r} \right)^2 \\
& - \frac{\sqrt{h}}{\sqrt{f}} G_{2,F} A_0' \left(\dot{W} - Q' + \frac{2Q}{r} \right) \delta A \\
& + (L-2) \left(\frac{\mathcal{F} Q^2}{4r^2 \sqrt{fh}} - \frac{\sqrt{fh}}{4r^2} \mathcal{G} W^2 \right) \\
& + \frac{1}{2\sqrt{fh}} G_{2,F} \left(\dot{\delta A}^2 - fh \delta A'^2 - \frac{Lf}{r^2} \delta A^2 \right),
\end{aligned} \tag{3.7}
$$

with

$$
\mathcal{H} \equiv 2G_4 + 2h\phi'^2 G_{4,X} - h\phi'^2 G_{5,\phi} - \frac{h^2 \phi'^3 G_{5,X}}{r}, \tag{3.8}
$$

$$
\mathcal{F} \equiv 2G_4 + h\phi'^2 G_{5,\phi} - h\phi'^2 \left(\frac{1}{2} h'\phi' + h\phi'' \right) G_{5,X}, \tag{3.9}
$$

$$
\mathcal{G} \equiv 2G_4 + 2h\phi'^2 G_{4,X} - h\phi'^2 \left(G_{5,\phi} + \frac{f' h \phi' G_{5,X}}{2f} \right). \tag{3.10}
$$

A. $l \geq 2$

We first derive linear stability conditions for the multipoles $l \geq 2$. To identify the dynamical DOFs, it is convenient to consider the following Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\rm odd} = {}& \frac{\sqrt{h}}{4\sqrt{f}} \mathcal{H} \left[2\chi \left(\dot{W} - Q' + \frac{2Q}{r} - \frac{2G_{2,F} A_0'}{\mathcal{H}} \delta A \right) - \chi^2 \right] \\
& - \frac{1}{\mathcal{H}} \frac{\sqrt{h}}{\sqrt{f}} G_{2,F}^2 A_0'^2 \delta A^2 + (L-2) \left(\frac{\mathcal{F} Q^2}{4r^2 \sqrt{fh}} - \frac{\sqrt{fh}}{4r^2} \mathcal{G} W^2 \right) \\
& + \frac{1}{2\sqrt{fh}} G_{2,F} \left(\dot{\delta A}^2 - fh \delta A'^2 - \frac{Lf}{r^2} \delta A^2 \right),
\end{aligned} \tag{3.11}
$$

where we introduced an auxiliary field χ. Variation of the Lagrangian (3.11) with respect to χ leads to

$$\chi = \dot{W} - Q' + \frac{2Q}{r} - \frac{2G_{2,F}A_0'}{\mathcal{H}}\delta A. \qquad (3.12)$$

Substituting Eq. (3.12) into Eq. (3.11), we find that the Lagrangian (3.11) is equivalent to (3.7). Varying (3.11) with respect to W and Q, respectively, we obtain

$$(L-2)f\mathcal{G}W + r^2\mathcal{H}\dot{\chi} = 0, \qquad (3.13)$$

$$[2(L-2)\mathcal{F}Q + 4rh\mathcal{H}\chi + 2r^2h\mathcal{H}\chi' + r^2(\mathcal{H}h' + 2h\mathcal{H}')\chi]f - r^2f'h\mathcal{H}\chi = 0. \qquad (3.14)$$

We solve Eqs. (3.13) and (3.14) for W and Q, and take the t and r derivatives of W and Q, respectively. Substituting them into Eq. (3.11) and integrating it by parts, the resulting second-order Lagrangian can be expressed in the form

$$\mathcal{L}_{\rm odd} = \dot{\vec{\mathcal{X}}}^t\boldsymbol{K}\dot{\vec{\mathcal{X}}} + \vec{\mathcal{X}}'^t\boldsymbol{G}\vec{\mathcal{X}}' + \vec{\mathcal{X}}^t\boldsymbol{M}\vec{\mathcal{X}}, \qquad (3.15)$$

where $\boldsymbol{K}, \boldsymbol{G}, \boldsymbol{M}$ are 2×2 symmetric matrices, and $\vec{\mathcal{X}}$ is a vector field defined by

$$\vec{\mathcal{X}} = \begin{pmatrix} \chi \\ \delta A \end{pmatrix}. \qquad (3.16)$$

Note that χ and δA correspond to the dynamical perturbations arising from the gravity and vector field sectors, respectively. The nonvanishing components of $\boldsymbol{K}, \boldsymbol{G}, \boldsymbol{M}$ are

$$K_{11} = \frac{r^2\sqrt{h}\mathcal{H}^2}{4(L-2)f^{3/2}\mathcal{G}}, \qquad K_{22} = \frac{G_{2,F}}{2\sqrt{fh}}, \qquad G_{11} = -fh\frac{\mathcal{G}}{\mathcal{F}}K_{11}, \qquad G_{22} = -fhK_{22},$$
$$M_{11} = -\frac{\sqrt{h}\mathcal{H}}{4(L-2)\sqrt{f}}\left(L-2+\alpha_M' - \frac{2}{r}\alpha_M\right), \qquad M_{22} = -\frac{G_{2,F}(Lf\mathcal{H} + 2r^2hA_0'^2G_{2,F})}{2r^2\mathcal{H}\sqrt{fh}},$$
$$M_{12} = -\frac{\sqrt{h}A_0'G_{2,F}}{2\sqrt{f}}, \qquad (3.17)$$

where

$$\alpha_M \equiv -\frac{r^2h\mathcal{H}}{\mathcal{F}}\left(\frac{\mathcal{H}'}{\mathcal{H}} - \frac{f'}{2f} + \frac{h'}{2h} + \frac{2}{r}\right). \qquad (3.18)$$

The ghosts are absent under the conditions $K_{11} > 0$ and $K_{22} > 0$. These translate to

$$\mathcal{G} > 0, \qquad (3.19)$$

$$G_{2,F} > 0, \qquad (3.20)$$

respectively, which correspond to the no-ghost conditions of gravitational and vector field perturbations in the odd-parity sector.

The perturbation equations of motion for χ and δA follow by varying the Lagrangian (3.15) with respect to these variables. For the propagation of χ and δA along the radial direction, we assume solutions to the perturbation equations in the form $\vec{\mathcal{X}}^t \propto e^{i(\omega t - kr)}$. In the short-wavelength limit $k\to\infty$, the dispersion relation is given by $\det(\omega^2\boldsymbol{K} + k^2\boldsymbol{G}) = 0$. The radial propagation speed c_r in proper time can be obtained by substituting $\omega = \sqrt{fh}c_rk$ into the dispersion relation. This gives the following two solutions:

$$c_{r1,\rm odd}^2 = -\frac{G_{11}}{fhK_{11}} = \frac{\mathcal{G}}{\mathcal{F}}, \qquad (3.21)$$

$$c_{r2,\rm odd}^2 = -\frac{G_{22}}{fhK_{22}} = 1, \qquad (3.22)$$

which are the squared propagation speeds of χ and δA, respectively. Under the no-ghost condition (3.19), the Laplacian stability of χ is ensured for

$$\mathcal{F} > 0. \qquad (3.23)$$

Since the second propagation speed squared (3.22) is luminal, there is no Laplacian instability for δA.

In the large multipole limit $L = l(l+1) \gg 1$, the matrix $\boldsymbol{M}$ gives contributions to the propagation speed c_Ω along the angular direction. In this limit, we have

$$M_{11} \simeq -\frac{\sqrt{h}\mathcal{H}}{4\sqrt{f}}, \qquad M_{22} \simeq -\frac{\sqrt{f}G_{2,F}}{2r^2\sqrt{h}}L. \qquad (3.24)$$

Substituting solutions of the form $\vec{\mathcal{X}}^t \propto e^{i(\omega t - l\theta)}$ into the perturbation equations, the dispersion relation yields $\det(\omega^2\boldsymbol{K} + \boldsymbol{M}) = 0$. The angular propagation speed in proper time is given by $c_\Omega = \hat{c}_\Omega/\sqrt{f}$, where $\hat{c}_\Omega = r{\rm d}\theta/{\rm d}t$. We substitute $\omega^2 = \hat{c}_\Omega^2 l^2/r^2 = c_\Omega^2 fl^2/r^2$ into the dispersion relation and solve it for c_Ω^2. In the limit $l \gg 1$, we obtain the following two solutions:

$$c_{\Omega 1,\mathrm{odd}}^2 = -\frac{r^2 M_{11}}{l^2 f K_{11}} = \frac{\mathcal{G}}{\mathcal{H}}, \tag{3.25}$$

$$c_{\Omega 2,\mathrm{odd}}^2 = -\frac{r^2 M_{22}}{l^2 f K_{22}} = 1, \tag{3.26}$$

which correspond to the squared angular propagation speeds of χ and δA, respectively. Under the no-ghost condition (3.19), the Laplacian instability of χ is absent for

$$\mathcal{H} > 0. \tag{3.27}$$

The angular propagation speed of δA is luminal, so there is no Laplacian instability.

We note that the stability conditions of χ are identical to those derived in Ref. [49] without a vector field A_μ. This means that the presence of A_μ coupled to the scalar field of the form $G_2(\phi, X, F)$ does not modify the odd-parity stability conditions in the gravitational sector. The odd-parity perturbation of A_μ propagates luminally, without a ghost for $G_{2,F} > 0$.

B. $l = 1$

We also study the odd-parity stability of dipolar perturbations ($l = 1$). Since $L = 2$ in this case, terms proportional to $L - 2$ in Eq. (3.7) vanish. Moreover, the metric components h_{ab} vanish identically and hence $U = 0$. To fix the residual gauge DOF, we choose the gauge

$$W = 0. \tag{3.28}$$

Varying the Lagrangian (3.7) with respect to W and Q, and setting $W = 0$ at the end, we obtain

$$\dot{\mathcal{E}} = 0, \qquad (r^2 \mathcal{E})' = 0, \tag{3.29}$$

where

$$\mathcal{E} = \mathcal{H}\sqrt{\frac{h}{f}}\left(Q' - \frac{2}{r}Q + \frac{2G_{2,F}A_0'}{\mathcal{H}}\delta A\right). \tag{3.30}$$

Integrating two differential equations in (3.29) leads to

$$Q' - \frac{2}{r}Q + \frac{2G_{2,F}A_0'}{\mathcal{H}}\delta A = \frac{1}{\mathcal{H}}\sqrt{\frac{f}{h}}\frac{\mathcal{C}}{r^2}, \tag{3.31}$$

where $\mathcal{C}$ is a constant. On using this relation to eliminate Q' from the Lagrangian (3.7), it follows that

$$\mathcal{L}_{\mathrm{odd}} = \frac{1}{2\sqrt{fh}}\left[G_{2,F}\dot{\delta A}^2 - G_{2,F} f h \delta A'^2 - \frac{2G_{2,F}(f\mathcal{H} + r^2 h G_{2,F} A_0'^2)}{r^2 \mathcal{H}}\delta A^2 + \frac{f\mathcal{C}^2}{2r^4\mathcal{H}}\right]. \tag{3.32}$$

Hence the propagating DOF is only the vector field perturbation δA. The ghost is absent so long as the first term in the square bracket of Eq. (3.32) is positive, i.e.,

$$G_{2,F} > 0, \tag{3.33}$$

which is the same as the no-ghost condition of δA derived for $l \geq 2$. In the short-wavelength limit, the dominant contributions to Eq. (3.32) are the first and second terms in the square bracket. Then, the radial propagation speed squared of δA in proper time is given by

$$c_{r,\mathrm{odd}}^2 = 1, \tag{3.34}$$

which is luminal. Thus, the stability of dipolar perturbations does not add any new conditions to those obtained for $l \geq 2$.

IV. EVEN-PARITY PERTURBATIONS

In this section, we derive the second-order action and perturbation equations of motion for the even-parity modes. On the background (2.3), the metric perturbations $h_{\mu\nu}$ in the even-parity sector are given by

$$h_{tt} = f(r)\sum_{l,m} H_0(t,r) Y_{lm}(\theta,\varphi), \qquad h_{tr} = h_{rt} = \sum_{l,m} H_1(t,r) Y_{lm}(\theta,\varphi), \qquad h_{rr} = h(r)^{-1}\sum_{l,m} H_2(t,r) Y_{lm}(\theta,\varphi),$$
$$h_{ta} = h_{at} = \sum_{l,m} h_0(t,r)\nabla_a Y_{lm}(\theta,\varphi), \qquad h_{ra} = h_{ar} = \sum_{l,m} h_1(t,r)\nabla_a Y_{lm}(\theta,\varphi),$$
$$h_{ab} = \sum_{l,m}[K(t,r) g_{ab} Y_{lm}(\theta,\varphi) + G(t,r)\nabla_a\nabla_b Y_{lm}(\theta,\varphi)], \tag{4.1}$$

where H_0, H_1, H_2, h_0, h_1, K, and G are scalar quantities depending on t and r. We also decompose the scalar and vector fields as

$$\phi = \bar{\phi}(r) + \sum_{l,m} \delta\phi(t,r) Y_{lm}(\theta,\varphi), \tag{4.2}$$

$$A_\mu = \bar{A}_\mu + \delta A_\mu, \tag{4.3}$$

with

$$\begin{aligned} \delta A_t &= \sum_{l,m} \delta A_0(t,r) Y_{lm}(\theta,\varphi), \\ \delta A_r &= \sum_{l,m} \delta A_1(t,r) Y_{lm}(\theta,\varphi), \\ \delta A_a &= \sum_{l,m} \delta A_2(t,r) \nabla_a Y_{lm}(\theta,\varphi), \end{aligned} \tag{4.4}$$

where $\delta\phi$, δA_0, δA_1, and δA_2 are functions of t and r.

Under the infinitesimal gauge transformation $x_\mu \to x_\mu + \xi_\mu$ with

$$\xi_t = \sum_{l,m} \mathcal{T}(t,r) Y_{lm}(\theta,\varphi), \qquad \xi_r = \sum_{l,m} \mathcal{R}(t,r) Y_{lm}(\theta,\varphi), \qquad \xi_a = \sum_{l,m} \Theta(t,r) \nabla_a Y_{lm}(\theta,\varphi), \tag{4.5}$$

the metric perturbations in Eq. (4.1) and $\delta\phi$ in Eq. (4.2) transform as

$$\begin{aligned} &H_0 \to H_0 + \frac{2}{f}\dot{\mathcal{T}} - \frac{f'h}{f}\mathcal{R}, \qquad H_1 \to H_1 + \dot{\mathcal{R}} + \mathcal{T}' - \frac{f'}{f}\mathcal{T}, \qquad H_2 \to H_2 + 2h\mathcal{R}' + h'\mathcal{R}, \\ &h_0 \to h_0 + \mathcal{T} + \dot{\Theta}, \qquad h_1 \to h_1 + \mathcal{R} + \Theta' - \frac{2}{r}\Theta, \qquad K \to K + \frac{2}{r}h\mathcal{R}, \qquad G \to G + \frac{2}{r^2}\Theta, \\ &\delta\phi \to \delta\phi - \phi' h \mathcal{R}. \end{aligned} \tag{4.6}$$

We can eliminate some of the perturbed variables on account of the gauge DOFs. For the multipoles $l \geq 2$, we choose the uniform curvature gauge given by

$$h_0 = 0, \qquad G = 0, \qquad K = 0, \tag{4.7}$$

under which $\mathcal{R}$, Θ, and $\mathcal{T}$ are fixed. In addition to the coordinate transformation (4.5), the action (2.2) is invariant under the $U(1)$ gauge transformation

$$\delta A_\mu \to \delta A_\mu + \partial_\mu \delta\chi \quad \text{with} \quad \delta\chi = \sum_{l,m} \tilde{\chi}(t,r) Y_{lm}(\theta,\varphi). \tag{4.8}$$

Under this transformation, the scalar quantities of vector field perturbations in Eq. (4.4) transform as

$$\delta A_0 \to \delta A_0 + \dot{\tilde{\chi}}, \qquad \delta A_1 \to \delta A_1 + \tilde{\chi}', \qquad \delta A_2 \to \delta A_2 + \tilde{\chi}. \tag{4.9}$$

We choose the gauge

$$\delta A_2 = 0, \tag{4.10}$$

under which $\tilde{\chi}$ is fixed.

A. Second-order action and perturbation equations of motion

We expand the action (2.2) up to second order with the gauge choices (4.7) and (4.10). As in the case of odd-parity modes, we set $m = 0$ without loss of generality. Performing the integration by parts and using the background equations of motion (2.7)–(2.9) and (2.14), the second-order action of even-parity perturbations can be expressed in the form

$$\mathcal{S}_{\rm even} = \sum_l \int {\rm d}t{\rm d}r (\mathcal{L}_u + \mathcal{L}_A), \tag{4.11}$$

where

$$\begin{aligned} \mathcal{L}_u = {}& H_0[a_1\delta\phi'' + a_2\delta\phi' + a_3 H_2' + La_4 h_1' + (a_5 + La_6)\delta\phi + (a_7 + La_8)H_2 + La_9 h_1] + Lb_1 H_1^2 \\ &+ H_1(b_2\dot{\delta\phi}' + b_3\dot{\delta\phi} + b_4\dot{H}_2 + Lb_5\dot{h}_1) + c_1\dot{\delta\phi}\dot{H}_2 + H_2[c_2\delta\phi' + (c_3 + Lc_4)\delta\phi + Lc_5 h_1] \\ &+ c_6 H_2^2 + Ld_1\dot{h}_1^2 + Lh_1(d_2\delta\phi' + d_3\delta\phi) + Ld_4 h_1^2 + e_1\dot{\delta\phi}^2 + e_2\delta\phi'^2 + (e_3 + Le_4)\delta\phi^2, \end{aligned} \tag{4.12}$$

$$\mathcal{L}_A = v_1(\delta A_0' - \dot{\delta A}_1)^2 + (\delta A_0' - \dot{\delta A}_1)(v_2 H_0 + v_3 H_2 + v_4\delta\phi' + v_5\delta\phi) + v_6 H_0^2 + L(v_7\delta A_0 h_1 + v_8\delta A_0^2 + v_9\delta A_1^2). \tag{4.13}$$

We recall that L is defined by $L = l(l+1)$. The coefficients $a_1, a_2, \ldots, v_9$ are given in Appendix A. In comparison to Horndeski theories without the Maxwell field, the Lagrangian $\mathcal{L}_u$ has a same structure with that derived in Refs. [50,51]. Still, the coefficients a_2, a_5, a_7, b_3, c_2, c_3, c_6, e_1, e_2 are subject to modifications by the presence of A_μ (see Appendix A). Moreover, the vector field perturbation gives rise to the new Lagrangian (4.13) whose contribution is absent in Refs. [50,51].

In what follows, we derive the perturbation equations of motion by varying the second-order action (4.11) with respect to H_0, H_1, H_2, h_1, δA_0, δA_1, $\delta\phi$, and eliminate nondynamical variables from the action by using their corresponding equations. The Lagrangian (4.13) shows that δA_0 is nondynamical since there is no quadratic term of its time derivative. Nevertheless, the perturbation equation of δA_0 cannot be explicitly solved for δA_0 due to the existence of the quadratic radial derivative term, i.e., $\delta A_0'^2$, in Eq. (4.13). This situation is similar to the case of odd-parity perturbations discussed in Sec. III. Thus, we introduce an auxiliary field $V(t,r)$ in analogy to the discussion in the odd-parity sector, and rewrite Eq. (4.13) in the form

$$\mathcal{L}_A = v_1\left[2V\left(\delta A_0' - \delta\dot{A}_1 + \frac{v_2 H_0 + v_3 H_2 + v_4\delta\phi' + v_5\delta\phi}{2v_1}\right) - V^2\right] - \frac{(v_2 H_0 + v_3 H_2 + v_4\delta\phi' + v_5\delta\phi)^2}{4v_1}$$
$$+ v_6 H_0^2 + L(v_7\delta A_0 h_1 + v_8\delta A_0^2 + v_9\delta A_1^2). \tag{4.14}$$

Varying this action with respect to V gives

$$V = \delta A_0' - \delta\dot{A}_1 + \frac{v_2 H_0 + v_3 H_2 + v_4\delta\phi' + v_5\delta\phi}{2v_1}. \tag{4.15}$$

Substituting Eq. (4.15) into Eq. (4.14), we find that Eq. (4.14) is equivalent to the original Lagrangian (4.13). By introducing the auxiliary field V, the quadratic terms $\delta A_0'^2$ and $\delta\dot{A}_1^2$ present in the original Lagrangian (4.13) are absent in the new Lagrangian (4.14). This allows us to solve the perturbation equations of δA_0 and δA_1 explicitly for themselves. Substituting such solutions into the Lagrangian, we will see later that the dynamical property of vector field perturbation is determined by the auxiliary field V.

We also note that the coefficients v_2 and v_6 have the following relation:

$$v_6 = \frac{v_2^2}{4v_1}. \tag{4.16}$$

This means that the two quadratic terms of H_0 appearing in Eq. (4.14), i.e., $-[v_2^2/(4v_1)]H_0^2$ and $v_6 H_0^2$ cancel each other as a result of introducing the auxiliary field V. Consequently, the total action (4.11) with the sum of Eqs. (4.12) and (4.14) depends on H_0 linearly. Hence the perturbation H_0 corresponds to a Lagrange multiplier and the variation of the action with respect to H_0 puts constraint on other perturbation variables.

Varying the total action (4.11) with Eqs. (4.12) and (4.14) with respect to H_0, H_1, H_2, h_1, δA_0, δA_1, and $\delta\phi$, we obtain the following linear perturbation equations:

$$a_1\delta\phi'' + a_3 H_2' + La_4 h_1' + \left(a_2 - \frac{v_2 v_4}{2v_1}\right)\delta\phi' + \left(a_5 + La_6 - \frac{v_2 v_5}{2v_1}\right)\delta\phi + \left(a_7 + La_8 - \frac{v_2 v_3}{2v_1}\right)H_2 + La_9 h_1 + v_2 V = 0, \tag{4.17}$$

$$2Lb_1 H_1 + b_2\delta\dot{\phi}' + b_3\delta\dot{\phi} + b_4\dot{H}_2 + Lb_5\dot{h}_1 = 0, \tag{4.18}$$

$$-c_1\delta\ddot{\phi} - b_4\dot{H}_1 + \left(c_2 - \frac{v_3 v_4}{2v_1}\right)\delta\phi' + \left(c_3 + Lc_4 - \frac{v_3 v_5}{2v_1}\right)\delta\phi + Lc_5 h_1 + \left(2c_6 - \frac{v_3^2}{2v_1}\right)H_2 - a_3 H_0'$$
$$+ \left(a_7 - a_3' + La_8 - \frac{v_2 v_3}{2v_1}\right)H_0 + v_3 V = 0, \tag{4.19}$$

$$-2d_1\ddot{h}_1 + d_2\delta\phi' + d_3\delta\phi + 2d_4 h_1 - a_4 H_0' + (a_9 - a_4')H_0 - b_5\dot{H}_1 + c_5 H_2 + v_7\delta A_0 = 0, \tag{4.20}$$

$$-2(v_1 V)' + Lv_7 h_1 + 2Lv_8\delta A_0 = 0, \tag{4.21}$$

$$2v_1\dot{V} + 2Lv_9\delta A_1 = 0, \tag{4.22}$$

$$
\begin{aligned}
&-2e_1\ddot{\delta\phi}-\left(2e_2-\frac{v_4^2}{2v_1}\right)\delta\phi''+\left[2e_3+2Le_4+\left(\frac{v_4v_5}{2v_1}\right)'-\frac{v_5^2}{2v_1}\right]\delta\phi+a_1H_0''+\left(2a_1'-a_2+\frac{v_2v_4}{2v_1}\right)H_0'\\
&+\left[a_1''-a_2'+a_5+La_6+\left(\frac{v_2v_4}{2v_1}\right)'-\frac{v_2v_5}{2v_1}\right]H_0+b_2\dot{H}_1'+(b_2'-b_3)\dot{H}_1-c_1\ddot{H}_2-\left(c_2-\frac{v_3v_4}{2v_1}\right)H_2'\\
&+\left[c_3-c_2'+Lc_4+\left(\frac{v_3v_4}{2v_1}\right)'-\frac{v_3v_5}{2v_1}\right]H_2-Ld_2h_1'+L(d_3-d_2')h_1-\left[2e_2'-\left(\frac{v_4^2}{2v_1}\right)'\right]\delta\phi'\\
&-v_4V'+(v_5-v_4')V=0, \qquad (4.23)
\end{aligned}
$$

where we used the relation (4.16).

B. Linear stability conditions

In order to derive the linear stability conditions of dynamical perturbations, we eliminate nondynamical variables from the total action (4.11) with Eqs. (4.12) and (4.14) by using some of the equations derived above. Since the number of dynamical perturbations is different depending on the values of l, we investigate the three cases (1) $l\geq 2$, (2) $l=0$, and (3) $l=1$, in turn.

1. $l\geq 2$

Among the eight variables (H_0, H_1, H_2, h_1, δA_0, δA_1, $\delta\phi$, V), we can eliminate H_1, δA_0, and δA_1 by using Eqs. (4.18), (4.21), and (4.22), respectively. This is due to the fact that the derivatives of H_1, δA_0, and δA_1 do not appear in their equations. We recall that H_0 corresponds to a Lagrange multiplier, so its perturbation equation (4.17) puts constraint on other variables. Introducing a new variable [50,51]

$$\psi=H_2+\frac{a_4}{a_3}Lh_1+\frac{a_1}{a_3}\delta\phi', \qquad (4.24)$$

we can write Eq. (4.17) in terms of ψ', ψ, $\delta\phi'$, $\delta\phi$, V, and h_1. We solve this equation for h_1 and take its time derivative. Terms H_2 and $\dot{H}_2$ in the action (4.11) can be expressed in terms of ψ, $\dot{\psi}$, $\delta\phi'$, $\dot{\delta\phi}'$, h_1, $\dot{h}_1$, where the latter two variables now depend on ψ, $\delta\phi$, V and their derivatives. Then, we can express the second-order action (4.11) in terms of the three dynamical perturbations ψ, $\delta\phi$, V and their derivatives. After the integration by parts, we obtain

$$\mathcal{S}_{\rm even}=\sum_l\int {\rm d}t{\rm d}r(\dot{\vec{\mathcal{X}}}^t\boldsymbol{K}\dot{\vec{\mathcal{X}}}+\vec{\mathcal{X}}'^t\boldsymbol{G}\vec{\mathcal{X}}'+\vec{\mathcal{X}}^t\boldsymbol{Q}\vec{\mathcal{X}}'+\vec{\mathcal{X}}^t\boldsymbol{M}\vec{\mathcal{X}}), \qquad (4.25)$$

where $\boldsymbol{K}$, $\boldsymbol{G}$, $\boldsymbol{M}$ are the 3×3 symmetric matrices while $\boldsymbol{Q}$ is antisymmetric, and the vector $\vec{\mathcal{X}}$ is defined as

$$\vec{\mathcal{X}}=\begin{pmatrix}\psi\\ \delta\phi\\ V\end{pmatrix}. \qquad (4.26)$$

Note that the derivative terms $\dot{\delta\phi}'$ and $\dot{\psi}'$ disappear from the final action (4.25).

The kinetic matrix $\boldsymbol{K}$ in the reduced action (4.25) must be positive definite for the absence of ghosts. This requires that the determinants of principal submatrices of $\boldsymbol{K}$ are positive, such that

$$K_{33}=\frac{v_1^2}{Lfhv_8}=\frac{2v_1^2}{L\sqrt{fh}G_{2,F}}>0, \qquad (4.27)$$

$$K_{11}K_{33}-K_{13}^2=\frac{(L\mathcal{P}_1-\mathcal{F})f^3\mathcal{P}_2^4v_1^2}{L^2h^3(rf'-2f)^4\mathcal{H}^2(\mathcal{P}_2+2rL\mathcal{H})^2G_{2,F}}>0, \qquad (4.28)$$

$$\det\boldsymbol{K}=\frac{(L-2)f^{5/2}\mathcal{F}v_1^2\mathcal{P}_2^4(2\mathcal{P}_1-\mathcal{F})}{2L^2h^{7/2}\mathcal{H}^2\phi'^2(\mathcal{P}_2+2rL\mathcal{H})^2(rf'-2f)^4G_{2,F}}>0, \qquad (4.29)$$

where we introduced the following combinations [50]:

$$\mathcal{P}_1\equiv\frac{h\mu}{2fr^2\mathcal{H}^2}\left(\frac{fr^4\mathcal{H}^4}{\mu^2h}\right)', \qquad \mathcal{P}_2\equiv\frac{h}{f}(rf'-2f)\mu,$$
$$\mu\equiv\frac{2(\phi'a_1+2ra_4)}{\sqrt{fh}}. \qquad (4.30)$$

Under the stability conditions (3.20) and (3.23) in the odd-parity sector, the quantities $G_{2,F}$ and $\mathcal{F}$ are positive. Then, the first condition (4.27) is automatically satisfied. Remembering that $L>2$, both the second and third inequalities (4.28)–(4.29) hold for

$$\mathcal{K}\equiv 2\mathcal{P}_1-\mathcal{F}>0, \qquad (4.31)$$

which coincides with the stability condition in Horndeski theories without the Maxwell field [50]. Consequently, the absence of ghost instabilities in the even-parity sector adds one condition (4.31) to the stability conditions in the odd-parity sector.

We proceed to derive the propagation speeds of even-parity perturbations along the radial direction. The equations of motion for three dynamical perturbations follow by varying the action (4.25) with respect to $\vec{\mathcal{X}}$. Assuming the solutions to those equations of the form $\vec{\mathcal{X}} \propto e^{i(\omega t - kr)}$, where ω and k are the frequency and wave number, respectively, we obtain the dispersion relation characterizing the radial propagation. In the limits of large ω and k, it is given by

$$\det|fhc_r^2\boldsymbol{K} + \boldsymbol{G}| = 0. \tag{4.32}$$

Here, the propagation speed c_r is defined by the rescaled radial coordinate $r_* = \int \mathrm{d}r/\sqrt{h}$ and proper time $\tau = \int \sqrt{f}\mathrm{d}t$, as $c_r = \mathrm{d}r_*/\mathrm{d}\tau = (fh)^{-1/2}(\mathrm{d}r/\mathrm{d}t) = (fh)^{-1/2}(\omega/k)$. The matrix components of $\boldsymbol{K}$ and $\boldsymbol{G}$ associated with the vector field perturbation V have the following relations:

$$\frac{G_{13}}{K_{13}} = \frac{G_{23}}{K_{23}} = \frac{G_{33}}{K_{33}} = -fh. \tag{4.33}$$

On using these relations, the radial propagation speed of V, which is decoupled from the other two, is simply given by

$$c_{r3,\mathrm{even}}^2 = 1, \tag{4.34}$$

which is equivalent to the radial propagation speed of vector field perturbation δA in the odd-parity sector (3.22).

The other components of matrices $\boldsymbol{K}$ and $\boldsymbol{G}$ are quite complicated, but we can resort to the following relation:

$$a_4' = \frac{1}{2f - rf'}\left[\left(rf'' - \frac{rf'^2}{f} + 2f' - \frac{2f}{r}\right)a_4 + \frac{f^{3/2}}{r\sqrt{h}}\mathcal{F} - 2rfhA_0'^2 v_8\right], \tag{4.35}$$

to eliminate the derivative a_4'. This relation follows by using the background Eqs. (2.7) and (2.9). As a consequence, the dispersion relation can be factorized in the form,

$$(c_r^2 - c_{r1,\mathrm{even}}^2)(c_r^2 - c_{r2,\mathrm{even}}^2) = 0, \tag{4.36}$$

where $c_{r1,\mathrm{even}}$ and $c_{r2,\mathrm{even}}$ correspond to the radial propagation speeds of ψ and $\delta\phi$, respectively, which are given by

$$c_{r1,\mathrm{even}}^2 = \frac{\mathcal{G}}{\mathcal{F}}, \tag{4.37}$$

$$c_{r2,\mathrm{even}}^2 = \frac{4\phi'}{(fh)^{3/2}(2\mathcal{P}_1 - \mathcal{F})\mu^2} \times \left[8r^2ha_4c_4(\phi' a_1 + ra_4) - \sqrt{fh}\phi' a_1^2\mathcal{G} + 2r^2a_4^2\left(\frac{f'}{f}a_1 + 2c_2 + A_0'v_4 + \frac{\phi' v_4^2}{2v_1}\right)\right]. \tag{4.38}$$

Notice that $c_{r1,\mathrm{even}}^2$ is equivalent to the squared propagation speed (3.21) of gravitational perturbation χ in the odd-parity sector, which is not directly affected by the presence of the vector field. On the other hand, the coupling between ϕ and A_μ modifies the value of $c_{r2,\mathrm{even}}$ due to the presence of the last two terms in Eq. (4.38) containing $v_4 = -r^2h^{3/2}\phi' A_0' G_{2,XF}/\sqrt{f}$. In the absence of the vector field, the result (4.38) is consistent with those derived in Refs. [50,51].

We will also obtain the propagation speeds of even-parity perturbations along the angular direction. For this purpose, we assume the solution to the equations of dynamical perturbations in the form $\vec{\mathcal{X}} \propto e^{i(\omega t - l\theta)}$. In the limit of large ω and l, the reduced Lagrangian (4.25) leads to the following dispersion relation along the angular direction:

$$\det|fl^2c_\Omega^2\boldsymbol{K} + r^2\boldsymbol{M}| = 0. \tag{4.39}$$

The propagation speed c_Ω is defined by using the proper time τ such that $c_\Omega = r\mathrm{d}\theta/\mathrm{d}\tau = (r/\sqrt{f})(\mathrm{d}\theta/\mathrm{d}t) = (r/\sqrt{f})(\omega/l)$. Expanding the components of $\boldsymbol{K}$ and $\boldsymbol{M}$ in the limit $l \to \infty$, we find that the leading-order matrix components have the following dependence[2]:

$$K_{11} = \frac{\tilde{K}_{11}}{l^4}, \quad K_{12} = \frac{\tilde{K}_{12}}{l^2}, \quad K_{13} = \frac{\tilde{K}_{13}}{l^4}, \quad K_{22} = \tilde{K}_{22}, \quad K_{23} = \frac{\tilde{K}_{23}}{l^2}, \quad K_{33} = \frac{\tilde{K}_{33}}{l^2},$$
$$M_{11} = \frac{\tilde{M}_{11}}{l^2}, \quad M_{12} = \tilde{M}_{12}, \quad M_{13} = \frac{\tilde{M}_{13}}{l^2}, \quad M_{22} = \tilde{M}_{22}l^2, \quad M_{23} = \tilde{M}_{23}, \quad M_{33} = \tilde{M}_{33}, \tag{4.41}$$

where the quantities with tildes do not contain the l dependence. Picking up the leading-order contributions to Eq. (4.39), it follows that

$$(f\tilde{K}_{33}c_\Omega^2 + r^2\tilde{M}_{33})[f^2(\tilde{K}_{11}\tilde{K}_{22} - \tilde{K}_{12}^2)c_\Omega^4 + r^2f(\tilde{K}_{11}\tilde{M}_{22} - 2\tilde{K}_{12}\tilde{M}_{12} + \tilde{K}_{22}\tilde{M}_{11})c_\Omega^2 + r^4(\tilde{M}_{11}\tilde{M}_{22} - \tilde{M}_{12}^2)] = 0. \tag{4.42}$$

[2]Nonvanishing components of the antisymmetric matrix $\boldsymbol{Q}$ have the following leading-order l-dependence:

$$Q_{12} = \frac{\tilde{Q}_{12}}{l^2}, \qquad Q_{13} = \frac{\tilde{Q}_{13}}{l^2}, \qquad Q_{23} = \frac{\tilde{Q}_{23}}{l^2}. \tag{4.40}$$

These do not contribute to the values of c_Ω in the large l limit.

Then, the propagation speed of V decouples from the other two, such that

$$c_{\Omega 3,\mathrm{even}}^2 = -\frac{r^2\tilde{M}_{33}}{f\tilde{K}_{33}} = \frac{r^2 h v_8}{v_1} = \frac{G_{2,F}}{G_{2,F}+2FG_{2,FF}}, \tag{4.43}$$

where F is given in Eq. (2.6). If the Lagrangian G_2 contains nonlinear functions of F, the propagation speed of V along the angular direction deviates from that of light. This property dose not necessarily hold in other spacetime since the propagation speed of perturbations generally depends on underlying symmetry of the background spacetime. Indeed, on the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological background, the propagation speeds of vector perturbations are luminal in theories with the coupling $G_2 = G_2(F)$ [105].

Under the no-ghost condition $G_{2,F} > 0$, the angular Laplacian stability of V is ensured for

$$G_{2,F} + 2FG_{2,FF} > 0. \tag{4.44}$$

The other two propagation speeds associated with the perturbations ψ and $\delta\phi$ are given by

$$c_{\Omega\pm,\mathrm{even}}^2 = -B_1 \pm \sqrt{B_1^2 - B_2}, \tag{4.45}$$

where

$$B_1 = \frac{r^2(\tilde{K}_{11}\tilde{M}_{22} - 2\tilde{K}_{12}\tilde{M}_{12} + \tilde{K}_{22}\tilde{M}_{11})}{2f(\tilde{K}_{11}\tilde{K}_{22} - \tilde{K}_{12}^2)},$$
$$B_2 = \frac{r^4(\tilde{M}_{11}\tilde{M}_{22} - \tilde{M}_{12}^2)}{f^2(\tilde{K}_{11}\tilde{K}_{22} - \tilde{K}_{12}^2)}. \tag{4.46}$$

While each matrix component of $\boldsymbol{K}$ and $\boldsymbol{M}$ is quite complicated, we can simplify the terms appearing in Eq. (4.46) by using relations among the coefficients given in Appendix A. We also exploit the following relation:

$$(2h\phi'' + h'\phi')\left[\frac{a_1 - r^2 h c_4}{r^2\sqrt{fh}} + \frac{f(\mathcal{G}-\mathcal{H})}{rh(rf'-2f)\phi'}\right] - \frac{h}{4f}\left[2f'' - \frac{(rf'-2f)f'}{rf} + \frac{(rf'-4f)h'}{rh}\right]\mathcal{H} - \frac{1}{r^2}\mathcal{F} - \frac{rh'-2h}{2r^2}\mathcal{G} - \frac{h(rf'-2f)\phi'}{2rf}\frac{\partial\mathcal{H}}{\partial\phi} + \frac{2h^{3/2}A_0'^2}{\sqrt{f}}v_8 = 0, \tag{4.47}$$

which is equivalent to the subtraction of Eq. (2.7) from Eq. (2.9). After lengthy but straightforward calculations, we find that the quantities B_1 and B_2 are of the same forms as those derived in Ref. [51] without a perfect fluid, i.e.,

$$B_1 = \frac{a_4 r^3[4h(\phi' a_1 + 2ra_4)\beta_1 + \beta_2 - 4\phi' a_1\beta_3] - 2fh\mathcal{G}[2ra_4(2\mathcal{P}_1 - \mathcal{F})(\phi' a_1 + ra_4) + \phi'^2 a_1^2\mathcal{P}_1]}{4\sqrt{fh}a_4(\phi' a_1 + 2ra_4)^2(2\mathcal{P}_1 - \mathcal{F})}, \tag{4.48}$$

$$B_2 = -r^2\frac{r^2h\beta_1[2fh\mathcal{F}\mathcal{G}(\phi' a_1 + 2ra_4) + r^2\beta_2] - r^4\beta_2\beta_3 - fh\mathcal{F}\mathcal{G}(\phi' fh\mathcal{F}\mathcal{G}a_1 + 4r^3 a_4\beta_3)}{fh\mathcal{F}\phi' a_1(\phi' a_1 + 2ra_4)^2(2\mathcal{P}_1 - \mathcal{F})}, \tag{4.49}$$

with

$$\beta_1 = \phi'^2 a_4 e_4 - 2\phi' c_4 a_4' + \left[\left(\frac{f'}{f} + \frac{h'}{h} - \frac{2}{r}\right)a_4 + \frac{\sqrt{fh}\mathcal{F}}{r}\right]\phi' c_4 + \frac{f\mathcal{F}\mathcal{G}}{2r^2}, \tag{4.50}$$

$$\beta_2 = \left[\frac{\sqrt{fh}\mathcal{F}}{r^2}\left(2hr\phi'^2 c_4 + \frac{rf'\phi' a_4}{f} - \sqrt{fh}\phi'\mathcal{G}\right) - \frac{2\sqrt{fh}\phi' a_4\mathcal{G}}{r}\left(\frac{\mathcal{G}'}{\mathcal{G}} - \frac{a_4'}{a_4} + \frac{f'}{f} + \frac{1}{2}\frac{h'}{h} - \frac{1}{r}\right)\right]a_1 - \frac{4\mathcal{F}\mathcal{G}fha_4}{r}, \tag{4.51}$$

$$\beta_3 = \left(hc_4' - \frac{d_3}{2} + \frac{1}{2}h'c_4\right)\phi' a_4 + \left(\frac{h'}{2h} - \frac{1}{r} + \frac{f'}{2f} - \frac{a_4'}{a_4}\right)\left(\frac{a_4 f'}{2f} + 2h\phi' c_4 + \frac{\sqrt{fh}\mathcal{G}}{2r}\right)a_4 + \frac{\sqrt{fh}\mathcal{F}}{4r}\left(\frac{f'}{f}a_4 + 2h\phi' c_4 + \frac{3\sqrt{fh}\mathcal{G}}{r}\right). \tag{4.52}$$

While B_1 and B_2 do not explicitly contain the vector field contribution, the quantity a_4' present in β_1, β_2, β_3 generally picks up such contributions, see Eq. (4.35). To ensure the Laplacian stabilities of perturbations ψ and $\delta\phi$, we require that $c_{\Omega\pm,\mathrm{even}}^2 > 0$. These conditions are satisfied if

$$B_1^2 \ge B_2 > 0 \quad \text{and} \quad B_1 < 0. \tag{4.53}$$

TABLE I. Summary of linear stability conditions.

	No ghosts	$c_r^2 > 0$	$c_\Omega^2 > 0$
Odd-parity modes	$\mathcal{G} > 0$, $G_{2,F} > 0$	$\mathcal{F} > 0$	$\mathcal{H} > 0$
Even-parity modes	$\mathcal{K} > 0$	$c_{r2,\rm even}^2 > 0$	$B_1^2 \geq B_2 > 0$, $B_1 < 0$, $G_{2,F} + 2FG_{2,FF} > 0$

In Table I, we summarize all the linear stability conditions in both odd- and even-parity sectors. The radial propagation speeds of vector field perturbations δA (odd-parity) and V (even-parity) are both luminal ($c_{r2,\rm odd}^2 = c_{r3,\rm even}^2 = 1$). In the gravitational sector, the radial propagation speeds of χ (odd-parity) and ψ (even-parity) are equivalent to each other ($c_{r1,\rm odd}^2 = c_{r1,\rm even}^2 = \mathcal{G}/\mathcal{F}$). We note that, on the FLRW background, the nonlinear term of F in G_2 does not affect the perturbation dynamics by reflecting the fact that the quantity F vanishes in $U(1)$ gauge-invariant theories [105]. In contrast, the quantity F does not vanish on the static and spherically symmetric background, and nonlinear terms of F affect linear stability conditions in the odd- and even-parity sectors.

2. $l = 0$

We proceed to the analysis of the monopole perturbation $l = 0$, i.e., $L = 0$. In this case, the perturbations h_0, h_1, and G vanish identically from the second-order action of even-parity perturbations [50,51]. While one can choose the gauge different from Eq. (4.7) to eliminate perturbations other than h_0 and G, we avoid doing so since the gauge DOFs are not completely fixed in such a case. For $l = 0$, the total action (4.11) reduces to

$$\mathcal{S}_{\rm even} = \sum_l \int {\rm d}t{\rm d}r\bigg\{v_1\bigg[2V\bigg(\delta A_0' - \dot{\delta A}_1 + \frac{v_3H_2 + v_4\delta\phi' + v_5\delta\phi}{2v_1}\bigg) - V^2\bigg] - \frac{(v_3H_2 + v_4\delta\phi' + v_5\delta\phi)^2}{4v_1}$$
$$+ (\Phi' + A_0'v_1V)H_0 - \frac{2}{f}\dot{\Phi}H_1 + c_1\dot{\delta\phi}\dot{H}_2 + (c_2\delta\phi' + c_3\delta\phi)H_2 + c_6H_2^2 + e_1\dot{\delta\phi}^2 + e_2\delta\phi'^2 + e_3\delta\phi^2\bigg\}, \quad (4.54)$$

where we introduced the combination

$$\Phi \equiv a_1\delta\phi' + \left(a_2 - a_1' - \frac{1}{2}A_0'v_4\right)\delta\phi + a_3H_2, \quad (4.55)$$

and used the relations among coefficients given in Appendix A. The quadratic terms of H_1, δA_0, and δA_1 present in the original Lagrangians (4.12) and (4.14) disappear in Eq. (4.54). This means that, in addition to H_0, each perturbation, H_1, δA_0, δA_1, plays a role of the Lagrange multiplier for $l = 0$, and their Euler-Lagrange equations put constraints on other variables. Indeed, varying (4.54) with respect to δA_0 and δA_1 leads to

$$(v_1V)' = 0, \qquad (v_1V)^{\cdot} = 0, \quad (4.56)$$

respectively. They are integrated to give

$$v_1V = \mathcal{C}_1, \quad (4.57)$$

where $\mathcal{C}_1$ is a constant. This shows that V depends only on r and hence it is nondynamical for $l = 0$. On the other hand, varying the action (4.54) with respect to H_0 and H_1 gives

$$\Phi' + A_0'v_1V = 0, \qquad \dot{\Phi} = 0, \quad (4.58)$$

respectively. Integrating these two equations with the use of Eq. (4.57), we obtain

$$\Phi = \mathcal{C}_2 - \mathcal{C}_1A_0, \quad (4.59)$$

where $\mathcal{C}_2$ is an integration constant. Combining Eq. (4.55) with Eq. (4.59), we find that the perturbation H_2 can be expressed by using other variables as

$$H_2 = -\frac{1}{a_3}\left[a_1\delta\phi' + \left(a_2 - a_1' - \frac{1}{2}A_0'v_4\right)\delta\phi - (\mathcal{C}_2 - \mathcal{C}_1A_0)\right]. \quad (4.60)$$

In the present case, the perturbation (4.24) reduces to $\psi = H_2 + a_1\delta\phi'/a_3$ and hence Eq. (4.60) gives a constraint on ψ. This means that, for $l = 0$, the gravitational perturbation ψ is not a propagating DOF.

Substituting Eqs. (4.57), (4.59), and (4.60) together with $\dot{H}_2$ into Eq. (4.54), the resulting second-order action consists of $\delta\phi$ and its derivatives. Since the integration constants $\mathcal{C}_1$ and $\mathcal{C}_2$ are irrelevant to the dynamics of perturbations, we simply set $\mathcal{C}_1 = 0 = \mathcal{C}_2$ in the following discussion. After the integration by parts, we obtain the second-order action in the form

$$\mathcal{S}_{\rm even} = \int {\rm d}t{\rm d}r(K_0\dot{\delta\phi}^2 + G_0\delta\phi'^2 + M_0\delta\phi^2), \quad (4.61)$$

where K_0, G_0, and M_0 are composed of the background quantities with the superscript representing $l=0$. This reduced action shows that the monopole perturbation possesses only one propagating DOF $\delta\phi$. The ghost is absent under the condition

$$K_0 = \frac{2\mathcal{P}_1 - \mathcal{F}}{\sqrt{fh}\phi'^2} > 0, \tag{4.62}$$

which is equivalent to the no-ghost condition (4.31) derived for $l \geq 2$. The squared propagation speed $c_{r,\rm even}^2 = -G_0/(fhK_0)$ also coincides with Eq. (4.38) obtained for $l \geq 2$. Consequently, the monopole perturbation $l=0$ does not give rise to additional stability conditions to those given in Table I.

3. $l=1$

For the dipole mode $l=1$, the perturbations K and G appear in the second-order action only in the form $G-K$ [50,51]. If we impose the gauge conditions $h_0=0$ and $K=G$, there is a residual gauge DOF associated with the transformation scalar $\mathcal{R}$. This can be fixed by choosing the gauge $\delta\phi=0$. Namely, for $l=1$, we choose the gauge conditions

$$h_0 = 0, \qquad K = G, \qquad \delta\phi = 0. \tag{4.63}$$

Eliminating nondynamical variables from the action (4.11) with the approach analogous to the case $l \geq 2$, the second-order action can be expressed in the form (4.25) with two dynamical perturbations

$$\vec{\mathcal{X}} = \begin{pmatrix} \psi \\ V \end{pmatrix}. \tag{4.64}$$

The ghosts are absent under the conditions

$$K_{22} > 0, \qquad K_{11}K_{22} - K_{12}^2 > 0, \tag{4.65}$$

which are equivalent to Eqs. (4.27) and (4.28), respectively, with the substitution of $L=2$. The propagation speeds c_r along the radial direction obey the same dispersion relation as Eq. (4.32). On using the properties $G_{12} = -fhK_{12}$ and $G_{22} = -fhK_{22}$, it follows that the squared propagation speeds of ψ and V are identical to Eqs. (4.38) and (4.34), respectively. Thus, the dipole perturbation possesses two propagating DOFs arising from the scalar and vector field perturbations. We do not have additional conditions to those given in Table I.

V. APPLICATION TO CONCRETE THEORIES WITH HAIRY BHS

Theories with the action (2.2) can accommodate a wide variety of hairy BH solutions known in the literature. In this section, we apply the linear stability conditions derived in Secs. III and IV to concrete BHs present in the framework of Maxwell-Horndeski theories. We will focus on the case $l \geq 2$, in which case five dynamical DOFs are present.

A. Nonminimally coupled k-essence with a gauge field

We begin with a subclass of Maxwell-Horndeski given by the action

$$\mathcal{S} = \int {\rm d}^4x\sqrt{-g}[G_2(\phi, X, F) + G_4(\phi)R], \tag{5.1}$$

where the nonminimal coupling G_4 is a function of ϕ only. The analysis of BH perturbations in this case was also addressed in Ref. [82], but the angular stability conditions of even-parity perturbations were missing. In the following, we will investigate all the linear stability conditions.

The stability of odd-parity perturbations requires that

$$\mathcal{G} = \mathcal{F} = \mathcal{H} = 2G_4 > 0, \qquad G_{2,F} > 0. \tag{5.2}$$

Hence all the propagation speeds of χ and δA are luminal in both radial and angular directions. In the even-parity sector, the quantity (4.31) yields

$$\mathcal{K} = \frac{2r^2\phi'^2 G_4(G_{2,X}G_4 + 3G_{4,\phi}^2)}{(2G_4 + r\phi' G_{4,\phi})^2}. \tag{5.3}$$

If the BH has a scalar hair, the field derivative ϕ' is nonvanishing. Under the first inequality (5.2), the no-ghost condition $\mathcal{K} > 0$ translates to

$$G_{2,X}G_4 + 3G_{4,\phi}^2 > 0. \tag{5.4}$$

For a minimally coupled scalar field ($G_4 = \text{constant} > 0$), this condition reduces to $G_{2,X} > 0$. The radial propagation speeds of both ψ and V are luminal ($c_{r1,\rm even}^2 = c_{r3,\rm even}^2 = 1$). To ensure the Laplacian stability of $\delta\phi$ along the radial direction, we require that

$$c_{r2,\rm even}^2 = 1 + \frac{2G_4 X[G_{2,XX}(G_{2,F} + 2FG_{2,FF}) - 2FG_{2,FX}^2]}{(G_{2,X}G_4 + 3G_{4,\phi}^2)(G_{2,F} + 2FG_{2,FF})} > 0, \tag{5.5}$$

which coincides with Eq. (5.26) of Ref. [82] (one needs to replace $G_2 \to f_2$ and $G_4 \to f_1/2$ for the notation used in [82]).

Along the angular direction, the quantities (4.48) and (4.49) reduce, respectively, to

$$B_1 = -1, \qquad B_2 = 1, \tag{5.6}$$

and hence

$$c^2_{\Omega+,\mathrm{even}} = 1, \qquad c^2_{\Omega-,\mathrm{even}} = 1. \tag{5.7}$$

Thus, there are no angular Laplacian instabilities for the perturbations ψ and $\delta\phi$. The angular stability of vector field perturbation V requires that

$$c^2_{\Omega 3,\mathrm{even}} = \frac{G_{2,F}}{G_{2,F} + 2FG_{2,FF}} > 0. \tag{5.8}$$

Under the no-ghost condition $G_{2,F} > 0$, the inequality (5.8) is satisfied for $G_{2,F} + 2FG_{2,FF} > 0$.

In the following, we will study the stability of hairy BHs in two subclasses of theories given by the action (5.1).

1. Einstein-Maxwell-dilaton theory

In bosonic heterotic string theory, the gauge field is coupled to a dilaton field ϕ with the Lagrangian $4e^{-2\phi}F$. In the Einstein frame, the corresponding effective four-dimensional action is given by

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}(R + 4X + 4e^{-2\phi}F), \tag{5.9}$$

where the unit $M_{\rm Pl}^2/2 = 1$ is used, with $M_{\rm Pl}$ being the reduced Planck mass. In this theory, there is an exact BH solution advocated by GM and GHS [56,57]. GHS derived the hairy BH solution for a static and spherically symmetric metric where r^2 in front of $\mathrm{d}\Omega^2$ in Eq. (2.3) is modified to a general function $\zeta^2(r)$. In Appendix B, we revisit the derivation of this exact solution. In terms of the coordinate (2.3), the GM-GHS solution is expressed as

$$f = 1 - \frac{2M}{r^2}\left(\sqrt{r^2 + r_q^2} - r_q\right), \qquad h = \left(1 + \frac{r_q^2}{r^2}\right)f,$$
$$\phi = \phi_0 + \frac{1}{2}\ln\left(\frac{\sqrt{r^2+r_q^2} - r_q}{\sqrt{r^2+r_q^2} + r_q}\right), \qquad A_0' = \frac{qre^{2\phi_0}}{\left(\sqrt{r^2+r_q^2}+r_q\right)^2\sqrt{r^2+r_q^2}}, \tag{5.10}$$

where M is a constant, ϕ_0 is an asymptotic value of ϕ at spatial infinity, and

$$r_q \equiv \frac{q^2 e^{2\phi_0}}{2M}. \tag{5.11}$$

Here, q is a constant corresponding to an electric charge. The radial derivative of ϕ is given by $\phi'(r) = r_q/[r\sqrt{r^2+r_q^2}]$, which behaves as $\phi'(r) \simeq q^2e^{2\phi_0}/(2Mr^2)$ at spatial infinity. The scalar field acquires a secondary hair q through a dilatonic coupling with the gauge field. In the limit that $q \to 0$, the solution (5.10) reduces to the Schwarzschild metric $f = h = 1–2M/r$ with $\phi = \phi_0$ and $A_0' = 0$. For $q \neq 0$, there is a single event horizon [57] located at

$$r_H = 2\sqrt{M(M - r_q)}, \tag{5.12}$$

whose existence requires that $r_q < M$. From Eq. (5.10), both ϕ' and A_0' are finite at $r = r_H$.

The action (5.9) corresponds to the coupling functions

$$G_2 = 4X + 4e^{-2\phi}F, \qquad G_4 = 1, \tag{5.13}$$

in Eq. (5.1). In this case we have $\mathcal{G} = \mathcal{F} = \mathcal{H} = 2$, $G_{2,F} = 4e^{-2\phi}$, and $G_{2,X}G_4 + 3G_{4,\phi}^2 = 4$, so the conditions (5.2) and (5.4) are automatically satisfied. The two squared propagation speeds (5.5) and (5.8) reduce, respectively, to

$$c^2_{r2,\mathrm{even}} = 1, \qquad c^2_{\Omega 3,\mathrm{even}} = 1. \tag{5.14}$$

Thus, all the linear stability conditions are consistently satisfied for the GM-GHS BH solution.

We can consider more general theories in which the dilatonic coupling $e^{-2\phi}$ is extended to an arbitrary function ξ of ϕ, i.e., $G_2 = 4X + 4\xi(\phi)F$ and $G_4 = 1$. This includes the case of spontaneous scalarized BHs which can be realized for even-power law functions of $\xi(\phi)$ [58–62]. In such Einstein-Maxwell-scalar theories, the difference of stability conditions from the dilatonic case appears only for the quantity $G_{2,F} = 4\xi(\phi)$. So long as $\xi(\phi) > 0$, hairy BH solutions are consistent with all the linear stability conditions.

2. Einstein-Born-Infeld-dilaton gravity

A BI-type action can arise as a low energy effective action describing the dynamics of vector fields in open string theory or on D-branes [70–72,106–109]. The Lagrangian of such a nonlinear BI theory is given by $\mathcal{L}_{\rm BI}(F) = (4/b^2)[1 - \sqrt{1 - 2b^2F}]$, where b is a coupling constant. The nonlinear BI vector field can be coupled to the dilaton field ϕ. The action of Einstein-BI-dilaton theory is given by

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[R + \eta X + \frac{4}{b^2\mu(\phi)} \times \left(1 - \sqrt{1 - 2b^2\mu^2(\phi)F}\right)\right], \tag{5.15}$$

where η is a constant, and $\mu(\phi) = e^{-2\phi}$. This theory corresponds to the coupling functions

$$G_2 = \eta X + \frac{4}{b^2\mu(\phi)}\left(1 - \sqrt{1 - 2b^2\mu^2(\phi)F}\right), \qquad G_4 = 1, \tag{5.16}$$

where $1 - 2b^2\mu^2(\phi)F > 0$ for theoretical consistency. In the limit that $b \to 0$ we have $G_2 \to \eta X + 4\mu(\phi)F$, so it recovers the theory discussed in Sec. V A 1. In the regime of small values of b, there should be hairy BHs similar to the exact solution (5.10). Indeed, the existence of regular BH solutions was shown for arbitrary couplings b [77–81].

Let us now discuss the linear stability of BHs in theories given by the coupling functions (5.16). First of all, we have $\mathcal{G} = \mathcal{F} = \mathcal{H} = 2 > 0$. Since $G_{2,F} = 4\mu(\phi)/\sqrt{1 - 2b^2\mu^2(\phi)F}$, the dilatonic coupling $\mu(\phi) = e^{-2\phi}$ satisfies the condition $G_{2,F} > 0$. The no-ghost condition (5.4) yields $G_{2,X}G_4 + 3G_{4,\phi}^2 = \eta > 0$. The radial propagation speed squared (5.5) reduces to the luminal value $c_{r2,\mathrm{even}}^2 = 1$. On the other hand, the angular propagation speed squared (5.8) yields

$$c_{\Omega 3,\mathrm{even}}^2 = 1 - b^2\mu^2(\phi)\frac{h}{f}A_0'^2. \tag{5.17}$$

From Eq. (2.15), we have

$$A_0' = \frac{q\sqrt{f}}{\sqrt{h(r^4 + b^2q^2)}\mu(\phi)}, \tag{5.18}$$

where $q = q_0/4$. Then, Eq. (5.17) reduces to

$$c_{\Omega 3,\mathrm{even}}^2 = \frac{r^4}{r^4 + b^2q^2}, \tag{5.19}$$

which is positive. Moreover, this vector field propagation speed is in the subluminal range $0 < c_{\Omega 3,\mathrm{even}}^2 < 1$. In summary, provided that $\eta > 0$, all the linear stability conditions are consistently satisfied.

Finally, there is a specific case in which the scalar field ϕ is absent, i.e.,

$$\eta = 0, \qquad \mu(\phi) = 1, \tag{5.20}$$

in the action (5.15). In this theory, there is an exact BH solution given by [74–76]

$$f = h = 1 - \frac{2M}{r} + \frac{2r^2}{3b^2} - \frac{2}{r}\int^r \frac{\sqrt{\tilde{r}^4 + b^2q^2}}{b^2}\mathrm{d}\tilde{r}, \tag{5.21}$$

where A_0' is given by Eq. (5.18) with $\mu(\phi) = 1$. The absence of the scalar field means that we do not have the no-ghost condition $\mathcal{K} > 0$ associated with the perturbation $\delta\phi$. Since the other stability conditions are the same as those derived for the action (5.15) with the replacement $\mu(\phi) \to 1$, there are neither ghost nor Laplacian instabilities. We note that $c_{\Omega 3,\mathrm{even}}$ is again subluminal.

B. Einstein-Maxwell-Dilaton-Gauss-Bonnet Theory

In low energy effective heterotic string theory, the dilaton field ϕ is not only coupled to the electromagnetic field strength F but also to a GB curvature invariant $R_{\mathrm{GB}}^2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ of the form $e^{-2\phi}R_{\mathrm{GB}}^2$, where $R_{\mu\nu}$ is the Ricci tensor and $R_{\mu\nu\rho\lambda}$ is the Riemann tensor. At leading order in the α' expansion, the low energy effective action of heterotic strings in the bosonic sector is given by [11,110,111]

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}[R + 4X + \alpha\xi(\phi)(4F + R_{\mathrm{GB}}^2)], \tag{5.22}$$

where $\alpha = \alpha'/8$ is a coupling constant, and

$$\xi(\phi) = e^{-2\phi}. \tag{5.23}$$

It is known that hairy BH solutions are present in this theory [63,67] (see also Refs. [112,113] for recent related works). The theory given by the action (5.22) belongs to a subclass of Horndeski theories with the coupling functions [37]

$$G_2 = 4X + 4\alpha\xi(\phi)F + 8\alpha\xi_{,\phi\phi\phi\phi}(\phi)X^2(3 - \ln|X|), \qquad G_3 = 4\alpha\xi_{,\phi\phi\phi}(\phi)X(7 - 3\ln|X|),$$
$$G_4 = 1 + 4\alpha\xi_{,\phi\phi}(\phi)X(2 - \ln|X|), \qquad G_5 = -4\alpha\xi_{,\phi}(\phi)\ln|X|. \tag{5.24}$$

In the limit that $\alpha \to 0$, the BH solution should reduce to the no-hair Schwarzschild metric. For arbitrary couplings α it is difficult to derive an exact BH solution, but we can obtain solutions for small α by using the expansions

$$f(r) = \left(1 - \frac{2M}{r}\right)\left[1 + \sum_{j\geq 1}\hat{f}_j(r)\alpha^j\right], \qquad h(r) = \left(1 - \frac{2M}{r}\right)\left[1 + \sum_{j\geq 1}\hat{h}_j(r)\alpha^j\right],$$
$$\phi(r) = \phi_0 + \sum_{j\geq 1}\hat{\phi}_j(r)\alpha^j, \tag{5.25}$$

where M and ϕ_0 are constants, $\hat{f}_j(r)$, $\hat{h}_j(r)$, and $\hat{\phi}_j(r)$ are functions of r. The temporal vector component obeys Eq. (2.15), i.e.,

$$A_0' = e^{2\phi}\sqrt{\frac{f}{h}}\frac{q}{r^2}, \tag{5.26}$$

where q is a constant. We substitute Eqs. (5.25) and (5.26), and their r derivatives into the background Eqs. (2.7)–(2.9) and (2.11). Then, we derive the solutions to $\hat{f}_j(r)$, $\hat{h}_j(r)$, and $\hat{\phi}_j(r)$ at each order in α.

At first order in α, the solutions regular on the horizon ($r = 2M$) are given by

$$\hat{f}_1(r) = -\frac{q^2 e^{2\phi_0}}{2M^2\hat{r}}, \qquad \hat{h}_1(r) = -\frac{q^2 e^{2\phi_0}}{2M^2\hat{r}}, \qquad \hat{\phi}_1(r) = \tilde{\phi}_1 - \frac{3e^{2\phi_0}q^2\hat{r}^2 + 2e^{-2\phi_0}(3\hat{r}^2 + 3\hat{r} + 4)}{6M^2\hat{r}^3}, \tag{5.27}$$

where $\hat{r} \equiv r/M$, and $\tilde{\phi}_1$ is a constant. For $q \to 0$, the vector field derivative (5.26) is vanishing and hence this corresponds to the limit in which only the dilaton-GB coupling $\alpha\xi(\phi)R^2_{\rm GB}$ is present. In this limit we have $\hat{f}_1(r) = 0 = \hat{h}_1(r)$, so the GB term does not contribute to the metric components at this order. This property is consistent with the findings in Refs. [44,53]. We note that $\hat{\phi}_1(r)$ is affected by both the GB term and the vector field.

At second order in α, we obtain the following regular solutions:

$$\hat{f}_2(r) = [3200 + 832\hat{r} + 112(5q^2e^{4\phi_0} - 1)\hat{r}^2 - 8\{137 + 5q^2e^{4\phi_0}(6\tilde{\phi}_1M^2\hat{r}^2e^{2\phi_0} + 5)\}\hat{r}^3 + 6(5q^4e^{8\phi_0} - 10q^2e^{4\phi_0} - 98)\hat{r}^4$$
$$+ 3(5q^4e^{8\phi_0} - 10q^2e^{4\phi_0} - 98)\hat{r}^5]/(240e^{4\phi_0}M^4\hat{r}^6), \tag{5.28}$$

$$\hat{h}_2(r) = [14720 + 6976\hat{r} + 16(125q^2e^{4\phi_0} + 203)\hat{r}^2 + 24(5q^2e^{4\phi_0} - 19)\hat{r}^3 + 6(15q^4e^{8\phi_0} + 30q^2e^{4\phi_0} - 58)\hat{r}^4$$
$$+ 3\{5q^4e^{8\phi_0} - 10q^2e^{4\phi_0}(8\tilde{\phi}_1M^2e^{2\phi_0} + 1) - 98\}\hat{r}^5]/(240e^{4\phi_0}M^4\hat{r}^6), \tag{5.29}$$

$$\hat{\phi}_2(r) = \tilde{\phi}_2 + \tilde{\phi}_1\frac{8 + 6\hat{r} + 3(2 - q^2e^{4\phi_0})\hat{r}^2}{3e^{2\phi_0}M^2\hat{r}^3} - [1600 + 2688\hat{r} + 60(73 - 10q^2e^{4\phi_0})\hat{r}^2 + 40(73 + 15q^2e^{4\phi_0})\hat{r}^3 + 2190\hat{r}^4$$
$$+ 15(146 - 30q^2e^{4\phi_0} - 15q^4e^{8\phi_0})\hat{r}^5]/(1800e^{4\phi_0}M^4\hat{r}^6), \tag{5.30}$$

where $\tilde{\phi}_2$ is a constant. At this order, both the GB term and the vector field contribute to the metric components. At spatial infinity, all of $\hat{f}_1(r)$, $\hat{h}_1(r)$, $\hat{f}_2(r)$, $\hat{f}_2(r)$ approach 0 with $\hat{\phi}_1(r) \to \tilde{\phi}_1$ and $\hat{\phi}_2(r) \to \tilde{\phi}_2$, so the above hairy BH solutions are asymptotically flat.

We derive the solutions (5.25) expanded up to the sixth order in α and use them to compute quantities associated with the linear stability of BHs. Since $G_{2,F} = 4\alpha e^{-2\phi}$, we require the condition

$$\alpha > 0, \tag{5.31}$$

to avoid ghost instabilities of vector field perturbations. The angular propagation speed of V is luminal, $c^2_{\Omega 3,\rm even} = 1$. In the odd-parity sector, we have

$$\mathcal{G} = 2 + \frac{8[q^2\hat{r}^2 + 2e^{-4\phi_0}(\hat{r}^2 + 2\hat{r} + 4)]}{M^4\hat{r}^6}\alpha^2 + \mathcal{O}(\alpha^3), \tag{5.32}$$

$$\mathcal{F} = 2 - \frac{8[q^2\hat{r}^2(2\hat{r} - 5) + 2e^{-4\phi_0}(2\hat{r}^3 + \hat{r}^2 + 2\hat{r} - 36)]}{M^4\hat{r}^6}\alpha^2$$
$$+ \mathcal{O}(\alpha^3), \tag{5.33}$$

$$\mathcal{H} = 2 + \frac{8(\hat{r} - 2)[q^2\hat{r}^2 + 2e^{-4\phi_0}(\hat{r}^2 + 2\hat{r} + 4)]}{M^4\hat{r}^6}\alpha^2 + \mathcal{O}(\alpha^3). \tag{5.34}$$

In the small coupling regime where terms of order α^2 in Eqs. (5.32)–(5.34) are smaller than the order 1, all of $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{H}$ are positive.

In the even-parity sector, the no-ghost parameter (4.31) yields

$$\mathcal{K} = \frac{[e^{2\phi_0}q^2\hat{r}^2 + 2e^{-2\phi_0}(\hat{r}^2 + 2\hat{r} + 4)]^2}{2M^4\hat{r}^6}\alpha^2 + \mathcal{O}(\alpha^3), \tag{5.35}$$

whose leading-order term is always positive. The radial squared propagation speed of $\delta\phi$ can be estimated as

$$c_{r2,\mathrm{even}}^2 = 1 - \frac{32(\hat{r}-2)[q^4\hat{r}^4 + 4q^2\hat{r}^2(\hat{r}^2+2\hat{r}+16)e^{-4\phi_0} + 4(\hat{r}^4+4\hat{r}^3+36\hat{r}^2+88\hat{r}+208)e^{-8\phi_0}]}{M^8\hat{r}^{12}}\alpha^4 + \mathcal{O}(\alpha^5). \tag{5.36}$$

To derive this result, we need to use the solutions (5.25) expanded up to the order $j=6$. The solutions expanded up to $j=7$ give the same coefficient of α^4 in $c_{r2,\mathrm{even}}^2$ as that appearing in Eq. (5.36). For small α, $c_{r2,\mathrm{even}}^2$ is close to 1 and hence there is no Laplacian stability of $\delta\phi$ along the radial direction. The squared propagation speeds of ψ and $\delta\phi$ along the angular direction are given by

$$c_{\Omega\pm,\mathrm{even}}^2 = 1 \pm \frac{24e^{-2\phi_0}}{M^2\hat{r}^3}\alpha + \mathcal{O}(\alpha^2), \tag{5.37}$$

where terms of order α arises from the dilaton-GB coupling. For small α, both $c_{\Omega+,\mathrm{even}}^2$ and $c_{\Omega-,\mathrm{even}}^2$ are positive. We have thus shown that all the linear stability conditions are consistently satisfied for hairy BH solutions present for small couplings α.

C. 4DEGB gravity

The GB curvature invariant R_{GB}^2 is a topological surface term, so the field equations of motion following from the action $\mathcal{S} = \int d^4x\sqrt{-g}\hat{\alpha}_{\mathrm{GB}}R_{\mathrm{GB}}^2$ vanish in 4 dimensions [114]. In a D-dimensional spacetime ($D>4$), rescaling the GB coupling constant as $\hat{\alpha}_{\mathrm{GB}} \to \alpha_{\mathrm{GB}}/(D-4)$ allows a possibility for extracting contributions of the higher-dimensional GB term [45]. One can perform a Kaluza-Klein reduction of the D-dimensional Einstein-GB gravity on a $(D-4)$-dimensional maximally symmetric space with a vanishing spatial curvature [46,47]. The size of a maximally symmetric space is characterized by a scalar field ϕ. Taking the Maxwell field into account, the four-dimensional action derived from the Kaluza-Klein reduction of D-dimensional Einstein-GB theory belongs to a subclass of shift-symmetric Maxwell-Horndeski theories given by the coupling functions [48]

$$G_2 = 8\alpha_{\mathrm{GB}}X^2 + 4F, \qquad G_3 = 8\alpha_{\mathrm{GB}}X, \qquad G_4 = 1 + 4\alpha_{\mathrm{GB}}X, \qquad G_5 = 4\alpha_{\mathrm{GB}}\ln|X|. \tag{5.38}$$

In this regularized 4DEGB gravity, it is known that there exists an exact hairy BH solution [66]. We first revisit the derivation of this exact BH solution and then study its linear stability. The background equations of motion for the line element (2.3) are expressed in the form

$$f' = -\frac{r^2 f(h-1) + \alpha_{\mathrm{GB}} f[h^2 - 2h(1-2j-2r\phi' j) + 1 - 4j + 3j^2] + hr^4 A_0'^2}{hr[r^2 - 2\alpha_{\mathrm{GB}}(h-1+j+r\phi' j)]}, \tag{5.39}$$

$$\frac{h'}{h} - \frac{f'}{f} = -\frac{4\alpha_{\mathrm{GB}} rj(\phi'^2 - \phi'')}{r^2 - 2\alpha_{\mathrm{GB}}(h-1+j+r\phi' j)}, \tag{5.40}$$

$$\sqrt{\frac{h}{f}}\alpha_{\mathrm{GB}}(f' + 2\phi' f)j = \mathcal{C}, \tag{5.41}$$

$$A_0' = \sqrt{\frac{f}{h}}\frac{q}{r^2}, \tag{5.42}$$

where $\mathcal{C}$ and q are constants, and

$$j \equiv 1 - h(1 + r\phi')^2. \tag{5.43}$$

We search for asymptotically flat BH solutions with $f = f_0 + f_1/r + f_2/r^2 + \cdots$, $h = 1 + h_1/r + h_2/r^2 + \cdots$, and $\phi = \phi_0 + \phi_1/r + \phi_2/r^2 + \cdots$ at spatial infinity, where f_j, h_j, and ϕ_j are constants. The left-hand side of Eq. (5.41) approaches 0 as $r \to \infty$, so the constant $\mathcal{C}$ is fixed to be 0.

Since f, h, f', and ϕ are finite outside the horizon, we require that $j=0$. Then, the scalar field solution with the behavior $\phi' \propto 1/r^2$ at large distances is given by

$$\phi' = \frac{1}{r}\left(\frac{1}{\sqrt{h}} - 1\right), \tag{5.44}$$

which diverges on the horizon ($h=0$). We note that the field kinetic term $X = -(1-\sqrt{h})^2/(2r^2)$ is finite at $h=0$. We can integrate Eq. (5.40) to give $h = \tilde{\mathcal{C}}f$, where $\tilde{\mathcal{C}}$ is a constant. Since $\tilde{\mathcal{C}}$ can be chosen 1 after the time reparametrization of f, it follows that $h=f$. Substituting $j=0$, $h=f$, and Eq. (5.42) into Eq. (5.39), we obtain

$$f' = -\frac{(f-1)[r^2 + \alpha_{\mathrm{GB}}(f-1)] + q^2}{r^3 - 2\alpha_{\mathrm{GB}} r(f-1)}. \tag{5.45}$$

The integrated solution to this equation, which is consistent with the boundary condition at spatial infinity, is given by

$$f = h = 1 + \frac{r^2}{2\alpha_{\rm GB}}\left[1 - \sqrt{1 + 4\alpha_{\rm GB}\left(\frac{2M}{r^3} - \frac{q^2}{r^4}\right)}\right], \quad (5.46)$$

where M is a constant. The vector field solution (5.42) reduces to $A_0' = q/r^2$. The horizons are located at

$$r_\pm = M \pm \sqrt{M^2 - q^2 - \alpha_{\rm GB}}. \quad (5.47)$$

The existence of horizons requires the condition $q^2 + \alpha_{\rm GB} \le M^2$.

On using Eqs. (5.42), (5.44), and (5.45) with $f = h$, it follows that

$$\mathcal{K} = 0, \quad (5.48)$$

at any distance r. Since the dynamical scalar field ϕ is present as a radion mode in the extra dimension, the vanishing kinetic term $\mathcal{K}$ means a strong coupling problem associated with the perturbation $\delta\phi$. The same strong coupling problem is also present for hairy BH solutions with $q = 0$ [52]. The denominator of Eq. (4.38) is proportional to $\mathcal{K} = 2\mathcal{P}_1 - \mathcal{F}$ and hence $c_{r2,\rm even}^2$ is divergent for arbitrary r. We also note that both B_1 and B_2 contain $\mathcal{K}$ in their denominators, so this generally leads to the divergences of $c_{\Omega\pm,\rm even}^2$ as well.

We can compute the ratio $c_{r2,\rm even}^2/B_2$ at large distances by using the following asymptotic solution of Eq. (5.46):

$$f = h = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \frac{4\alpha_{\rm GB}M^2}{r^4} - \frac{4\alpha_{\rm GB}q^2M}{r^5} + \mathcal{O}(r^{-6}). \quad (5.49)$$

Then, we obtain the following asymptotic behavior:

$$\frac{c_{r2,\rm even}^2}{B_2} = -2 - \frac{3(3M^2 + q^2)}{2Mr} + \mathcal{O}(r^{-2}). \quad (5.50)$$

This means that both $c_{r2,\rm even}^2$ and B_2 cannot be simultaneously positive at large distances, so either of the linear stability conditions $c_{r2,\rm even}^2 > 0$ or $B_2 > 0$ is violated. The same instability was also found for hairy BH solutions with $q = 0$ [52].

In the vicinity of the outer horizon $r_+ = M + \sqrt{M^2 - q^2 - \alpha_{\rm GB}}$, the metric components can be expanded as

$$f = h = \frac{r_+^2 - q^2 - \alpha_{\rm GB}}{r_+(r_+^2 + 2\alpha_{\rm GB})}(r - r_+) - \frac{r_+^6 - (2q^2 + 3\alpha_{\rm GB})r_+^4 - 2\alpha_{\rm GB}(q^2 + 3\alpha_{\rm GB})r_+^2 - \alpha_{\rm GB}(q^2 + \alpha_{\rm GB})^2}{r_+^2(r_+^2 + 2\alpha_{\rm GB})^3}(r - r_+)^2 + \mathcal{O}(r - r_+)^3. \quad (5.51)$$

On using this expanded solution with Eqs. (5.42) and (5.44), the product $\mathcal{F}\mathcal{K}B_2$ yields

$$\mathcal{F}\mathcal{K}B_2 = -\frac{16\alpha_{\rm GB}^2[r_+^4 + (r_+^2 + q^2)\alpha_{\rm GB} + \alpha_{\rm GB}^2]^2}{r_+^2(r_+^2 + 2\alpha_{\rm GB})^4}(r - r_+)^{-2} + \mathcal{O}((r - r_+)^{-3/2}). \quad (5.52)$$

Hence the leading-order term of $\mathcal{F}\mathcal{K}B_2$ is negative around $r = r_+$. This means that either $\mathcal{F}$, $\mathcal{K}$, or B_2 must be negative, so the hairy BH is unstable in the vicinity of the outer horizon. For $q = 0$, the leading-order term of $\mathcal{F}\mathcal{K}B_2$ coincides with the one derived in Ref. [52].

For the specific case $r_+^2 = q^2 + \alpha_{\rm GB}$, there is a single horizon located at $r_+ = M$. Since the first term on the right-hand side of Eq. (5.51) vanishes in such a case, the leading-order contribution to $\mathcal{F}\mathcal{K}B_2$ is not necessarily negative. However the properties (5.48) and (5.50) still hold, so the problems of strong coupling as well as large-distance Laplacian instability are unavoidable for the hairy BH present in regularized 4DEGB gravity.

VI. CONCLUSIONS

In Maxwell-Horndeski theories given by the action (2.2), we derived BH linear stability conditions on the static and spherically symmetric background (2.3). We incorporated a $U(1)$ gauge-invariant vector field A_μ coupled to a scalar field ϕ with the Lagrangian $G_2(\phi, X, F)$, where $F = -F_{\mu\nu}F^{\mu\nu}/4$ is the gauge field strength. Due to the gauge invariance, the vector field equation has an integrated solution of the form (2.15), where q_0 corresponds to an electric charge. As we observe in Eqs. (2.7) and (2.8), the temporal component of A_μ modifies the background equations in the gravitational sector. The scalar field Eq. (2.11) can be also generally affected by the coupling with A_μ through the existence of G_2-dependent terms such as $G_{2,\phi}$.

In Sec. III, we first showed that the second-order Lagrangian of odd-parity perturbations is expressed as Eq. (3.7). Introducing an auxiliary field χ defined by Eq. (3.12), the Lagrangian for the multipoles $l \ge 2$ can be expressed in terms of the two dynamical fields χ and δA.

They correspond to the perturbations arising from the gravitational and vector field sectors, respectively. We found that the propagation of χ is analogous to the case of Horndeski theories without the vector field [49]. In the limit $l \gg 1$, the propagation speeds of δA are luminal in both radial and angular directions. In the odd-parity sector, there are neither ghost nor Laplacian instabilities under the conditions $\mathcal{G} > 0$, $G_{2,F} > 0$, $\mathcal{F} > 0$, and $\mathcal{H} > 0$. For $l = 1$, δA is the only propagating DOF, whose stability does not require additional conditions.

In Sec. IV, we obtained the second-order action of even-parity perturbations in the form (4.11) with the sum of (4.12) and (4.13). The auxiliary field V introduced in Eq. (4.15) plays a role of the dynamical vector field perturbation in the even-parity sector. There are also two dynamical perturbations ψ and $\delta\phi$ arising from the gravitational and scalar field sectors, respectively. For $l \geq 2$, we showed that the second-order action can be expressed in the form (4.25) with $\vec{\mathcal{X}} = {}^t(\psi, \delta\phi, V)$. Under the stability conditions $\mathcal{F} > 0$ and $G_{2,F} > 0$ for odd-parity perturbations, the ghosts in the even-parity sector are absent under the condition $\mathcal{K} = 2\mathcal{P}_1 - \mathcal{F} > 0$, where $\mathcal{P}_1$ is defined in Eq. (4.30).

The squared propagation speeds of even-parity perturbations ψ, $\delta\phi$, V along the radial direction are given, respectively, by Eqs. (4.34), (4.37), and (4.38). While the expression of $c_{r1,\mathrm{even}}$ coincides with that derived in Ref. [50,51], the vector field coupled to ϕ modifies the propagation speed $c_{r2,\mathrm{even}}$ of $\delta\phi$. Along the angular direction, the squared propagation speed of V in the large l limit is given by Eq. (4.43), which is different from 1 in the presence of nonlinear functions of F in G_2. The angular squared propagation speeds $c^2_{\Omega\pm,\mathrm{even}}$ of ψ and $\delta\phi$ are expressed as Eq. (4.45) with B_1 and B_2 given by Eqs. (4.48) and (4.49). These expressions of B_1 and B_2 are of the same forms as those derived in Ref. [51] without a perfect fluid, but there are modifications to $c^2_{\Omega\pm,\mathrm{even}}$ arising from the vector field through the term a_4' [see Eq. (4.35)]. We also studied the dynamics of monopole ($l = 0$) and dipole ($l = 1$) perturbations and showed that there are no additional conditions to those derived for $l \geq 2$. In Table I, we summarized all the linear stability conditions of odd- and even-parity perturbations.

In Sec. V, we applied the linear stability conditions to hairy BH solutions present in Maxwell-Horndeski theories. In Einstein-Maxwell-dilaton theory, which is given by the action (5.9), there is the exact solution (5.10) where the dilaton acquires a secondary hair through a coupling with the vector field. In this case, we showed that all the linear stability conditions are satisfied with luminal propagation speeds of odd- and even-parity perturbations. In Einstein-BI-dilaton gravity with the action (5.15), the angular propagation speed of vector field perturbation is subluminal without a scalar ghost for $\eta > 0$. The exact BH solution (5.21) present in Einstein-BI theory ($\eta = 0$, $\mu(\phi) = 1$) has neither ghost nor Laplacian instabilities. In Einstein-Maxwell-dilaton-GB theory with the action (5.22), we showed that the hairy BH solution derived under a small α expansion can be consistent with all the linear stability conditions. In regularized 4DEGB gravity, however, the exact BH solution (5.46) is prone to the strong coupling and instability problems. As shown in Ref. [52], this conclusion also holds for an uncharged exact BH solution present in the same theory without the Maxwell field.

We thus showed that the linear stability conditions derived in this paper are useful to exclude some BH solutions or to put constraints on stable parameter spaces. It will be of interest to apply our general framework of BH perturbations to the computation of quasinormal modes of BHs. The analysis can be also extended to the case in which a perfect fluid is present in Maxwell-Horndeski theories. This will allow us to study the stability of hairy neutron stars along the line of Refs. [51,83–85]. The scalar field coupling with the $U(1)$ gauge-invariant vector field can be further generalized by preserving the second-order property of equations of motion. Such theories are known as $U(1)$ gauge-invariant scalar-vector-tensor theories [115], in which vector and scalar fields have nonminimal and derivative couplings to gravity. Hairy BH solutions in these theories were derived in Refs. [116,117] and the odd-parity stability was studied in Refs. [96,118]. It would be also of interest to address the stability of even-parity BH perturbations in such theories. These issues are left for future works.

ACKNOWLEDGMENTS

R. K. is supported by the Grant-in-Aid for Young Scientists of the JSPS No. 20K14471. S. T. is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 19K03854 and No. 22K03642.

APPENDIX A: COEFFICIENTS IN THE SECOND-ORDER ACTION OF EVEN-PARITY PERTURBATIONS

The coefficients in Eqs. (4.12) and (4.13) are given by

$$
\begin{aligned}
a_1 &= \sqrt{fh}\bigg[\left\{G_{4,\phi}+\frac{1}{2}h(G_{3,X}-2G_{4,\phi X})\phi'^2\right\}r^2+2h\phi'\left\{G_{4,X}-G_{5,\phi}-\frac{1}{2}h(2G_{4,XX}-G_{5,\phi X})\phi'^2\right\}r \\
&\quad +\frac{1}{2}G_{5,XX}h^3\phi'^4-\frac{1}{2}G_{5,X}h(3h-1)\phi'^2\bigg],\\
a_2 &= \sqrt{fh}\left(\frac{a_1}{\sqrt{fh}}\right)'-\left(\frac{\phi''}{\phi'}-\frac{1}{2}\frac{f'}{f}\right)a_1+\frac{r}{\phi'}\left(\frac{f'}{f}-\frac{h'}{h}\right)a_4+\frac{A_0'}{2}v_4, \qquad a_3=-\frac{1}{2}\phi' a_1-ra_4,\\
a_4 &= \frac{\sqrt{fh}}{2}\mathcal{H}, \qquad a_5=a_2'-a_1''-\left(\frac{A_0'}{2}v_4\right)'+\frac{A_0'}{2}v_5, \qquad a_6=-\frac{\sqrt{f}}{2\sqrt{h}\phi'}\left(\mathcal{H}'+\frac{\mathcal{H}}{r}-\frac{\mathcal{F}}{r}\right),\\
a_7 &= a_3'-\frac{A_0'^2}{2}v_1-\frac{\phi' A_0'}{4}v_4, \qquad a_8=-\frac{1}{2}\frac{a_4}{h}, \qquad a_9=a_4'+\left(\frac{1}{r}-\frac{1}{2}\frac{f'}{f}\right)a_4,\\
b_1 &= \frac{1}{2f}a_4, \qquad b_2=-\frac{2}{f}a_1, \qquad b_3=-\frac{2}{f}(a_2-a_1')+\frac{A_0'}{f}v_4, \qquad b_4=-\frac{2}{f}a_3, \qquad b_5=-2b_1,\\
c_1 &= -\frac{1}{fh}a_1,\\
c_2 &= \sqrt{fh}\bigg[\bigg\{\frac{1}{2f}\left(-\frac{1}{2}h(3G_{3,X}-8G_{4,\phi X})\phi'^2+\frac{1}{2}h^2(G_{3,XX}-2G_{4,\phi XX})\phi'^4-G_{4,\phi}\right)r^2\\
&\quad -\frac{h\phi'}{f}\left(\frac{1}{2}h^2(2G_{4,XXX}-G_{5,\phi XX})\phi'^4-\frac{1}{2}h(12G_{4,XX}-7G_{5,\phi X})\phi'^2+3(G_{4,X}-G_{5,\phi})\right)r\\
&\quad +\frac{h\phi'^2}{4f}(G_{5,XXX}h^3\phi'^4-G_{5,XX}h(10h-1)\phi'^2+3G_{5,X}(5h-1))\bigg\}f'\\
&\quad +\phi'\left\{\frac{1}{2}G_{2,X}-G_{3,\phi}-\frac{1}{2}h(G_{2,XX}-G_{3,\phi X})\phi'^2+\frac{hA_0'^2}{2f}G_{2,XF}\right\}r^2\\
&\quad +2\left\{-\frac{1}{2}h(3G_{3,X}-8G_{4,\phi X})\phi'^2+\frac{1}{2}h^2(G_{3,XX}-2G_{4,\phi XX})\phi'^4-G_{4,\phi}\right\}r\\
&\quad -\frac{1}{2}h^3(2G_{4,XXX}-G_{5,\phi XX})\phi'^5+\frac{1}{2}h\{2(6h-1)G_{4,XX}+(1-7h)G_{5,\phi X}\}\phi'^3-(3h-1)(G_{4,X}-G_{5,\phi})\phi'\bigg],\\
c_3 &= -\frac{1}{2}\frac{\sqrt{f}r^2}{\sqrt{h}}\frac{\partial\mathcal{E}_{11}}{\partial\phi},\\
c_4 &= \frac{1}{4}\frac{\sqrt{f}}{\sqrt{h}}\bigg[\frac{h\phi'}{f}\left\{2G_{4,X}-2G_{5,\phi}-h(2G_{4,XX}-G_{5,\phi X})\phi'^2-\frac{h\phi'(3G_{5,X}-G_{5,XX}\phi'^2h)}{r}\right\}f'\\
&\quad +4G_{4,\phi}+2h(G_{3,X}-2G_{4,\phi X})\phi'^2+\frac{4h(G_{4,X}-G_{5,\phi})\phi'-2h^2(2G_{4,XX}-G_{5,\phi X})\phi'^3}{r}\bigg],\\
c_5 &= -h\phi' c_4-\frac{1}{2}\frac{\sqrt{fh}}{r}\mathcal{G}-\frac{1}{2}\frac{f'}{f}a_4,\\
c_6 &= \frac{1}{8}\frac{f'\phi'}{f}a_1+\frac{1}{2}\frac{f'r}{f}a_4-\frac{1}{4}\phi' c_2+\frac{1}{2}h\phi' rc_4+\frac{1}{4}\sqrt{fh}\mathcal{G}+\frac{A_0'^2}{4}v_1+\frac{\phi' A_0'}{8}v_4,
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= \frac{1}{2f}a_4, \qquad d_2 = 2hc_4, \\
d_3 &= -\frac{1}{r^2}\left(\frac{2\phi''}{\phi'}+\frac{h'}{h}\right)a_1 + \frac{2f}{(f'r-2f)\phi'}\left(\frac{2\phi''}{h\phi' r}+\frac{f'^2}{f^2}-\frac{f'h'}{fh}-\frac{2f'}{fr}+\frac{2h'}{hr}+\frac{h'}{h^2 r}\right)a_4 \\
&\quad + \frac{f'r-2f}{fr}\frac{\partial a_4}{\partial\phi} + \frac{\sqrt{f}}{\phi'\sqrt{h}r^2}\mathcal{F} - \frac{f^{3/2}}{\sqrt{h}(f'r-2f)\phi'}\left(\frac{f'}{fr}+\frac{2\phi''}{\phi' r}+\frac{h'}{hr}-\frac{2}{r^2}\right)\mathcal{G}, \\
d_4 &= \frac{1}{2}\frac{\sqrt{fh}}{r^2}\mathcal{G}, \\
e_1 &= \frac{1}{\phi' f h}\left[\left(\frac{f'}{f}+\frac{1}{2}\frac{h'}{h}\right)a_1 - 2a_1' + a_2 - 2rha_6 - \frac{A_0'}{2}v_4\right], \\
e_2 &= -\frac{1}{2\phi'}\left(\frac{f'}{f}a_1 + 2c_2 + 4hrc_4 + A_0' v_4\right), \qquad e_3 = \frac{1}{4}\frac{\sqrt{f}r^2}{\sqrt{h}}\frac{\partial \mathcal{E}_\phi}{\partial\phi}, \\
e_4 &= \frac{1}{\phi'}c_4' - \frac{1}{2}\frac{f'}{f\phi'^2 h}a_4' - \frac{1}{2}\frac{\sqrt{f}}{\phi'^2\sqrt{h}r}\mathcal{G}' + \frac{1}{h\phi' r^2}\left(\frac{\phi''}{\phi'}+\frac{1}{2}\frac{h'}{h}\right)a_1 \\
&\quad + \frac{1}{4h\phi'^2}\left[\frac{(f'r-6f)f'}{f^2 r} + \frac{h'(f'r+4f)}{hrf} - \frac{4f(2\phi'' h + h'\phi')}{\phi' h^2 r(f'r-2f)}\right]a_4 + \frac{1}{2}\frac{h'}{h\phi'}c_4 - \frac{1}{2}\frac{f'r-2f}{fhr\phi'}\frac{\partial a_4}{\partial\phi} \\
&\quad + \frac{1}{2}\frac{f'hr-f}{r^2\sqrt{f}\phi'^2 h^{3/2}}\mathcal{F} + \frac{1}{2}\frac{\sqrt{f}}{r\phi'^2 h^{3/2}}\left[\frac{f(2\phi'' h + h'\phi')}{h\phi'(f'r-2f)} + \frac{1}{2}\frac{2f-f'hr}{fr}\right]\mathcal{G}, \\
v_1 &= \frac{r^2}{2}\sqrt{\frac{h}{f}}\left(G_{2,F} + \frac{hA_0'^2}{f}G_{2,FF}\right), \qquad v_2 = A_0' v_1, \qquad v_3 = -A_0' v_1 - \frac{\phi'}{2}v_4, \qquad v_4 = -\frac{r^2 h^{3/2}\phi' A_0' G_{2,XF}}{\sqrt{f}}, \\
v_5 &= \frac{r^2\sqrt{h}A_0' G_{2,\phi F}}{\sqrt{f}}, \qquad v_6 = \frac{A_0'^2}{4}v_1, \qquad v_7 = -2hA_0' v_8, \qquad v_8 = \frac{G_{2,F}}{2\sqrt{fh}}, \qquad v_9 = -fhv_8, \qquad \text{(A1)}
\end{aligned}
$$

where $\mathcal{E}_{11}$ and $\mathcal{E}_\phi$ are defined in Eqs. (2.8) and (2.16), respectively.

APPENDIX B: DERIVATION OF THE GM-GHS BH SOLUTION

In theories given by the action (5.9), GHS [57] derived a static and spherically symmetric BH solution given by the line element

$$ds^2 = -f(\hat{r})dt^2 + h^{-1}(\hat{r})d\hat{r}^2 + \zeta^2(\hat{r})d\Omega^2, \qquad \text{(B1)}$$

where $\zeta(\hat{r})$ is a function of $\hat{r}$. Varying the action (5.9) with respect to f and h, we obtain the following two equations:

$$\zeta\zeta' hf' = f + fh(\phi'^2\zeta^2 - \zeta'^2) - e^{-2\phi}hA_0'^2\zeta^2, \qquad \text{(B2)}$$

$$\frac{f'}{f} - \frac{h'}{h} = \frac{2(\zeta'' + \phi'^2\zeta)}{\zeta'}, \qquad \text{(B3)}$$

where a prime in this Appendix B represents the derivative with respect to $\hat{r}$. We search for a BH solution satisfying the relation

$$f = h, \qquad \text{(B4)}$$

under which Eq. (B3) gives

$$\zeta'' + \phi'^2\zeta = 0. \qquad \text{(B5)}$$

Exploiting this relation for the equation of motion of A_0', we obtain the integrated solution

$$A_0' = \frac{qe^{2\phi}}{\zeta^2}, \qquad \text{(B6)}$$

where q is a constant corresponding to an electric charge. Varying the action (5.9) with respect to ζ and ϕ and using the equations derived above, it follows that

$$(f\zeta^2)'' = 2, \qquad \text{(B7)}$$

$$(2f\zeta^2\phi' - f'\zeta^2)' = 0. \qquad \text{(B8)}$$

These equations are integrated to give

$$f\zeta^2 = (\hat{r} - 2M)^2 + \mathcal{C}_1(\hat{r} - 2M), \qquad \text{(B9)}$$

$$\phi' = \frac{f'}{2f} + \frac{\mathcal{C}_2}{2f\zeta^2}, \tag{B10}$$

where M, $\mathcal{C}_1$, and $\mathcal{C}_2$ are constants. The BH event horizon corresponds to $\hat{r} = 2M$, at which $f = h = 0$. Taking the $\hat{r}$ derivative of Eq. (B9) and using Eq. (B10), we obtain

$$f\zeta^2\phi' = \hat{r} - 2M - f\zeta\zeta' + \frac{1}{2}(\mathcal{C}_1 + \mathcal{C}_2). \tag{B11}$$

Provided that ϕ' is finite on the horizon, the consistency of Eq. (B11) at $\hat{r} = 2M$ requires that

$$\mathcal{C}_2 = -\mathcal{C}_1. \tag{B12}$$

Now, we can eliminate A_0', ζ, ζ', and ϕ' in Eq. (B2) by using Eqs. (B6), (B9), and (B10) with Eq. (B12). Then, we find that the metric components

$$f = h = 1 - \frac{2M}{\hat{r}}, \qquad \zeta^2 = \hat{r}(\hat{r} - 2r_q) \tag{B13}$$

are the solutions to the above system for

$$\mathcal{C}_1 = 2M - 2r_q, \tag{B14}$$

where r_q is defined by Eq. (5.11). Integrating Eq. (B10) with respect to $\hat{r}$ and substituting the integrated solution into Eq. (B6), the solutions to the scalar and vector fields are given by

$$\phi = \phi_0 + \frac{1}{2}\ln\left(1 - \frac{2r_q}{\hat{r}}\right), \qquad \frac{\mathrm{d}A_0}{\mathrm{d}\hat{r}} = \frac{qe^{2\phi_0}}{\hat{r}^2}, \tag{B15}$$

where ϕ_0 is the value of ϕ at spatial infinity. In the case of a magnetic charge q, we just need to change the sign of the scalar field, i.e., $\phi \to -\phi$ and $\phi_0 \to -\phi_0$ [57,119,120].

To express this GM-GHS BH solution with respect to the line element (2.3), we perform the transformation

$$\zeta^2 = \hat{r}(\hat{r} - 2r_q) \to r^2. \tag{B16}$$

We choose the branch $\hat{r} = \sqrt{r^2 + r_q^2} + r_q$ to have the property $\hat{r} \to +\infty$ as $r \to +\infty$. Then, the above BH solution is expressed in the form (5.10) for the coordinate (2.3).

[1] W. Israel, Phys. Rev. **164**, 1776 (1967).
[2] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).
[3] R. Ruffini and J. A. Wheeler, Phys. Today **24**, 30 (1971).
[4] S. W. Hawking, Commun. Math. Phys. **25**, 152 (1972).
[5] J. D. Bekenstein, Phys. Rev. Lett. **28**, 452 (1972).
[6] A. A. H. Graham and R. Jha, Phys. Rev. D **89**, 084056 (2014); **92**, 069901(E) (2015).
[7] S. W. Hawking, Commun. Math. Phys. **25**, 167 (1972).
[8] J. D. Bekenstein, Phys. Rev. D **51**, R6608 (1995).
[9] T. P. Sotiriou and V. Faraoni, Phys. Rev. Lett. **108**, 081103 (2012).
[10] V. Faraoni, Phys. Rev. D **95**, 124013 (2017).
[11] B. Zwiebach, Phys. Lett. **156B**, 315 (1985).
[12] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. **B415**, 497 (1994).
[13] M. Gasperini, M. Maggiore, and G. Veneziano, Nucl. Phys. **B494**, 315 (1997).
[14] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Phys. Rev. D **54**, 5049 (1996).
[15] S. O. Alexeev and M. V. Pomazanov, Phys. Rev. D **55**, 2110 (1997).
[16] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Phys. Rev. D **57**, 6255 (1998).
[17] C.-M. Chen, D. V. Gal'tsov, and D. G. Orlov, Phys. Rev. D **75**, 084030 (2007).
[18] Z.-K. Guo, N. Ohta, and T. Torii, Prog. Theor. Phys. **120**, 581 (2008).
[19] Z.-K. Guo, N. Ohta, and T. Torii, Prog. Theor. Phys. **121**, 253 (2009).
[20] P. Pani and V. Cardoso, Phys. Rev. D **79**, 084031 (2009).
[21] D. Ayzenberg and N. Yunes, Phys. Rev. D **90**, 044066 (2014); **91**, 069905(E) (2015).
[22] A. Maselli, P. Pani, L. Gualtieri, and V. Ferrari, Phys. Rev. D **92**, 083014 (2015).
[23] B. Kleihaus, J. Kunz, and E. Radu, Phys. Rev. Lett. **106**, 151104 (2011).
[24] B. Kleihaus, J. Kunz, S. Mojica, and E. Radu, Phys. Rev. D **93**, 044047 (2016).
[25] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. Lett. **112**, 251102 (2014).
[26] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. D **90**, 124063 (2014).
[27] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. **120**, 131103 (2018).
[28] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, Phys. Rev. Lett. **120**, 131104 (2018).
[29] G. Antoniou, A. Bakopoulos, and P. Kanti, Phys. Rev. Lett. **120**, 131102 (2018).
[30] J. L. Blázquez-Salcedo, D. D. Doneva, J. Kunz, and S. S. Yazadjiev, Phys. Rev. D **98**, 084011 (2018).
[31] M. Minamitsuji and T. Ikeda, Phys. Rev. D **99**, 044017 (2019).
[32] H. O. Silva, C. F. B. Macedo, T. P. Sotiriou, L. Gualtieri, J. Sakstein, and E. Berti, Phys. Rev. D **99**, 064011 (2019).
[33] C. F. B. Macedo, J. Sakstein, E. Berti, L. Gualtieri, H. O. Silva, and T. P. Sotiriou, Phys. Rev. D **99**, 104041 (2019).
[34] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. D **105**, L041502 (2022).

[35] T. Damour and G. Esposito-Farese, Phys. Rev. Lett. **70**, 2220 (1993).
[36] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[37] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Prog. Theor. Phys. **126**, 511 (2011).
[38] M. Rinaldi, Phys. Rev. D **86**, 084048 (2012).
[39] A. Anabalon, A. Cisterna, and J. Oliva, Phys. Rev. D **89**, 084050 (2014).
[40] M. Minamitsuji, Phys. Rev. D **89**, 064017 (2014).
[41] A. Cisterna and C. Erices, Phys. Rev. D **89**, 084038 (2014).
[42] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos, and G. Siopsis, Classical Quantum Gravity **29**, 205011 (2012).
[43] M. Minamitsuji, Gen. Relativ. Gravit. **46**, 1785 (2014).
[44] M. Minamitsuji, K. Takahashi, and S. Tsujikawa, Phys. Rev. D **105**, 104001 (2022).
[45] D. Glavan and C. Lin, Phys. Rev. Lett. **124**, 081301 (2020).
[46] H. Lu and Y. Pang, Phys. Lett. B **809**, 135717 (2020).
[47] T. Kobayashi, J. Cosmol. Astropart. Phys. 07 (2020) 013.
[48] P. G. S. Fernandes, P. Carrilho, T. Clifton, and D. J. Mulryne, Classical Quantum Gravity **39**, 063001 (2022).
[49] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. D **85**, 084025 (2012); **96**, 109903(E) (2017).
[50] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. D **89**, 084042 (2014).
[51] R. Kase and S. Tsujikawa, Phys. Rev. D **105**, 024059 (2022).
[52] S. Tsujikawa, Phys. Lett. B **833**, 137329 (2022).
[53] M. Minamitsuji, K. Takahashi, and S. Tsujikawa, Phys. Rev. D **106**, 044003 (2022).
[54] E. Babichev and C. Charmousis, J. High Energy Phys. 08 (2014) 106.
[55] T. Kobayashi and N. Tanahashi, Prog. Theor. Exp. Phys. **2014**, 073E02 (2014).
[56] G. W. Gibbons and K.-i. Maeda, Nucl. Phys. **B298**, 741 (1988).
[57] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991); **45**, 3888(E) (1992).
[58] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual, and J. A. Font, Phys. Rev. Lett. **121**, 101102 (2018).
[59] P. G. S. Fernandes, C. A. R. Herdeiro, A. M. Pombo, E. Radu, and N. Sanchis-Gual, Classical Quantum Gravity **36**, 134002 (2019); **37**, 049501(E) (2020).
[60] Y. S. Myung and D.-C. Zou, Phys. Lett. B **790**, 400 (2019).
[61] J. L. Blázquez-Salcedo, C. A. R. Herdeiro, J. Kunz, A. M. Pombo, and E. Radu, Phys. Lett. B **806**, 135493 (2020).
[62] R. A. Konoplya and A. Zhidenko, Phys. Rev. D **100**, 044015 (2019).
[63] S. Mignemi and N. R. Stewart, Phys. Rev. D **47**, 5259 (1993).
[64] T. Torii, H. Yajima, and K.-i. Maeda, Phys. Rev. D **55**, 739 (1997).
[65] S. Alexeyev, A. Barrau, and K. A. Rannu, Phys. Rev. D **79**, 067503 (2009).
[66] P. G. S. Fernandes, Phys. Lett. B **805**, 135468 (2020).
[67] D. L. Wiltshire, Phys. Lett. **169B**, 36 (1986).
[68] D. L. Wiltshire, Phys. Rev. D **38**, 2445 (1988).
[69] R.-G. Cai, Phys. Rev. D **65**, 084014 (2002).
[70] A. A. Tseytlin, Nucl. Phys. **B276**, 391 (1986); **B291**, 876(E) (1987).
[71] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **163B**, 123 (1985).
[72] A. Abouelsaood, C. G. Callan, Jr., C. R. Nappi, and S. A. Yost, Nucl. Phys. **B280**, 599 (1987).
[73] M. Born and L. Infeld, Proc. R. Soc. A **144**, 425 (1934).
[74] S. Fernando and D. Krug, Gen. Relativ. Gravit. **35**, 129 (2003).
[75] R.-G. Cai, D.-W. Pang, and A. Wang, Phys. Rev. D **70**, 124034 (2004).
[76] T. K. Dey, Phys. Lett. B **595**, 484 (2004).
[77] G. Clement and D. Gal'tsov, Phys. Rev. D **62**, 124013 (2000).
[78] T. Tamaki and T. Torii, Phys. Rev. D **64**, 024027 (2001).
[79] S. S. Yazadjiev, Phys. Rev. D **72**, 044006 (2005).
[80] A. Sheykhi, N. Riazi, and M. H. Mahzoon, Phys. Rev. D **74**, 044025 (2006).
[81] I. Z. Stefanov, S. S. Yazadjiev, and M. D. Todorov, Phys. Rev. D **75**, 084036 (2007).
[82] R. Gannouji and Y. R. Baez, J. High Energy Phys. 02 (2022) 020.
[83] R. Kase, R. Kimura, S. Sato, and S. Tsujikawa, Phys. Rev. D **102**, 084037 (2020).
[84] R. Kase and S. Tsujikawa, J. Cosmol. Astropart. Phys. 01 (2021) 008.
[85] M. Minamitsuji and S. Tsujikawa, Phys. Rev. D **106**, 064008 (2022).
[86] L. Heisenberg, J. Cosmol. Astropart. Phys. 05 (2014) 015.
[87] G. Tasinato, J. High Energy Phys. 04 (2014) 067.
[88] J. Beltran Jimenez and L. Heisenberg, Phys. Lett. B **757**, 405 (2016).
[89] J. Chagoya, G. Niz, and G. Tasinato, Classical Quantum Gravity **33**, 175007 (2016).
[90] M. Minamitsuji, Phys. Rev. D **94**, 084039 (2016).
[91] E. Babichev, C. Charmousis, and M. Hassaine, J. High Energy Phys. 05 (2017) 114.
[92] L. Heisenberg, R. Kase, M. Minamitsuji, and S. Tsujikawa, Phys. Rev. D **96**, 084049 (2017).
[93] L. Heisenberg, R. Kase, M. Minamitsuji, and S. Tsujikawa, J. Cosmol. Astropart. Phys. 08 (2017) 024.
[94] R. Kase, M. Minamitsuji, S. Tsujikawa, and Y.-L. Zhang, J. Cosmol. Astropart. Phys. 02 (2018) 048.
[95] R. Kase, M. Minamitsuji, and S. Tsujikawa, Phys. Lett. B **782**, 541 (2018).
[96] Y. R. Baez and M. Gonzalez-Espinoza, Classical Quantum Gravity **40**, 085017 (2023).
[97] S. Garcia-Saenz, A. Held, and J. Zhang, Phys. Rev. Lett. **127**, 131104 (2021).
[98] A. De Felice, L. Heisenberg, R. Kase, S. Tsujikawa, Y.-l. Zhang, and G.-B. Zhao, Phys. Rev. D **93**, 104016 (2016).
[99] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[100] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).
[101] A. De Felice, T. Suyama, and T. Tanaka, Phys. Rev. D **83**, 104035 (2011).
[102] H. Motohashi and T. Suyama, Phys. Rev. D **84**, 084041 (2011).
[103] R. Kase, L. A. Gergely, and S. Tsujikawa, Phys. Rev. D **90**, 124019 (2014).
[104] C. de Rham, J. Francfort, and J. Zhang, Phys. Rev. D **102**, 024079 (2020).
[105] R. Kase and S. Tsujikawa, J. Cosmol. Astropart. Phys. 11 (2018) 024.

[106] R. G. Leigh, Mod. Phys. Lett. A **04**, 2767 (1989).
[107] M. Cederwall, A. von Gussich, A. R. Mikovic, B. E. W. Nilsson, and A. Westerberg, Phys. Lett. B **390**, 148 (1997).
[108] G. W. Gibbons and D. A. Rasheed, Nucl. Phys. **B454**, 185 (1995).
[109] J. Beltran Jimenez, L. Heisenberg, G. J. Olmo, and D. Rubiera-Garcia, Phys. Rep. **727**, 1 (2018).
[110] D. J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
[111] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **158B**, 316 (1985).
[112] D. D. Doneva, S. Kiorpelidi, P. G. Nedkova, E. Papantonopoulos, and S. S. Yazadjiev, Phys. Rev. D **98**, 104056 (2018).
[113] C. L. Hunter and D. J. Smith, Int. J. Mod. Phys. A **37**, 2250045 (2022).
[114] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971).
[115] L. Heisenberg, J. Cosmol. Astropart. Phys. 10 (2018) 054.
[116] L. Heisenberg and S. Tsujikawa, Phys. Lett. B **780**, 638 (2018).
[117] T. Ikeda, T. Nakamura, and M. Minamitsuji, Phys. Rev. D **100**, 104014 (2019).
[118] L. Heisenberg, R. Kase, and S. Tsujikawa, Phys. Rev. D **97**, 124043 (2018).
[119] R. Gregory and J. A. Harvey, Phys. Rev. D **47**, 2411 (1993).
[120] J. H. Horne and G. T. Horowitz, Nucl. Phys. **B399**, 169 (1993).