Scalar fluid in scalar-tensor gravity from an equivalent picture
of thermodynamic and fluid descriptions of gravitational dynamics

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We revisit the thermodynamic description of fluid, represented by scalar field in scalar-tensor gravity theory through a general approach to study the thermodynamics of relativistic fluids. In order to identify the fluid energy-momentum tensor, contrary to the existing way, we use an equivalent description of gravitational dynamics both in Jordan and Einstein frames as a thermodynamical identity and fluid equation on a generic null surface. Such an approach provides the energy-momentum tensors for scalar fluid as that of an ideal fluid in both the frames. We then mention few issues in the existing way of using Eckart's formalism for an ideal fluid. Our investigation suggests that the Eckart's frame may not be suitable to consider an ideal fluid. Consequently a possible alternative description, valid in something other than Eckart's frame, is being suggested for a general ideal fluid. Based on this we obtain a unified thermodynamic description in both Jordan and Einstein frames from our identified energy-momentum tensors. Finally, the relations between thermodynamic entities on different frames are being put forwarded. Thus, the description in either of the frames is enough to provide the complete picture.

I. INTRODUCTION AND MOTIVATION

Among the four fundamental forces of nature, gravity still remains the most difficult one to understand at the very fundamental level. The present understanding of gravity, as provided by Einstein's theory of general relativity (GR) is based upon the description of spacetime geometry due to presence of matter. Irrespective of huge success of GR, both in the theoretical as well as experimental fronts, a quantum theory of gravity still remains elusive. On another frontier, it was found that [1–6] the dynamical equations of GR show similarity to the equations governing thermodynamic and fluid-dynamic systems. It is widely believed that in the absence of quantum theory of gravitation, the thermodynamic and the fluid-dynamic aspects of gravity may provide us with some alternative approaches/viewpoints to understand gravity in a more deeper sense.

As already mentioned above, despite huge successes of Einstein's GR, theories with modifications of GR appears to be important to incorporate various situations, like strong gravitational regime, present observational data, etc. Among them scalar-tensor (ST) theory of gravity is one of the popular ones and attracted a huge attention [7–12]. The theory is described by both metric tensor g_{ab} and a scalar field ϕ that is nonminimally coupled with Ricci

scalar R, known as the Jordan frame description. The gravitational action is given by

$$
\begin{aligned}
S_J &= \int d^4x \sqrt{-g} L \\
&= \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega(\phi)}{\phi} \nabla_a \phi \nabla^a \phi - V(\phi) \right).
\end{aligned} \tag{1.1}
$$

In the above, $\omega(\phi)$ is the Brans-Dicke parameter and $V(\phi)$ is an arbitrary potential. Such an action can be expressed in minimally coupled structure by using the following transformations:

$$
g_{ab} \to \tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \text{with} \quad \Omega = \sqrt{\phi}, \tag{1.2}
$$

and

$$
\phi \to \tilde{\phi} \quad \text{with} \quad d\tilde{\phi} = \sqrt{\frac{2\omega(\phi) + 3}{16\pi}} \frac{d\phi}{\phi}. \tag{1.3}
$$

This is known as the Einstein frame description, and in this case the action takes the form

$$
\begin{aligned}
S_{\text{EF}} &= \int d^4x \sqrt{-\tilde{g}} \tilde{L} \\
&= \int d^4x \sqrt{-\tilde{g}} \left(\frac{\tilde{\mathcal{R}}}{16\pi} - \frac{1}{2} \tilde{\nabla}_i \tilde{\phi} \tilde{\nabla}^i \tilde{\phi} - U(\tilde{\phi}) \right), \tag{1.4}
\end{aligned}
$$

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with the nonzero potential $U = V(\phi)/16\pi\phi^2$. The Lagrangians (densities) in both the frames are related to each other by a total derivative term:

$$\sqrt{-\tilde{g}}\tilde{L} = \sqrt{-g}L - (3/16\pi)(\sqrt{-g}\Box\phi),$$

which is unimportant as far as dynamics are concerned. Consequently, the equations of motion for metric and scalar field appears to be equivalent in both the frames (a detailed discussion can be found in [13–17]). Despite this mathematical equivalence, it appears that certain physical properties in different frames disagree with each other, while others agree (see, e.g., [14,15,17–24]). Therefore it remains a "dilemma" to give preference to a particular frame [13,25,26].

Recently, a series of papers [27–29] have appeared in the literature that treat the ST theory in terms of the thermodynamics of an effective scalar ϕ fluid. The original discussion [27,28] was done in the Jordan frame where the energy-momentum (EM) tensor for fluid has been identified by comparing the equation of motion for metric tensor with that in nonvacuum GR theory. Such a tensor depends on the higher derivative of scalar field and so can be compared with that of a dissipative fluid. The authors constructed the thermodynamics of this case by using Eckart's first order thermodynamic formalism [30] for a dissipative fluid. The central idea about this investigation is as follows: it is known that (vacuum) GR corresponds to the "equilibrium thermodynamic state" as it does not incorporate any dissipative term when it is viewed as an emergent theory from a thermodynamic description [5]. While a modified theory of gravity is usually described through dissipative phenomenon [e.g., see [6] for $f(R)$ gravity]. So usually the modified gravity theories belongs to the non-equilibrium thermodynamical state. Hence there must exists a thermodynamic limit that describes how a modified theory propagates to GR. In this formalism it is found that the vanishing temperature of ϕ fluid leads to a GR equilibrium state [28]. Following this idea the same set of authors investigated the thermodynamics of scalar fluid in Einstein's frame as well [29]. Their approach is inherently under the inequivalent picture, and therefore the EM tensor in the latter case is different from that in the Jordan frame. They showed that in the Einstein frame the GR limit is obtained by imposing vanishing chemical potential. These investigations naturally deliver a question to ask: Can such an aspect of ST theory can have a frame-independent description? If so then a universal limit can be fixed to approach the GR equilibrium description. In this paper we aim to investigate this possibility through a general approach to the thermodynamics of relativistic fluids. In this regard we mention that an earlier analysis was done within Eckart's formalism. Unfortunately this one suffers from an acausal nature. An important improvement has been done by Israel and Stewart [31,32] to make the

theory consistent with causality. The defined parameters in both the formalisms differ by terms related to the viscous and heat flux of EM tensor. However for an ideal fluid EM tensor these two formalisms coincide and hence we do not need to worry about this issue. Therefore, for our present case, as the EM tensor will be as that of an ideal fluid, we can use any formalism without the loss of any physical restriction.

In order to have an equivalent description, the central aim will be to identity an effective scalar field fluid EM tensor that satisfies the usual relation (i.e., $\tilde{T}_{ab}^{(\tilde{\phi})} = T_{ab}^{(\phi)}/\phi$) under the transformations (1.2) and (1.4). Naturally, an earlier way to read off such a quantity cannot fulfil the requirement. So we adopt a different way to find the fluid description. Within various aspects of ST theory, the thermodynamic description of gravity on a Killing horizon (e.g., event horizon of a stationary black hole) appears to be equivalent in Jordan and Einstein frames (see [33] for initial attempt and [15] for an improved approach). The geometrical definitions of thermodynamic entities (like temperature, energy, entropy, work term) in one frame boils down to their counterparts in the other frame by the transformations (1.2) and (1.3). Spirited by this fact, a thermodynamic description of field equation for metric tensor on a generic null surface (generated through normal as well as tangent vector l^a) has been investigated within the equivalent picture. A particular projection of this equation together with a virtual displacement along the auxiliary null vector captures a thermodynamic identity in both the frames, which is of the form

$$\int_{S_t} d^2x T\delta_\lambda s = \delta_\lambda E + \delta_\lambda W, \qquad (1.5)$$

where T, s, E are temperature, entropy density, and energy, respectively. S_t denotes the cross section of the null surface and λ parametrizes the auxiliary null vector k^a (for details, see [24]; for a review on this topic, see [17]). The last one is the work term and is defined through $\delta_\lambda W = -\int_{S_t} \sqrt{q}\delta\lambda\tilde{T}_{ab}^{(\tilde{\phi})}\tilde{l}^a\tilde{k}^b$ in the Einstein frame.[1] Here $\tilde{T}_{ab}^{(\tilde{\phi})}$ is given by

$$\tilde{T}_{ab}^{(\tilde{\phi})} = \tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{2}\tilde{g}_{ab}\tilde{\nabla}_i\tilde{\phi}\tilde{\nabla}^i\tilde{\phi} - \tilde{g}_{ab}U(\tilde{\phi}). \qquad (1.6)$$

In this equivalent description the same work term appears in a Jordan frame as well, and in this case the energy-momentum tensor for the scalar field takes the form

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[1] Here we denote the tilde variables as Einstein frame quantities and without tilde ones as Jordan frame quantities. This notation, wherever necessary, will be followed.

$$T_{ab}^{(\phi)} = \phi \tilde{T}_{ab}^{(\tilde{\phi})}$$
$$= \phi \left[\left(\frac{2\omega + 3}{16\pi} \right) \left\{ \nabla_a (\ln \phi) \nabla_b (\ln \phi) \right. \right.$$
$$\left. \left. - \frac{1}{2} g_{ab} \nabla_i (\ln \phi) \nabla^i (\ln \phi) \right\} - g_{ab} \frac{V(\phi)}{16\pi\phi} \right]. \quad (1.7)$$

It may be noted that up to an overall multiplicative factor ϕ, (1.6) and (1.7) are same under the transformations (1.2) and (1.3).

We now note that the work term in the thermodynamic identity is usually defined through the external pressure $P = -T_{ab}^{(m)} l^a k^b$ (see discussion in [34,35]), where $T_{ab}^{(m)}$ is the EM tensor for the external matter field as it contributes in the work term. Therefore following the above discussion it is natural to think (1.6) or (1.7) corresponds to the ϕ fluid in ST theory. Interestingly, such an analogy can be well supported by following the fluid interpretation of dynamical equation of ST theory. It is well known that in GR a specific projection of Einstein's equation of motion on a generic null surface yields a Naiver-Stokes-like equation [3,36–39]. The same has been investigated in ST theory as well [23]. It has been observed that if one demands an equivalent picture for fluid variables in both the frames then (1.6) and (1.7) provides the external forcing terms in Einstein and Jordan frames, respectively (see [17,23] for details). Looking at these facts we consider (1.6) or (1.7) as the EM tensor for the ϕ fluid in ST theory. Moreover, these forms are equivalent and applied in both the frames. Therefore a thermodynamic description of this fluid captures the spirit of the equivalent picture.

It is worth to mention that in [24] the authors discussed the thermodynamic interpretation in Einstein frame in which their approach identifies the EM tensor as (1.6). Therefore according to our above discussion such an analysis can be considered as an equivalent description. However we have a few observations on the previous analysis that need to be further investigated. We mention them in the next discussion. This will formulate the basis of our present investigation and hopefully will lead to a fruitful description of thermodynamics of ϕ fluid in a frame independent setup.

Therefore, in order to set up the plot, in next couple of sections, we first briefly discuss the thermodynamics of relativistic fluids and then mention our observations on the previous works [27–29]. In the third section, we present the present status of ST theories via fluid thermodynamic viewpoints. In Sec. IV, in order to further proceed towards our goal, we motivate the reader towards a new EM tensor. In the next section, we develop a general formalism to describe a mapping between a minimally coupled scalar field and corresponding fluid/thermodynamic quantities. Using the same mapping, the fluid/thermodynamic quantities for our case (in both frames) are found in this section

as well. Finally, in Sec. VI, we conclude the paper with some future directions.

II. SUMMARY OF THE THERMODYNAMICS OF RELATIVISTIC FLUIDS

Consider a fluid with four-velocity u^a satisfying $u_a u^a = -1$ moving on a spacetime manifold $(\mathcal{M}, g_{ab}, \nabla)$. We choose the fluid velocity along time direction and foliate the spacetime into (spacelike) hypersurfaces orthogonal to flow lines. The projector onto these hyper-surfaces as well as the induced metric is then given by

$$h_{ab} = g_{ab} + u_a u_b. \quad (2.1)$$

The acceleration $\dot{u}_a = u^b \nabla_b u^a$ is perpendicular to the velocity, i.e., $\dot{u}_a u^a = 0$. In general a viscous fluid is described as the following energy-momentum tensor:

$$T_{ab} = \underbrace{\rho u_a u_b + p_{\text{iso}} h_{ab}}_{T_{ab}^{\text{ideal}}} + \underbrace{q_a u_b + q_a u_b}_{T_{ab}^{\text{heat}}}$$
$$+ \underbrace{p_{\text{vis}} h_{ab} + \pi_{ab}}_{T_{ab}^{\text{vis}}}, \quad (2.2)$$

which is covariantly conserved, i.e., $\nabla_a T^{ab} = 0$. Here $\rho, p_{\text{iso}}, q_a, p_{\text{vis}}$, and π_{ab} are the energy density, isotropic pressure, heat flux, viscous pressure, and traceless parts of the viscous stress tensor (i.e., $\pi_a^a = 0$), respectively. q_a is chosen to be a spatial vector, i.e., $u^a q_a = 0$, and also we have $u^a \pi_{ab} = 0$. Choosing the viscous stress tensor is linear in the velocity gradients and proportional to the (scalar) coefficient of viscosity η; the suggested form for π_{ab} is given by

$$\pi_{ab} = -\eta \left\{ h_{ap} h_{bq} (\nabla^q u^p + \nabla^p u^q) - \frac{2}{3} h_{ab} \nabla_p u^p \right\}$$
$$= -2\eta \sigma_{ab}. \quad (2.3)$$

In the above σ_{ab} is known as shear. The "bulk viscosity" ζ is defined through p_{vis}, which is related to the trace part of the viscous stress tensor:

$$p_{\text{vis}} = -\zeta \theta, \quad (2.4)$$

where $\theta = \nabla_a u^a$, which is called an expansion. Apart from the conservation energy-momentum tensor (2.2), the fluid dynamics is supplemented by the conservation of the mass current ($\nabla_a m^a = 0$) and the conservation of the total number current ($\nabla_a N^a = 0$), where $m^a = m u^a$ with $m = \sqrt{-m_a m^a}$ is the mass density and

$$N^a = n^a + \nu^a. \quad (2.5)$$

In the above, one has $n^a = n u^a$ and ν^a is known as a diffusion flux. By construction $u_a \nu^a = 0$; i.e., the diffusion flux ν^a is

spatial, so the number density can be obtained as $n = -u_a N^a$. The total number current is conserved, i.e., $\nabla_a N^a = 0$, so we can define the fluid chemical potential μ.

An approach to understand the (first order) thermodynamics of an imperfect fluid is using Eckart's formalism [30]. This formalism provides a relation between the thermodynamic parameters and the fluid parameters under the input that the viscous fluid satisfies the usual laws of thermodynamics. Although originally the formalism was developed in a flat background we work in some arbitrary curved spacetime with metric signature $(-+++)$. Also we will include a particle exchange that was missing in the original analysis. The whole generalization has been done in [40]. Here, we will briefly state the results. The entropy current in the system can be written as $s^a = su^a + \beta q^a - \lambda \nu^a$, where s is the entropy density and β, λ are unknown parameters. These parameters are fixed by imposing the second law $\nabla_a s^a \geq 0$ along with the use of thermodynamic identity (written in per unit volume)

$$\nabla_a \rho = T \nabla_a s + \mu \nabla_a n, \qquad (2.6)$$

and Euler's relation

$$\rho + p_{\text{iso}} = Ts + \mu n. \qquad (2.7)$$

Here T is the temperature of the fluid. Such an imposition fixes those unknown parameters as

$$\beta = \frac{1}{T}, \qquad \lambda = \frac{p+\rho}{nT} - \frac{s}{n} = \frac{\mu}{T}, \qquad (2.8)$$

along with a particular choices of q_a and ν_a. Thus, the salient features of extended-Eckart's formalism are summarized as follows:

$$q^a = -\kappa h^{ab}(\nabla_b T + \dot{u}_b T), \qquad (2.9)$$

$$\pi^{ab} = -2\eta \sigma^{ab}, \qquad (2.10)$$

$$p_{\text{vis}} = -\zeta \theta, \qquad (2.11)$$

$$\nu^a = -\sigma T^2 h^{ab} \nabla_b \left(\frac{\mu}{T}\right). \qquad (2.12)$$

Here κ is heat conductivity and σ is known as a diffusion coefficient. Also remember that $\kappa, \sigma \geq 0$. The above ones relate the thermodynamic parameters of a viscous fluid with its fluid parameters (the details are given in [40]). In the original construction [30], the bulk viscosity ζ was taken to be zero and the velocity u^a was taken to be the velocity of the matter, i.e., the fluid. This is a particular choice of a local rest frame (the Eckart frame), and in this choice $\nu^a = 0$. If we choose a particular rest frame, then we may have to impose conditions on the above quantities that arise from the physical

restrictions of the rest frames. However, as of yet the above description and (2.9)–(2.12) are much more general.

Although Eckart's formalism is a fundamental approach to studying the thermodynamics of relativistic fluids, the theory has limitations since it does not account for causality and exhibits instability issues. Extensions of this theory, such as those by Israel and Stewart [31,32], aim to fix these issues. The Israel-Stewart formalism is then a stable, causal model to study relativistic fluids. In this approach, the fundamental fluid quantities (2.9)–(2.12) are modified to include second order perturbations. This modification comes from terms describing dissipation in the fluid itself. In principle, one must use this more general approach when studying relativistic fluids. However in the absence of dissipation, i.e., when studying ideal fluids, Eckart's formalism is sufficient as both the formalisms coincide in this limit. In our analysis we consider an EM tensor that has ideal fluid structure [see Eqs. (1.6) and (1.7)]. Therefore although the explicit relationship between temperature and the dissipative quantities may be modified by using the Israel-Stewart formalism (see [32]), in our analysis these modifications will vanish since they will depend on dissipative quantities that are absent in our problem. Therefore, the two approaches yield identical results and, hence, the relations (2.9)–(2.12) can be used in this context.

III. STATUS QUO OF SCALAR-TENSOR GRAVITY VIA FLUID THERMODYNAMICS

In a series of recent papers by Faraoni *et al.* [27–29], scalar-tensor gravity is described through the thermodynamics of a fluid, employing Eckart's first order thermodynamics formalism [30]. In these papers, they use Eckart's formalism to consider scalar-tensor gravity both in the Jordan and the Einstein frame and provide one approach to reconcile the differences between the two frames. However, this analysis requires a few inputs and logistics that may be put under a scanner. We elucidate them below.

(1) Fundamental "discrepancy" between the fluid energy-momentum tensors: Beginning with the appropriate action in either frame, the authors write down the field equations in such a way so as to treat any new terms apart from the Einstein tensor in the field equations to be a part of the fluid energy-momentum tensor. That is, the field equations are written as $G_{ab} = T_{ab}^{(\phi)}$ and $\tilde{G}_{ab} = \tilde{T}_{ab}^{(\phi)}$, where we use tilde variables to indicate the Einstein frame quantities. Here, G_{ab} is the usual Einstein tensor and any extra terms that appear due to the terms apart from the Einstein Hilbert action are grouped into $T_{ab}^{(\phi)}$. However, this naively motivated way of identifying an EM tensor for the fluid lacks any physically motivated argument, and moreover the EM tensor in both the frames appears to be inequivalent; i.e., they are not related by the transformations (1.2) and (1.3)

between the Einstein and the Jordan frames. Furthermore, this inequivalence manifests into the thermodynamic description that follows, as we will point out in the next discussion.

(2) Absence of a single thermodynamic description between the frames: In [27,28], the authors discuss scalar-tensor gravity in the Jordan frame. The aforesaid way of identification of the EM tensor for the ϕ fluid incorporates a viscous part. In this process, they obtain a nonzero heat flux [41,42] from which they eventually obtain the "temperature of gravity" using Eckart's heat flux. However, a crucial assumption used in this analysis is the vanishing spatial temperature gradient, i.e., $h^{ab}\nabla_b T = 0$, which is used to find out the specific conductivity and the temperature, and ultimately shows that the GR limit corresponds to a $T = 0$ state for the fluid. In this analysis, the number density n, the chemical potential μ, and the entropy density s are not discussed and therefore they are not computed.

However, in [29], the authors apply Eckart's formalism and use the existing dictionary between perfect fluids and minimally coupled scalar fields in [43] to study the Einstein frame description of scalar-tensor gravity. Here, they demand that the temperature must vanish identically for Eckart's formalism to be consistent with the vanishing heat flux for the perfect fluid. Now to provide a mechanism in obtaining GR as some stable thermodynamic limit, they introduce the chemical potential μ, which leads to a vanishing diffusion flux:

$$\nu^a = -\mathcal{D}h^{ab}[\nabla_b\mu + \dot{u}_b\mu] = 0, \qquad (3.1)$$

where $\mathcal{D}$ is the diffusion coefficient. Inspired by Eckart's heat flux (2.9) and the work in [44], the authors chose the above form for ν^a. Note that this form is different from (2.12) by the last term. The chemical potential is such that the GR limit corresponds to vanishing chemical potential. Thus, the GR limit corresponds to $T = 0$ in the Jordan frame, while $\mu = 0$ in the Einstein frame.

The above two points, therefore, indicate an inequivalent description of thermodynamics between the two frames.

(3) Ambiguity in the choice of ν^a: Within this inequivalent description a comment can be made on the choice (3.1) for the diffusion flux. This form of the diffusion flux does not guarantee consistency with the second law of thermodynamics except the Eckart's frame in which $\nu^a = 0$. In calculating the divergence of the entropy current $\nabla_a s^a$, the contribution due to the diffusion flux is given by $-\nu^a\nabla_a\lambda$ [see Eq. (282) of [40]]. Then for the choice (3.1) it yields

$$-\nu^a\nabla_a\lambda = \frac{\nu^a\nu_a}{\mathcal{D}} + \mathcal{D}\mu\nu^a\dot{u}_a. \qquad (3.2)$$

While the first term on the right-hand side is always positive since ν^a is spacelike, the second term is ambiguous. Although the two vectors are individually spacelike, the contraction may not be positive. While a vanishing ν^a can be considered (as done in Eckart's frame [40]), this also may face an issue as described below, particularly for a perfect fluid. Moreover, this choice dose not guarantee the positivity of $\nabla_a s^a$ in other frames.

(4) Eckart frame and a vanishing diffusion current: Conventionally, choosing the Eckart frame implies that the velocity is taken to be along the flow of the fluid particles. Thus, the diffusion current must vanish, i.e., $\nu^a = 0$. With the choice of μ in [29] [i.e., Eq. (3.1)] this is indeed the case. However, the vanishing of the diffusion flux places restrictions on the theory under consideration, particularly in Einstein's frame. In the Einstein frame, if one considers the number currently defined naively as nu^a, then it is no longer covariantly conserved due to the dynamics of the field ϕ. Consider an action of the form

$$S = \int \sqrt{-g}d^4x\mathcal{L}(\psi, X), \qquad (3.3)$$

with $X = -(1/2)\nabla_i\psi\nabla^i\psi$. The Einstein frame ST gravity action is an example of this form of action. For this class of actions, the energy momentum tensor is given by

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = \mathcal{L}_X\nabla_a\psi\nabla_b\psi + \mathcal{L}g_{ab} \qquad (3.4)$$

and is covariantly conserved, i.e., $\nabla^a T_{ab} = 0$. Then, we can write down the Euler Lagrange equation for the field ψ as

$$\nabla_a(nu^a) = -\mathcal{L}_\phi, \qquad (3.5)$$

where $\mathcal{L}_\phi = \partial\mathcal{L}/\partial\phi$. The chemical potential defined in this context is ambiguous. This can be remedied by modifying the number current to include the diffusion flux [40], such that $N^a = nu^a + \nu^a$ with $\nabla_a N^a = 0$. Thus one finds

$$\nabla_a(nu^a) = -\nabla_a\nu^a = -\mathcal{L}_\phi, \qquad (3.6)$$

and therefore the imposition of $\nu^a = 0$ leads to $\mathcal{L}_\phi = 0$. This suggests that the Lagrangian must be independent of ϕ. In other words, the conservation of nu^a is connected to the fact that the

Lagrangian must be independent of ϕ. This type of Lagrangian usually must be independent of the potential term and then we could have a Lagrangian that only contains the kinetic terms involving X. However, ST theory in the Einstein frame in a general framework is given by the action (1.4). This gives an energy momentum tensor for ϕ as (1.6). Note that this one resemblances that of a perfect fluid. In that case, if $\mathcal{L}_\phi = 0$, then the Lagrangian being independent of ϕ would imply that the above potential term in the Lagrangian must not exist. It would refer to describing a minimally coupled free scalar field in curved spacetime. Hence, this is incompatible with considering a general ST theory in the Einstein frame.

Observe that we obtain this restriction on Lagrangian directly comes from imposition of $\nu^a = 0$, which amounts to choosing the Eckart frame. Thus it seems that, in general to describe minimally coupled scalar fields whose EM tensor is identical to that of a perfect fluid, the Eckart frame is not the perfect choice to describe its thermodynamics.

Thus, the description presented so far has a few limitations and needs to revisited so as to obtain an equivalent picture from a thermodynamic and gravitational perspective.

IV. MOTIVATING A NEW EM TENSOR

We want a description of the thermodynamics of ST gravity that is equivalent between the two frames so as to provide such an interpretation for the gravity side. To do so, we first realize that the scalar field in ST theory contributes to the description of gravity, as well as to describe the ϕ fluid. We adopt the approach used in [17,24] where we use the fact that projecting the gravitational field equations onto a null surface can give us an equation similar in structure to the first law, from which we identify an EM tensor such that the individual thermodynamic quantities between the two frames are in correspondence. Doing so provides us with the EM tensor (1.7). The details of the derivation of this are given in [24]. Such an identification, as mentioned in the introduction, is also bolstered from fluid description of ST gravitational theory in an equivalent platform [23]. Interestingly, such an EM tensor up to an over all factor boils down to (1.6) under the transformations (1.2) and (1.3), and moreover the Einstein frame EM tensor structure can be derived from the last two terms of (1.4) and hence is covariantly conserved ($\tilde{\nabla}^a \tilde{T}_{ab}^{(\bar{\phi})} = 0$). What is important is that this is the EM tensor that emerges in both frames if we define it while keeping in mind the thermodynamic equivalence of the gravitational theory; any analysis performed using this $\tilde{T}_{ab}^{(\bar{\phi})}$ holds in both frames. Thus, relying on thermodynamic equivalence naturally leads with a hint of gravitational equivalence, at least at the level of the EM tensor. Further, note that the structure of (1.6) is similar to that of a perfect fluid. Since this way of identified EM tensor is the same in both frames, it suffices us to perform all our analysis in the Einstein frame description. Also note that the action for this EM tensor can be considered as the last two terms in (1.4), which falls within the general class of actions considered in [29]. However, performing the same mapping will still not be useful, even with this modified $T_{ab}^{(\phi)}$ due to a vanishing diffusion flux. As mentioned earlier, since $\nabla_a \nu^a = \mathcal{L}_\phi$, a vanishing ν^a would mean that the Lagrangian is independent of ϕ. Thus, the Eckart frame may not be the ideal choice to describe a perfect fluid.

In the following section we will develop a general formalism to describe a mapping between a minimally coupled scalar field and corresponding fluid/thermodynamic quantities. This can then be used to study the particular example of the ϕ fluid in the Einstein frame ST theory. We will then discuss how we can also obtain the Jordan frame picture without additional work.

V. A POSSIBLE ALTERNATE WAY OUT

We continue to consider the general class of actions for a minimally coupled scalar field since it is sufficient to demonstrate this approach at least in the Einstein frame. Moreover, our identified tensor (1.6) belongs to this class as well. We therefore consider a minimally coupled scalar field ψ with a general Lagrangian density $\mathcal{L}$ whose action is (3.3). The energy momentum tensor for this field can be derived as (3.4) with the notation $\mathcal{L}_X = \partial \mathcal{L}/\partial X$ and $\mathcal{L}_\psi = \partial \mathcal{L}/\partial \psi$. Note that T_{ab} is conserved.

Comparing (3.4) with (2.2), we have the following preliminary dictionary:

$$p = p_{\text{iso}} = \mathcal{L}; \quad u_a = \frac{\nabla_a \psi}{\sqrt{2X}}; \quad \rho = 2X\mathcal{L}_X - \mathcal{L}. \quad (5.1)$$

We also see that there is no term corresponding to a heat flux or viscous stress, i.e., $p_{\text{vis}} = 0$, $\pi_{ab} = 0$, and $q_a = 0$. This is reasonable because we do not expect a minimally coupled scalar field to generate any dissipation. Thus, the scalar field is mapped to a perfect fluid. Now from thermodynamics, the enthalpy is defined as $h = \frac{\rho + p}{n}$. We choose the flow potential normalization through the enthalpy as done in [43]. That is to say, we choose

$$h u_a = \nabla_a \psi. \quad (5.2)$$

Then (5.1) yields

$$n = \sqrt{2X}\mathcal{L}_X. \quad (5.3)$$

So far, what we have mentioned about the fluid variables in terms of field ψ is exactly identical to that in [29]. If we

now look at the dynamics of ψ we might be able to describe the dynamics of the fluid through our growing dictionary. From the dynamical equations of the field, we have $\nabla_a(\mathcal{L}_X \nabla^a \psi) = \nabla_a(nu^a) = -\mathcal{L}_\psi$. Thus, we see that particle number is not conserved in this fluid. This suggests that there is some diffusion, and our definition of the chemical potential for the fluid must be connected to this diffusion. Such a situation has been taken into account in [40] to provide a general version of Eckart's formulation. This we already mentioned in Sec. II. In this case the number current is given by (2.5). Then the conservation it yields is

$$\nabla_a \nu^a = -\nabla_a(nu^a) = \mathcal{L}_\psi. \tag{5.4}$$

Moreover, we can now follow the dictionary (2.9)–(2.12) to find the thermodynamic quantities of our scalar fluid. Then since the heat flux q^a vanishes, we have

$$q^a = -\kappa h^{ab}[\nabla_b T + \dot{u}_b T] = 0. \tag{5.5}$$

To satisfy the above criteria, in [29] $T = 0$ has been chosen. If we choose $T = 0$, then the present choice of ν^a [given by Eq. (2.12)] yields

$$\nu^a = \sigma\mu h^{ab}\nabla_b T - \sigma T h^{ab}\nabla_b \mu = 0. \tag{5.6}$$

Remember that even the choice (3.1), adopted in [29], also implies the vanishing of ν^a. But as discussed earlier $\nu^a = 0$ with (5.4) implies that the Lagrangian is independent of ψ, which is in contradiction to the present situation as the Lagrangian explicitly depends on ψ. Moreover in Eckart's frame one has $\nu^a = 0$. Therefore this discussion seems to indicate that the present system cannot be well described in this frame and $T = 0$ may not be a good choice. Hence we propose an alternative choice for T to satisfy (5.5). In order to work in the paradigm of equilibrium thermodynamics, we demand that the temperature be constant and positive definite. Hence, we take $T = T_0$, a constant, and $T_0 > 0$. From (5.5), this would mean that

$$-\kappa \dot{u}^a T_0 = 0. \tag{5.7}$$

We choose $\kappa = 0$, i.e., vanishing thermal conductivity. This would mean that we consider a fluid that is at constant temperature but is a perfect insulator. This can be considered as a consequence of the fact that we have a perfect fluid that does not allow a heat flux. However, with a nonzero temperature, we can proceed to define a nonvanishing diffusion flux. Since the temperature is a constant, (2.12) implies

$$\nu^a = -\mathcal{D}h^{ab}\nabla_a \mu, \tag{5.8}$$

where $\mathcal{D} = \sigma T$ is taken to be the (scaled) diffusion coefficient. Note that this is the first term in the choice (3.1).

With this choice, we now want to find a closed form expression for the chemical potential. To do so, we try to set up a differential equation for the chemical potential through the Euler relation (2.7). A simple computation yields

$$h^{ab}\nabla_a \mu = h^{ab}\nabla_a(\sqrt{2X}). \tag{5.9}$$

A detailed derivation of the above equation is presented in the Appendix. A solution for μ can be taken as

$$\mu = \sqrt{2X} - f(\psi). \tag{5.10}$$

for some arbitrary f which is function of ψ only. Using the facts $\nabla_a \psi \propto u_a$ and $h^{ab}u_a = 0$ one can check that it satisfies (5.9). For $f = 0$ this yields $\mu = \sqrt{2X}$, which was obtained in [29]. But in the present case we can not choose $f = 0$ as this is not consistent with a nonzero temperature as shown below. When $f = 0$ then we have $\mu = \sqrt{2X}$ and then the Euler relation (2.7) along with (5.1) and (5.3) imply $Ts = 0$. Therefore, when $\mu = \sqrt{2X}$ and $s \neq 0$, we must have $T = 0$. Hence, we instead determine the chemical potential up to the function $f(\psi)$, given by (5.10). Also note that the chemical potential, depending on the values of $\sqrt{2X}$ and f, can be positive or negative. Then the Euler relation (2.7) fixes the entropy density as

$$s = \frac{\sqrt{2X}\mathcal{L}_X}{T}f(\psi) = \frac{\sqrt{2X}\mathcal{L}_X}{T_0}f(\psi). \tag{5.11}$$

To ensure that the entropy density is positive, we impose $f > 0$.

Finally, from the divergence of the diffusion, one can find the diffusion constant $\mathcal{D} = \sigma T_0$. This is done using Eq. (5.8). This yields

$$\nabla_a \nu^a = -\nabla_a(\mathcal{D}h^{ab}\nabla_b\mu)$$
$$= -\nabla_a[\mathcal{D}h^{ab}\nabla_a(\sqrt{2X})] = \mathcal{L}_\psi, \tag{5.12}$$

where the last equation is obtained using (5.4). Then the diffusion coefficient $\mathcal{D}$ is at least formally obtained as $\sigma(\psi, X) = \mathcal{D}(\psi, X)/T_0$ by solving the above equation. These results are summarized in Table I.

This dictionary is a general prescription. To work in the Einstein frame with the Einstein variables, one would simply have to let $\psi = \tilde{\phi}$ and $\nabla_a = \tilde{\nabla}_a$. All thermodynamic quantities are then computed as above with

$$\tilde{\mathcal{L}} = -(1/2)\tilde{\nabla}_i\tilde{\phi}\tilde{\nabla}^i\tilde{\phi} - U(\tilde{\phi}). \tag{5.13}$$

Using this Lagrangian and Table I, one obtains for the Einstein frame quantities as shown in Table II.

In this table we have defined $X_J = (-1/2) \times \nabla_a(\ln\phi)\nabla^a(\ln\phi)$ and for the Lagrangian $\tilde{\mathcal{L}}$ one has

TABLE I.   Mapping between fluid and scalar field variables.

Fluid and thermodynamic variable	Field theory variable
X	$(-1/2)\nabla_a\psi\nabla^a\psi$
p	$\mathcal{L}$
u_a	$\nabla_a\psi/\sqrt{2X}$
ρ	$2X\mathcal{L}_X - \mathcal{L}$
n	$\sqrt{2X}\mathcal{L}_X$
h	$\sqrt{2X}$
μ	$\sqrt{2X} - f(\psi)$
s	$\sqrt{2X}\mathcal{L}_X f(\psi)/T$
T	T_0 (a constant)

TABLE II.   Fluid quantities in the Einstein frame.

$$\tilde{u}_a = \tilde{\nabla}_a\tilde{\phi}/\sqrt{2\tilde{X}} = \sqrt{\phi}\nabla_a(\ln\phi)/\sqrt{2X_J}$$
$$\tilde{p} = \widetilde{\mathcal{L}} = (2\omega+3)/(16\pi\phi)X_J - V(\phi)/16\pi\phi^2$$
$$\tilde{\rho} = 2\tilde{X}\widetilde{\mathcal{L}}_{\tilde{X}} - \widetilde{\mathcal{L}} = (2\omega+3)/(16\pi\phi)X_J + V(\phi)/16\pi\phi^2$$
$$\tilde{n} = \sqrt{2\tilde{X}}\widetilde{\mathcal{L}}_{\tilde{X}} = \sqrt{(2\omega+3)/(16\pi\phi)2X_J}$$
$$\tilde{h} = \sqrt{2\tilde{X}} = \sqrt{(2\omega+3)/(16\pi\phi)2X_J}$$
$$\tilde{\mu} = \sqrt{2\tilde{X}} - \tilde{f}(\tilde{\phi}) = \sqrt{(2\omega+3)/(16\pi\phi)2X_J} - \tilde{f}(\phi)$$
$$\tilde{s} = \sqrt{2\tilde{X}}\widetilde{\mathcal{L}}_{\tilde{X}}\tilde{f}(\tilde{\phi})/\tilde{T} = \sqrt{(2\omega+3)/(16\pi\phi)2X_J}\tilde{f}(\phi)/T_0$$
$$\tilde{T} = \tilde{T}_0 \text{ (constant)}$$

$\widetilde{\mathcal{L}}_X = 1$. The second equality in this table is obtained using the conformal transformations (1.2) and (1.3).

We can use the dictionary given in Table I to determine the fluid variables in the Jordan frame as well. To do so, we first require a Lagrangian for the ϕ fluid in the Jordan frame. This can be achieved by the requirement that the corresponding action is same in both the frames, i.e.,

$$\int d^4x\sqrt{\tilde{g}}\widetilde{\mathcal{L}} = \int d^4x\sqrt{-g}\mathcal{L}_J. \tag{5.14}$$

This yields $\mathcal{L}_J = \phi^2\widetilde{\mathcal{L}}$. Hence from (5.13) we have

$$\mathcal{L}_J = -\frac{\phi}{2}\left(\frac{2\omega+3}{16\pi}\right)\nabla_a(\ln\phi)\nabla^a(\ln\phi) - \frac{V(\phi)}{16\pi}. \tag{5.15}$$

With this Lagrangian, we can also derive the EM tensor $T_{ab}^{(\phi)}$ in (1.7) by considering variations with respect to g^{ab}, the Jordan frame metric. Therefore, we identify this as the ϕ fluid Lagrangian in the Jordan frame. Using this Lagrangian and the general dictionary (shown in Table I), we can now compute the various thermodynamic quantities in the Jordan frame. This is done in Table III below.

First, in our formulation, we have chosen $T_J = T_0 = \tilde{T}_0$, which is a constant. This is because in our formulation we choose the temperature of scalar fluid as a constant value. Since $\tilde{T}_0$ is the temperature in Einstein frame, the same in Jordan frame should not be scaled by ϕ; otherwise it would

TABLE III.   Fluid quantities in the Jordan frame.

$$u_a^J = \nabla_a(\ln\phi)/\sqrt{2X_J}$$
$$p_J = \mathcal{L}_J$$
$$\rho_J = \phi(2\omega+3)/(16\pi)X_J + V(\phi)/16\pi$$
$$n_J = \sqrt{2X_J}\phi(2\omega+3)/(16\pi)$$
$$\mu_J = \sqrt{2X_J} - f(\phi)$$
$$h_J = \sqrt{2X_J}$$
$$s_J = \sqrt{2X_J}\phi(2\omega+3)/(16\pi)f(\phi)/T_0$$
$$T_J = T_0 = \tilde{T}_0$$

not be constant in the latter frame. Now, consider the following quantity:

$$\begin{aligned}
\tilde{X} &= -\frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} \\
&= -\frac{1}{2}\left(\frac{2\omega+3}{16\pi}\right)\frac{g^{ab}}{\phi}\nabla_a(\ln\phi)\nabla_b(\ln\phi) \\
&= \left(\frac{2\omega+3}{16\pi\phi}\right)X_J.
\end{aligned} \tag{5.16}$$

In a similar way, we can relate the other quantities:

$$u_a^J = \frac{\nabla_a(\ln\phi)}{\sqrt{2X_J}} = \frac{\tilde{u}_a}{\sqrt{\phi}}, \tag{5.17}$$

$$p_J = \mathcal{L}_J = \phi^2\widetilde{\mathcal{L}} = \phi^2\tilde{p}, \tag{5.18}$$

$$\rho_J = \phi\left(\frac{2\omega+3}{16\pi}\right)X_J + \frac{V(\phi)}{16\pi} = \phi^2\tilde{\rho}, \tag{5.19}$$

$$\begin{aligned}
n_J &= \sqrt{2X_J}\phi\left(\frac{2\omega+3}{16\pi}\right) \\
&= \phi\sqrt{\phi\left(\frac{2\omega+3}{16\pi}\right)}\sqrt{\left(\frac{2\omega+3}{16\pi\phi}\right)2X_J} \\
&= \phi\sqrt{\phi\left(\frac{2\omega+3}{16\pi}\right)}\tilde{n}.
\end{aligned} \tag{5.20}$$

Next in order to the validity of the Euler relations $\tilde{\rho} + \tilde{p} = \tilde{T}_0\tilde{s} + \tilde{\mu}\tilde{n}$ and $\rho_J + p_J = T_0s_J + \mu_Jn_J$ in the respective frames one must have $s_J = \phi^2\tilde{s}$ and $\mu_Jn_J = \phi^2\tilde{\mu}\tilde{n}$. From Tables II and III and (5.20) we have

$$\begin{aligned}
\mu_Jn_J &= \{\sqrt{2X_J} - f(\phi)\}\left\{\phi\sqrt{\phi\left(\frac{2\omega+3}{16\pi}\right)}\tilde{n}\right\} \\
&= \phi^2\left\{\sqrt{\left(\frac{2\omega+3}{16\pi\phi}\right)2X_J} - \sqrt{\left(\frac{2\omega+3}{16\pi\phi}\right)}f(\phi)\right\}\tilde{n}, \\
&= \phi^2\left\{\sqrt{2\tilde{X}} - \sqrt{\left(\frac{2\omega+3}{16\pi\phi}\right)}f(\phi)\right\}\tilde{n}.
\end{aligned} \tag{5.21}$$

Now the above will satisfy $\mu_J n_J = \phi^2 \tilde{\mu}\,\tilde{n}$ if the functions $\tilde{f}(\phi)$ and $f(\phi)$ are related as

$$\tilde{f}(\phi) = \sqrt{\frac{(2\omega + 3)}{(16\pi\phi)}} f(\phi). \qquad (5.22)$$

Thus, this fixes the relationship between the functions $\tilde{f}(\phi)$ and $f(\phi)$. Then the chemical potentials are related as

$$\begin{aligned} \mu_J &= \sqrt{2X_J} - f(\phi) \\ &= \left(\frac{2\omega + 3}{16\pi\phi}\right)^{-1/2} \left\{\sqrt{2\tilde{X}} - \tilde{f}(\phi)\right\} \\ &= \left(\frac{2\omega + 3}{16\pi\phi}\right)^{-1/2} \tilde{\mu}, \qquad (5.23) \end{aligned}$$

where we have used (5.16), (5.22), and Table II. Finally one can check, with the above identification for $f(\phi)$ that the entropy density satisfies the predicted relation $s_J = \phi^2 \tilde{s}$. We summarize these results in Table IV.

Having determined all the thermodynamic variables, we can consider the GR limit of the action. This would correspond to a constant field $\psi = \psi_0$. This we will discuss in the general frame work, i.e., with respect to the ψ field. Einstein or Jordan frame quantities can be obtained by using the above relations. In this case, we would have

$$\begin{aligned} p &= \mathcal{L} = \mathcal{L}_0, \qquad \rho = -p = -\mathcal{L}_0, \qquad s = 0, \\ T &= \text{constant by construction}; \qquad \mu = -f(\psi_0) = -f_0 < 0, \end{aligned}$$
$$(5.24)$$

where $\mathcal{L}_0$ is the value of Lagrangian density corresponding to action (3.3) at $\psi = \psi_0$. A couple of comments are in order. The chemical potential appears negative, which needs to be explained. As mentioned earlier, we must ensure that the function $f(\psi)$ is positive so that the entropy density obtained is also positive (since $T > 0$ by construction). Thus, this means that the GR chemical potential is a constant but a negative constant. Physically, the chemical potential is the energy cost of adding another particle to the system at constant entropy and volume. For a thermodynamic system, given the total energy and the total number of particles, there is a definite number of possible

TABLE IV. Relationship between Jordan and Einstein frame thermodynamic variables.

$p_J = \phi^2 \tilde{p}$
$\rho_J = \phi^2 \tilde{\rho}$
$n_J = \phi\sqrt{(2\omega + 3)\phi/(16\pi)}\,\tilde{n}$
$\mu_J = \{(2\omega + 3)/(16\pi\phi)\}^{-1/2}\tilde{\mu}$
$h_J = \{(2\omega + 3)/(16\pi\phi)\}^{-1/2}\tilde{h}$
$s_J = \phi^2 \tilde{s}$

arrangements of the particles that fixes the entropy through Boltzmann's definition. If we are to add a particle to this system that results in an increase of the total energy, then the number of arrangements must necessarily increase, which is to say that the entropy must increase. If we want to add a particle to the system such that the number of arrangements (and hence the entropy) is held constant, then the net energy of the system must decrease. Since the change in energy only comes from the energy of the particle added, then this must mean that the energy of the particle added at constant entropy must be negative. But this is precisely how the chemical potential is defined. In this regard, recall that the chemical potential for an ideal gas in classical thermodynamics is also negative for this reason [45,46]. The chemical potential for the ideal gas is given by [Eq. (2.14) of [45] or Eq. (45.5) of [46]]

$$\mu = T \log\left[\frac{N}{V}\left(\frac{2\pi\hbar^2}{mT}\right)^{\frac{3}{2}}\right] = T \log\left[\frac{\lambda^3}{V/N}\right], \qquad (5.25)$$

where

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mT}} \qquad (5.26)$$

is the thermal de-Broglie wavelength. However, the classical ideal gas law holds as long as the thermal de-Broglie wavelength λ is much smaller than the volume per unit particle, so we can ignore corrections from quantum mechanics, i.e.,

$$\lambda^3 \ll \frac{V}{N} \Rightarrow \mu < 0. \qquad (5.27)$$

Similarly, for a (noninteracting) Bose gas the chemical potential turns out to be negative [see Eq. (54) of [46]]. Thus, the negative chemical potential in the GR limit of the theory is not unusual, especially when we take into account the fact that we are mapping the theory to a classical perfect fluid in thermal equilibrium at constant temperature and pressure.

Finally, we close this section with the following comments. Note that the dictionary presented in Table I is valid only for perfect fluid case. The perfect fluid structure is usually guaranteed for a gravity-matter theory in which the matter is minimally coupled to gravity. Therefore this dictionary is useful to such a theory of gravity. However the applicability of Table I, in principle, can be generalized to any theory that provides an ideal fluid EM tensor. Therefore in summary such a dictionary is viable to any perfect fluid. In the present discussion we have used this to a general scalar-tensor theory with a nonminimally coupled scalar field, which is described in the Jordan frame. The results are summarized in Tables II–IV. Here we have been able to use Table I for such a theory because our identified

EM tensors, both in Jordan and Einstein frames, are similar in structure to that of an ideal fluid. In that sense the outcomes, presented in Tables II–IV, as far as our way of identifying the EM tensor for scalar fluid is concerned, are valid for any general scalar-tensor theory in which the scalar field is nonminimally coupled to gravity in Jordan frame, and hence they are not restricted to Brans-Dicke type.

VI. CONCLUSION

We revisited the thermodynamic description of scalar fluid in ST theory of gravity using a general approach to the thermodynamics of relativistic fluids, identifying different thermodynamic and fluid parameters. Contrary to the existing works [27–29], the fluid EM tensor has been identified through the description of equation of motion for metrics in ST theory as thermodynamic identity as well as fluid equation. To find such EM the main emphasis has been given on the equivalent analysis of thermodynamics and fluid descriptions of gravitational dynamics on a generic null surface both in Jordan and Einstein frames (see, e.g., [17,23,24]). It appeared that the EM tensors in both the frames are connected by a conformal factor [see Eq. (1.7)] and both take ideal fluid structure. Before identifying the fluid parameters and thermodynamic quantities of these fluids, we reinvestigated the existing formulation of the same for ideal fluid. It appeared that certain alternatives can be put forward so that few existing shortcomings can be improved. Importantly we found that Eckart's frame may not be the best one to analyze perfect fluid.

In this paper, we have provided a possible alternate approach to the thermodynamic description of ideal scalar fluid that is valid in something other than Eckart's frame. Using this the fluid and thermodynamic parameters were obtained in both the frames. Moreover, the relations among the quantities on different frames were given. This gives us an equivalent picture with a unique relation between thermodynamic quantities in both frames. Thus the analysis in either of the frames is sufficient to get a complete description. Importantly, we can identify a unique GR limit as the zero entropy state for the fluid; i.e., the fluid entropy density vanishes in the limit of Einstein gravity. This holds in both the frames. Interestingly, in this limit the chemical potential reduces to a constant negative value, much like the chemical potential for an ideal gas in classical thermodynamics.

Finally, we mention that one should treat our investigation as one of the possibilities as far as the ideal fluid is concerned. Therefore one needs to further study to find a concrete analogy between the thermodynamic and fluid description within the perfect fluid level as well as the scalar fluid in ST theory. Hence we hope the present study is one step forward towards this and may help in future for further progress.

APPENDIX: DERIVATION OF EQ. (5.9)

Euler relation (2.7). This yields

$$
\begin{aligned}
\nabla_a \mu &= \nabla_a \left(\frac{\rho + p}{n} \right) - T_0 \nabla_a \left(\frac{s}{n} \right) \\
&= \nabla_a \left(\frac{\rho + p}{n} \right) - T_0 \left[\frac{\nabla_a \rho}{n T_0} - \frac{p + \rho}{n^2 T_0} \nabla_a n \right] \\
&= \frac{\nabla_a p}{n},
\end{aligned} \tag{A1}
$$

where in the second equality we have used the following identity:

$$
n \nabla_a \left(\frac{s}{n} \right) = \frac{\nabla_a \rho}{T_0} - \frac{p + \rho}{n T_0} \nabla_a n, \tag{A2}
$$

which is derived as follows. From the first law of thermodynamics (per unit volume) one finds

$$
\begin{aligned}
\nabla_a \rho &= T_0 \nabla_a s + \mu \nabla_a n = T_0 \nabla_a \left(n \frac{s}{n} \right) + \mu \nabla_a n, \\
&= n T \nabla_a \left(\frac{s}{n} \right) + \left(\mu + T \frac{s}{n} \right) \nabla_a n.
\end{aligned} \tag{A3}
$$

Next we use Euler's relation (2.7), which yields (A2). Now using (5.1) and (5.3) we find $\nabla_a p = \nabla_a \mathcal{L} = \mathcal{L}_\psi \nabla_a \psi + \mathcal{L}_X \nabla_a X$ and so one has

$$
\begin{aligned}
\frac{\nabla_a p}{n} &= \frac{1}{n} (\sqrt{2X} \mathcal{L}_X u_a + \mathcal{L}_X \nabla_a X), \\
&= \left(\frac{\mathcal{L}_\psi}{\mathcal{L}_X} \right) u_a + \nabla_a (\sqrt{2X}).
\end{aligned} \tag{A4}
$$

Then combining (A1) and (A4) we can take the spatial projection to obtain (5.9).

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