Slowly rotating Kerr metric derived from the Einstein equations in affine-null coordinates

Thomas Mädler [*]
Escuela de Obras Civiles and Instituto de Estudios Astrofísicos, Facultad de Ingeniería y Ciencias, Universidad Diego Portales, Avenida Ejército Libertador 441, Casilla 298-V, Santiago, Chile

Emanuel Gallo [†]
FaMAF, UNC; Instituto de Fsica Enrique Gaviola (IFEG), CONICET, Ciudad Universitaria, (5000) Córdoba, Argentina

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Using a quasispherical approximation of an affine-null metric adapted to an asymptotic Bondi inertial frame, we present high order approximations of the metric functions in terms of the specific angular momentum for a slowly rotating stationary and axisymmetric vacuum spacetime. The metric is obtained by following the procedure of integrating the hierarchy of Einstein equations in a characteristic formulation utilizing master functions for the perturbations. It further verified its equivalence with the Kerr metric in the slowly rotating approximation by carrying out an explicit transformation between the Boyer-Lindquist coordinates to the employed affine-null coordinates. A peculiar feature of the derivation is that in the solution of the perturbation equations for every order a new integration constant appears which cannot be set to zero using asymptotical flatness or regularity arguments. However, these additional integration constants can be absorbed into the overall Komar mass and Komar angular momentum of a slowly rotating black hole.

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I. INTRODUCTION

At the dawn of the "golden era of general relativity" in the 1960s of the last century, two important spacetime metrics were found, the Bondi-Sachs metric [1–3] and the Kerr metric [4,5]. The first settled the question that an isolated system loses mass via gravitational radiation and that this effect is a nonlinear effect of general relativity; while the second describes a stationary and rotating isolated black hole that is expected to be the end product of a gravitational collapse of a massive star or a merger of two compact objects.

One of the defining features of the Bondi-Sachs metric is that one coordinate is constant along a family of null hypersurfaces while a radial coordinate along these null hypersurfaces is an areal distance that can be related to a luminosity distance [6]. Indeed, the first long term stable evolution of black hole spacetimes were made using such families of null hypersurfaces in a null cone-world tube formalism [7]; also see [8,9] for review. Apart from usage in numerical relativity simulations, the Bondi-Sachs metric is now frequently used in high energy physics addressing questions of the AdS/CFT correspondence [10] (and citations thereof). It also became popular to discuss gravitational wave memory effects [11–15]. A pleasant property of the Bondi-Sachs formalism is that the Einstein equations can be solved in a hierarchical manner when initial data on a null hypersurface and boundary conditions at a null hypersurface [16], world tube [17], or vertex [18–20] are given. However, the radial coordinate of the Bondi-Sachs metric has the unpleasant property that it breaks down when an apparent horizon forms due to the focusing of the surface-forming null rays and their vanishing expansion. This can be overcome in choosing an affine parameter as the radial coordinate because an affine parameter only becomes singular at a caustic. But, the Einstein equations resulting from an affine-null metric do not provide the hierarchical structure as the Bondi-Sachs metric [9] and the hierarchical structure needs to be reestablished by various new definitions of variables [21–23]. Moreover, it turns out that the hierarchy of equations in the affine-null metric formulation also breaks down in the events of apparent horizon formation, but fortunately the equations can be regularized so that it is possible to follow the formation of black holes up to singularity [24,25].

Despite the success and popularity of the Bondi-Sachs metric in the various areas, an explicit closed analytical representation of the Kerr metric in Bondi-Sachs form

[*]thomas.maedler@mail.udp.cl
[†]egallo@unc.edu.ar

without bad behavior in the exterior region or related metrics with one or two null coordinates is missing. Various attempts have been made to derive a null metric representation numerically [26,27] as well as analytically [28–30]. In *all* of the approaches, the authors start out with the Kerr metric and then calculate the respective null metric via a coordinate transformation. After these transformations the resulting metric can still possess a conical singularity at the axis of symmetry (see [27] for a complete discussion). In addition, the final metric is determined by integrals of nonelementary functions. The approach of Argañaraz and Moreschi [27] differs from the aforementioned ones that the authors aim to find a double-null representation of the Kerr metric by geometrically adopting the coordinates to in- and outgoing null geodesics adapted to the center of mass [31]. In this way, the authors were successful in finding null coordinates that are not only regular at every point of the external communication region (unlike the previous formulations) but also that they are regular at the event horizon, thus allowing a way to study the evolution of different matter fields (as scalar fields) in such background even when they cross the event horizon [32]. Unfortunately, even in their construction arises a differential equation that needs to be solved numerically and an explicit closed form representation of the double null version of the Kerr metric is not possible. The work of Bai and collaborators [33,34] also starts with the Kerr metric (in Boyer-Lindquist coordinates) and then makes coordinate transformation to a Bondi-Sachs metric valid near future null infinity (in a compactified version of the metric). The authors are able to calculate the Newman-Penrose quantities and multipoles at large distances and show the peeling property of the Weyl tensor at large radii and the vanishing of the so-called Newman-Penrose constants.

In this article, *in contrast* to all the previous works which start with the Kerr metric expressed in Boyer-Lindquist coordinates and attempt to find a null coordinate version of it, we will *directly solve* the Einstein equations in a characteristic formulation based on an affine-null metric formulation of the Einstein equations. In addition, inspired by the Hartle-Thorne methods for obtaining solutions for slowly rotating compact stars [35], we will employ a quasispherical approximation of the field equations to find a high order approximation of the Kerr metric in outgoing polar null coordinates. To obtain our solution, we assume stationarity and axial symmetry. We further require an asymptotic inertial observer as well as that Weyl scalar Ψ_0 is regular everywhere where the background solution is regular. A study of vacuum stationary metrics with a smooth future null infinity in affine-null coordinates has recently been carried out by Tafel in [36] by considering power series of the metric components in terms of the inverse affine distance.

Throughout the article, we will use signature $+2$, units $G = c = 1$, and the Einstein sum convention for indices as well as products of associated Legendre polynomials.

The article is organized as follows: Sec. II recalls the affine-null metric formulation, makes the necessary symmetry assumptions for archiving our goal, and defines the perturbative variables; in Sec. III, we determine the background model (Sec. III A), define useful recursively reappearing functions in the perturbation analysis (Sec. III B), and solve the perturbation equations (Sec. III C–III F); in Sec. III G the affine-null metric functions for the null are expressed in terms of the mass and specific angular momentum; in Sec. IV, to verify our results, we calculate the affine-null version of the Kerr metric in a Bondi frame via a coordinate transformation with a method adopted from [33]; in Sec. V the position of the outer ergosphere and (past) event horizon of the black hole are discussed; and Sec. VI contains the final discussion of our work. The article finishes with two appendixes: Appendix A lists relations between associated Legendre polynomials and Appendix B presents a derivation of the expression of the Komar charges relevant for this work.

II. AFFINE-NULL METRIC FORMULATION FOR STATIONARY AND AXIAL SYMMETRIC SPACETIMES

Here we review the necessary properties of characteristic initial value formulation of the Einstein equations in affine-null coordinates, discuss the implications of the imposed symmetry assumptions and present the notation used in our analysis.

Taking coordinates $x^a = (u, \lambda, x^A)$, where u is an outgoing null coordinate, λ is an affine parameter, and x^A are angular coordinates, a generic line element for an affine-null metric defined with respect to a family of outgoing null hypersurfaces $u = \text{const}$ is [21–23,37]

$$g_{ab}dx^a dx^b = -W du^2 - 2 du d\lambda + R^2 h_{AB}(dx^A - W^A du)(dx^B - W^B du). \quad (2.1)$$

The determinant $\det(h_{AB}) = \det(q_{AB}) = \sin^2\theta$ is the determinant of a round unit sphere metric q_{AB}. We remark that the affine parameter λ is chosen along the outgoing null hypersurfaces $u = \text{const}$ such that $\nabla^a u \nabla_a \lambda = -1$ everywhere along the rays generating the null hypersurfaces $u = \text{const}$ [37]. Consequently h_{AB} is transverse-traceless and has only 2 degrees of freedom. Thus, the function R relates to the area of cuts $du = d\lambda = 0$. The nonzero components of the inverse metric are given by

$$g^{u\lambda} = -1, \quad g^{\lambda\lambda} = W, \quad g^{\lambda A} = -W^A, \quad g^{AB} = \frac{h^{AB}}{R^2}, \quad (2.2)$$

where $W^A = (W^\theta, W^\phi)$ and $h_{AB}h^{BC} = \delta_A^C$ and in particular [38]

$$h_{AB}dx^A dx^B = \left(e^{2\gamma}d\theta^2 + \frac{\sin^2\theta}{e^{2\gamma}}d\phi^2\right)\cosh(2\delta) + 2\sin\theta\sinh(2\delta)d\theta d\phi. \tag{2.3}$$

A complex null dyad to represent the 2-metric h_{AB} such as $h_{AB} = m_{(A}\bar{m}_{B)}$ with $m^A m^B h_{AB} = m^A \bar{m}^B h_{AB} - 1 = 0$ is

$$m^A\partial_A = \frac{1}{\sqrt{2}e^\gamma}(\cosh\delta - i\sinh\delta)\partial_\theta + \frac{ie^\gamma}{\sqrt{2}\sin\theta}(\cosh\delta + i\sinh\delta)\partial_\phi, \tag{2.4}$$

As in any Bondi-Sachs type metric [9], the vacuum field equations $R_{ab} = 0$ with R_{ab} being the Ricci tensor can be grouped into supplementary equations $S_i = 0$ with

$$S_i = (R_{uu}, R_{u\theta}, R_{u\phi}), \tag{2.5}$$

one trivial equation, $R_{u\lambda} = 0$, and the six main equations $H_K^{(\gamma)} = 0$, $K \in (1,2,3,4)$ and $H_k^{(\delta)} = 0$, $k \in (1,2)$ with

$$H_K^{(\gamma)} = (R_{\lambda\lambda}, R_{\lambda\theta}, h^{AB}R_{AB}, \mathfrak{Re}(m^A m^B R_{AB})),$$
$$H_k^{(\delta)} = (R_{\lambda\phi}, \mathfrak{Im}(m^A m^B R_{AB})), \tag{2.6}$$

with $\mathfrak{Re}(x)$ and $\mathfrak{Im}(x)$ the real and imaginary parts of x, respectively. We assume that the spacetime is axisymmetric and stationary with associated Killing vector fields ∂_u and ∂_ϕ. Therefore the metric functions do not depend on u and ϕ. The Killing symmetries imply two conserved quantities, the Komar mass, K_m, and the Komar angular momentum, K_L, which can be calculated from their respective integrals (also see Appendix B)

$$K_m \coloneqq K(\partial_u) = \frac{1}{8\pi}\lim_{\lambda\to\infty}\oint (W_{,\lambda} - R^2 h_{AB}W^A W^B_{,\lambda})R^2 d^2q, \tag{2.7}$$

while for the axial Killing vector we have

$$K_L \coloneqq K(\partial_\phi) = -\frac{1}{16\pi}\lim_{\lambda\to\infty}\oint (R^4 h_{\phi B}W^B_{,\lambda})d^2q, \tag{2.8}$$

where $dq = \sin\theta d\theta d\phi$ is the surface area element of the unit sphere.

Let us assume there is a smooth one parameter family of stationary and axially symmetric metrics $g_{ab}(\varepsilon)$, where ε is a small parameter such that $\varepsilon = 0$ corresponds to a (static) spherically symmetric spacetime solution of the vacuum Einstein equations. Then there is an expansion of the metric fields such as

$$R(\lambda,\theta) = r(\lambda) + R_{[1]}(\lambda,\theta)\varepsilon + R_{[2]}(\lambda,\theta)\varepsilon^2 + R_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \tag{2.9a}$$

$$W(\lambda,\theta) = V(\lambda) + W_{[1]}(\lambda,\theta)\varepsilon + W_{[2]}(\lambda,\theta)\varepsilon^2 + W_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \tag{2.9b}$$

$$W^A(\lambda,\theta) = W^A_{[1]}(\lambda,\theta)\varepsilon + W^A_{[2]}(\lambda,\theta)\varepsilon^2 + W^A_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \tag{2.9c}$$

$$\gamma(\lambda,\theta) = \gamma_{[1]}(\lambda,\theta)\varepsilon + \gamma_{[2]}(\lambda,\theta)\varepsilon^2 + \gamma_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \tag{2.9d}$$

$$\delta(\lambda,\theta) = \delta_{[1]}(\lambda,\theta)\varepsilon + \delta_{[2]}(\lambda,\theta)\varepsilon^2 + \delta_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4). \tag{2.9e}$$

Inserting (2.9) in (2.7) and (2.8) implies $K_m = O(\varepsilon^0)$ and $K_L = O(\varepsilon)$. We make the requirements

$$K_m(\varepsilon) = K_m(-\varepsilon), \qquad K_L(\varepsilon) = -K_L(-\varepsilon). \tag{2.10}$$

These conditions imply that under the change $\varepsilon \to -\varepsilon$ the sense of rotation is reversed [recall that $K(\partial_\phi) = -K(\partial_{(-\phi)})$]. From the metric (2.1), we see that the 2-surfaces with $u = u_0$ and $\lambda = \lambda_0$, defined such that $R(u_0, \lambda_0, \theta) = \text{const}$ have the induced metric $R^2 h_{AB}dx^A dx^B$ with area $4\pi R^2(u_0, \lambda_0)$. We assume that the area of these 2-surfaces is invariant under the change $\varepsilon \to -\varepsilon$, which implies that R^2 is an even function of ε. Therefore R is either an even or an odd function of ε. However, if R were an odd function, we had $R(\varepsilon = 0) = 0$, which is a nonadmissible solution. In addition, $ds^2(\partial_\phi, \partial_\phi)$ and $ds^2(\partial_\theta, \partial_\theta)$ must be independent of the sense of rotation implying that $h_{\phi\phi}$ and $h_{\theta\theta}$ are even. However, due to the frame dragging effect $ds^2(\partial_\theta, \partial_\phi)$ must depend on the sense of rotation. Therefore $h_{\theta\phi}$ is an odd function of ε. Using similar arguments, because the Komar angular momentum K_L is an odd function of ε and taking into account (2.8) and the parity behavior of h_{AB} and R^2, we have that W^θ is even and W^ϕ odd. Similarly, since K_m must be an even function of ε, W must be even in ε. Therefore,

$$R_{[2n+1]} = W_{[2n+1]} = 0, \tag{2.11a}$$

$$W^{\theta}_{[2n+1]} = 0, \tag{2.11b}$$

$$W^{\phi}_{[2n]} = 0, \tag{2.11c}$$

$$\gamma_{[2n+1]} = \delta_{[2n]} = 0. \tag{2.11d}$$

To arrive at the last conditions (2.11d) we have taken into account the odd parity of $h_{\theta\phi}$, which gives us $\sinh(\delta(\varepsilon)) = -\sinh(\delta(-\varepsilon))$. Hence, δ must be odd in ε. Similarly, for $h_{\theta\theta}$ and $h_{\phi\phi}$ to be even, $\gamma(\varepsilon)$ must satisfy $e^{2\gamma(\varepsilon)} = e^{2\gamma(-\varepsilon)}$, which implies that γ is a even function of ε.

We conclude with

$$R = r + R_{[2]}\varepsilon^2 + R_{[4]}\varepsilon^4 + O(\varepsilon^6), \tag{2.12a}$$

$$W = V + W_{[2]}\varepsilon^2 + W_{[4]}\varepsilon^4 + O(\varepsilon^6), \tag{2.12b}$$

$$W^{\theta} = W^{\theta}_{[2]}\varepsilon^2 + W^{\theta}_{[4]}\varepsilon^4 + O(\varepsilon^4), \tag{2.12c}$$

$$W^{\phi} = W^{\phi}_{[1]}\varepsilon + W^{\phi}_{[3]}\varepsilon^3 + O(\varepsilon^5), \tag{2.12d}$$

$$\gamma = \gamma_{[2]}\varepsilon^2 + \gamma_{[4]}\varepsilon^4 + O(\varepsilon^6), \tag{2.12e}$$

$$\delta = \delta_{[1]}\varepsilon + \delta_{[3]}\varepsilon^3 + O(\varepsilon^5). \tag{2.12f}$$

A similar expansion was made by Hartle [35] in the derivation of a metric for slowly rotating stars using a $3+1$ decomposition of the metric. From (2.9) follows that the Ricci tensor has the expansions

$$R_{ab} = R_{[0]ab} + R_{[1]ab}\varepsilon + R_{[2]ab}\varepsilon^2 + R_{[3]ab}\varepsilon^3 + \cdots. \tag{2.13}$$

In fact, with the notation $f_{[i]} \in \{\gamma_{[i]}, \delta_{[i]}, R_{[i]}, W^A_{[i]}, W_{[i]}\}$, it turns out for a perturbation at order $n > 1$ that

$$S_{[n]i} = \hat{S}_i(f_{[n]}) + s_{[i]}(f_{[m<n]}), \tag{2.14}$$

$$H^{(\gamma)}_K = \hat{H}^{(\gamma)}_K(f_{[n]}) + h^{(\gamma)}_K(f_{[m<n]}), \tag{2.15}$$

$$H^{(\delta)}_k = \hat{H}^{(\delta)}_k(f_{[n]}) + h^{(\delta)}_k(f_{[m<n]}), \tag{2.16}$$

where $\hat{S}_i$, $\hat{H}^{(\gamma)}_K$, and $\hat{H}^{(\delta)}_k$ are linear differential operators of the indicated arguments. The functions $s_{[i]}$, $h^{(\gamma)}_K$, and $h^{(\delta)}_k$ are nonlinear functions of the lower order perturbations $f_{[m]}$ for $m < n$.

For the computations, it is useful to change the angular coordinate according to $y = -\cos\theta$, introduce $s(y) = \sqrt{1-y^2}$ and transform $W^{\theta} = s^{-1}W^{y}$. In addition, for a perturbation at order n it will be useful to make the following decomposition of the perturbation $f_{[n]}$ in terms of associated Legendre polynomials, $P^m_{\ell}(y)$:

$$R_{[n]}(\lambda, y) = R_{[n.\ell]}(\lambda)P^0_{\ell}(y), \tag{2.17}$$

$$W^{y}_{[n]}(\lambda, y) = W^{\phi}_{[n.\ell]}(\lambda)[s(y)P^1_{\ell}(y)], \tag{2.18}$$

$$W^{\phi}_{[n]}(\lambda, y) = W^{\phi}_{[n.\ell]}(\lambda)\left[\frac{P^1_{\ell}(y)}{s(y)}\right], \tag{2.19}$$

$$W_{[n]}(\lambda, y) = W_{[n,\ell]}(\lambda)P^0_{\ell}(y), \tag{2.20}$$

$$\gamma_{[n]}(\lambda, y) = \gamma_{[n.\ell]}(\lambda)P^2_{\ell}(y), \tag{2.21}$$

$$\delta_{[n]}(\lambda, y) = \delta_{[n.\ell]}(\lambda)P^2_{\ell}(y), \tag{2.22}$$

in which we also apply the Einstein sum convention over ℓ, the respective harmonics of the associated Legendre polynomials. We remark that this decomposition with respect to the associated Legendre polynomials is in fact a decomposition in terms of axisymmetric spin-weighed harmonics (up to normalization) obtained by setting $m = 0$ in the standard ${}_sY_{\ell m}(y, \phi)$.

III. SOLUTION OF THE BACKGROUND AND PERTURBATION EQUATIONS

A. Solution background equations

The main equations for the background model are

$$0 = \frac{r_{,\lambda\lambda}}{r}, \tag{3.1a}$$

$$0 = [(r^2)_{,\lambda}V - 2\lambda]_{,\lambda}, \tag{3.1b}$$

from which we deduce

$$r(\lambda) = r_1\lambda + r_0, \tag{3.2}$$

where r_1 and r_0 are integration constants; however, since we have the freedom of rescaling the affine parameter $\lambda \to \alpha\lambda + \beta$, so that we can take without loss of generality

$$r = \lambda. \tag{3.3a}$$

The next integration of (3.1b) yields

$$V(\lambda) = 1 - \frac{A}{2\lambda} \tag{3.3b}$$

with A an integration constant.

The resulting spacetime is the Schwarzschild metric in outgoing Eddington-Finkelstein coordinates, with a total Bondi mass m_0 related to the integration constant A by $A = 4m_0$. Moreover, $\lambda = A/2$ corresponds to the location of the past event horizon of the Schwarzschild horizon.

B. Recurrent operators in the equations of the perturbations

The principal part $\hat{S}_i(f_n)$ of the supplementary equations in (2.14) while recalling the notation $s^2(y) = 1 - y^2$ are

$$\hat{S}_1(R_{[n]}, W_{[n]}, W^y_{[n]}) = \frac{1}{2\lambda^2}\left(1 - \frac{A}{2\lambda}\right)\left(\lambda^2 W_{[n],\lambda} + \frac{AR_{[n]}}{\lambda}\right)_{,\lambda} + \frac{(s^2 W_{[n],y})_{,y}}{2\lambda^2} - \frac{A}{4\lambda^2} W^y_{[n],y}, \tag{3.4a}$$

$$\hat{S}_2(W_{[n]}, W^y_{[n]}) = \frac{1}{2\lambda^2}\left(1 - \frac{A}{2\lambda}\right)\frac{(\lambda^4 W^y_{[n]})_{,\lambda}}{s} + \frac{s}{2} W_{[n],\lambda y} + \frac{W^y_{[n]}}{s}, \tag{3.4b}$$

$$\hat{S}_3(W^\phi_{[n]}) = \frac{s^2}{2\lambda^2}\left(1 - \frac{A}{2\lambda}\right)(\lambda^4 W^\phi_{[n],\lambda})_{,\lambda} + \frac{1}{2}(s^4 W^\phi_{[n],y})_{,y}. \tag{3.4c}$$

Those for $\hat{H}^{(\gamma)}_K$ in (2.15) are

$$\hat{H}^{(\gamma)}_1(R_{[n]}) = -\frac{2}{\lambda} R_{[n],\lambda\lambda}, \tag{3.5a}$$

$$\hat{H}^{(\gamma)}_2(R_{[n]}, \gamma_{[n]}, W^y_{[n]}) = \frac{1}{2\lambda^2}\frac{(\lambda^4 W^y_{[n],\lambda})_{,\lambda}}{s} - s\left(\frac{R_{[n],y}}{\lambda}\right)_{,\lambda} + \frac{(\gamma_{[n],\lambda} s^2)_{,y}}{s}, \tag{3.5b}$$

$$\begin{aligned}\hat{H}^{(\gamma)}_3(R_{[n]}, \gamma_{[n]}, W^y_{[n]}, W_{[n]}) = &-(\lambda W_{[n]})_{,\lambda} - \left[\left(1 - \frac{A}{2\lambda}\right)(\lambda R_{[n]})_{,\lambda}\right]_{,\lambda} - \frac{(\lambda_{[n],y} s^2)_{,y}}{\lambda} + \frac{(\lambda^4 W^y_{[n]})_{,\lambda y}}{2\lambda^2} \\ &+ \frac{(\gamma_{[n],y} s^4)_{,y}}{s^2} - 2\gamma_{[n]},\end{aligned} \tag{3.5c}$$

$$\hat{H}^{(\gamma)}_4(\gamma_{[n]}, W^y_{[n]}) = -\left[\lambda\left(\lambda - \frac{A}{2}\right)\gamma_{[n],\lambda}\right]_{,\lambda} + \frac{s^2}{2}\left(\frac{\lambda^2 W^y_{[n]}}{s^2}\right)_{,\lambda y}, \tag{3.5d}$$

and those for $\hat{H}^{(\delta)}_k$ of (2.16) are

$$\hat{H}^{(\delta)}_1(\delta_{[n]}, W^\phi_{[n]}) = \frac{s^2}{2\lambda^2}(\lambda^4 W^\phi_{[n],\lambda})_{,\lambda} + (\delta_{[n],\lambda} s^2)_{,y}, \tag{3.6a}$$

$$\hat{H}^{(\delta)}_2(\delta_{[n]}, W^\phi_{[n]}) = -\left[\lambda\left(\lambda - \frac{A}{2}\right)\delta_{[n],\lambda}\right]_{,\lambda} - \frac{s^2}{2}(\lambda^2 W^\phi_{[n]})_{,\lambda y}. \tag{3.6b}$$

We observe that (3.5b) and (3.5d) as well as (3.6a) and (3.6b) can be combined (see, e.g., in [39]) to two fourth order (master) equations

$$0 = \mathcal{M}(\gamma_{[n]}) - s^2 R_{[n],\lambda\lambda yy}, \tag{3.7a}$$

$$0 = \mathcal{M}(\delta_{[n]}), \tag{3.7b}$$

where

$$\begin{aligned}\mathcal{M}(F) \coloneqq &\frac{1}{\lambda^2}[\lambda^4(\lambda F)_{,\lambda\lambda\lambda}]_{,\lambda} + [(\lambda F)_{,\lambda\lambda y} s^2]_{,y} \\ &+ \left(\frac{A}{2\lambda} + 2 - \frac{4}{s^2}\right)(\lambda F)_{,\lambda\lambda} - \frac{A}{2}[\lambda(\lambda F)_{,\lambda\lambda\lambda}]_{,\lambda}.\end{aligned} \tag{3.8}$$

We emphasize that Eqs. (3.7a) and (3.7b) (similar to the Teukolsky master equations in $3+1$ perturbation theory) are the key equations to solve the system, because they provide the initial data $\gamma_{[n]}$ or $\delta_{[n]}$ needed to integrate the hypersurface equations of the characteristic initial value problem.

C. First order perturbations

Since $\gamma_{[1]}$, $R_{[1]}$, $W^{y}_{[1]}$, and $W_{[1]}$ are zero, we only have to consider the equations

$$0 = \hat{S}_3(\delta_{[1]}, W^{\phi}_{[1]}), \tag{3.9}$$

$$0 = \hat{H}^{(\delta)}_1(\delta_{[1]}, W^{\phi}_{[1]}), \tag{3.10}$$

$$0 = \hat{H}^{(\delta)}_2(\delta_{[1]}, W^{\phi}_{[1]}), \tag{3.11}$$

whose explicit form can be read off from (3.4c), (3.6a), and (3.6b). The corresponding master equation is

$$0 = \frac{1}{\lambda^2}[\lambda^4(\lambda\delta_{[1]})_{,\lambda\lambda\lambda}]_{,\lambda} + [(\lambda\delta_{[1]})_{,\lambda\lambda y}s^2]_{,y} + \left(\frac{A}{2\lambda} + 2 - \frac{4}{s^2}\right)(\lambda\delta_{[1]})_{,\lambda\lambda} - \frac{A}{2}[\lambda(\lambda\delta_{[1]})_{,\lambda\lambda\lambda}]_{,\lambda}, \tag{3.12}$$

which is in fact a second order equation for the variable

$$\psi_{[1]} \coloneqq (\lambda\delta_{[1]})_{,\lambda\lambda}, \tag{3.13}$$

namely

$$0 = \lambda(2\lambda - A)\psi_{[1],\lambda\lambda} + (8\lambda - A)\psi_{[1],\lambda} + \frac{A\psi_{[1]}}{\lambda} + 2\left\{(s^2\psi_{[1],y})_{,y} + \left[\left(2 - \frac{4}{s^2}\right)\right]\psi_{[1]}\right\}, \tag{3.14}$$

which admit a solution by separation of variables by setting $\psi_{[1]}(\lambda, y) = p_{[1]}(\lambda)S(y)$,

$$0 = \lambda(2\lambda - A)p_{[1],\lambda\lambda} + (8\lambda - A)p_{[1],\lambda} + \left(\frac{A}{\lambda} + 2k\right)p_{[1]}, \tag{3.15}$$

$$0 = \frac{d}{dy}\left[s^2\frac{dS}{dy}\right] + \left(2 - k - \frac{4}{s^2}\right)S, \tag{3.16}$$

with k a constant. Identifying $2 - k = \ell(\ell + 1)$, we see that (3.16) is an associated Legendre differential equation (A1), whose general solution is

$$S_\ell(y) = B_{0k}P(\ell, 2, y) + B_{1k}Q(\ell, 2, y), \tag{3.17}$$

where $P(\cdot)$ and $Q(\cdot)$ are the Legendre functions of the first kind and of the second kind, respectively.

Requiring a regular solution at the poles $y = \pm 1$ imposes that ℓ must be a non-negative integer and $B_{1k} = 0$, because $P(\ell, 2, -1)$ blows up at the pole $y = -1$ and $Q(\ell, 2, \pm 1)$ blows up at the poles $y = \pm 1$. Then the remaining Legendre function $P(\cdot)$ is the associated Legendre polynomial $P^2_\ell(y)$.

To find a solution for (3.13) and (3.14), we set

$$\psi_{[1]}(\lambda, y) = \psi_{[1.\ell]}(\lambda)P^2_\ell(y), \qquad \delta_{[1]}(\lambda, y) = \delta_{[1.\ell]}(\lambda)P^2_\ell(y), \tag{3.18}$$

where a sum in ℓ is understood. Note that $P^2_0(y) = P^2_1(y) = 0$, and consequently, $\delta_{[1.0]} = \delta_{[1.1]} = 0$ without loss of generality. Subsequent insertion into (3.14) while using Eq. (A1) gives us

$$0 = -\frac{1}{2}\lambda(A - 2\lambda)\frac{d^2\psi_{[1.\ell]}}{d\lambda^2} + \left(4\lambda - \frac{A}{2}\right)\frac{d\psi_{[1.\ell]}}{d\lambda} + \left[2 - \ell(\ell + 1) + \frac{A}{2\lambda}\right]\psi_{[1.\ell]}. \tag{3.19}$$

Using the parameter transformation $x = \frac{4\lambda}{A} - 1$, similar to [35], we find

$$0 = (1 - x)\frac{d^2\psi_{[1.\ell]}}{dx^2} - \frac{4x + 2}{x + 1}\frac{d\psi_{[1.\ell]}}{dx} + \frac{\ell(\ell + 1)(x + 1) - 2x - 4}{(x + 1)^2}\psi_{[1.\ell]}, \tag{3.20}$$

which can also be written as

$$0 = \frac{d}{dy}\left[(1 - x^2)\frac{d}{dx}(1 - x)\psi_{[1.\ell]}\right] + \left[\ell(\ell + 1) - \frac{4}{1 - x^2}\right](1 - x)\psi_{[1.\ell]}. \tag{3.21}$$

Equation (3.21) is an associated Legendre differential equation, such as (A1), with the general solution

$$\psi_{[1.\ell]}(x) = \frac{B_{[1.\ell]}P^2_\ell(x) + B_{[2.\ell]}Q^2_\ell(x)}{1 - x}. \tag{3.22}$$

Inverting the parameter transformation from x to λ yields the general solution of (3.19) so that using (3.18)

$$\psi_{[1]}(\lambda, y) = \left(\frac{AB_{[1.\ell]}}{2A - 4\lambda}\right)P^2_\ell\left(\frac{4\lambda}{A} - 1\right)P^2_\ell(y) + \left(\frac{AB_{[2.\ell]}}{2A - 4\lambda}\right)Q^2_\ell\left(\frac{4\lambda}{A} - 1\right)P^2_\ell(y). \tag{3.23}$$

The field $\psi_{[1]}$ is related to the Weyl scalar Ψ_0,

$$\Psi_0 = -\frac{i\psi_{[1]}}{\lambda}\varepsilon + O(\varepsilon^2). \tag{3.24}$$

An inspection of (3.22) shows that Ψ becomes infinite for $\lambda \to A/2$ and for $\lambda \to \infty$ if $\ell \geq 2$. The first case corresponds to the unperturbed location of the horizon while the second one corresponds to the asymptotic region. Consequently, Ψ_0 also becomes infinite in these cases. We require regularity of the scalar curvature Ψ_0 at these locations, which implies $B_{[1.\ell]} = B_{[2.\ell]} = 0$. This leaves us with the trivial solution $\psi_{[1]} = 0$.

Integration of (3.13) with this trivial solution while using (3.18) yields

$$\delta_{[1.\ell]}(\lambda) = B^{\delta}_{[0.\ell]} + \frac{B^{\delta}_{[1.\ell]}}{\lambda}, \tag{3.25}$$

where as aforementioned, since $\delta_{[1.0]} = \delta_{[1.1]} = 0$, we get that $B^{\delta}_{[0.0]} = B^{\delta}_{[0.1]} = B^{\delta}_{[1.0]} = B^{\delta}_{[1.1]} = 0$. These modes are physically irrelevant because $\delta_{[1]}$ is expressed by the angular base of P^2_ℓ-associated Legendre polynomials and $P^2_0 = P^2_1 = 0$. Since $\delta_{[1]}$ is now known, we are now in position to integrate the hypersurface equation (3.10). We insert (3.25) into (3.10), while using (A3), to find

$$(\lambda^4 W^{\phi}_{[1],\lambda})_{,\lambda} = 2\lambda^2 \left(\frac{d\delta_{[1.\ell]}}{d\lambda}\right) \frac{K_\ell P^1_\ell(y)}{s}, \tag{3.26}$$

where

$$K_\ell = 2 - \ell(\ell+1) = (1-\ell)(2+\ell). \tag{3.27}$$

Then setting

$$W^{\phi}_{[1]}(\lambda, y) = W^{\phi}_{[1.\ell]}(\lambda) \frac{P^1_\ell(y)}{s} \tag{3.28}$$

gives us

$$\frac{d}{d\lambda}\left(\lambda^4 \frac{W^{\phi}_{[1.\ell]}}{d\lambda}\right) = 2B^{\delta}_{[1.\ell]} K_\ell; \tag{3.29}$$

or after integration

$$W^{\phi}_{[1.\ell]} = B^{\phi}_{[0.\ell]} - \frac{K_\ell B^{\delta}_{[1.\ell]}}{\lambda^2} - \frac{B^{\phi}_{[3.\ell]}}{3\lambda^3}. \tag{3.30}$$

This gives us for the first order axisymmetric perturbations

$$\delta_{[1]}(\lambda, y) = \left(B^{\delta}_{[0.\ell]} + \frac{B^{\delta}_{[1.\ell]}}{\lambda}\right) P^2_\ell(y), \tag{3.31}$$

$$W^{\phi}_{[1]}(\lambda, y) = \left[B^{\phi}_{[0.\ell]} - \frac{K_\ell B^{\delta}_{[1.\ell]}}{\lambda^2} - \frac{B^{\phi}_{[3.\ell]}}{3\lambda^3}\right] \frac{P^1_\ell(y)}{s}. \tag{3.32}$$

Again we set the unphysical modes $B^{\phi}_{[0.0]} = B^{\phi}_{[3.0]} = 0$, because of behavior of the angular base functions of $W^{\phi}_{[1]}$.

Inserting the obtained solutions into (3.11) yields while using (A6)

$$0 = \frac{A}{2}[\lambda \delta_{[1.\ell],\lambda}]_{,\lambda} - [\lambda^2 \delta_{[1.\ell],\lambda}]_{,\lambda} - \frac{1}{2}(\lambda^2 W^{\phi}_{[1.\ell]})_{,\lambda}, \tag{3.33}$$

and together with (3.25) and (3.30) this gives us for any $\ell \geq 2$

$$0 = -B^{\phi}_{[0.\ell]}\lambda + \frac{1}{\lambda^2}\left(\frac{A}{2}B^{\delta}_{[0.\ell]} - \frac{B^{\phi}_{[3.\ell]}}{6}\right). \tag{3.34}$$

Hence for any $\ell \geq 2$,

$$B^{\phi}_{[0.\ell]} = 0, \qquad B^{\delta}_{[0.\ell]} = \frac{B^{\phi}_{[3.\ell]}}{3A}. \tag{3.35}$$

Moreover, inserting the obtained solution into the supplementary equation (3.9) while using (A7), we find

$$\hat{S}_3 = \left\{\frac{B^{\phi}_{[0.\ell]}}{2} - \frac{\ell(\ell+1)B^{\delta}_{[1.\ell]}}{2\lambda^2} + \frac{3AB^{\delta}_{[1.\ell]} - B^{\phi}_{[3.\ell]}}{6\lambda^3}\right\} \times K_\ell \times s(y) P^1_\ell(y). \tag{3.36}$$

Considering (3.36) for the various modes of ℓ gives us the following: $\ell = 0$ is trivial because $P^1_0 = 0$; the $\ell = 1$ coefficient vanishes since $K_1 = 0$. Therefore the coefficients $B^{\phi}_{[0.1]}$ and $B^{\phi}_{[3.1]}$ are *unconstrained* by the supplementary equation $\hat{S}_3$. Finally considering $\hat{S}_3 = 0$ for the $\ell > 1$ coefficients while using (3.35) gives us

$$0 = \ell(\ell+1)B^{\delta}_{[1.\ell]}, \tag{3.37}$$

which implies

$$B^{\delta}_{[1.\ell]} = 0: \quad \forall\ \ell > 1. \tag{3.38}$$

Furthermore, requiring an asymptotic Bondi frame (a nonrotating inertial observer at large distances), i.e.

$$g_{ab}dx^a dx^b \to -du^2 - d\lambda du + \lambda^2 q_{AB} dx^A dx^B, \tag{3.39}$$

annuls the integration constants,

$$W^{\phi}_{[0.1]} = B^{\delta}_{[0.\ell]} = 0. \tag{3.40}$$

From the above requirements, the final solution of the linear perturbations are

$$\delta_{[1]}(y, \lambda) = 0,$$
$$W^{\phi}_{[1]}(y, \lambda) = -\frac{B}{3\lambda^3}\frac{P^1_\ell(y)}{s} = -\frac{B}{3\lambda^3}\frac{y}{s}, \tag{3.41}$$

where we redefined $B \coloneqq B^{\phi}_{[3,1]}$ for notational convenience because it is the only remaining integration constant.

D. Quadratic perturbations

Using the notation of Sec. III B, the relevant main equations (i.e. only those containing $\gamma_{[2]}$, $R_{[2]}$, $W^{y}_{[2]}$, and $W_{[2]}$) for the quadratic perturbations are found to be

$$0 = \hat{S}_1(W_{[2]}, W^{y}_{[2]}) + \frac{B^2 s^2}{2\lambda^6}\left(1 - \frac{A}{2\lambda}\right), \tag{3.42a}$$

$$0 = \hat{S}_2(W_{[2]}, W^{y}_{[2]}), \tag{3.42b}$$

$$0 = \hat{H}^{(\gamma)}_1(R_{[2]}), \tag{3.42c}$$

$$0 = \hat{H}^{(\gamma)}_2(R_{[2]}, \gamma_{[2]}, W^{y}_{[2]}), \tag{3.42d}$$

$$0 = \hat{H}^{(\gamma)}_3(W_{[2]}, R_{[2]}, \gamma_{[2]}, W^{y}_{[2]}) - \frac{B^2 s^2}{4\lambda^4}, \tag{3.42e}$$

$$0 = \hat{H}^{(\gamma)}_4(\gamma_{[2]}, W^{y}_{[2]}) + \frac{B^2 s^2}{4\lambda^4}. \tag{3.42f}$$

The first hypersurface equation (3.42c) is readily integrated,

$$R_{[2]} = C_{R20}(y) + C_{R11}(y)\lambda, \tag{3.43}$$

where $C_{R20}(y)$ and $C_{R11}(y)$ are free functions of y. Similar to (3.7b), we can deduce a master equation for $\gamma_{[2]}$,

$$0 = \mathcal{M}(\gamma_{[2]}) - s^2 R_{2,\lambda\lambda yy} - \frac{5B^2}{2\lambda^5} s^2. \tag{3.44}$$

For finding a solution of the remaining fields $\gamma_{[2]}$, $W^{y}_{[2]}$, and $W_{[2]}$, we need to solve the master equation (3.44). Defining

$$\psi_{[2]} = (\lambda\gamma_{[2]})_{,\lambda\lambda} \tag{3.45}$$

with Legendre decomposition

$$\psi_{[2]} = \psi_{[2.\ell]}(\lambda) P^2_{\ell}(y) \tag{3.46}$$

while using (3.42c) gives us after insertion of (3.43), (3.45), and (3.46) into (3.44)

$$0 = \left\{-\frac{1}{2} r(A - 2\lambda)\frac{d^2\psi_{[2.\ell]}}{d\lambda^2} + \left(4r - \frac{A}{2}\right)\frac{d\psi_{[2.\ell]}}{dy} + \left[2 - \ell(\ell+1) + \frac{A}{2\lambda}\right]\psi_{[2.\ell]}\right\} P^2_{\ell} - \frac{5B^2}{2\lambda^5} s^2. \tag{3.47}$$

To fully factor out the Legendre polynomials P^2_{ℓ}, we recall that $P^2_2(y) = 3s^2$. This allows us to write

$$0 = \left[\left\{-\frac{1}{2} r(A - 2\lambda)\frac{d^2\psi_{[2.\ell]}}{d\lambda^2} + \left(4r - \frac{A}{2}\right)\frac{d\psi_{[2.\ell]}}{dy} + \left[2 - \ell(\ell+1) + \frac{A}{2\lambda}\right]\psi_{[2.\ell]}\right\}\delta^{\ell'}_{\ell} - \frac{5B^2}{6\lambda^5}\delta^{\ell'}_{2}\right] P^2_{\ell'}(y). \tag{3.48}$$

We can see that (3.48) resembles (3.19) if $B = 0$. It is in fact an inhomogeneous version of (3.19). We seek solutions of (3.48) as a superposition of a homogeneous solution, $\psi^{(\mathrm{hom})}_{[2.\ell]}$ for $B = 0$, and a particular solution $\psi^{(\mathrm{part})}_{[2.\ell]}$ for $B \neq 0$, i.e.

$$\psi_{[2.\ell]} = \psi^{(\mathrm{hom})}_{[2.\ell]} + \psi^{(\mathrm{part})}_{[2.\ell]}. \tag{3.49}$$

The homogeneous solution $\psi^{(\mathrm{hom})}_{[2.\ell]}$ will be like (3.23). Also note that a particular solution needs to be found for only the $\ell = 2$ mode. We find $\psi^{(\mathrm{part})}_{[2.2]} = -B^2/(9A\lambda^4)$. Hence,

$$\psi_{[2.\ell]}(\lambda) = A\left\{\frac{C_{[1.\ell]} P^2_{\ell}(\frac{4\lambda}{A} - 1) + C_{[2.\ell]} Q^2_{\ell}(\frac{4\lambda}{A} - 1)}{2A - 4\lambda}\right\} + \left(-\frac{B^2}{9A\lambda^4}\right)\delta^2_{\ell}. \tag{3.50}$$

It follows by the same regularity arguments as in the discussion for (3.23) that so the Weyl curvature scalar Ψ_0 does not blow up at the horizon of the unperturbed solution and toward null infinity we must set $C_{[1.\ell]} = C_{[2.\ell]} = 0$.

Consequently a solution for the $\psi_{[2.\ell]}$ modes is

$$\psi_{[2.\ell]}(\lambda) = \left(-\frac{B^2}{9A\lambda^4}\right)\delta^2_{\ell}. \tag{3.51}$$

Setting

$$\gamma_{[2]}(\lambda, y) = \gamma_{[2.\ell]}(\lambda) P^2_{\ell}(y), \tag{3.52}$$

we find after the integration of (3.45)

$$\gamma_{[2.\ell]}(\lambda, y) = C^{\gamma}_{[0.\ell]} + \frac{C^{\gamma}_{[1.\ell]}}{\lambda} - \frac{B^2}{54A\lambda^3}\delta^2_{\ell}. \tag{3.53}$$

Insertion of (3.53) and (3.43) into (3.42d) gives us

$$0 = \frac{(\lambda^4 W^{y}_{[2],r})}{2s\lambda^2} + \frac{sC_{R20,y}}{\lambda^2} + \left(\frac{d\gamma_{[2.\ell]}}{d\lambda}\right)\frac{1}{s}\frac{d}{dy}[s^2 P^2_{\ell}], \tag{3.54}$$

and using (A3) we find

$$0 = \left(\lambda^4 \frac{W^y_{[2],r}}{s}\right)_{,\lambda} + 2sC_{R20,y} - 2\lambda^2 K_\ell \left(\frac{d\gamma_{[2.\ell]}}{d\lambda}\right) P^1_\ell(y), \tag{3.55}$$

which indicates that the angular behavior of $W^y_{[2]}/s$ and $sC_{R20,y}$ are dictated by the associated Legendre polynomials $P^1_\ell(y)$. As of (A5), we set [note $P_\ell(y) = P^0_\ell(y)$]

$$R_{[2]}(\lambda, y) = R_{[2.\ell]}(\lambda) P_\ell(y) = [C^R_{[20.\ell]} + C^R_{[21.\ell]}\lambda] P_\ell(y), \tag{3.56}$$

$$W^y_{[2]}(\lambda, y) = W^y_{[2.\ell]}(\lambda) s(y) P^1_\ell(y). \tag{3.57}$$

This gives us

$$0 = \frac{d}{d\lambda}\left(\lambda^4 \frac{d}{d\lambda} W^y_{[2.\ell]}\right) - 2C^R_{[20.\ell]} - 2\lambda^2 K_\ell \left(\frac{d\gamma_{[2.\ell]}}{d\lambda}\right). \tag{3.58}$$

Integrating (3.58) yields

$$W^y_{[2.\ell]} = C^y_{[0.\ell]} + \frac{K_\ell C^\gamma_{[1.\ell]} - C^R_{[20.\ell]}}{\lambda^2} - \frac{C^y_{[3.\ell]}}{3\lambda^3} - \frac{B^2}{9A\lambda^4}\delta^2_\ell, \tag{3.59}$$

where we set the integration constants $C^y_{[0.0]} = C^y_{[3.0]} = 0$, because $P^1_0(y) = 0$. Considering (3.5d) with (3.52), (3.57), (A6), and $s^2 = P^2_\ell(y)/3$ gives us

$$\left[\lambda^2\left(1 - \frac{A}{2\lambda}\right)\gamma_{[2.\ell],r}\right]_{,\lambda} = \frac{1}{2}(\lambda^2 W^y_{[2.\ell]})_{,\lambda} + \frac{B^2}{12\lambda^4}\delta^2_\ell \tag{3.60}$$

so that after the insertion of (3.53) and (3.59), we obtain

$$\lambda C^y_{[0.\ell]} = \frac{A}{2\lambda^2}\left(C^\gamma_{[1.\ell]} - \frac{C^y_{[3.\ell]}}{3A}\right) + \frac{B^2}{9A\lambda^3}\left(1 + \frac{K_\ell}{4}\right)\delta^2_\ell \tag{3.61}$$

implying for any $\ell \geq 2$

$$C^y_{[0.\ell]} = 0, \qquad C^y_{[3.\ell]} = 3AC^\gamma_{[1.\ell]}. \tag{3.62}$$

Next, proceed with the hypersurface equation (3.42e) for $W_{[2]}$. Insertion of (3.52), (3.56), and (3.57) into (3.42e) gives us

$$\begin{aligned}(\lambda W_{[2]})_{,\lambda} &= \left\{-\left[\left(1 - \frac{A}{2\lambda}\right)(\lambda R_{[2.\ell]})_{,\lambda}\right]_{,\lambda}\right. \\ &\quad \left. + \ell(\ell+1)\left[\frac{R_{[2.\ell]}}{\lambda} + \frac{(\lambda^4 W^y_{[2.\ell]})_{,\lambda}}{2\lambda^2} - K_\ell \gamma_{[2.\ell]}\right]\right\} P^0_\ell(y) \\ &\quad - \frac{B^2 s^2}{4\lambda^4}.\end{aligned} \tag{3.63}$$

Using

$$s^2 = 1 - y^2 = \frac{2}{3}[P^0_\ell(y) - P^0_2(y)] \tag{3.64}$$

as well as setting

$$W_{[2]}(\lambda, y) = W_{[2.\ell]}(\lambda) P^0_\ell(y) \tag{3.65}$$

yields

$$\begin{aligned}(\lambda W_{[2.\ell]})_{,\lambda} &= -\left[\left(1 - \frac{A}{2\lambda}\right)(\lambda R_{[2.\ell]})_{,\lambda}\right]_{,\lambda} \\ &\quad + \ell(\ell+1)\left[\frac{R_{[2.\ell]}}{\lambda} + \frac{(\lambda^4 W^y_{[2.\ell]})_{,\lambda}}{2\lambda^2} - K_\ell \gamma_{[2,\ell]}\right] \\ &\quad - \frac{B^2}{6\lambda^4}(\delta^0_\ell - \delta^2_\ell).\end{aligned} \tag{3.66}$$

Since $R_{[2.\ell]}$, $W^y_{[2.\ell]}$, and $\gamma_{[2.\ell]}$ are known, we find after integration

$$\begin{aligned}W_{[2.\ell]} &= -K_\ell C^R_{[21.\ell]} - \ell(\ell+1) K_\ell C^\gamma_{[0.\ell]} + \frac{C^W_{[1.\ell]}}{\lambda} + \frac{AC^R_{[20.\ell]}}{2\lambda^2} \\ &\quad + \frac{\ell(\ell+1) C^y_{[3.\ell]}}{6\lambda^2} + \left(\frac{2B^2}{9A\lambda^3} - \frac{B^2}{18\lambda^4}\right)\delta^2_\ell + \frac{B^2}{18\lambda^4}\delta^0_\ell,\end{aligned} \tag{3.67}$$

where $C^W_{[1.\ell]}$ are integration constants.

The calculation of (3.42a) and (3.42b) while using (3.56), (3.57), (3.64), (A1) (for $\hat{m} = 0$), and (A8) gives us

$$\begin{aligned}0 &= \left(1 - \frac{A}{2\lambda}\right)\left(\lambda^2 W_{[2.\ell],r} + \frac{AR_{[2.\ell]}}{\lambda}\right)_{,\lambda} \\ &\quad - \ell(\ell+1)\left(W_{[2.\ell]} + \frac{A}{2} W^y_{[2,\ell]}\right),\end{aligned} \tag{3.68}$$

$$0 = \frac{1}{2\lambda^2}\left(1 - \frac{A}{2\lambda}\right)(\lambda^4 W^y_{[2.\ell]})_{,\lambda} - \frac{1}{2} W_{[2.\ell],r} + W^y_{[2.\ell]}, \tag{3.69}$$

and the insertion of the respective coefficient solutions (3.56), (3.59), and (3.67) yields

$$0=\ell(\ell+1)\left\{-\frac{C^W_{[1.\ell]}}{\lambda}+K_\ell[C^\gamma_{[1.\ell]}-C^R_{[21.\ell]}]+\frac{C^y_{[3.\ell]}-3AC^\gamma_{[1.\ell]}}{6\lambda^2}\right\}, \tag{3.70}$$

$$0=\frac{C^W_{[1.\ell]}}{2\lambda^2}+\frac{(C^y_{[3.\ell]}-3AC^\gamma_{[1.\ell]})K_\ell}{6\lambda^3},\quad \forall\ \ell\geq 1. \tag{3.71}$$

Therefore,

$$C^W_{[1.\ell]}=0,\quad \forall\ \ell\geq 1, \tag{3.72a}$$

$$C^y_{[3.\ell]}=3AC^\gamma_{[1.\ell]},\quad \forall\ \ell\geq 2, \tag{3.72b}$$

$$C^R_{[21.\ell]}=C^\gamma_{[1.\ell]},\quad \forall\ \ell\geq 2. \tag{3.72c}$$

Note that (3.72b) is consistent with (3.62). The requirement of an asymptotic inertial observer leads to

$$C^\gamma_{[0.\ell]}=C^R_{[21.\ell]}=0, \tag{3.73}$$

which gives with (3.72) that $C^\gamma_{[1.\ell]}=C^y_{[3.\ell]}=0$. Thus, redefining $C\coloneqq C^W_{[1.1]}$, the quadratic perturbations are

$$\gamma_{[2]}(\lambda,y)=\left(-\frac{B^2}{54A\lambda^3}\delta^2_\ell\right)P^2_\ell(y), \tag{3.74}$$

$$R_{[2]}(\lambda,y)=0, \tag{3.75}$$

$$W^y_2(\lambda,y)=\left(-\frac{B^2}{9A\lambda^4}\delta^2_\ell\right)s(y)P^1_\ell(y), \tag{3.76}$$

$$W_{[2]}(\lambda,y)=\frac{C}{\lambda}+\frac{B^2}{18\lambda^4}+\left(\frac{2B^2}{9A\lambda^3}-\frac{B^2}{18\lambda^4}\right)P^0_2(y). \tag{3.77}$$

E. Third order perturbations

Similarly, expressions for the higher order perturbation quantities $f_{[i]}$ can be obtained using the same procedure as in the previous sections. In this and in the next subsection we show the fundamental results without repeating intermediate steps.

The relevant equations for the third perturbations are

$$0=\hat{S}_3(\delta_{[3]},W^\phi_{[3]})-\frac{B^3s^4}{6A\lambda^6}, \tag{3.78}$$

$$0=\hat{H}^{(\delta)}_1(\delta_{[3]},W^\phi_{[3]})-\frac{B^3s^4}{6A\lambda^6}, \tag{3.79}$$

$$0=\hat{H}^{(\delta)}_2(\delta_{[3]},W^\phi_{[3]})+\frac{2B^3ys^2}{3A\lambda^5}. \tag{3.80}$$

Similar to (3.7b), we can deduce a master equation for $\delta_{[3]}$,

$$0=\mathcal{M}(\delta_{[3]})-\frac{40B^3}{3A\lambda^6}s^2y. \tag{3.81a}$$

Using $P^2_3(y)=15ys^2$ and following the steps of Sec. III C, we find

$$\delta_{[3]}(\lambda,y)=\left(-\frac{B^3}{162A^2\lambda^4}\right)P^2_3(y), \tag{3.82}$$

$$W^\phi_{[3]}(\lambda,y)=\left[-\frac{D}{3\lambda^3}-\frac{2B^3}{135A\lambda^6}\right]\frac{P^1_1(y)}{s(y)}+\left[\frac{B^3}{405A\lambda^6}-\frac{4B^3}{81A^2\lambda^5}\right]\frac{P^1_3(y)}{s(y)}, \tag{3.83}$$

where D is the only free new remaining integration constant that appears at this order.

F. Fourth order perturbations

Here the relevant main equations are those containing $\gamma_{[4]}$, $R_{[4]}$, $W^y_{[4]}$, and $W_{[4]}$ which are

$$0=\hat{S}_1(W_{[4]},W^y_{4]})+\left(\frac{14}{9A\lambda}-\frac{1}{12\lambda^2}-\frac{35}{6A^2}\right)\frac{B^4s^4}{3\lambda^8}+\left[\left(1-\frac{A}{2\lambda}\right)D-\frac{CB}{2A}+\left(\frac{16}{A\lambda^2}-\frac{7}{3\lambda^3}\right)\frac{B^3}{9A}\right]\frac{Bs^2}{\lambda^6}-\frac{8B^4}{27\lambda^8A^2}+\frac{CB^2}{3A\lambda^6}, \tag{3.84a}$$

$$0=\hat{S}_2(W_{[4]},W^y_{[4]})+\left[\frac{(7A+120\lambda)s^2}{12\lambda}-8\right]\frac{B^4ys}{9A^2\lambda^7}+\frac{2ysCB^2}{3A\lambda^5}, \tag{3.84b}$$

$$0=\hat{H}^{(\gamma)}_1(R_{[4]})-\frac{B^4s^4}{18A^2\lambda^8}, \tag{3.84c}$$

$$0=\hat{H}^{(\gamma)}_2(R_{[2]},\gamma_{[2]},W^y_{[2]})+\frac{B^4ys^3}{27A^2\lambda^7}, \tag{3.84d}$$

$$0=\hat{H}^{(\gamma)}_3(W_{[2]},R_{[2]},\gamma_{[2]},W^y_{[2]})+\frac{(A-14r)B^4s^4}{36A^2\lambda^7}+\frac{2B^4s^2}{9A^2\lambda^6}+\frac{DBs^4}{2\lambda^4}, \tag{3.84e}$$

$$0=\hat{H}_4^{(\gamma)}(\gamma_{[2]},W^y_{[2]})+\left(14+\frac{A}{2r}\right)\frac{B^4s^4}{9A^2\lambda^6}+\left(\frac{B^2C}{2A\lambda^4}-\frac{BD}{2\lambda^4}-\frac{38B^2}{27A^2\lambda^6}\right)s^2. \tag{3.84f}$$

The first hypersurface equation (3.84c) is readily integrated,

$$R_{[4]}(\lambda,y)=E_{R0}(y)+E_{R1}(y)\lambda-\frac{s^4B^4}{1080A^2\lambda^5}, \tag{3.85}$$

or expressed in terms of the Legendre polynomials $P^0_\ell(y)$,

$$R_{[4]}(\lambda,y)=(E^R_{[0.\ell]}+E^R_{[1.\ell]}\lambda)P^0_\ell(y)-\frac{B^4}{135A^2\lambda^5}\left(\frac{P^0_0(y)}{15}-\frac{2P^0_2(y)}{21}+\frac{P^0_4(y)}{35}\right). \tag{3.86}$$

Similar to (3.7a) we can deduce a master equation for $\gamma_{[4]}$:

$$0=\mathcal{M}(\gamma_{[4]})-s^2R_{[4],\lambda\lambda yy}-\frac{5B^2Cs^2}{A\lambda^5}-\frac{5BDs^2}{\lambda^5}+\left[358-s^2\left(397+\frac{A}{\lambda}\right)\right]\frac{B^4s^2}{9A^2\lambda^7}. \tag{3.87a}$$

Using the methods of Sec. III D together with the inverted Legendre relations

$$1=P^0_0(y)=-\frac{P^1_1(y)}{s}, \tag{3.88a}$$

$$y=P^0_1(y)=-\frac{P^1_2(y)}{3s}, \tag{3.88b}$$

$$y^2=\frac{1}{3}-\frac{2}{3}P^0_2(y)=1-\frac{1}{3}P^2_2(y), \tag{3.88c}$$

$$y^3=-\frac{2P^1_4(y)}{35s}+\frac{P^1_2(y)}{7s}, \tag{3.88d}$$

$$y^4=\frac{1}{5}-\frac{4P^0_2(y)}{7}+\frac{8P^0_4(y)}{35}=1-\frac{8}{21}P^2_2(y)-\frac{2}{105}P^2_4(y), \tag{3.88e}$$

we deduce the following solution for the fourth order perturbation:

$$R_{[4]}=-\frac{B^4}{135A^2\lambda^5}\left(\frac{P^0_0}{15}-\frac{2P^0_2}{21}+\frac{P^0_4}{35}\right), \tag{3.89a}$$

$$\gamma_{[4]}=\left(\frac{BD}{27A\lambda^3}-\frac{B^2C}{27A^2\lambda^3}-\frac{B^4}{1134A^2\lambda^6}\right)P^2_2+\left(\frac{B^4}{405A^3\lambda^5}+\frac{B^4}{17010A^2\lambda^6}\right)P^2_4, \tag{3.89b}$$

$$W^y_{[4]}=\left(\frac{2BD}{9A\lambda^4}-\frac{2B^2C}{9A^2\lambda^4}-\frac{2B^4}{2835A^2\lambda^7}\right)P^1_2+\left(\frac{2B^4}{81A^3\lambda^6}+\frac{B^4}{4725A^2\lambda^7}\right)P^1_4, \tag{3.89c}$$

$$W_{[4]}=\frac{E}{\lambda}-\frac{BD}{9\lambda^4}+\frac{4B^4}{405A^2\lambda^6}-\frac{B^4}{675A\lambda^7}+\left(\frac{2B^4}{945A\lambda^7}-\frac{2B^4}{81A^2\lambda^6}+\frac{4B^2C}{9A^2\lambda^3}-\frac{4BD}{9A\lambda^3}+\frac{BD}{9\lambda^4}\right)P^0_2+\left(-\frac{8B^4}{81A^3\lambda^5}+\frac{2B^4}{135A^2\lambda^6}-\frac{B^4}{1575A\lambda^7}\right)P^0_4. \tag{3.89d}$$

Note that E is the only remaining new integration constant; all others vanish because of the reasons mentioned in Sec. III D.

G. Perturbations in terms of Komar quantities

The solution of the perturbation involves the free integration constants A, B, C, D, and E. These free constants determine the Komar mass, K_m, and the Komar angular momentum, K_L, which can be found by the calculation of (2.7) and (2.8),

$$m\coloneqq K_m=\frac{A}{4}-\frac{C}{2}\varepsilon^2-\frac{E}{2}\varepsilon^4+O(\varepsilon^5), \tag{3.90}$$

$$L\coloneqq K_L=-\frac{B}{6}\varepsilon+\frac{D}{6}\varepsilon^3+O(\varepsilon^5). \tag{3.91}$$

If $\varepsilon=0$, $K_m=A/4$ corresponds to the mass m_0 of the unperturbed system. Furthermore, we can see that $L=O(\varepsilon)$. This allows us to relate ε with the angular momentum L of the system. To do that we have to solve the cubic equation

$$0=\frac{D}{6}\varepsilon^3-\frac{B}{6}\varepsilon+L \tag{3.92}$$

for ε. This equation also shows that in order to make the substitution of ε by L, we seek the solution $\varepsilon(L)=O(L)$. The root of (3.92) which fulfills this requirement is

$$\varepsilon=-\frac{6}{B}L-\frac{216D}{B^4}L^3+O(L^5). \tag{3.93}$$

Subsequent insertion of this expansion into (3.90) and solving for A gives us

$$A = 4m + \frac{72C}{B^2}L^2 + 2592\frac{EB-2CD}{B^5}L^4 + O(L^6). \tag{3.94}$$

The relations (3.93) and (3.94) allow us to substitute A and ε, by the physical quantities m and L. Insertion of (3.93) and (3.94) into the solution of the perturbations and subsequent expansion up to $O(L^4)$ allows us to eliminate the integration constants C, D, and E from the perturbations. This means that all integration constants are absorbed into the Komar mass, m and the Komar angular momentum L. Thus the final solution is uniquely described by the two physical quantities m and L. This gives us

$$R(\lambda, y) = \lambda - \left(\frac{P_0^0}{5} - \frac{2P_2^0}{7} + \frac{3P_4^0}{35}\right)\frac{L^4}{5m^2\lambda^5} + O(L^6), \tag{3.95a}$$

$$W(\lambda, y) = 1 - \frac{2m}{\lambda} + \left[\frac{2}{\lambda^4} + \left(\frac{2}{m\lambda^3} - \frac{2}{\lambda^4}\right)P_2^0\right]L^2 + \left[\frac{4}{5m^2\lambda^6} - \frac{12}{25\lambda^7} + \left(\frac{24}{35m\lambda^7} - \frac{2L^4}{m^2\lambda^6}\right)P_2^0 + \left(-\frac{2}{m^3\lambda^5} + \frac{6}{5m^2\lambda^6} - \frac{36}{175m\lambda^7}\right)P_4^0\right]L^4 + O(L^6), \tag{3.95b}$$

$$W^y(\lambda, y) = \left(-\frac{1}{m\lambda^4}(sP_2^1)\right)L^2 + \left[-\frac{2}{35m^2\lambda^7}(sP_2^1) + \left(\frac{1}{2m^3\lambda^6} + \frac{3}{175m^2\lambda^7}\right)(sP_4^1)\right]L^4 + O(L^6), \tag{3.95c}$$

$$W^\phi(\lambda, y) = \left(-\frac{2}{\lambda^3}\frac{P_1^1}{s}\right)L + \left[\frac{4}{5m\lambda^6}\frac{P_1^1}{s} + \left(\frac{2}{3m^2\lambda^5} - \frac{2}{15m\lambda^6}\right)\frac{P_3^1}{s}\right]L^3 + O(L^5), \tag{3.95d}$$

$$\gamma(\lambda, y) = \left(-\frac{1}{6m\lambda^3}P_2^2\right)L^2 + \left[-\frac{1}{14m^2\lambda^6}P_2^2 + \left(\frac{1}{20m^3\lambda^5} + \frac{1}{210m^2\lambda^6}\right)P_4^2\right]L^4 + O(L^6), \tag{3.95e}$$

$$\delta(\lambda, y) = \left(\frac{1}{12m^2\lambda^4}P_3^2\right)L^3 + O(L^5). \tag{3.95f}$$

We see in (3.95) that the perturbations are determined by the mass and angular momentum; i.e. the solution has two hairs. To show that this solution represents the Kerr solution in affine-null coordinates, we introduce the specific angular momentum, $a \coloneqq L/m$. In terms of a, Eqs. (3.95) read after changing to the angular coordinate θ

$$R(\lambda, \theta) = \lambda - \frac{3m^2 \sin^4\theta}{40\lambda^5}a^4 + O(a^6), \tag{3.96a}$$

$$W(\lambda, \theta) = 1 - \frac{2m}{\lambda} + \left[\frac{2m}{\lambda^3} + \left(-\frac{3m}{\lambda^3} + \frac{3m^2}{\lambda^4}\right)\sin^2\theta\right]a^2 + \left[-\frac{2m}{\lambda^5} + \left(\frac{10m}{\lambda^5} - \frac{3m^2}{\lambda^6}\right)\sin^2\theta + \left(-\frac{35m}{4\lambda^5} + \frac{21m^2}{4\lambda^6} - \frac{9m^3}{10\lambda^7}\right)\sin^4\theta\right]a^4 + O(a^6), \tag{3.96b}$$

$$W^\theta(\lambda, \theta) = \left\{-\frac{3m}{\lambda^4}a^2 + \left[\frac{5m}{\lambda^6} - \left(\frac{35m}{4\lambda^6} + \frac{3m^2}{10\lambda^7}\right)\sin^2\theta\right]a^4\right\}\sin\theta\cos\theta + O(a^6), \tag{3.96c}$$

$$W^\phi(\lambda, \theta) = \frac{2m}{\lambda^3}a + \left[-\frac{4m}{\lambda^5} + \left(\frac{5m}{\lambda^5} - \frac{m^2}{\lambda^6}\right)\sin^2\theta\right]a^3 + O(a^5), \tag{3.96d}$$

$$\gamma(\lambda, \theta) = \left(-\frac{m\sin^2\theta}{2\lambda^3}\right)a^2 + \left[\frac{9m\sin^2\theta}{4\lambda^5} + \left(-\frac{21m}{8\lambda^5} - \frac{m^2}{4\lambda^6}\right)\sin^4\theta\right]a^4 + O(a^6), \tag{3.96e}$$

$$\delta(\lambda, \theta) = -\frac{5m\cos\theta\sin^2\theta}{4\lambda^4}a^3 + O(a^5). \tag{3.96f}$$

Comparing with [33], we find agreement for R which corresponds to their areal coordinate r. Calculation of the metric components g_{ab} using (3.96) gives us

$$g_{uu}(\lambda,\theta) = -1 + \frac{2m}{\lambda} + \left[\left(\frac{3m}{\lambda^3} + \frac{m^2}{\lambda^4}\right)\sin^2\theta - \frac{2m}{\lambda^3}\right]a^2 + \left[\frac{2m}{\lambda^5} - \left(\frac{10m}{\lambda^5} + \frac{4m^2}{\lambda^6}\right)\sin^2\theta + \left(\frac{35m}{4\lambda^5} + \frac{23m^2}{4\lambda^6} + \frac{9m^3}{10\lambda^7}\right)\sin^4\theta\right]a^4 + O(a^6), \tag{3.97a}$$

$$g_{u\lambda}(\lambda,\theta) = -1, \tag{3.97b}$$

$$g_{u\theta}(\lambda,\theta) = \left\{\left(\frac{3m}{\lambda^2}\right)a^2 + \left[-\frac{5m}{\lambda^4} + \left(\frac{35m}{\lambda^4} + \frac{23m^2}{10\lambda^5}\right)\sin^2\theta\right]a^4\right\}\sin\theta\cos\theta + O(a^6), \tag{3.97c}$$

$$g_{u\phi}(\lambda,\theta) = \left\{\left(-\frac{2m}{\lambda}\right)a + \left[\frac{4m}{\lambda^3} - \left(\frac{5m}{\lambda^3} + \frac{m^2}{\lambda^4}\right)\sin^2\theta\right]a^3\right\}\sin^2\theta + O(a^5), \tag{3.97d}$$

$$g_{\theta\theta}(\lambda,\theta) = \lambda^2 + \left(-\frac{m\sin^2\theta}{\lambda}\right)a^2 + \left[\frac{9m}{2\lambda^3}\sin^2\theta - \left(\frac{21m}{4\lambda^3} + \frac{3m^2}{20\lambda^4}\right)\sin^4\theta\right]a^4 + O(a^6), \tag{3.97e}$$

$$g_{\theta\phi}(\lambda,\theta) = \left(-\frac{5m\sin^3\theta\cos\theta}{2\lambda^2}\right)a^3 + O(a^5), \tag{3.97f}$$

$$g_{\phi\phi}(\lambda,\theta) = \left\{\lambda^2 + \left(\frac{m\sin^2\theta}{\lambda}\right)a^2 + \left[-\frac{9m}{2\lambda^3}\sin^2\theta + \left(\frac{21m}{4\lambda^3} + \frac{17m^2}{20\lambda^4}\right)\sin^4\theta\right]a^4\right\}\sin^2\theta + O(a^6). \tag{3.97g}$$

Equations (3.97) constitute our final expression for the slowly rotating stationary and axially symmetric (Kerr) metric adapted to null coordinates which asymptotically match an inertial Bondi frame. At the difference of all previous approaches, it was obtained as *an explicit solution* of the Einstein equations. After comparison with Ref. [33], we find agreement up to a typo in their equation for $g_{\theta\phi}$. We also note care should be taken when comparing the expressions of Ref. [33] with ours. First, the authors of Ref. [33] present a Bondi-Sachs form of the metric, while we have an affine-null metric approaching a Bondi frame; the difference is in the choice of radial coordinate, and the two agree with one another only up to $O(\lambda^{-4})$. Second, the authors of Ref. [33] make a large λ expansion while we make a small a expansion, which results in powers of λ^{-k} absorbed by order symbols in [33]. A slowly rotating version of the Kerr metric in null affine coordinates at second order in a was also obtained by Dozmorov who made null tetrad rotations starting with the Kerr metric as expressed in Boyer-Lindquist coordinates [40]. In the next section we show an alternative procedure to recover the slowly rotating Kerr metric components as expressed in (3.97) by doing appropriate coordinate transformations.

IV. APPROXIMATED AFFINE-NULL METRIC DERIVED FROM THE KERR METRIC

Here, starting with the Kerr metric expressed in Boyer-Lindquist coordinates (BL) $\{\hat{t},\hat{r},\hat{\theta},\hat{\phi}\}$, we present an explicit transformation to affine-null coordinates up to fourth order in a. The Kerr metric in BL coordinates reads

$$ds^2 = g_{\hat{t}\hat{t}}d\hat{t}^2 + g_{\hat{t}\hat{\phi}}d\hat{t}d\hat{\phi} + g_{\hat{r}\hat{r}}d\hat{r}^2 + g_{\hat{r}\hat{r}}d\hat{r}^2 + g_{\hat{\theta}\hat{\theta}}d\hat{\theta}^2 + g_{\hat{\phi}\hat{\phi}}d\hat{\phi}^2; \tag{4.1}$$

with

$$g_{\hat{t}\hat{t}} = -\left(1 - \frac{2m\hat{r}}{\Sigma}\right), \tag{4.2}$$

$$g_{\hat{t}\hat{\phi}} = -\frac{2ma\hat{r}\sin^2\hat{\theta}}{\Sigma}, \tag{4.3}$$

$$g_{\hat{r}\hat{r}} = \frac{\Sigma}{\Delta}, \tag{4.4}$$

$$g_{\hat{\theta}\hat{\theta}} = \Sigma, \tag{4.5}$$

$$g_{\hat{\phi}\hat{\phi}} = \left(\hat{r}^2 + a^2 + \frac{2ma^2\hat{r}\sin^2\hat{\theta}}{\Sigma}\right)\sin^2\hat{\theta}, \tag{4.6}$$

with $\Delta = \hat{r}^2 - 2m\hat{r} + a^2$ and $\Sigma = \hat{r}^2 + a^2\cos^2\hat{\theta}$. The u null coordinate must satisfy the eikonal equation

$$g^{ab}\nabla_a u\nabla_b u = 0. \tag{4.7}$$

Inspired by [33], we propose the following expansion for u;

$$u = \hat{t} - \hat{r} - 2m \ln\left(\frac{\hat{r} - 2m}{2m}\right) + \sum_{i=1}^{\infty} f_i(\hat{r}, \hat{\theta}) a^i. \tag{4.8}$$

Note that for $a = 0$ this expression reduces to the standard outgoing Schwarzschild null coordinate. By replacing (4.8) into (4.7), we obtain a set of differential equations for $f_i(\hat{r}, \hat{\theta})$ that can be solved iteratively. Conserving terms up to fourth order in a we find that only the even coefficients $f_{2n}(\hat{r}, \hat{\theta})$ are nonvanishing with

$$f_2(\hat{r}, \hat{\theta}) = \frac{5\hat{r} - 2m}{4\hat{r}(2m - \hat{r})} + \frac{\cos 2\hat{\theta}}{4\hat{r}} - \frac{\ln(1 - \frac{2m}{\hat{r}})}{2m}, \tag{4.9}$$

$$f_4(\hat{r}, \hat{\theta}) = \frac{(2\hat{r} + m)}{16\hat{r}^4} \sin^4(2\hat{\theta}) - \frac{3\ln(1 - \frac{2m}{\hat{r}})}{8m^3} - \frac{4m^2 - 9m\hat{r} + 3\hat{r}^2}{4m^2\hat{r}(\hat{r} - 2m)^2}. \tag{4.10}$$

Similarly, affine-null coordinates $\{\lambda, \theta, \phi\}$ can be obtained from the requirements

$$g^{ab}\nabla_a u \nabla_b \lambda = -1, \tag{4.11a}$$

$$g^{ab}\nabla_a u \nabla_b \theta = g^{ab}\nabla_a u \nabla_b \phi = 0, \tag{4.11b}$$

by assuming relations of the form

$$\lambda = \hat{r} + \sum_{i=1}^{\infty} \hat{\Lambda}_i(\hat{\theta}, \hat{r}) a^i, \tag{4.12}$$

$$\theta = \hat{\theta} + \sum_{i=1}^{\infty} \hat{\Theta}_i(\hat{\theta}, \hat{r}) a^i, \tag{4.13}$$

$$\phi = \hat{\phi} + \sum_{i=1}^{\infty} \hat{\Phi}_i(\hat{\theta}, \hat{r}) a^i, \tag{4.14}$$

and replacing into the set (4.11), the coefficient functions $\hat{\Lambda}_i, \hat{\Theta}_i, \hat{\Phi}_i$ can be obtained in the same way as u. After that, the resulting relations can be inverted in order to express the BL coordinates in terms of the affine-null coordinates. Following these steps up to fourth order, the final transformation coordinates read

$$\begin{aligned}\hat{t} = {} & u + \lambda + 2m \ln\left(\frac{\lambda}{2m} - 1\right) + \left[\frac{\ln(1 - \frac{2m}{\lambda})}{2m} + \frac{3m\cos(2\theta) + 4\lambda - 3m}{(4\lambda - 8m)\lambda}\right] a^2 \\ & + \left[-\frac{m(175\lambda^2 - 224m\lambda - 72m^2)(\cos(2\theta))^2}{320(\lambda - 2m)^2\lambda^4} - \frac{m(25\lambda^2 + 64m\lambda + 72m^2)\cos(2\theta)}{160(\lambda - 2m)^2\lambda^4}\right. \\ & \left. + \frac{3\ln(1 - \frac{2m}{\lambda})}{8m^3} + \frac{240\lambda^5 - 720\lambda^4 m + 320\lambda^3 m^2 + 225\lambda^2 m^3 - 96\lambda m^4 + 72m^5}{320m^2\lambda^4(\lambda - 2m)^2}\right] a^4 + O(a^6),\end{aligned} \tag{4.15}$$

$$\hat{r} = \lambda - \frac{(\lambda + m)\sin^2\theta}{2\lambda^2} a^2 + \left[\frac{\sin^2\theta(5\cos 2\theta + 3))}{16\lambda^3} + \frac{m\sin^2\theta(7\cos 2\theta + 1)}{16\lambda^4} - \frac{m^2\sin^4\theta}{5\lambda^5}\right] a^4 + O(a^6), \tag{4.16}$$

$$\hat{\theta} = \theta - \frac{\sin(2\theta)}{4\lambda^2} a^2 + \frac{\sin(2\theta)(3\lambda\cos(2\theta) + m\cos(2\theta) - m)}{16\lambda^5} a^4 + O(a^6), \tag{4.17}$$

$$\begin{aligned}\hat{\phi} = {} & \phi + \left[\frac{1}{\lambda} + \frac{\ln(1 - \frac{2m}{\lambda})}{2m}\right] a + \left[\frac{\ln(1 - \frac{2m}{\lambda})}{4m} + \frac{m(2m + 5\lambda)\cos 2\theta}{8(\lambda - 2m)\lambda^4}\right. \\ & \left. - \frac{6m^4 - m^3\lambda + 8m^2\lambda^2 + 12m\lambda^3 - 12\lambda^4}{24m^2(\lambda - 2m)\lambda^4}\right] a^3 + O(a^5).\end{aligned} \tag{4.18}$$

Finally, with these transformations in hand, we obtain the same metric components in affine-null coordinates up to fourth order in a as given in (3.97) in the previous section.

V. LOCALIZING THE EVENT HORIZON AND ERGOSPHERE IN AFFINE-NULL COORDINATES

In this section we show that the affine-null coordinates for the slowly rotating Kerr metric cover the ergosphere and the (past) event horizon r_+. In order to find them in a consistent way, they must be localized at $O(a^4)$. Recall that in BL coordinates the Kerr metric has the external ergosphere placed at

$$\begin{aligned}\hat{r}_{\rm erg} &= m + \sqrt{m^2 - a^2\cos^2\hat{\theta}} \\ &= 2m - \frac{a^2\cos^2\hat{\theta}}{2m} - \frac{a^4\cos^4\hat{\theta}}{8m^3} + O(a^6),\end{aligned} \tag{5.1}$$

and the event horizon at

$$r_+ = m + \sqrt{m^2 - a^2} = 2m - \frac{a^2}{2m} - \frac{a^4}{8m^3} + O(a^6). \quad (5.2)$$

The boundary of the external ergosphere is obtained by looking for the timelike surface Γ where the stationary Killing vector field ∂_u becomes a null vector field that is where

$$g_{uu}|_\Gamma = 0. \quad (5.3)$$

Taking into account the expression for g_{uu} as found in the first equation of (3.97), the ergosphere will be located at a given $\lambda = \lambda_{\rm erg}(\theta)$, with

$$\lambda_{\rm erg}(\theta) = \sum_{i=0}^{2} \lambda_{{\rm erg}[2i]}(\theta) a^{2i} + O(a^6), \quad (5.4)$$

where the even expansion is a consequence of the symmetry assumption of Sec. II. Inserting (5.4) into (5.3), and after reexpanding in powers of a we find

$$\lambda_{\rm erg}(\theta) = 2m - \frac{(7\cos^2\theta - 3)}{8m} a^2 - \frac{(51\cos^4\theta - 2\cos^2\theta + 31)}{640m^3} a^4 + O(a^6), \quad (5.5)$$

which gives the location of the (external) ergosphere in affine-null coordinates. By replacing (5.5) into (4.16) [using the inverse of (4.17) to relate θ with $\hat{\theta}$], and after a reexpansion in powers of a it can be checked that the standard fourth order expression for the BL expression of the ergosphere as given by (5.1) is recovered.

Similarly, for the (Killing) event horizon we search a null surface Σ described in affine-null coordinates by $\Sigma(\lambda, \theta) = \lambda - \lambda_H(\theta) = 0$. Hence, its normal vector $N_a = \nabla_a \Sigma$ must satisfy $N^a N_a = 0$ which implies the following differential equation for $\lambda_H(\theta) = 0$,

$$g^{ab} N_a N_b = W + 2W^\theta \frac{\partial \lambda_H(\theta)}{\partial \theta} + \frac{h^{\theta\theta}}{R^2} \left(\frac{\partial \lambda_H(\theta)}{\partial \theta}\right)^2 = 0. \quad (5.6)$$

Let us assume an expansion for $\lambda_H(\theta)$ similar to (5.4), i.e.

$$\lambda_H(\theta) = \sum_{i=0}^{2} \lambda_{H[2i]}(\theta) a^{2i} + O(a^6) \quad (5.7)$$

with $\lambda_{H[0]} = 2m$ (the Schwarzschild value for the location of the horizon). Introducing (5.7) into (5.6); reexpanding again in powers of a, we find [omitting the $O(a^6)$ term]

$$0 = \left(\frac{\lambda_{H[2]}}{2m} + \frac{3\cos^2\theta + 1}{16m^2}\right) a^2 + \left[\frac{(\lambda_{H[2],\theta})^2}{4m^2} - \frac{3\sin\theta\cos\theta}{8m^3}\lambda_{H[2],\theta} - \frac{\lambda_{H[2]}^2}{4m^2} - \frac{3(\cos^2\theta + 1)}{16m^3}\lambda_{H[2]} - \frac{127\cos^4\theta - 320m^3\lambda_{H[4]} - 84\cos^2\theta - 3}{640m^4}\right] a^4. \quad (5.8)$$

So that solving for the coefficient $\lambda_{H[2]}$ and $\lambda_{H[4]}$ gives us

$$\lambda_H(\theta) = 2m - \frac{(1 + 3\cos^2\theta)}{8m} a^2 + \frac{(29\cos^4\theta - 78\cos^2\theta - 31)}{640m^3} a^4 + O(a^6), \quad (5.9)$$

which gives the location of the (past) event horizon in affine-null coordinates. By replacing into (4.16) and after a reexpansion in a up to fourth order, the well-known value (5.2) for the BL radial coordinate of the event horizon is recovered. At this location, the affine-null coordinate system is regular.

VI. SUMMARY

We have derived high order slow rotation approximation of the Kerr metric in affine-null coordinates. To achieve this aim a metric in affine-null coordinates was expanded off a spherically symmetric background metric that corresponds to a Schwarzschild metric in outgoing Eddington Finkelstein coordinates. This quasispherical expansion was done with respect to a general smallness parameter ε. Subject to stationarity and axial symmetry the perturbations did not depend on the u and ϕ coordinates. Moreover, requiring even parity of the Komar integral of stationary (giving the mass of the system) and odd parity of the Komar integral of axial symmetry (giving the angular momentum of the system), we argued that, on the one hand, the metric functions γ, R, W^θ, and W have only even perturbations in ε while, on the other hand, the metric fields δ and W^ϕ have only odd perturbations in ε. This fact significantly simplifies the integration of the perturbation equations resulting from the quasispherical expansion of the Ricci tensor. In addition, we find that the integration of the perturbation equations follows an alternative hierarchical structure. This means that with the spherically symmetric background solution at hand, the linear perturbations only involve the functions δ and W^{ϕ}, and its integration provides (after application of the boundary condition of an asymptotic inertial observer) one free integration constant B. At next order, the quadratic perturbations turn out to be a linear combination of the derivatives of functions γ, R, W^{θ}, and W together with nonlinear terms containing the integration constants A of the background model and the free integration constant B of the linear perturbation. Their integration also yields a free integration constant C. Following up the next order, there are only differential equations involving the cubic perturbations of δ and W^{ϕ} as well as the integration constants A, B, and C characterizing the lower order perturbations. This alternating scheme between the perturbations of (δ, W^{ϕ}) and those of $(\gamma, R, W^{\theta}, W)$ continues up to any order and is in fact a result of the symmetry assumptions. A common feature in solving for the even and odd-parity modes of ε, is that at any order there is a fourth order master equation for either the perturbation in γ or the perturbation in δ. With the solution of this master equation, the remaining perturbations can be solved by mere integration. After having obtained the perturbed solution and calculation of the Komar mass and Komar angular momentum, the arising free integration constants $A, B, C, \ldots$, can be expressed by the Komar mass and Komar angular momentum or by the mass and specific angular momentum. Hence, the solution depends only on two free physical parameters. The fact that the derived solution is depending only on two parameters goes along with the black holes uniqueness theorems stating that any stationary and axially symmetric vacuum solution of Einstein equations is uniquely determined by two parameters characterizing the mass and angular momentum of the black hole. Here we have required the solutions of occurring master equations to be finite [see discussion around (3.23)] at the affine parameter value $\lambda = A/2$. This is the position of the past event horizon of the nonrotating solution and similar to Carter's requirement of having a nondegenerate horizon [41–43]. Since the Komar angular momentum is $O(\varepsilon)$, it turns out that the formal expansion parameter ε relates to the specific angular momentum, and the previously made quasi-spherical approximation is in fact a slow rotation approximation, like those of Hartle and Thorne [35,44]. By successively solving Einstein equations, we thus derived a slow rotation approximation of the Kerr metric up to fourth order in the specific angular momentum. This solution is further verified for correctness using a "standard" approach by obtaining a different representation of a given metric in another coordinate chart via a coordinate transformation. The slowly rotating Kerr metric presented here also obeys the peeling property, which can be seen considering the Weyl scalars in (6.1),

$$\Psi_0 = \left(\frac{3ma^2}{\lambda^5} + i\frac{15ma^3}{\lambda^6}\cos\theta\right)\sin^2\theta + O(\lambda^{-7}), \tag{6.1a}$$

$$\Psi_1 = i\frac{3\sqrt{2}ma}{2\lambda^4}\sin\theta + O(\lambda^{-5}), \tag{6.1b}$$

$$\Psi_2 = -\frac{m}{\lambda^3} - i\frac{3ma\cos\theta}{\lambda^4} + O(\lambda^{-5}), \tag{6.1c}$$

$$\Psi_3 = -i\frac{3\sqrt{2}ma}{4\lambda^4}\sin\theta + O(\lambda^{-5}), \tag{6.1d}$$

$$\Psi_4 = \frac{3ma^2}{4\lambda^5}\sin^2\theta + O(\lambda^{-6}). \tag{6.1e}$$

We can see in (6.1) that Ψ_4 and Ψ_3 have a stronger falloff as required by the peeling property stating that $\Psi_n \sim \lambda^{5-n}$ at large radii. This stronger falloff is because of the requirement of stationarity, the metric is not depending on u and the multipole structure of the solution [45]. We recall that for a most general spacetime satisfying the peeling property $\psi_4 \sim (\partial_u^2\sigma)/\lambda$ where σ is the gravitational strain (e.g. the gravitational wave) as measured by an asymptotic observer.

Moreover it is easily checked that the (only) conserved Newman Penrose constant [26] vanishes [33].

What is interesting to remark is that up to the considered order of approximation of our work and those of [33], the small a expansion and the large λ expansion coincide. It would be interesting to see up until which order this is the case. Such an analysis might give insight on the validity and universality of general small parameter expansions of the Kerr spacetime in relation to null coordinates. It may also give insight if a closed form solution of the Kerr metric with a surface forming null coordinate can be obtained at all. The method presented here offers the possibility to calculate any type of approximate rotating null-metric solution that is stationary, is axially symmetric, and has a known spherically symmetric background, e.g. those describing compact matter systems or having a cosmological constant. Indeed, the study presented here (solving the characteristic equations in this affine-null, metric formulation for vacuum spacetimes) is the natural starting point for further studying the matter system under the given symmetry assumptions. Some of such questions we are currently investigating.

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APPENDIX A: USEFUL RELATIONS BETWEEN LEGENDRE POLYNOMIALS

For completeness, we list some properties of the associated Legendre differential equations and relations between the Legendre polynomials. The associated Legendre differential equation is

$$\frac{d}{dy}\left[(1-y^2)\frac{dP_\ell^{\hat{m}}}{dy}\right]+\left[\ell(\ell+1)-\frac{\hat{m}^2}{1-y^2}\right]P_\ell^{\hat{m}}=0, \quad \text{(A1)}$$

where $P_\ell^{\hat{m}}(y)$ are the associated Legendre polynomials, defined via

$$P_\ell^{\hat{m}}(y)=\frac{(-)^{\hat{m}}}{2^\ell \ell!}(1-y^2)^{\hat{m}/2}\frac{d^{\ell+\hat{m}}}{dy^{\ell+\hat{m}}}(y^2-1)^\ell. \quad \text{(A2)}$$

In particular, if $\hat{m}=0$ we have $P_\ell^0(y)=P_\ell(y)$, which are the well-known Legendre polynomials. From these definitions, some useful identities can be derived

$$\frac{d}{dy}[(1-y^2)P_\ell^2(y)]=[\ell(\ell+1)-2](1-y^2)^{1/2}P_\ell^1(y), \quad \text{(A3)}$$

$$\frac{\frac{d}{dy}[(1-y^2)^2\frac{dP_\ell^2}{dy}]}{1-y^2}-2P_\ell^2=\ell(\ell+1)(\ell+2)(\ell-1)P_\ell(y), \quad \text{(A4)}$$

$$P_\ell^1=-(1-y^2)^{1/2}\frac{dP_\ell^0}{dy}, \quad \text{(A5)}$$

$$P_\ell^2=(1-y^2)\frac{d^2P_\ell^0}{dy^2}, \quad \text{(A6)}$$

$$\frac{d}{dy}\left[(1-y^2)^2\frac{d}{dy}\frac{P_\ell^1}{(1-y^2)^{1/2}}\right] =[2-\ell(\ell+1)](1-y^2)^{1/2}P_\ell^1, \quad \text{(A7)}$$

$$\frac{d}{dy}(1-y^2)^{1/2}P_\ell^1=\ell(\ell+1)P_\ell^0. \quad \text{(A8)}$$

APPENDIX B: KOMAR CHARGES

Depending on the Killing vector $X^a\in\{\partial_u,\partial_\phi\}$, we take the Komar charges to be

$$K_X=-\frac{k_X}{8\pi}\oint\nabla^{[a}X^{b]}d\Sigma_{ab} \quad \text{(B1)}$$

with $k_X=1,-1/2$ for a timelike (e.g. ∂_u) or rotational Killing vector (e.g. ∂_ϕ), respectively. Consider the general null metric with the nonzero contravariant components g^{01}, g^{11}, g^{1A}, and g^{AB}. The corresponding line element is

$$g_{ab}dx^adx^b=(g^{11}+g_{AB}g^{1A}g^{1B})\left(\frac{dx^0}{g^{01}}\right)^2+2\left(\frac{dx^0}{g^{01}}\right)dx^1 -2g_{AB}g^{1A}dx^B\left(\frac{dx^0}{g^{01}}\right)+g_{AB}dx^Adx^B, \quad \text{(B2)}$$

where $g_{AC}g^{CB}=\delta_A^B$. We define the null vectors l and n which obey $l^an_a+1=l^al_a=n_an^a=0$ as

$$l=l^a\partial_a=-g^{01}\partial_1, \quad n=n^a\partial_a=\partial_0+\frac{1}{2}\frac{g^{11}}{g^{01}}\partial_1+\frac{g^{1A}}{g^{01}}\partial_A. \quad \text{(B3)}$$

We note that l points into the future. The associated covariant components are

$$l_adx^a=-dx^0, \qquad n_adx^a=-\frac{1}{2}\frac{g^{11}}{(g^{01})^2}dx^0+\frac{dx^1}{g^{01}}, \quad \text{(B4)}$$

respectively. The surface element $d\Sigma_{ab}$ follows as

$$d\Sigma_{ab}=2l_{[a}n_{b]}\sqrt{\det(g_{AB})}dx^2dx^3 \quad \text{(B5)}$$

with $x^A=(x^2,x^3)$ being any angular coordinates for the units sphere. Setting $g_{AB}=R^2h_{AB}$ with h_{AB} having the determinant of the unit sphere metric q_{AB}, $q(x^C)\coloneqq\det(h_{AB})$. The corresponding volume element is defined as $d^2q\coloneqq\sqrt{q}dx^2dx^3$, and we have $\oint d^2q=4\pi$. Hence,

$$d\Sigma_{ab}=2l_{[a}n_{b]}R^2d^2q.$$

This allows us to write the Komar integral as

$$K(X)=-\frac{k_X}{8\pi}\oint(2l^an^b\partial_{[a}X_{b]})R^2d^2q. \quad \text{(B6)}$$

Since

$$2l^an^b\partial_{[a}X_{b]}=2l^{[a}n^{b]}X_{b,a} \quad \text{(B7)}$$

$$=(l^an^b-l^bn^a)X_{b,a} \quad \text{(B8)}$$

$$=l^1(n^bX_{b,1})-l^1(n^bX_{1,b}) \quad \text{(B9)}$$

$$=-g^{01}[(n^bX_{b,1})-(n^bX_{1,b})], \quad \text{(B10)}$$

we have

$$K(X)=\frac{k_X}{8\pi}\oint[(n^bX_{b,1})-(n^bX_{1,b})]g^{01}R^2d^2q. \quad \text{(B11)}$$

Taking the Killing vector to be $X=X^a\partial_a$ and specification to an affine null metric

$$g^{01} = \epsilon, \qquad g^{1A} = \epsilon W^A, \qquad g^{11} = W,$$
$$g_{0A} = -R^2 h_{AB} W^B, \qquad g_{AB} = R^2 h_{AB}, \tag{B12}$$

and $\epsilon^2 = 1$ gives us

$$\begin{aligned} 2l^a n^b \partial_{[a} X_{b]} = & -[W_{,1} - R^2 h_{AB} W^A W^B_{,1}] X^0 \\ & + R^2 (h_{AB} W^B_{,1} X^A - 2 h_{AB} W^B X^A_{,1}) \\ & + \epsilon (X^1_{,1} - X^0_{,0}) - W X^0_{,1} - W^A X^0_{,A}. \end{aligned} \tag{B13}$$

Assuming the timelike Killing vector $X = \partial_0$ gives us

$$2l^a n^b \partial_{[a} X_{b]} = -[W_{,1} - R^2 h_{AB} W^A W^B_{,1}].$$

Thus for the above form of the Killing vector we have the related Komar charge using $k_X = 1$,

$$K(\partial_0) = \frac{1}{8\pi} \oint (-\epsilon [W_{,1} - R^2 h_{AB} W^A W^B_{,1}]) R^2 d^2 q. \tag{B14}$$

With the rotational Killing $X = \partial_3$, we have

$$2l^a n^b \partial_{[a} X_{b]} = R^2 h_{3B} W^B_{,1}$$

so that the Komar charge is with $k_X = -\frac{1}{2}$

$$K(\partial_3) = -\frac{\epsilon}{16\pi} \oint (R^4 h_{3B} W^B_{,1}) d^2 q. \tag{B15}$$

[1] H. Bondi, Nature (London) **186**, 535 (1960).
[2] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. A **269**, 21 (1962).
[3] R. K. Sachs, Proc. R. Soc. A **270**, 103 (1962).
[4] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[5] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. (N.Y.) **6**, 918 (1965).
[6] P. Jordan, J. Ehlers, and R. K. Sachs, Gen. Relativ. Gravit. **45**, 2691 (2013).
[7] R. Gómez, L. Lehner, R. L. Marsa, and J. Winicour, Phys. Rev. D **57**, 4778 (1998).
[8] J. Winicour, Living Rev. Relativity **15**, 2 (2012).
[9] T. Mädler and J. Winicour, Scholarpedia **11**, 33528 (2016).
[10] G. Barnich and C. Troessaert, J. High Energy Phys. 05 (2010) 062.
[11] S. Pasterski, A. Strominger, and A. Zhiboedov, J. High Energy Phys. 12 (2016) 053.
[12] T. Mädler and J. Winicour, Classical Quantum Gravity **33**, 175006 (2016).
[13] D. A. Nichols, Phys. Rev. D **95**, 084048 (2017).
[14] T. Mädler and J. Winicour, Classical Quantum Gravity **35**, 035009 (2018).
[15] T. Mädler and J. Winicour, Classical Quantum Gravity **36**, 095009 (2019).
[16] P. Papadopoulos, Phys. Rev. D **65**, 084016 (2002).
[17] N. T. Bishop, R. Gómez, L. Lehner, and J. Winicour, Phys. Rev. D **54**, 6153 (1996).
[18] R. Gómez, P. Papadopoulos, and J. Winicour, J. Math. Phys. (N.Y.) **35**, 4184 (1994).
[19] F. Siebel, J. A. Font, E. Müller, and P. Papadopoulos, Phys. Rev. D **65**, 064038 (2002).
[20] T. Mädler and E. Müller, Classical Quantum Gravity **30**, 055019 (2013).
[21] J. Winicour, Phys. Rev. D **87**, 124027 (2013).
[22] T. Mädler, Phys. Rev. D **99**, 104048 (2019).
[23] E. Gallo, C. Kozameh, T. Mädler, O. M. Moreschi, and A. Perez, Phys. Rev. D **104**, 084048 (2021).
[24] J. A. Crespo, H. P. de Oliveira, and J. Winicour, Phys. Rev. D **100**, 104017 (2019).
[25] O. Baake and T. Mädler (to be published).
[26] N. T. Bishop and L. R. Venter, Phys. Rev. D **73**, 084023 (2006).
[27] M. A. Argañaraz and O. M. Moreschi, Phys. Rev. D **104**, 024049 (2021).
[28] S. J. Fletcher and A. W. C. Lun, Classical Quantum Gravity **20**, 4153 (2003).
[29] S. Jahanur Hoque and A. Virmani, Classical Quantum Gravity **38**, 225002 (2021).
[30] S. A. Hayward, Phys. Rev. Lett. **92**, 191101 (2004).
[31] E. Gallo and O. M. Moreschi, Phys. Rev. D **89**, 084009 (2014).
[32] M. A. Argañaraz and O. M. Moreschi, Phys. Rev. D **105**, 084012 (2022).
[33] S. Bai, Z. Cao, X. Gong, Y. Shang, X. Wu, and Y. K. Lau, Phys. Rev. D **75**, 044003 (2007).
[34] X. Gong, Y. Shang, S. Bai, Z. Cao, Z. Luo, and Y. K. Lau, Phys. Rev. D **76**, 107501 (2007).
[35] J. B. Hartle, Astrophys. J. **150**, 1005 (1967).
[36] J. Tafel, Classical Quantum Gravity **39**, 115013 (2022).
[37] R. Gómez, S. Husa, and J. Winicour, Phys. Rev. D **64**, 024010 (2001).
[38] M. G. J. van der Burg, Proc. R. Soc. A **294**, 112 (1966).
[39] T. Mädler, Phys. Rev. D **87**, 104016 (2013).
[40] I. M. Dozmorov, Fizika **18**, 95 (1975).
[41] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).
[42] D. C. Robinson, Phys. Rev. Lett. **34**, 905 (1975).
[43] M. Heusler, P. Goddard, and J. Yeomans, *Black Hole Uniqueness Theorems*, Cambridge Lecture Notes in Physics (Cambridge University Press, Cambridge, England, 1996), ISBN 9780521567350.
[44] J. B. Hartle and K. S. Thorne, Astrophys. J. **153**, 807 (1968).
[45] A. I. Janis and E. T. Newman, J. Math. Phys. (N.Y.) **6**, 902 (1965).