Nonlinear scattering of classical gravitons

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The nonlinear classical graviton solution of the Einstein equations is given to second order in the null surface formulation (NSF). We solve the NSF field equations in a perturbative scheme that gives geometrical and finite results at each order of the expansion obtaining the main variables of the formalism to second order. Then, using an algebraic relationship with the NSF variables, we obtain the metric components and the nonlinear scattering of incoming gravitons. We also analyze the relevance of this result at a classical and quantum level.

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I. INTRODUCTION

The classical graviton is a vacuum, singularity free spacetime which is asymptotically flat along future and past directed null geodesics. The picture that emerges for such a spacetime is a bounded manifold with a regular conformal metric and two null boundaries connected via a suitable definition of spacelike infinity [1]. This spacetime represents the nonlinear interaction of incoming gravitational waves that end up at future null infinity. A typical behavior of such interaction would be that even when the incoming waves have compact support, the outgoing waves would exhibit gravitational tails. One must give a word of caution with the regularity structure given to spacelike infinity; it is only for a flat spacetime a regular point for the conformal metric. For all other known asymptotically flat solutions, this region is not regular if one assumes spacelike infinity to be a point and, despite several attempts to find a suitable regularization procedure, one still does not have a definite answer. There are however, some suitable assumptions that can be given to regularize the direction dependent limits that one gets for certain components of the Weyl tensor [1–3]. In this work we will assume the two null boundaries are connected by a suitable spacelike infinity so that there is a diffeomorphism between the coordinates of future and past null infinities.

Such a spacetime would be the ideal framework to attempt a quantization procedure along the lines proposed by Ashtekar [4,5]. The so-called, asymptotic quantization procedure gives free field quantization of the gravitational radiation at future or past null infinity. This formalism uses the physical degrees of freedom of general relativity given as radiation data at past or future null infinity to construct the phase space of the theory, the Fock space and the commutation relations for the radiation fields. However, given this free-field quantization one still needs a method to construct the metric operator for the quantum fields or the scattering of incoming quantum gravitons.

In this work we present a perturbation approach based on the null surface formulation (NSF) of general relativity [6] to obtain the metric of a classical graviton spacetime and the scattering of incoming gravitational waves. We first solve the NSF equations for the main variables of the formalism to second order. Depending on whether the free data at future or past null infinities is selected, the NSF solutions are called advanced or retarded and labeled by a $+$ or $-$ sign. We first show that the solution is regular up to second order and obtain a relationship between the metric and the main variables of the NSF. We then impose the condition that the advanced and retarded solutions of the NSF yield the same metric to find the nontrivial scattering of incoming and outgoing gravitational waves.

It is worth asking why one should use the NSF formulation, as opposed to any other perturbation approach, to consider the nontrivial interaction of incoming gravitational waves on a regular spacetime. The first reason is that NSF is the only approach where the two gravitational degrees of freedom are identified and given explicitly as free data at either future or past null infinity, i.e., there are no constraint conditions on the data as opposed to any $3+1$ formulation. The second reason is that only NSF offers a regular, i.e., nondiverging, approach to consider perturbative solutions to any order since at each order of the calculation all the variables have geometrical meaning and this amounts to a modification of the conformal structure of the spacetime at each order of the calculation. This is not done in other perturbative approaches where the conformal structure is kept fixed

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throughout the perturbation and this is a source of diverging solutions.

In NSF the main variables are two nonlocal scalars, one gives the conformal structure of the spacetime and the other one the conformal factor of the underlying metric. Thus, the perturbation procedure that arises naturally in the NSF is one where the conformal structure and conformal factor are corrected at each order of the expansion but the null boundaries remain fixed. As a result one keeps changing the internal metric at each order of the perturbation. Since, by assumption, the conformal spacetime is bounded along null directions any perturbation procedure gives finite terms and thus one does not have divergences in the perturbed solutions.

One must distinguish the above mentioned approach with a different construction that has essentially the same name but it is very different construction, namely, the nonlinear graviton in twistor theory. This is done using the self dual solutions of the vacuum equations via a twistor approach [7,8]. Although we believe a complex version of our approach plus a self-duality condition should be equivalent to the twistor method, we will not be addressing this issue now.

There is also available in the literature other perturbative approaches [9] pursuing the goal of obtaining an S matrix for quantum gravity. In particular there are some results [10] that should be related to our recent work using the NSF formulation [11].

It is also worth mentioning that NSF can be extended to spacetimes containing black holes via a regularization procedure of the null cone cuts since in those spacetimes the cuts are not closed two surfaces, some null geodesics of the future null cone from any point are lost when they cross the event horizon. Nevertheless, the regularized null cones can be used to define global variables like the Bondi 4-momentum, relativistic angular momentum and compare with results in the literature of gravitational-wave radiation where there exists an abundant work with many perturbation procedures using radiative coordinates [12,13]. In this work we only concentrate on spacetimes without black holes for a very important reason. A quantum theory of gravity will not have singularities, i.e., black holes will no longer be part of the theory, although there will be regions with strong curvature. Thus, in this preliminary work we only consider regular spacetimes without singularities and this preclude ourselves to consider not only black holes spacetimes but also other relevant solutions like the pp waves [14,15]. However, we give as an example a construction that is very close to a pp wave collision but keeping regular incoming and outgoing gravitational waves.

The paper is organized as follows. In Sec. II we give a brief review of the NSF and develop the field equations that are valid up to second order in a perturbation procedure. In Sec. III we derive the relation between the metric and the NSF variables for the linear case and obtain the map between the past and future null radiation data. As expected the map is the identity but it allows us to check the implemented approach. In Sec. IV we derive the advanced and retarded solutions to second order. They are then used to obtain the scattering of gravitons showing a nontrivial interaction that can be explicitly displayed. In particular, we consider incoming radiation that can be thought of a head on collision of regularized pp waves and obtain the outgoing radiation after they interact with each other. Finally, we finish the work with a summary and conclusions in Sec. V.

II. THE NULL SURFACE FORMULATION

The null surface formulation recasts general relativity as a theory of null surfaces with field equations that are equivalent to the Einstein equations [16]. The classical graviton construction in NSF was developed to obtain a real generalization of self-dual spacetimes [6]. In this work we will be using the field equations given in 2016 [6]. This approach could be of importance when dealing with quantization of pure gravity and hints of this quantization were given [17,18]. Below we present some results of NSF that are relevant for this work.

Introducing a bundle of null directions for the points of the spacetime with local coordinates $(x^a, \zeta, \bar{\zeta})$, the main variables of the NSF are two scalars functions, the null cone cut $Z(x^a, \zeta, \bar{\zeta})$ which yields the conformal structure of the spacetime and the conformal factor $\Omega(x^a, \zeta, \bar{\zeta})$ that depends on the matter fields of the spacetime as well as the Ricci tensor of the conformal metric. In addition these scalars satisfy two real differential equations that are purely kinematical in nature and are called "metricity conditions".

Mathematically speaking, the ten Einstein equations with components depending of the spacetime points x^a are exchanged by one equation for the scalars $\Omega(x^a, \zeta, \bar{\zeta})$ and $Z(x^a, \zeta, \bar{\zeta})$ along with the two metricity conditions [19]. The new system of differential equations depends on six variables instead of the four spacetime points x^a. This differential system equation continues being nonlinear and thus numerical and perturbative approaches must be employed in order to find solutions.

In NSF the Lorentzian metric of the spacetime g_{ab} is constructed from a conformally invariant part h_{ab} and a conformal factor Ω. The conformal metric is obtained from knowledge of special null surfaces obtained from a real function Z given on the sphere of null directions bundle. Given any metric g_{ab} for the spacetime, the function Z is constructed from the intersection of N_x^+, the future null cone from a point x with future null infinity. Using Bondi coordinates $(u, \zeta, \bar{\zeta})$ this intersection is denoted by $u = Z(x^a, \zeta, \bar{\zeta})$. Moreover, one can show that for fixed values of $(u, \zeta, \bar{\zeta})$, the condition $Z = \text{const}$ yields a null surface on the spacetime. From knowledge of Z one then constructs a conformal metric $h_{ab}[Z]$. In addition, one gives a second real function $\Omega = \Omega(x^a, \zeta, \bar{\zeta})$, a conformal factor, that plays a dual role: it is used to obtain a conformal metric that only depends on the spacetime points and it yields an Einstein metric via the field equations. From knowledge of these two functions one obtains

$$g_{ab}(x^a) = \Omega^2 h_{ab}[Z]. \tag{1}$$

Since $Z = \text{const}$ are null hypersurfaces they satisfy

$$g^{ab}\partial_a Z \partial_b Z = 0. \tag{2}$$

It is clear from (2) that the conformal structure does not depend on Ω. It also follows from this equation that the null vector $Z^a = g^{ab}\partial_b Z$, satisfies the homogeneous geodesic equation, thus defining an affine length s.

Directly from (2), and taking $\eth$ and $\bar{\eth}$ derivatives [19], one obtains the components of the conformal metric. The nontrivial coefficients of h^{ab} are functions of a single scalar Λ defined as $\Lambda = \eth^2 Z$. Once the conformal metric coefficients have been obtained one finds a condition on Λ, namely

$$\eth^3(g^{ab}\partial_a Z\partial_b Z) = 0 \Rightarrow g^{ab}(3\partial_a \eth Z \partial_b \Lambda + \partial_a Z \partial_b \eth\Lambda) = 0,$$

which can be rewritten as

$$\frac{\partial \eth \Lambda}{\partial s} + 3\eth Z^b \partial_b \Lambda = 0. \tag{3}$$

Only for functions Λ that satisfy condition (3) it is possible to obtain a conformal metric. In what follows we assume condition (3) is satisfied.

One can also show that directly from

$$\eth^2\bar{\eth}^2(g^{ab}\partial_a Z\partial_b Z) = 0,$$

one obtains a relationship between Ω and Λ, namely,

$$\frac{\partial \bar{\eth}^2\Lambda}{\partial s} = \eth\bar{\eth}(\Omega^2) + g^{ab}\partial_a\Lambda\partial_b\bar{\Lambda}, \tag{4}$$

which can be formally integrated giving [6],

$$\bar{\eth}^2\eth^2 Z = \eth^2\bar{\sigma}(Z,\zeta,\bar{\zeta}) + \bar{\eth}^2\sigma(Z,\zeta,\bar{\zeta}) + \int_{-\infty}^{Z} \dot{\sigma}\,\dot{\bar{\sigma}}\,du$$
$$- \int_s^{\infty} (\eth\bar{\eth}(\Omega^2) + g^{ab}\partial_a\Lambda\partial_b\bar{\Lambda}) ds', \tag{5}$$

where $\sigma(u,\zeta,\bar{\zeta})$ is the Bondi shear given at future null infinity and $\dot{\sigma}$ is the derivative with respect to the Bondi time u. The complex shear represents the two degrees of freedom of the gravitational field and the solutions of the above equations are functionally dependent on $\sigma(u,\zeta,\bar{\zeta})$.

Conditions (3) and (5) are necessary and sufficient to construct a metric for the space time. In this approach the function Λ plays an important role since the conformal metric is completely given in terms of this function and its vanishing yields a flat conformal metric. Thus, one can implement a perturbation procedure directly from knowledge of Λ and write down the lowest nontrivial formulation from a linearized approximation.

The scalar $Z(x,\zeta,\bar{\zeta})$ is also used to introduce a $(\zeta,\bar{\zeta})$-dependent coordinate system with a geometrical meaning. Introducing $u = Z$, $\omega = \eth Z$, $\bar{\omega} = \bar{\eth} Z$, $r = \bar{\eth}\eth Z$, the condition $u = \text{const}$ yields the past light cone from the point $(u,\zeta,\bar{\zeta})$ at future null infinity. The condition $\omega = \text{const}$ selects a particular null geodesic and $r = \text{const}$ a point on that geodesic. The coordinate r is also an affine parameter for the conformal metric h_{ab}. The scalars r and s are related via

$$\frac{\partial r}{\partial s} = \Omega^2.$$

Thus, r is an injective function of s. As vector fields, they are related via

$$\frac{\partial}{\partial s} = \Omega^2 \frac{\partial}{\partial r}.$$

One can also write down the metricity condition (5) in terms of r using the above relationship.

The field equations for NSF take a simple form in terms of this coordinate system. Directly from the relationship between g_{ab} and h_{ab}, the trace-free Ricci flat equations read [6,20],

$$2\partial_r^2\Omega = R_{rr}[h]\Omega, \tag{6}$$

with the component R_{rr} given by

$$R_{rr}[h] = \frac{1}{4q}\partial_r^2\Lambda\partial_r^2\bar{\Lambda} + \frac{3}{8q^2}(\partial_r q)^2 - \frac{1}{4q}\partial_r^2 q,$$
$$q = 1 - \partial_r\Lambda\partial_r\bar{\Lambda}, \tag{7}$$

and the scalar r is defined as $r = \eth\bar{\eth}Z$. It is clear from the above equations that $R_{ab}[h]$ vanishes when $\Lambda = 0$.

The three scalar equations (3), (4), and (6) are completely equivalent to the vacuum Einstein equations for a metric g_{ab}. If we substitute (5) for (4), and ask for regularity conditions on the integrals of (5), then we get the vacuum, asymptotically flat solutions, the so called classical gravitons. They represent the nonlinear interaction of incoming gravitational waves that are freely given at future null infinity. This backwards time direction can be thought of the advanced solution of a generalized wave equation. One can also obtain the retarded graviton solutions where outgoing gravitational waves are freely given at past null infinity and evolve to the future giving a Ricci flat metric.

The linearized version of the last two equations is obtained by first giving the zeroth-order solution that yields a flat metric, namely,

$$Z_0 = x^a l_a, \qquad \Omega_0 = 1, \tag{8}$$

with l^a a null vector defined as $l^a = \frac{1}{\sqrt{2}}(1,\hat{r}^i)$ and $\hat{r}^i$ the unit vector on the sphere of null directions, and x^a a point in the flat spacetime.

One then writes down a linearized departure from the zeroth-order solution as

$$Z = x^a l_a + Z_1, \qquad \Omega = 1 + \Omega_1. \tag{9}$$

The equation of motion for Ω_1,

$$2\frac{\partial^2 \Omega_1}{\partial s^2} = 0, \tag{10}$$

yields a trivial solution when regularity conditions are imposed while the equation for Z_1 is given by

$$\bar{\eth}^2 \eth^2 Z_1 = \bar{\eth}^2 \sigma(Z_0, \zeta, \bar{\zeta}) + \eth^2 \bar{\sigma}(Z_0, \zeta, \bar{\zeta}) + \mathcal{O}(\Lambda^2). \tag{11}$$

Equation (11) is a nonhomogeneous fourth-order elliptic equation on the sphere and its solution can be found by convoluting the inhomogeneity with the corresponding Green function [21]. It is worth pointing out that (11) is given at future null infinity and the points x^a in the equation are simply constants of integration. The connection with the underlying spacetime is given when Z_0 is interpreted as the null cone cut for a flat spacetime and for that one presupposes the existence of this null boundary. The same happens when giving a similar construction for past null infinity. Moreover, the two null boundaries are connected via spacelike infinity which is a point for a flat spacetime.

The second-order Z_2, Ω_2 satisfy the following equations:

$$\bar{\eth}^2 \eth^2 Z_2 = \eth^2 \bar{\sigma}(Z_1, \zeta, \bar{\zeta}) + \bar{\eth}^2 \sigma(Z_1, \zeta, \bar{\zeta}) + \int_{-\infty}^{Z_0} \dot{\sigma}\,\dot{\bar{\sigma}}\, du - 2\int_r^{\infty} (\eth\bar{\eth}(\Omega_2) + \eta^{ab}\partial_a \Lambda_1 \partial_b \overline{\Lambda_1}) dr, \tag{12}$$

$$\partial_r^2 (8\Omega_2 - \partial_r \Lambda_1 \partial_r \bar{\Lambda}_1) = \partial_r^2 \Lambda_1 \partial_r^2 \bar{\Lambda}_1. \tag{13}$$

Equation (13) can be integrated directly to obtain

$$8\Omega_2 = \partial_r \Lambda_1 \partial_r \bar{\Lambda}_1 + \int_r^{\infty} dr' \int_{r'}^{\infty} dr'' \partial_{r''}^2 \Lambda_1 \partial_{r''}^2 \bar{\Lambda}_1. \tag{14}$$

Note that there are two different contributions in (12), a sphere integral on the first-order cut Z_1, and two integral contributions that are quadratic on the free data σ and therefore one is using the zeroth-order coordinate system to perform the integrals. One could have used a first-order coordinate system to perform the integrals but the difference between these two approaches would be of order σ^3. Likewise, the double-integral contribution in (14) is performed by selecting a null geodesic in flat space.

There is also a conceptual difference between the first- and second-order contribution regarding the geometrical meaning of the null boundaries in the two cases. Whereas in the linear case spacelike infinity is a point, we have to modifiy the conformal geometry in the quadratic term since the spacetime is no longer flat and thus spacelike infinity is no longer a point. We assume to have an S^2 completion, together with a matching condition for fields defined in past and future null infinities [1].

A. The metric tensor and null surfaces

As it was mentioned before, $\partial_a Z$ is a null covector and satisfies,

$$g^{ab}\partial_a Z \partial_b Z = 0.$$

It also follows from the field equations that Z has a functional dependence on the free null data σ. Thus, assuming the free data is small, one can write down a perturbation series for Eq. (2) relating g^{ab} with the perturbed solutions of Z. We thus write

$$\sum_{n=0}^{\infty} \sum_{r+s=0}^{n} g_{n-r-s}^{ab} \partial_a Z_r \partial_b Z_s = 0, \tag{15}$$

where

$$\sum_{n=0}^{\infty} g_n^{ab} = g_0^{ab} + g_1^{ab} + g_2^{ab} + \cdots = (1 + \Omega_1 + \Omega_2 + \cdots)^2 (\eta^{ab} + h_1^{ab} + h_2^{ab} + \cdots) \tag{16}$$

with η^{ab} the flat metric and the labels $_{1,2,\ldots}$ the different orders of the NSF variables.

(i) Taking $n = 0$ in (15) we have

$$\eta^{ab}\partial_a Z_0 \partial_b Z_0 = 0. \tag{17}$$

Taking ∂_a on Eq. (8) we obtain $\partial_a Z_0 = l_a$. then the expression (17) can be written as

$$\eta^{ab} l_a l_b = 0. \tag{18}$$

Taking $\eth$ and $\bar{\eth}$ on (18) one gets all the metric components in the flat null tetrad [22].

(ii) Taking $n = 1$ in (15) one gets,

$$h_1^{ab} l_a l_b + 2\eta^{ab} l_a \partial_b Z_1 = 0, \tag{19}$$

since Ω_1 vanishes at the linearized approximation. The above expression can be rewritten as

$$h_{1ab} l^a l^b + 2 l^a \partial_a Z_1 = 0,$$

from which one can obtain all the components of h_{1ab}. Note that l^a is not a null vector of h_{1ab} but nevertheless it is useful to determine all the components of the linearized metric.

(iii) Taking $n = 2$ on (15), we get the second-order term,

$$h_{2ab} l^a l^b + 2 h_1^{ab} l_a \partial_b Z_1 + 2 l^a \partial_a Z_2 = 0, \tag{20}$$

from which h_{2ab} can be obtained by repeated $\eth$ and $\bar{\eth}$ operations on (20).

Up to second order, the metric of the spacetime can be written as

$$g_{ab} = \eta_{ab} + h_{1ab} + 2\Omega_2 \eta_{ab} + h_{2ab}, \tag{21}$$

where Ω_2 and Z_2 are obtained from the second-order field equations and h_{2ab} is algebraically related to Z_2 via Eq. (20).

The above construction can be done with free data given at future or past null infinity. In each case the solution is labeled as Z^+ or Z^- respectively and it is analogous to the advanced or retarded solutions that can be constructed for the solutions of the wave equation. Following the algebraic relationship between the pair (Z, Ω) and the metric of the spacetime one can construct an advanced, g^+_{ab}, or retarded, g^+_{ab}, solution of the vacuum equations.

B. Scattering of gravitational waves

If we impose the condition

$$g^+_{ab} = g^-_{ab}, \tag{22}$$

then there is a correlation between the incoming and outgoing radiation, they are no longer free. As one is used to a causal picture, we can assume that we have free incoming gravitational radiation and want to know the effect on the outgoing radiation. A linear theory would immediately show that they are unscattered but Einstein equations, being highly nonlinear, should exhibit gravitational tails for the outgoing radiation.

III. THE LINEARIZED NSF

Using the results obtained in the previous section we derive the relation between the metric of the spacetime and the linearized solution to the NSF equations. We first obtain the advanced and retarded solutions and then the algebraic relationship to the metric in terms of the free data at future or past null infinity. By demanding that the metric tensor constructed from the advanced or retarded solution is the same we obtain a relationship from the incoming and outgoing gravitational waves. This trivial scattering nevertheless has some interesting features between the radiation fields that are given at future and past null infinity.

We will use the superscripts $+$ and $-$ to denote quantities associated to future and past null infinity, respectively. In this sense, σ^+ will stand for outgoing radiation at future null infinity and σ^- for ingoing radiation coming from past null infinity.

A. Tensorial spin-s harmonics

Given a Newman-Penrose (NP) null tetrad $(l^a, n^a, m^a, \bar{m}^a)$ and keeping just the spatial part of the vectors $l^a, m^a, \bar{m}^a$ (or n^a instead of l^a) we can obtain an orthonormal vector base to any tangent space of a point p in S^2. This orthonormal base can be used to set up a general n-dimensional tensor base by doing the exterior product of the vectors $l^a, m^a, \bar{m}^a$ (keeping only the spatial part). This concept is what the tensor spin-s harmonics embrace. We mention in this subsection only a few main properties of the construction but the reader may find more in the following Refs. [22,23]. The tensorial spin-l harmonics, denoted Y^l_{lI} with I being a set of l indices, are defined as the l-times product of the spatial part of the vector m^a. Then, the complex covariant derivative on the manifold defined in S^2, known as ð, along with its conjugate $\bar{ð}$ behave as "ladder operators" with respect to the spin index (the number in the superscript). These ladder operators allow to obtain the whole set of representations of the tensors Y^s_{lI}. A relevant property of these tensors is

$$\bar{Y}^s_{lI} = Y^{-s}_{lI}.$$

In this work we are interested in Y^2_{2ij} which is a spin-2 tensor harmonic and is the one needed for expanding gravitational radiation, σ.

B. Antipodal transformations on the sphere

To link incoming and outgoing radiation at null infinity, we need to introduce the notion of antipodal points on the sphere. More specifically, the antipodal points on the sphere are those diametrically opposite to each other. Then, we define the antipodal transformation to be the one that carries a point in the sphere to its antipodal point. In the usual spherical chart (θ, ϕ), the antipodal transformation reads

$$(\theta, \phi) \to (\pi - \theta, \pi + \phi),$$

or in stereographic coordinates

$$(\zeta, \bar{\zeta}) \to (-1/\bar{\zeta}, -1/\zeta).$$

We denote the antipodal transformation with the symbol $\hat{}$, i.e, $\hat{\zeta} = -1/\bar{\zeta}$. Given a tensorial spin-s harmonic $Y^s_{lI}(\zeta, \bar{\zeta})$, we have

$$Y^s_{lI}(\hat{\zeta}, \hat{\bar{\zeta}}) = (-1)^l Y^{-s}_{lI}(\zeta, \bar{\zeta}). \tag{23}$$

In particular, if we write $l^a_- = \frac{1}{\sqrt{2}}(-1, r^i)$ with r^i the corresponding spatial vector, the antipodal transformation is

$$\hat{l}^a_- = \frac{1}{\sqrt{2}}(-1, \hat{r}^i) = \frac{1}{\sqrt{2}}(-1, -r^i) = -\frac{1}{\sqrt{2}}(1, r^i) = -l^a_+. \tag{24}$$

Equation (24) is the antipodal transformation applied to a null vector l^a_- defined at past null infinity. Hence, this vector has its antipodal point at minus the position of a vector defined at future null infinity.

The antipodal transformation on the derivative operator, which will appear later in the calculations, can be proven to be

$$\hat{ð} = -\bar{ð}.$$

C. Relation between null cuts at $\mathcal{I}^+$ and $\mathcal{I}^-$

As discussed in Sec. II, the condition of the null cone cuts to be null at $n = 1$, is given by

$$h_1^{ab} l_a l_b + 2\eta^{ab} l_a \partial_b Z_1 = 0, \tag{25}$$

since the conformal factor vanishes at that order. Up to first order, we can use the Minkowski metric to raise or lower indices, thus, $\eta^{ab} l_a = l^b$ and,

$$h_{1ab}(x) l^a l^b = -2 l^a \partial_a Z_1(x^a, \zeta, \bar{\zeta}), \tag{26}$$

relating the first-order correction h_{1ab} with the first-order correction to the null foliation Z_1. Note that Z_1 plays the role of a potential for h_{1ab}. As we will see below the solution to the linearized equation for Z_1 shows its explicit dependence on the gravitational radiation σ and it is completely equivalent to the solution of the wave equation using an advanced or retarded Green function.

D. The linear graviton solution

The first-order correction to the null flat foliation Z_0 can be obtained by solving Eq. (5) to first order, i.e.,

$$\bar{\eth}^2 \eth^2 Z_1(x^a, \zeta, \bar{\zeta}) = \bar{\eth}^2 \sigma(Z_0, \zeta, \bar{\zeta}) + \eth^2 \bar{\sigma}(Z_0, \zeta, \bar{\zeta}). \tag{27}$$

The solution can be found by convoluting the inhomogeneity of the equation with the Green function on the sphere [21],

$$\begin{aligned} G_{00'}(\zeta, \bar{\zeta}, \zeta', \bar{\zeta}') &= \frac{1}{4\pi} l^{+a} l'_{+a} ln(l^{+a} l'_{+a}) \\ &= \frac{1}{4\pi} l^{-a} l'_{-a} ln(l^{-a} l'_{-a}). \end{aligned} \tag{28}$$

The solution reads

$$\begin{aligned} Z_1^+(x^a, \zeta, \bar{\zeta}) = \oint_{S^2} & G_{00'}(\zeta, \bar{\zeta}, \zeta', \bar{\zeta}')(\bar{\eth}'^2 \sigma^+(x_a^{al+\prime}, \zeta', \bar{\zeta}') \\ & + \eth'^2 \bar{\sigma}^+(x_a^{al+\prime}, \zeta', \bar{\zeta}')) dS', \end{aligned} \tag{29}$$

where we have used that $Z_0^+ = u = x^a l_a^+$. The superscript in Z_1^+ is used to denote that the future null directed foliation has been used to construct the future null cone cut $x^a l_a^+$. Similarly, σ^+ denotes the outgoing radiation at future null infinity.

Using the explicit form of $G_{00'}$ and integrating by parts one writes (29) as

$$Z_1^+(x^a, \zeta, \bar{\zeta}) = \frac{1}{4\pi} \oint_{S^2} \left(\frac{(l^a m'_a)^2}{l^b l'_b} \bar{\sigma}^+(x_c^{cl+\prime}, \zeta', \bar{\zeta}') + \text{c.c.} \right) dS'. \tag{30}$$

Directly from (25) one gets

$$h_{1ab}^+(x) l^a l^b = \frac{-1}{2\pi} l^a l^b \oint_{S^2} (m'_a m'_b \dot{\bar{\sigma}}^+(x_c^{cl+\prime}, \zeta', \bar{\zeta}') + \text{c.c.}) dS'. \tag{31}$$

Note that the vector $l^a(\zeta, \bar{\zeta})$ spans the sphere of null directions at each point of the spacetime. Thus, one can obtain from the above equation the conformal metric of the spacetime, i.e.,

$$h_{1ab}^+(x) = \frac{-1}{2\pi} \oint_{S^2} (m'_a m'_b \dot{\bar{\sigma}}^+(x_c^{cl+\prime}, \zeta', \bar{\zeta}') + \text{c.c.}) dS'. \tag{32}$$

The integrand of Eq. (32) is

$$N_{ab} = m_a m_b \dot{\bar{\sigma}}^+(u, \zeta, \bar{\zeta}) + \bar{m}_a \bar{m}_b \dot{\sigma}^+(u, \zeta, \bar{\zeta}),$$

is called the News tensor and constitutes the null free data of the vacuum Einstein equations.

Another solution to Eq. (11) can be found in terms of ingoing radiation

$$\begin{aligned} Z_1^-(x^a, \zeta, \bar{\zeta}) = \oint_{S^2} & G_{00'}(\bar{\eth}'^2 \sigma^-(-x^a l'^-_a, \zeta', \bar{\zeta}') \\ & + \eth'^2 \bar{\sigma}^-(-x^a l'^-_a, \zeta', \bar{\zeta}')) dS'. \end{aligned} \tag{33}$$

where we have used the null cone cut $Z_0^- = -x^a l_a^-$, given by the intersection of the past null cone from x^a with past null infinity. Note also that here $Z_0^- = v$, with v the usual advanced time coordinate. This solution can also be written as

$$Z_1^-(x^a, \zeta, \bar{\zeta}) = \frac{1}{4\pi} \oint_{S^2} \left(\frac{(l^{-a} m'_a)^2}{l^{-b} l'_{-b}} \bar{\sigma}^-(-x^c l'^-_c, \zeta', \bar{\zeta}') + \text{c.c.} \right) dS', \tag{34}$$

since $m'_a = m'_{-a}$. One also has

$$h_{1ab}^-(x) l^{-a} l^{-b} = \frac{1}{2\pi} \oint_{S^2} ((l^{-a} m'_a)^2 \dot{\bar{\sigma}}^-(-x_c^{cl-\prime}, \zeta', \bar{\zeta}') + \text{c.c.}) dS', \tag{35}$$

and

$$h_{1ab}^-(x) = \frac{1}{2\pi} \oint_{S^2} (m'_a m'_b \dot{\bar{\sigma}}^-(-x_c^{cl-\prime}, \zeta', \bar{\zeta}') + \text{c.c.}) dS'. \tag{36}$$

Given a point in the spacetime x^a and the metric $g^{ab}(x^c)$ at that point, we can write the metric in terms of the ingoing (σ^-) or outgoing radiation (σ^+) using the scalar field Z. On the other hand, from the uniqueness of the metric tensor $g^{ab}(x^c)$ both descriptions of the metric must coincide, i.e., $g_-^{ab}(x^c) = g_+^{ab}(x^c)$. Thus, to first order we have

$$h_{1ab}^+(x^c) = h_{1ab}^-(x^c). \tag{37}$$

One can obtain the relationship between the free data at future and past null infinity by doing a coordinate transformation $\zeta \to \hat{\zeta}$ on Eq. (36) obtaining

$$h^{-}_{1ab}(x) = \frac{1}{2\pi}\oint_{S^2}(m'_a m'_b \dot{\sigma}^{-}(x^{al+\prime}_a, \hat{\zeta}', \hat{\bar{\zeta}}') + \text{c.c.})dS', \quad (38)$$

identifying the integrands in each expression for the linearized metric one obtains

$$\dot{\sigma}^{+}(u,\zeta,\bar{\zeta}) + \dot{\bar{\sigma}}^{-}(u,\hat{\zeta},\hat{\bar{\zeta}}') = 0, \quad (39)$$

relating the incoming and outgoing gravitational radiation. The functional dependence of $\dot{\sigma}^{+}(u,\zeta,\bar{\zeta})$ on the antipodal value of $\dot{\bar{\sigma}}^{-}(u,\zeta,\bar{\zeta})$ follows directly from the fact that a null ray coming from a given direction in spherical coordinates goes to the antipodal direction after passing from the origin.

The above expression can also be obtained from a slightly different approach which can also be applied to obtain the nontrivial scattering. We now present the second approach to obtain the metric coefficients and scattering amplitudes.

Note that the first-order deviation h^{ab}_1 only depends on x^a. Using Eqs. (37) and (26) we find a relation between first-order outgoing and ingoing null surfaces

$$l^{+a}\partial_a Z^{+} = \hat{l}^{+a}\partial_a \widehat{Z^{-}}, \quad (40)$$

which can also be written as

$$l^{+a}\partial_a (Z^{+} + \widehat{Z^{-}}) = 0. \quad (41)$$

The solution to can be stated as

$$Z^{+}_1 + \widehat{Z^{-}_1} = F_0(\zeta,\bar{\zeta}) + F_1(\hat{l}^{+a}x_a, \zeta, \bar{\zeta}),$$

with F_0 and F_1 arbitrary functions of their arguments. F_0 can be gauged away by a supertranslation whereas F_1 can be absorbed since it represents a particular form of free data. Alternatively, one can keep the gauge functions and they disappear when the physical quantities are constructed since they all depend on $l^{+a}\partial_a Z^{+}$. For simplicity we solve

$$Z^{+}_1 + \widehat{Z^{-}_1} = 0, \quad (42)$$

in a particular gauge and take a dot derivative at the end to obtain a gauge independent result.

E. Modes relation

To study the relation between the different modes of gravitational radiation, we expand the dependence on the stereographic coordinates in all the physical scalar quantities. That is to say, given a quantity $\eta(u,\zeta,\bar{\zeta})$ with spin weight s, we have

$$\eta(u,\zeta,\bar{\zeta}) = \sum_l \eta(u)^I Y^s_{lI},$$

where $s = 2$ if $\eta = \sigma^0$.

It is also convenient to expand the Green function of Eq. (28) in a spherical harmonics decomposition as

$$G_{00'} = \sum_{l=2}^{\infty} \frac{4\pi}{2l+1} Y^0_{lI}(\zeta,\bar{\zeta}) Y'^0_{lI}(\zeta',\bar{\zeta}'). \quad (43)$$

Replacing the above formula in Eq. (29) yields the lI contribution to Z, i.e.,

$$Z^{+}_{lI} = \oint (Y^{-2}_{lI}\sigma^{+}(u,\zeta,\bar{\zeta}) + Y^{2}_{lI}\bar{\sigma}^{+}(u,\zeta,\bar{\zeta}))d^2S, \quad (44)$$

where the factor $\frac{4\pi}{2l+1}$ has been absorbed in the definition of Z^{+}_{lI}. A similar calculation can be carried out to obtain the right-hand side in Eq. (40) but we must also apply the antipodal transformation from Sec. III B, then

$$\hat{Z}^{-}_{lI} = (-1)^l \oint (Y^{-2}_{lI}\sigma^{-}(v,\zeta,\bar{\zeta}) + Y^{2}_{lI}\bar{\sigma}^{-}(v,\zeta,\bar{\zeta}))d^2S. \quad (45)$$

To obtain the relation between the different modes from outgoing radiation σ^{+}_{lI} and incoming radiation σ^{-}_{lI} one expands both functions in a harmonic decomposition and makes a change of variables so that $v' = u$. The details of the derivation are shown in the Appendix. The coefficients of the ingoing and outgoing radiation are given by the following relations

$$\sigma^{+}_{lI}(w) + (-1)^l \bar{\sigma}^{-}_{lI}(w) = 0, \quad (46)$$

using a positive frequency w or,

$$\sigma^{+}(u,\zeta,\bar{\zeta}) + \bar{\sigma}^{-}(u,\hat{\zeta},\hat{\bar{\zeta}}) = 0, \quad (47)$$

as a function of the Bondi time u and the stereographic coordinate on the sphere $(\zeta,\bar{\zeta})$ in a particular BMS gauge.

IV. NONTRIVIAL SCATTERING OF GRAVITATIONAL RADIATION

Following Eq. (12) and (13) derived in Sec. II, there is an additional term affecting the field equation for Z. These extra terms yield an additional complication since the calculations are technically more involved. Nevertheless, it is possible to derive an explicit formula for each mode. We first obtain the advanced solution Z^{+}.

Using Eq. (43) one can write Z^{+}_{2lI} directly from Eq. (12) as

$$Z^{+}_{2lI} = Z^{+}_{2lI,\text{cut}} + Z^{+}_{2lI,\text{cone}}, \quad (48)$$

with

$$Z^{+}_{2lI,\text{cut}} = \oint d^2S(Y^{-2}_{lI}\sigma^{+}(Z_1,\zeta,\bar{\zeta}) + Y^{2}_{lI}\bar{\sigma}^{+}(Z_1,\zeta,\bar{\zeta}) + Y^{0}_{lI}\Sigma(Z_0,\zeta,\bar{\zeta})), \quad (49)$$

with $\Sigma(Z,\zeta)$ a quadratic function in the gravitational radiation defined as

$$\Sigma^{+}(Z,\zeta,\bar{\zeta}) = \int_{-\infty}^{Z} \dot{\sigma}\,\dot{\bar{\sigma}}\,du.$$

One also has a contribution from the integral on the future null cone from x^a, namely

$$Z^{+}_{2lI,\mathrm{cone}} = -\oint d^2 S Y^0_{lI} \int_0^\infty ds (2\eth\bar{\eth}\Omega_2(y^c,\zeta,\bar{\zeta}) + \eta^{ab}\partial_a\Lambda_1\partial_b\overline{\Lambda_1}(y^c,\zeta,\bar{\zeta})), \tag{50}$$

with $y^c = x^c + sl^{+c}$. Defining N^{+}_x as the future null cone from the point x^c and C^{+}_x as the intersection of N^{+}_x with future null infinity, the 2-surface integral in Eq. (49) is given on C^{+}_x while the 3-dim integral is given on future null infinity and its boundary is C^{+}_x. The 3-dim integral of Eq. (50) is given on N^{+}_x. An analogous formula can be given for Z^{-}_{2lI}.

One can then obtain h_{2ab} from Z_1 and Z_2 via Eq. (20). The explicit form of the metric coefficients is not needed in this work but it can be obtained from repeated $\eth$ and $\bar{\eth}$ operations on Eq. (20).

A. The conformal factor Ω_2

Equation (13) can be integrated directly to obtain

$$8\Omega^{+}_2 = \partial_r\Lambda^{+}_1\partial_r\bar{\Lambda}^{+}_2 + \int_r^\infty dr' \int_{r'}^\infty dr'' \partial^2_{r''}\Lambda^{+}_1\partial^2_{r''}\bar{\Lambda}^{+}_1. \tag{51}$$

Note that conformal factor depends on Λ^{+}_1, whose harmonic decomposition is given as

$$\Lambda^{+}_{1\,lI}(x) = Z^{+}_{1\,lI}(x) = \int d^3k[Y^{-2}_{lI}(\hat{k})\sigma^{+}(k)e^{-ix^a k_a} + Y^2_{lI}(\hat{k})\bar{\sigma}^{+}(k)e^{ix^a k_a}], \tag{52}$$

thus, Z^{+}_1 and Λ^{+}_1 contain identical information. They only differ in their spin weight, namely,

$$Z^{+}_1(x,\zeta,\bar{\zeta}) = Y^0_{lI}(\zeta,\bar{\zeta})Z^{+}_{1\,lI}(x), \tag{53}$$

whereas

$$\Lambda^{+}_1(x,\zeta,\bar{\zeta}) = Y^2_{lI}(\zeta,\bar{\zeta})Z^{+}_{1\,lI}(x). \tag{54}$$

Defining the Fourier transform of Ω^{+}_2 as

$$8\Omega^{+}_2(k^a,\zeta,\bar{\zeta}) = \int d^4x e^{ix^a k_a}\left[\partial_r\Lambda^{+}_1\partial_r\bar{\Lambda}^{+}_1 + \int_r^\infty dr' \int_{r'}^\infty dr'' \partial^2_{r''}\Lambda^{+}_1\partial^2_{r''}\bar{\Lambda}^{+}_1\right], \tag{55}$$

and following a calculation derived in the Appendix, we obtain

$$8\Omega^{+}_2(k^a,\zeta,\bar{\zeta}) = \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\delta^4(k^a-(k_1+k_2)^a)\sigma^{+}(k_1)\bar{\sigma}^{+}(k_2)\mathcal{S}^{+}_{\Omega}(k_1,k_2,\zeta,\bar{\zeta}), \tag{56}$$

with

$$\mathcal{S}^{+}_{\Omega} = G_{2,2'}(\zeta,\bar{\zeta},\hat{k}_1)G_{-2,-2'}(\zeta,\bar{\zeta},\hat{k}_2)\left(l^{+a}k_{1a}l^{+b}k_{2b} + \frac{(l^{+a}k_{1a}l^{+b}k_{2b})^2}{(l^{+a}(k_1+k_2)_a)^2}\right), \tag{57}$$

and $l^{+a} = l^{+a}(\zeta,\bar{\zeta})$, $G_{2,2'}(\zeta,\bar{\zeta},\zeta',\bar{\zeta}') = \eth^2\eth'^2 G_{0,0'}$, $G_{-2,-2'} = \bar{\eth}^2\bar{\eth}'^2 G_{0,0'}$. Inverting Eq. (56) we obtain,

$$8\Omega^{+}_2(x^a,\zeta,\bar{\zeta}) = \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2} e^{-ix^c(k_1+k_2)_c}\sigma^{+}(k_1)\bar{\sigma}^{+}(k_2)\mathcal{S}^{+}_{\Omega}(k_1,k_2,\zeta,\bar{\zeta}). \tag{58}$$

The details of the interaction, with the explicit dependence on $(\zeta,\bar{\zeta})$, is given in $\mathcal{S}_{\Omega}$. A similar calculation can be done with the retarded solution and an analogous formula can be obtained for $\Omega^{-}_2(x^a,\zeta,\bar{\zeta})$.

B. The map between the in and out fields

Directly from Eq. (22) we get

$$g^{+}_{2ab}l^{+a}l^{+b} = g^{-}_{2ab}\hat{l}^{-a}\hat{l}^{-b}, \tag{59}$$

since $l^{+a} = -\hat{l}^{-a}$. Substituting Eqs. (37) and (39) in Eq. (20) and inserting in Eq. (59) one gets

$$l^{+a}\partial_a(Z^{+}_2 + \widehat{Z^{-}_2}) = 0, \tag{60}$$

i.e., and identical equation than Eq. (40) but for the second-order term. This relation yields a map from the incoming and outgoing gravitational radiation. To obtain the explicit relationship it is convenient to assume that $\sigma^{+}(u,\zeta,\bar{\zeta})$ is a small deviation from the unscattered incoming field described by Eq. (39),

$$\sigma^{+}(u,\zeta,\bar{\zeta}) = \sigma^{+}_1(u,\zeta,\bar{\zeta}) + \sigma^{+}_2(u,\zeta,\bar{\zeta}), \tag{61}$$

where σ_1^+ is equal to the incoming field in Eq. (39). The second term σ_2^+ is the nontrivial deviation due to the nonlinear interaction of the incoming waves. It is responsible for the gravitational tail that is observed in the nonlinear scattering of gravitational waves and can be obtained from Eq. (60).

To compute $Z^+_{2lJ,\text{cone}}$ it is better to express the integrands in its Fourier decomposition.

$$\bar{\eth}\eth\Omega_2(x^c,\zeta,\bar{\zeta}) = \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\sigma^+(k_1)\bar{\sigma}^+(k_2) \times e^{-ix^c(k_1+k_2)_c}\bar{\eth}\eth\mathcal{S}_\Omega,$$

and

$$\eta^{ab}\partial_a\Lambda\partial_b\bar{\Lambda} = \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\sigma^+(k_1)\bar{\sigma}^+(k_2)e^{-ix^c(k_1+k_2)_c}\mathcal{S}_\Lambda,$$

with

$$\mathcal{S}_\Lambda(k_1,k_2,\zeta,\bar{\zeta}) = G_{2,2'}(\zeta,\bar{\zeta},\hat{k}_1)G_{-2,-2'}(\zeta,\bar{\zeta},\hat{k}_2)\eta^{ab}k_{1a}k_{2b}. \tag{62}$$

Following a calculation derived in the Appendix one can show that

$$Z^+_{2lI,\text{cone}}(x^a) = i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}e^{-ix^c(k_1+k_2)_c}\sigma_1^+(k_1)\bar{\sigma}_1^+(k_2) \times \mathcal{S}^+_{lI}(k_1,k_2), \tag{63}$$

with

$$\mathcal{S}^+_{lI}(k_1,k_2) = \oint d^2SY^0_{l,I}(\zeta,\bar{\zeta}) \times \frac{\bar{\eth}\eth\mathcal{S}_\Omega(k_1,k_2,\zeta,\bar{\zeta}) + \mathcal{S}_\Lambda(k_1,k_2,\zeta,\bar{\zeta})}{l^{+c}(\zeta,\bar{\zeta})(k_1+k_2)_c}.$$

Thus, it gives a second-order correction term to the radiation data on the null cone cut and its time derivative, which is related to the metric, is simply the quadratic integrand evaluated at the null cone cut. Any $l=1,2$ part of $\Sigma(Z,\zeta,\bar{\zeta})$ is eliminated when multiplied by $G_{00'}$, thus the mass and linear momentum of the system do not contribute directly to the scattering.

Hence, each second-order advanced mode solution to the NSF equation reads

$$Z^+_{2lI,\text{cut}} = \oint\Big(Y^{-2}_{lI}(\hat{k})\Big[\sigma_1^+(Z_1,\hat{k}) + \sigma_2^+(Z_0^+,\hat{k})\Big] + \text{c.c.} + Y^0_{lI}(\hat{k})\Sigma^+(Z_0^+,\hat{k})\Big)d^2\hat{k}, \tag{64}$$

$$Z^+_{2lI,\text{cone}} = i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\Big(e^{-ix^c(k_1+k_2)_c}\sigma^+(k_1)\bar{\sigma}^+(k_2) \times \mathcal{S}^+_{lI}(k_1,k_2) + \text{c.c.}\Big). \tag{65}$$

Likewise, the retarded solution for Z yields

$$Z^-_{2lI,\text{cut}} = (-1)^l\oint\Big(Y^{-2}_{lI}(\hat{k})\sigma^-(Z_1^-,\hat{k}) + Y^0_{lI}(\hat{k})\Sigma^-(Z_0^-,\hat{k}) + \text{c.c.}\Big)d^2\hat{k}, \tag{66}$$

$$Z^-_{2lI,\text{cone}} = i(-1)^l\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\Big(\sigma_1^-(k_1)\bar{\sigma}_1^-(k_2) \times e^{-ix^c(k_1+k_2)_c}\mathcal{S}^-_{lI}(k_1,k_2) + \text{c.c.}\Big). \tag{67}$$

The relationship between the incoming an outgoing radiation then follows from $Z^+(x^a,\zeta,\bar{\zeta})+Z^-(x^a,\hat{\zeta},\hat{\bar{\zeta}})=0$. It follows from $Z_1^+(x^a,\zeta,\bar{\zeta})+Z_1^-(x^a,\hat{\zeta},\hat{\bar{\zeta}})=0$ and $\sigma_1^+(u,\zeta,\bar{\zeta})+\bar{\sigma}^-(u,\hat{\zeta},\hat{\bar{\zeta}})=0$ that the terms $\sigma_1^+(Z_1,\hat{k})$ and $\Sigma^+(Z_0^+,\hat{k})$ in $Z^+_{2lI,\text{cut}}$ cancel out with the analogous terms in $Z^-_{2lI,\text{cut}}$ since they correspond to gravitational waves that have the same functional form with the same cuts at both ends.

One then obtains

$$\oint\Big(Y^{-2}_{lI}(\hat{k})\sigma_2^+(Z_0^+,\hat{k}) + \text{c.c.}\Big)d^2\hat{k} + Z^+_{2lI,\text{cone}} + (-1)^l Z^-_{2lI,\text{cone}} = 0, \tag{68}$$

where $\sigma_2^+(u,\hat{k})$ is the unknown, and $Z^+_{2lI,\text{cone}}$, $Z^-_{2lI,\text{cone}}$, are integrals on the future and past null cones from x^a respectively and depend on $\sigma^-(v,\zeta,\bar{\zeta})$. To obtain an explicit result we consider the case where $\sigma^-(v,\zeta,\bar{\zeta})$ has compact support. Selecting appropriate times v_i and v_f so that there are no incoming waves for $v<v_i$ or $v_f<v$, then we would like to obtain the gravitational tail for the radiation at future null infinity at sufficient large value of v so that the "free" incoming gravitational waves have died out. Identifying the values $u_f=v_f$, this corresponds to values $u>u_f$. For this case the corresponding scattering formula reads

$$\int \frac{d^3k}{2w}Y^{-2}_{lI}(\hat{k})\sigma_2^+(w,\hat{k})e^{-ix^ck_c} + i(-1)^l \times \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\sigma_1^-(k_1)\bar{\sigma}_1^-(k_2)e^{-ix^c(k_1+k_2)_c}\mathcal{S}^-_{lI} + \text{c.c.} = 0. \tag{69}$$

One can see in Eq. (69) the explicit quadratic dependence on the incoming field and the details of the interaction are contained in Eq. (71). Following a calculation given in the Appendix we obtain,

$$\sigma_2^+(u,\zeta,\bar{\zeta}) = i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2} e^{-iu|\vec{k}_1+\vec{k}_2|}\sigma_1^-(\vec{k}_1)\bar{\sigma}_1^-(\vec{k}_2) \times \mathcal{S}^-(\vec{k}_1,\vec{k}_2,\hat{\zeta},\hat{\bar{\zeta}}), \quad (70)$$

with

$$\mathcal{S}^-(\vec{k}_1,\vec{k}_2,\hat{\zeta},\hat{\bar{\zeta}}) = Y^2_{lI}(\hat{\zeta},\hat{\bar{\zeta}})\mathcal{S}^-_{lI}(k_1,k_2), \quad (71)$$

and we have used $l^{-a} = -\hat{l}^a$, $G_{00'} = \hat{G}_{00'}$ together with the coordinate transformation $\zeta \to \hat{\zeta}$ to get the final form of Eq. (70). Note that $\sigma^-(u,\zeta,\bar{\zeta})$ is absent in the above equation since it has compact support and vanishes for $u = x^a l_a^+ > u_f$.

It is worth making some remarks regarding these results:

(i) Equation (70) exhibits the tail of the gravitational wave due to the self-interaction of the incoming radiation since it is evaluated for times $u > u_f$. Gravitational tails are usually produced by back-scattering of the outgoing gravitational radiation emitted by an isolated system [12], and in this case we find an analogous behavior.

(ii) The equation for the frequency decomposition of (70) is also interesting since it shows that one can obtain different helicity values for the outgoing gravitational wave that may not be present in the incoming wave.

(iii) Equation (70) is also interesting in a quantum field theory approach since it gives the probability scattering amplitude that an incoming graviton with a given value of quantum numbers end up with a different set of outgoing quantum numbers.

One can perform a second check on the above equation, assuming again that the incoming radiation has compact support but now $\sigma^-(v,\zeta,\bar{\zeta})$ vanishes for times $v < v_i$. For those Bondi times the retarded null cone cuts correspond to a flat metric and therefore, the integral terms vanish in Eq. (68) since the incoming radiation has not been turned on yet. In this situation one gets a similar formula with a different meaning,

$$\sigma_2^+(u,\zeta,\bar{\zeta}) = i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2} e^{-iwu}\sigma_1^-(\vec{k}_1)\bar{\sigma}_1^-(\vec{k}_2) \times \mathcal{S}^+(\vec{k}_1,\vec{k}_2,\hat{\zeta},\hat{\bar{\zeta}}). \quad (72)$$

The term $Z^+_{2lI,\mathrm{cone}}$ in Eq. (68) represents the contribution of the future null cone from a point x^a such that $u \geq x^a l_a$. It is nonvanishing since it contains the radiation that has gone from past to future null infinity. Therefore, the outgoing radiation $\sigma_2^+(u,\zeta,\bar{\zeta})$ cannot vanish. This formula appears to contradict causality but in general relativity both the advanced and retarded solutions are equally valid to describe the underlying metric of the spacetime. The formula Eq. (72) is telling us that even when the future null cone cut from x^a is flat since the metric of the spacetime is still η_{ab}, the nonlinearity of the theory is responsible for a nontrivial radiation $\sigma_2^+(u,\zeta,\bar{\zeta})$ that compensates the integral term on the future null cone from x^a. One could call this situation a gravitational trailer, it tells us what is going to happen in the future since the movie has already been filmed.

The general formula, valid up to second order, is given by

$$\sigma^+(u,\zeta,\bar{\zeta}) + \bar{\sigma}^-(u,\hat{\zeta},\hat{\bar{\zeta}}) - i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2} e^{-iwu}\sigma^-(\vec{k}_1) \times \bar{\sigma}^-(\vec{k}_2)(\mathcal{S}^+ + \mathcal{S}^-) = 0, \quad (73)$$

with $w = |\vec{k}_1 + \vec{k}_2|$ and $\mathcal{S}^+$, $\mathcal{S}^-$ defined before.

C. A working example

Although pp waves cannot be analyzed in this formulation since by assumption the spacetime has no singularities, it is nevertheless interesting to assume the incoming gravitational waves are plane fronted. This is accomplished assuming the Bondi shear at past null infinity has the form

$$\sigma^-(v,\zeta,\bar{\zeta}) = \sigma^-(v)_{lI} Y^2_{lI}(\zeta,\bar{\zeta})\delta(\zeta-\zeta_o)\delta(\bar{\zeta}-\bar{\zeta}_o), \quad (74)$$

where $(\zeta_o,\bar{\zeta}_o)$ is a fixed point on the sphere of the Bondi coordinates, and $\sigma^-(v)_{lI}$ is assumed to have compact support. Using Eqs. (38) and (39) one can write the metric as

$$h_{ab} = \widehat{m_{oa}}\widehat{m_{ob}}\dot{\bar{\sigma}}^-(u = x_a\hat{l}^a_o, \widehat{\zeta_o}, \hat{\bar{\zeta}}_o) + \mathrm{c.c.},$$

which represents a transverse plane fronted spin-2 field propagating along the direction $\hat{l}^{+a}_o$ that ends up at the point $(u = x_a\hat{l}^a_o, \hat{\zeta}_o, \hat{\bar{\zeta}}_o)$. A similar construction to (74) can be given with another plane fronted wave from the null direction $(\hat{\zeta}_o, \hat{\bar{\zeta}}_o)$ at past null infinity, and the idea is to obtain the nontrivial scattering of these two waves.

For simplicity we will assume that the incoming gravitons with antipodal directions only have a quadrupolar component, i.e., $l = 2$ and they have exactly the same energy and frequency decomposition. The corresponding Bondi shear can be written as

$$\sigma^-(v,\zeta,\bar{\zeta}) = \sigma^-(v)_{ij} Y^2_{2ij}(\zeta,\bar{\zeta})\Big[\delta(\zeta-\zeta_o)\delta(\bar{\zeta}-\bar{\zeta}_o) + \delta(\zeta-\hat{\zeta}_o)\delta(\bar{\zeta}-\hat{\bar{\zeta}}_o)\Big], \quad (75)$$

and we assume that $\sigma^-(v)_{ij}$ admits a Fourier decomposition

$$\sigma^-(v)_{ij} = \int_\infty^\infty \sigma^-(w)_{ij} e^{iwv} dw. \quad (76)$$

Since, $\sigma^-(v)_{ij}$ is assumed to have compact support then $\sigma^-(w)_{ij}$ cannot have compact support. An example of the above is $\sigma_{ij}\delta(v)$ with σ_{ij} a constant symmetric trace free matrix. The corresponding transform is given by σ_{ij}. In general, the Fourier transform of a function of compact support goes to zero when $w \to \pm\infty$.

Finally, we give some identifications that are useful to apply the formulas derived before.

$$\sigma^-(w,\zeta,\bar{\zeta}) = \sigma^-(w,\hat{k}) = \sigma^-(\vec{k}), \tag{77}$$

with $w = \sqrt{\vec{k}.\vec{k}}$, since we are assuming spherical coordinates are used in the description of the data given in momentum space. It follows from the construction that the outgoing wave will also exhibit a singular behavior, the nontrivial part is quadratic on the incoming data. Thus, it is best to perform the calculation *ab initio* to see where are the conflicting terms. It is also convenient to regularize the delta function to avoid ill-defined terms like the squared of a delta function. This is equivalent to say that the incoming data is peaked around a generator $(\zeta_o, \overline{\zeta_o})$.

Following Eq. (70) we write,

$$\sigma_2^+(u,\zeta,\bar{\zeta}) = iY^2_{lI}(\hat{\zeta},\hat{\bar{\zeta}})\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}e^{-iu|\vec{k}_1+\vec{k}_2|}\sigma_1^-(\vec{k}_1)\bar{\sigma}_1^-(\vec{k}_2) \times \mathcal{S}^-_{lI}(k_1,k_2), \tag{78}$$

from which we can extract the component $\sigma_2^+(u)_{lI}$ as

$$\sigma_2^+(u)_{lI} = i(-1)^l \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}e^{-iu|\vec{k}_1+\vec{k}_2|}\sigma_1^-(\vec{k}_1)\bar{\sigma}_1^-(\vec{k}_2) \times \mathcal{S}^-_{lI}(k_1,k_2), \tag{79}$$

$$\sigma^+(u,\zeta,\bar{\zeta}) = \sigma^+_{00} + \sigma^+_{\hat{0}\hat{0}} + \sigma^+_{0\hat{0}} + \sigma^+_{\hat{0}0}, \tag{80}$$

where each contribution is labeled according to the self interacting and the interference terms. An explicit calculation yields

$$\begin{aligned}\sigma^+_{00} &= i\int_0^\infty \frac{w_1}{2}dw_1 \times \int_0^\infty \frac{w_2}{2}dw_2 e^{-iu(w_1+w_2)}\sigma^-(w_1,\hat{k}_o) \times \bar{\sigma}^-(w_2,\hat{k}_o)\mathcal{S}^-_{0,0}(\vec{k}_1,\vec{k}_2,\hat{\zeta},\hat{\bar{\zeta}}),\\
\sigma^-(w_1,\hat{k}_o)\bar{\sigma}^-(w_2,\hat{k}_o) &= \sigma^-(w_1)_{ij}\bar{\sigma}^-(w_2)_{kl} \times \left(\frac{1}{5}\delta_{ik}\delta_{jl} - \frac{1}{7}\delta_{ik}Y^o_{2jl} + \frac{1}{1680}Y^o_{4ijkl}\right),\\
\mathcal{S}^-_{0,0}(\vec{k}_1,\vec{k}_2,\hat{\zeta},\hat{\bar{\zeta}}) &= \frac{w_1w_2}{(w_1+w_2)}\left[1+\frac{w_1w_2}{(w_1+w_2)^2}\right] \times \frac{\bar{\eth}\eth\mathcal{S}_\Omega(\hat{k}_o,\hat{k}_o,\hat{\zeta},\hat{\bar{\zeta}})}{\hat{l}^c l_{o^c}},\end{aligned} \tag{81}$$

since $\mathcal{S}_\Lambda = 0$ for the self-interaction term. Likewise,

$$\sigma^+_{\hat{0}\hat{0}} = \sigma^+_{00}(\hat{k}_o \to -\hat{k}_o), \tag{82}$$

i.e., one replaces the incoming direction by its antipodal angles.

The last two terms represent the non trivial part of the scattering between the two waves and they are more involved formulas,

$$\sigma^+_{0\hat{0}} = i\int_0^\infty \frac{w_1}{2}dw_1\int_0^\infty \frac{w_2}{2}dw_2 e^{-iu|w_1-w_2|}\sigma^-(w_1,\hat{k}_o) \times \bar{\sigma}^-(w_2,-\hat{k}_o)\mathcal{S}^-_{0\hat{0}}, \tag{83}$$

$$\begin{aligned}\sigma^-(w_1,\hat{k}_o)\bar{\sigma}^-(w_2,-\hat{k}_o) &= \sigma^-(w_1)_{ij}\bar{\sigma}^-(w_2)_{kl}Y^o_{4ijkl}\\
\mathcal{S}^-_{0\hat{0}}(\vec{k}_1,\vec{k}_2,\hat{\zeta},\hat{\bar{\zeta}}) &= \frac{\bar{\eth}\eth\mathcal{S}_\Omega + \mathcal{S}_\Lambda}{w_1\hat{l}^c l_{oc} + w_2\hat{l}^c n_{oc}},\end{aligned} \tag{84}$$

with

$$\mathcal{S}_\Lambda = w_1w_2G_{2,2'}(\hat{\zeta},\hat{\bar{\zeta}},\hat{k}_o)G_{-2-2'}(\hat{\zeta},\hat{\bar{\zeta}},-\hat{k}_o), \tag{85}$$

$$\mathcal{S}^+_\Omega = \mathcal{S}_\Lambda\left(l^a_o\hat{l}_a n^b_o\hat{l}_b + \frac{w_1w_2(l^a_o\hat{l}_a n^b_o\hat{l}_b)^2}{(w_1\hat{l}^c l_{oc} + w_2\hat{l}^c n_{oc})^2}\right). \tag{86}$$

One can make several comments regarding the scattered waves:

(i) The first two terms in Eq. (80) are simply the self interaction of each incoming plane wave whereas the last two represent the interaction between the two waves.

(ii) All these terms have non vanishing amplitudes for angles $\zeta \neq \zeta_o$. Thus, the nontrivial outgoing radiation reaches the whole celestial sphere.

(iii) All these terms are nondiverging when $w_{1,2} \to 0$ and/or $\zeta \to \zeta_o$. To show this we first use $l^a_o\hat{l}_a \to 0$, $n^a_o\hat{l}_a \to 1$ in the above expressions and the factor w^2 in the denominator is canceled by an identical term in the numerator. Thus, there are no infrared divergences in the above results. The Green functions involved and their derivatives give finite expressions when $\zeta \to \zeta_o$. This can be seen directly from Eq. (28) by taking $\eth$ and $\eth'$ derivatives to (28) and then taking $\zeta \to \zeta_o$. One thus shows, that there are no infrared divergences in the above results.

(iv) They also pick up other $l \geq 2$ components other than the original $l = 2$ for the incoming field.

(v) Note also that the interaction terms $\sigma^+_{0\hat{0}}$ and $\sigma^+_{\hat{0}0}$ are different from the first two terms since $\mathcal{S}_\Lambda$ vanishes for σ^+_{00} and $\sigma^+_{\hat{0}\hat{0}}$.

(vi) Although the scattering of monochromatic waves cannot be dealt with on this formulation, it nevertheless is interesting to observe that if the incoming waves have the same amplitude and frequency the interaction term is not an outgoing wave. This feature is analogous to the addition of electromagnetic waves traveling in opposite directions and it follows from setting $\sigma^-(w_1) = \sigma^-(w_2) = \sigma(w_o)\delta(w - w_o)$.

V. SUMMARY AND CONCLUSIONS

In this work we use the NSF approach to study the nonlinear interaction of incoming/outgoing gravitational radiation given on the null boundaries of an asymptotically-flat spacetime.

Using the available vacuum field equation for the main variable of the formalism we solve the equations by a perturbation scheme valid up to second order and obtain advanced and retarded solutions that depend on the future or past radiation data representing outgoing or incoming gravitational waves.

We then consider the dispersion relation between incoming and outgoing gravitational waves at first and second order of a perturbation procedure.

In Sec. III we solve the linearized equations obtaining a relation between the incoming Z^- and outgoing Z^+ null foliations. This relation directly follows from the uniqueness of the first-order metric. Formal solutions to the elliptic equation on S^2 are given for null radiation data giving, by means of Eq. (42), a relation for every mode in the decomposition of the gravitational shear σ. Although this relation gives the trivial scattering in every mode of the gravitational radiation, it is important to obtain since it fixes the one to one correspondence between the incoming and outgoing radiation fields.

A similar relation can be derived for every mode at the second order in the perturbation procedure. Although the calculations are technically more involved than in the linear case, a relation between the incoming and outgoing radiation is given in Eq. (68). The two checks done on Eq. (68) to understand the validity of the result give us formulas for the gravitational tail and, what we called, a gravitational trailer.

Finally, we analyze the scattering of two plane-fronted waves coming from opposite directions. The main features of the diffracted waves is that they are scattered throughout the celestial sphere. Also, there is a clear distinction between the self interacting terms and the one that is responsible for the interaction between the two waves. Another feature of this scattering is that the scattered waves have different spherical harmonic decomposition than the incoming waves.

As a final comment we would like to give an idea of future developments following this approach. One of our main goals is to obtain a quantum scattering theory for the asymptotic quantization procedure first presented by Ashtekar in 1986 [4], where free fields are quantized at future or past null infinity. In our formalism the same free data is used for any order of the perturbation procedure, i.e., the phase space is constructed once and for all in our approach. This is extremely important at a classical or quantum level since one can introduce either a canonical form with Poisson brackets or quantum commutation relations for fields given on the null boundaries that will not be modified as one proceeds with a perturbation calculation. The calculations performed at second order do not have divergences if one assumes peeling for the gravitational field. Thus, a future area of research is to derive "Feynman-like" formulas for the higher-order terms and analyze for possible divergences, compute radiative corrections, etc.

There is also an assumption that has been used in this work, namely, the existence of a map between the past and future Bondi coordinates mediated by a regularity assumption of spacelike infinity [1,9]. If one does not have such a map then the scattering matrix construction would have an S^2 gauge freedom. Although some progress has been made to understand the regularity structure of spacelike infinity on a generic situation, there is still much work to be done to get a full understanding of its conformal completion [2,3].

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APPENDIX: DERIVATION OF σ MODES RELATION

We first derive the relationship for the "free" part of Z, that satisfies the wave equation, since it corresponds to undisturbed gravitational waves that propagate from minus to plus null infinity. Z^+_{lI} is given by

$$Z^+_{lI,\mathrm{cut}} = \oint (Y^{-2}_{lI}(\zeta,\bar{\zeta})\sigma^+(u,\zeta,\bar{\zeta}) + Y^2_{lI}(\zeta,\bar{\zeta})\bar{\sigma}^+(u,\zeta,\bar{\zeta}))d^2S, \tag{A1}$$

and we assume the free data $\sigma^+(u,\zeta,\bar{\zeta})$ admits a positive frequency decomposition,

$$\sigma^+(u,\zeta,\bar{\zeta}) = \int_0^\infty \frac{w}{2} dw \sigma^+(w,\zeta,\bar{\zeta}) e^{-iwu}. \tag{A2}$$

We then rewrite the free data part of Z as

$$Z(x)^{+}_{lI,\mathrm{cut}} = \int_0^\infty \frac{w}{2}dw \oint d^2S\Big[Y^{-2}_{lI}(\zeta,\bar\zeta)\sigma^{+}(w,\zeta,\bar\zeta)e^{-iwx^a l_a} + Y^{2}_{lI}(\zeta,\bar\zeta)\bar\sigma^{+}(w,\zeta,\bar\zeta)e^{iwx^a l_a}\Big], \tag{A3}$$

which can be rewritten as

$$Z(x)^{+}_{lI,\mathrm{cut}} = \int \frac{d^3k}{2w}\Big[Y^{-2}_{lI}(\hat k)\sigma^{+}(k)e^{-ix^a k_a} + Y^{2}_{lI}(\hat k)\bar\sigma^{+}(k)e^{ix^a k_a}\Big], \tag{A4}$$

using a spherical decomposition of the 3D momentum space. In the above equation the coordinates $(\zeta,\bar\zeta)$ on the sphere have been rewritten as $(\hat k)$, and $k^a = wl^a(\hat k)$. Note that $k^a k_a = 0$, and we have a field that satisfies the wave equation. Using the eigenfunctions $e^{ix^a k_a}$ we obtain the Fourier transform of the above equation as

$$i\int d^3x e^{ix^a k_a}\overleftrightarrow{\partial}_t Z^{+}_{lI} = Y^{-2}_{lI}(\hat k)\sigma^{+}(w,\hat k). \tag{A5}$$

Finally, integrating on the sphere yields

$$i\oint d^2\hat k\int d^3x e^{ix^a k_a}\overleftrightarrow{\partial}_t Z^{+}_{lI} = \sigma^{+}(w)_{lI}. \tag{A6}$$

$\widehat{Z^{-}}_{lI}$, on the other hand, is given by

$$\widehat{Z^{-}}_{lI} = (-1)^l\oint\Big(Y^{-2}_{lI}(\zeta,\bar\zeta)\sigma^{-}(v,\zeta,\bar\zeta) + Y^{2}_{lI}(\zeta,\bar\zeta)\bar\sigma^{-}(v,\zeta,\bar\zeta)\Big)d^2S, \tag{A7}$$

thus, to compare both expressions one must first rewrite Eq. (A7) as a function of $u = x^a l^{+}_a$ instead of $v = -x^a l^{-}_a$. From $l^{-}_a = -\hat{l^{+}}_a$ we obtain $v = \hat u$. performing a change of variables $\zeta \to \hat\zeta$ in Eq. (A7) we obtain

$$\widehat{Z^{-}}_{lI} = \oint\Big(Y^{-2}_{lI}(\zeta,\bar\zeta)\bar\sigma^{-}(u,\hat\zeta,\hat{\bar\zeta}) + Y^{2}_{lI}(\zeta,\bar\zeta)\sigma^{-}(u,\hat\zeta,\hat{\bar\zeta})\Big)d^2S, \tag{A8}$$

where we have used $Y^{2}_{lI}(\hat\zeta,\hat{\bar\zeta}) = (-1)^l Y^{-2}_{lI}(\zeta,\bar\zeta)$. Note that $\bar\sigma^{-}(u,\hat\zeta,\hat{\bar\zeta})$ is a s.w. 2 function and plays the same role as $\sigma^{+}(u,\zeta,\bar\zeta)$ in Z^{+}_{lI}. Thus,

$$\widehat{Z^{-}}_{lI} = \int_0^\infty \frac{w}{2}dw \oint d^2S\Big[Y^{-2}_{lI}(\zeta,\bar\zeta)\bar\sigma^{-}(w,\hat\zeta,\hat{\bar\zeta})e^{-iwx^a l_a} + Y^{2}_{lI}(\zeta,\bar\zeta)\sigma^{-}(w,\hat\zeta,\hat{\bar\zeta})e^{iwx^a l_a}\Big]. \tag{A9}$$

Finally,

$$i\oint d^2\hat k\int d^3x e^{ix^a k_a}\overleftrightarrow{\partial}_t\widehat{Z^{-}}_{lI} = (-1)^l\overline{\sigma}^{-}(w)_{lI}, \tag{A10}$$

and from $Z^{+}_{lI} + \hat{Z}^{-}_{lI} = 0$ we get

$$\sigma^{+}(w)_{lI} + (-1)^l\bar\sigma^{-}(w)_{lI} = 0. \tag{A11}$$

We now address the mode decomposition of the integration on the future or past null cones from the point x^a starting with the advanced solution, i.e.,

$$Z^{+}_{2lI,\mathrm{cone}} = -\oint d^2S Y^{0}_{lI}(\zeta,\bar\zeta)\int_0^\infty ds\Big(2\eth\bar\eth\Omega_2(y^c,\zeta,\bar\zeta) + \eta^{ab}\partial_a\Lambda_1\partial_b\overline{\Lambda_1}(y^c,\zeta,\bar\zeta)\Big), \tag{A12}$$

where the rhs is integrated on the future null cone from x^a. To express Eq. (A12) in terms of the null free data it is better to rewrite Eq. (A4) as

$$Z^{+}_{1\,lI}(x) = \int d^4k\delta(k^a k_a)\theta(k^o)(Y^{-2}_{lI}(\hat k)\sigma^{+}(k)e^{-ix^a k_a} + \mathrm{c.c.}), \tag{A13}$$

where the second equality follows from the reality condition of $Z^{+}_{1\,lI}(x)$. One can check by a direct calculations that Eq. (A13) is equal to Eq. (A4). One can also write the Fourier transform of Eq. (A13),

$$Z^{+}_{1\,lI}(k) = Y^{-2}_{lI}(\hat k)\sigma^{+}(k)\delta(k^a k_a), \tag{A14}$$

and

$$\Lambda^{+}_1(k,\zeta,\bar\zeta) = Y^{2}_{lI}(\zeta,\bar\zeta)Y^{-2}_{lI}(\hat k)\sigma^{+}(k)\delta(k^a k_a). \tag{A15}$$

Defining the Fourier transform of Ω_2 as

$$8\Omega_2(k^a,\zeta,\bar\zeta) = \int d^4x e^{ix^a k_a}\Big[\partial_r\Lambda_1\partial_r\bar\Lambda_1 + \int_r^\infty dr'\int_{r'}^\infty dr''\partial^2_{r''}\Lambda_1\partial^2_{r''}\bar\Lambda_1\Big], \tag{A16}$$

and using Eq. (A15), we obtain

$$8\Omega_2(k^a,\zeta,\bar{\zeta}) = \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\delta^4(k^a-(k_1+k_2)^a)\sigma^+(k_1)\bar{\sigma}^+(k_2)\mathcal{S}^+_\Omega(k_1,k_2,\zeta,\bar{\zeta}), \tag{A17}$$

with

$$\mathcal{S}^+_\Omega(k_1,k_2,\zeta,\bar{\zeta}) = G_{2,2'}(\zeta,\bar{\zeta},\hat{k}_1)G_{-2,-2'}(\zeta,\bar{\zeta},\hat{k}_2)\left(l^{+a}(\zeta,\bar{\zeta})k_{1a}l^{+b}(\zeta,\bar{\zeta})k_{2b} + \frac{(l^{+a}(\zeta,\bar{\zeta})k_{1a}l^{+b}(\zeta,\bar{\zeta})k_{2b})^2}{(l^{+a}(\zeta,\bar{\zeta})k_a)^2}\right). \tag{A18}$$

Inverting $\Omega_2(k^a,\zeta,\bar{\zeta})$ one gets

$$8\Omega_2(x^a,\zeta,\bar{\zeta}) = \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}e^{-ix^a(k_1+k_2)_a}\sigma^+(k_1)\bar{\sigma}^+(k_2)\mathcal{S}^+_\Omega(k_1,k_2,\zeta,\bar{\zeta}). \tag{A19}$$

A similar calculation gives

$$\eta^{ab}\partial_a\Lambda\partial_b\Lambda = \int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\sigma^+(k_1)\bar{\sigma}^+(k_2)e^{-ix^c(k_1+k_2)_c}\mathcal{S}_\Lambda,$$

with

$$\mathcal{S}_\Lambda(k_1,k_2,\zeta,\bar{\zeta}) = G_{2,2'}(\zeta,\bar{\zeta},\hat{k}_1)G_{-2,-2'}(\zeta,\bar{\zeta},\hat{k}_2)\eta^{ab}k_{1a}k_{2b}. \tag{A20}$$

The above results are then used to compute Eq. (A12) giving

$$Z(x^a)^+_{2lI,\mathrm{cone}} = i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}e^{-ix^c(k_1+k_2)_c}\sigma^+(k_1)\bar{\sigma}^+(k_2)\mathcal{S}^+_{lI}(k_1,k_2), \tag{A21}$$

with

$$\mathcal{S}^+_{lI}(k_1,k_2) = \oint d^2SY^0_{lI}(\zeta,\bar{\zeta})\frac{\bar{\eth}\eth\mathcal{S}^+_\Omega(k_1,k_2,\zeta,\bar{\zeta}) + \mathcal{S}_\Lambda(k_1,k_2,\zeta,\bar{\zeta})}{l^{+c}(\zeta,\bar{\zeta})(k_1+k_2)_c}.$$

Note that $Z(x^a)^+_{2lI,\mathrm{cone}}$ does not satisfy the wave equation since $(k_1+k_2)^c$ is not a null vector.

Taking the Fourier transform of $Z^+_{2\,lI,\mathrm{cut}}(x) + Z^+_{2\,lI,\mathrm{cone}}(x)$, using Eqs. (A4) and (A21), and integrating on the frequency and on the sphere in momentum space we get

$$Z^+_{2\,lI}(\vec{k}) = \frac{Y^{-2}_{lI}(\hat{k})\sigma(\vec{k})}{2w} + i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\delta^3(\vec{k}-(\overrightarrow{k_1}+\overrightarrow{k_2})\sigma^+(k_1)\overline{\sigma}^+(k_2)\mathcal{S}^+_{lI}(k_1,k_2)), \tag{A22}$$

with $\vec{k} = w\hat{k}$ since we are using spherical coordinates in momentum space. Integrating in the angular coordinates gives

$$Z^+_{lI}(w) = \frac{w}{2}\sigma^+_{lI}(w) + i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}\delta(w-|\overrightarrow{k_1}+\overrightarrow{k_2}|)\sigma^+(k_1)\overline{\sigma}^+(k_2)\mathcal{S}^+_{lI}(k_1,k_2). \tag{A23}$$

One can also take the inverse Fourier transform of this equation. Multiplying by e^{-iuw}, integrating on the positive frequencies and using Eq. (A2) we get

$$Z(u)^+_{lI} = \sigma(u)^+_{lI} + i\int \frac{d^3k_1}{2w_1}\frac{d^3k_2}{2w_2}e^{-iu|\overrightarrow{k_1}+\overrightarrow{k_2}|}\sigma^+(k_1)\overline{\sigma}^+(k_2)\mathcal{S}^+_{lI}(k_1,k_2) + \mathrm{c.c.} \tag{A24}$$

To obtain the contribution of the retarded solution we first write

$$Z^-_{\mathrm{cut}}(x^a,\zeta,\bar{\zeta}) = \oint_{S^2}(\bar{\eth}'^2G^-_{00}(\zeta,\bar{\zeta},\zeta',\bar{\zeta}')\sigma^-(v,\zeta',\bar{\zeta}') + \mathrm{c.c.})dS', \tag{A25}$$

with $v = -x^a l^-_a = -x^a\hat{l}^+_a$. In order to compare with $Z^+(x^a,\zeta,\bar{\zeta})$ we perform a change of variables $\zeta' \to \hat{\zeta}'$ giving

$$Z^-_{\mathrm{cut}}(x^a,\hat{\zeta},\hat{\bar{\zeta}}) = \oint_{S^2}(\bar{\eth}'^2G^-_{00}(\hat{\zeta},\hat{\bar{\zeta}},\widehat{\zeta'},\widehat{\bar{\zeta}'})\sigma^-(-u,\widehat{\zeta'},\widehat{\bar{\zeta}'}) + \mathrm{c.c.})dS'. \tag{A26}$$

Thus, the positive frequency decomposition of $Z^{-}_{\rm cut}(x^a,\hat{\zeta},\hat{\bar{\zeta}})$ is given by

$$Z^{-}_{\rm cut}(x^a,\hat{\zeta},\hat{\bar{\zeta}}) = \int_0^\infty dw \oint_{S^2} dS'(\bar{\eth}'^2 G^{-}_{00}(\hat{\zeta},\hat{\bar{\zeta}},\widehat{\zeta'},\widehat{\bar{\zeta}'})\sigma^{-}(w,\widehat{\zeta'},\hat{\bar{\zeta}}')e^{iwx^a l'_a} + \text{c.c.}), \tag{A27}$$

and,

$$Z^{-}_{\rm cut}(x^a)_{lJ} = (-1)^l \int_0^\infty dw \oint d^2S'[Y'^2_{lJ}(\zeta',\bar{\zeta}')\bar{\sigma}^{-}(w,\zeta',\bar{\zeta}')e^{-iwx^a l'_a} + Y'^{-2}_{lJ}(\zeta',\bar{\zeta}')\sigma^{-}(w,\zeta',\bar{\zeta}')e^{iwx^a l'_a}], \tag{A28}$$

where we have used $\widehat{Y^2}_{lJ} = (-1)^l Y^{-2}_{lJ}$ and $\widehat{Y^2_{lJ}Y^{-2}_{l'J}} = Y^2_{lJ}Y^{-2}_{l'J}$. Thus, directly from

$$Z^{+}_{\rm cut}(x^a,\zeta,\bar{\zeta}) + Z^{-}_{\rm cut}(x^a,\hat{\zeta},\hat{\bar{\zeta}}) = 0,$$

we get

$$\sigma^{+}_{lJ}(w) + (-1)^l \bar{\sigma}^{-}_{lJ}(w) = 0.$$

To obtain the past null cone contribution we have to replace $y^{+c} = x^c + sl^{+c}$ by $y^{-c} = x^c + sl^{-c}$. Thus,

$$Z^{-}_{2lI,\text{cone}}(x^a) = i(-1)^l \int \frac{d^3k}{2w}\frac{d^3k'}{2w'} e^{-ix^c(k+k')_c}\sigma^{-}(k)\bar{\sigma}^{-}(k')\mathcal{S}^{-}_{lI}(k,k'). \tag{A29}$$

Finally, from $Z^{+}_2(x^a,\zeta,\bar{\zeta}) + Z^{-}_2(x^a,\hat{\zeta},\hat{\bar{\zeta}}) = 0$, we get

$$\sigma^{+}(u)_{lI} + (-1)^l\bar{\sigma}^{-}(u)_{lI} + i\int \frac{d^3k}{2w}\frac{d^3k'}{2w'} e^{-iu|\vec{k}+\vec{k'}|}\sigma^{+}(k)\bar{\sigma}^{+}(k')(\mathcal{S}^{+}_{lI} + (-1)^l\mathcal{S}^{-}_{lI}) = 0. \tag{A30}$$

Multiplying the above equation by $Y^2_{lI}(\zeta,\bar{\zeta})$ and using $(-1)^l Y^2_{lI}(\zeta,\bar{\zeta}) = Y^{-2}_{lI}(\hat{\zeta},\hat{\bar{\zeta}})$ gives

$$\sigma^{+}(u,\zeta,\bar{\zeta}) + \bar{\sigma}^{-}(u,\hat{\zeta},\hat{\bar{\zeta}}) - i\int \frac{d^3k}{2w}\frac{d^3k'}{2w'} e^{-iu|\vec{k}+\vec{k'}|}\sigma^{-}(\vec{k})\bar{\sigma}^{-}(\vec{k'})(\mathcal{S}^{+}(\vec{k},\vec{k'},\zeta,\bar{\zeta}) + \mathcal{S}^{-}(\vec{k},\vec{k'},\hat{\zeta},\hat{\bar{\zeta}})) = 0, \tag{A31}$$

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