Beyond Schwarzschild–de Sitter spacetimes. III. A perturbative vacuum with nonconstant scalar curvature in $R + R^2$ gravity

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In violation of the generalized Lichnerowicz theorem advocated in the past, quadratic gravity admits vacua with *nonconstant* scalar curvature. In a recent publication, we revitalized a program—that Buchdahl originated but prematurely abandoned circa 1962—and uncovered a novel exhaustive class of static spherically symmetric vacua for pure $\mathcal{R}^2$ gravity. The Buchdahl-inspired metrics we obtained therein are *exact* solutions which exhibit nonconstant scalar curvature. A product of fourth-order gravity, the metrics entail a new (Buchdahl) parameter k which allows the Ricci scalar to vary on the manifold. The metrics are able to *defeat* the generalized Lichnerowicz theorem by evading an overly strong restriction on the asymptotic falloff in the spatial derivatives of the Ricci scalar as assumed in the earlier theorem. The Buchdahl parameter k is a new characteristic of pure $\mathcal{R}^2$ gravity, a higher-derivative theory. By venturing that the Buchdahl parameter should be a universal hallmark of higher-derivative gravity at large, in this paper we seek to extend the concept to the quadratic action $\mathcal{R}^2 + \gamma(\mathcal{R} - 2\Lambda)$. We determine that, *up to the order* $\mathcal{O}(k^2)$, the quadratic field equation admits the following vacuum solution $ds^2 = e^{k\varphi(r)}[-\Psi(r)dt^2 + \frac{dr^2}{\Psi(r)} + r^2 d\Omega^2]$ in which $\Psi(r) := 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2$ and the function $\varphi(r)$ obeys a linear second-order ordinary differential equation, per $6(r^2\Psi(r)\varphi'(r))' = \gamma r^2 \varphi(r)$ subject to $\varphi(r \to \infty) = 0$. Conforming with our intuition, the Ricci scalar carries the footprint of a higher-derivative characteristic k, given by $\mathcal{R}(r) = 4\Lambda - k(4\Lambda + \frac{\gamma}{2})\varphi(r) + \mathcal{O}(k^2)$. The Ricci scalar is nonconstant, including the asymptotically flat case (i.e., $\Lambda = 0$) as long as $k \neq 0$ and $\gamma \neq 0$. The existence of such an asymptotically flat vacuo with nonconstant scalar curvature defeats the generalized Lichnerowicz theorem in *its entirety*. Our finding thus warrants restoring the $\mathcal{R}^2$ term in the full quadratic action, $\gamma\mathcal{R} + \beta\mathcal{R}^2 - \alpha\mathcal{C}^{\mu\nu\rho\sigma}\mathcal{C}_{\mu\nu\rho\sigma}$, when applying the Lü-Perkins-Pope-Stelle ansatz. Implications to the Lü-Perkins-Pope-Stelle solution are discussed herein.

I. MOTIVATION: ESCAPES FROM THE GENERALIZED LICHNEROWICZ THEOREM

In [1] Nelson offered a proof concerning the vacua of the quadratic action (with α, β, γ all non-negative)

$$\gamma\mathcal{R} + \beta\mathcal{R}^2 - \alpha\mathcal{C}^{\mu\nu\rho\sigma}\mathcal{C}_{\mu\nu\rho\sigma}. \tag{1}$$

His proof contains two parts: the trace part and the nontrace part. The trace part concluded that static vacua of quadratic gravity must be Ricci-scalar-flat, viz. $\mathcal{R} \equiv 0$, if $\gamma > 0$. The nontrace part enforced an even stronger result; it posited that these vacua must be Ricci-flat, viz. $\mathcal{R}_{\mu\nu} \equiv 0$. In [2,3] Lü, Perkins, Pope, and Stelle identified crucial sign errors which invalidate the nontrace part of Nelson's proof. However, these authors still retained the validity of the trace part, i.e., $\mathcal{R} \equiv 0$ identically, which has tentatively become known as the generalized Lichnerowicz theorem.

If this "no-go"-type theorem is valid, then the constraint $\mathcal{R} = 0$ that follows would nullify the contributions that stem from the $\mathcal{R}^2$ term to the quadratic-gravity field equation. Thus, for the *sole* purpose of finding static vacuo configurations, the $\mathcal{R}^2$ term may be suppressed. With β set equal to zero and action (1) reduced to an Einstein-Weyl gravity, Lü *et al.* then proceeded to discovering the Lü-Perkins-Pope-Stelle numerical solution, which represents a second branch of static, spherically symmetric, and asymptotically flat spacetimes over and above the Schwarzschild branch. This rather surprising result has generated considerable interest [4–9]; notably, Podolský *et al.* have identified an exact infinite-series solution in place of the numerical approach [4].

The generalized Lichnerowicz theorem was "proved" again by Kehagias *et al.* [10] in a more limited situation, the pure quadratic action, viz. $\gamma = 0$. These authors concluded that $\mathcal{R} \equiv$ const everywhere for this setup.

However, these conclusions, put forth by Nelson [1] and reenforced by Lü *et al.* [2,3] and by Kehagias *et al.* [10],

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are under serious challenge. Inspired by a seminal—yet obscure—sixty-year-old work by Buchdahl [11], we recently uncovered a new class of metrics that project *nonconstant* scalar curvature in pure $\mathcal{R}^2$ gravity; the derivation was detailed in our companion paper [12] and the result shall be summarized momentarily below. Leaving no stone unturned, we successfully checked these Buchdahl-inspired metrics directly against the pure $\mathcal{R}^2$ field equation, thereby affirming their validity. The existence of the Buchdahl-inspired metrics is in stark defiance of the generalized Lichnerowicz theorem.

What has gone astray with the "proofs" of the generalized Lichnerowicz theorem, which at first sight seem to be water-tight, then? It turns out that the proofs advocated in [1–3,10] contain detrimental gaps which render the theorem vulnerable for violations. Let us first expose the gaps in these proofs.

A. The gaps in the generalized Lichnerowicz theorem

We shall adopt the derivation of Lü *et al.* in [2,3]. With α, β, γ all non-negative, the quadratic action (1) produces the vacuo field equation

$$\gamma\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\right) - 4\alpha B_{\mu\nu}$$
$$+ 2\beta\left[\mathcal{R}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathcal{R}\right] = 0, \quad (2)$$

where the Bach tensor $B_{\mu\nu} := (\nabla^\rho\nabla^\sigma + \frac{1}{2}\mathcal{R}^{\rho\sigma})\mathcal{C}_{\mu\nu\rho\sigma}$ is trace-free. The trace of (2) yields

$$-\gamma\mathcal{R} + 6\beta\Box\mathcal{R} = 0. \quad (3)$$

Lü *et al.* then considered a static black hole metric of the form $ds^2 = -\lambda^2 dt^2 + h_{ij}dx^i dx^j$, where λ and h_{ij} are functions of the spatial coordinates, and $\lambda \geq 0$ is supposed to vanish on the horizon. With the aid of (3), for $\beta > 0$, one obtains the following identity:

$$\int_V d^3x\sqrt{h}D^i(\lambda\mathcal{R}D_i\mathcal{R}) = \int_V d^3x\sqrt{h}\lambda\left[(D_i\mathcal{R})^2 + \frac{\gamma}{6\beta}\mathcal{R}^2\right]. \quad (4)$$

If the left-hand side of (4) could be made vanish, then the non-negativity of the right-hand side of (4) would readily enforce that

$$\mathcal{R}(r) \equiv \begin{cases} 0 & \text{if } \gamma > 0 \\ \text{const} & \text{if } \gamma = 0. \end{cases} \quad (5)$$

This constraint on static vacuo configurations for action has tentatively become known as the generalized Lichnerowicz theorem. Furthermore, restricted to the case

of $\alpha = 0$, the field equation (2), subject to constraint (5), leads to

$$\mathcal{R}_{\mu\nu} = \begin{cases} 0 & \text{if } \gamma > 0 \\ \frac{1}{4}g_{\mu\nu} \times \text{const} & \text{if } \gamma = 0, \end{cases} \quad (6)$$

meaning that the only static spherically symmetric vacuo admissible for the $\mathcal{R} + \mathcal{R}^2$ action is Schwarzschild, and likewise, the only static spherically symmetric vacuo admissible for the pure $\mathcal{R}^2$ action is Schwarzschild–de Sitter.

To make the left-hand side of (4) vanish, in [1], with the integrand therein being a total derivative, Nelson applied the 3D divergence theorem to turn the integral $\int_V d^3x\sqrt{h}D^i(\lambda\mathcal{R}D_i\mathcal{R})$ into a surface term at spatial infinity. Next, he assumed that the derivatives $D_i\mathcal{R}$ go to zero sufficiently rapidly so that the surface term would vanish.

Nelson's reasoning contains *two* gaps, however. First, an actual vacuo configuration may *not* guarantee that $D_i\mathcal{R}$ decay sufficiently rapidly to make the surface term vanish. This excessively strong requirement can be susceptible for violations. Second, and more seriously as this point is often overlooked, to apply the divergence theorem, the integrand $D^i(\lambda\mathcal{R}D_i\mathcal{R})$ must be a continuous function *everywhere* within the integration volume V. If the integrand diverges anywhere inside V, the divergence theorem would cease to hold, and Nelson's proof would fall apart. There is ample evidence that the second gap is pertinent: close to a black hole, spacetimes often project singularities at the origin and/or at the horizon.

In [2,3] Lü *et al.* bypassed the second gap in Nelson's proof by restricting the integration volume to the *exterior* of the black hole, viz. from the horizon outward to the spatial infinity. In the exterior region, it is reasonable to expect the integrand $D^i(\lambda\mathcal{R}D_i\mathcal{R})$ to be well behaved. As such, the left-hand side of (4) becomes the difference between *two* surface terms, one at infinity (denoted by S_∞) and one at the horizon (denoted by S_h), namely,

$$\oint_{S_\infty} d^iS(\lambda\mathcal{R}D_i\mathcal{R}) - \oint_{S_h} d^iS(\lambda\mathcal{R}D_i\mathcal{R}). \quad (7)$$

Lü *et al.*'s maneuver would seem to rescue Nelson's proof from the abyss. However, their walk around introduces a *third* gap: whereas $\lambda \to 0$ on the horizon, the terms $\mathcal{R}D_i\mathcal{R}$ may diverge there and overwhelm λ, forcing the surface term at the horizon S_h to be finite or even divergent.

In sum, due to the first gap and the third gap, the left-hand side of Eq. (4) in principle may—and in practice can—deviate from zero, rendering the generalized Lichnerowicz theorem invalid.

As we shall see right below, there exists a class of non-Schwarzschild vacua, the Buchdahl-inspired vacua, that project *nonconstant* scalar curvature in stark violation of the *would-be* conclusion in Eq. (5) of the generalized Lichnerowicz theorem.

II. THE BUCHDAHL-INSPIRED METRIC

Circa 1962 Buchdahl spearheaded a program seeking vacuum configurations for pure $\mathcal{R}^2$ gravity, namely, the action (1) with $\alpha = \gamma = 0$ [11]. Therein, he was able to show that the pure $\mathcal{R}^2$ vacua in general can acquire nonconstant scalar curvature. Proceeding further, he arrived at a nonlinear second-order ordinary differential equation (ODE) which would prescribe *all* nontrivial static spherically symmetric solutions admissible in the pure $\mathcal{R}^2$ theory. Unfortunately, Buchdahl deemed his ODE intractable and prematurely discontinued his pursuit for an analytical solution. To this day, his ODE remains untackled and his 1962 work has largely gone unnoticed by the gravitation research community.[1] Recently, in [12] we revisited Buchdahl's program, broke this outstanding six-decades-old impasse, and uncovered a novel exhaustive class of vacua for pure $\mathcal{R}^2$ gravity, to be summarized below.

The Buchdahl-inspired metric, as we called it as such, takes the following compact expression

$$ds^2 = e^{k\int \frac{dr}{rq(r)}}\left\{p(r)\left[-\frac{q(r)}{r}dt^2 + \frac{r}{q(r)}dr^2\right] + r^2 d\Omega^2\right\}, \quad (8)$$

in which the pair of functions $\{p(r), q(r)\}$ obey the "evolution" rules

$$\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2}, \quad (9)$$

$$\frac{dq}{dr} = (1 - \Lambda r^2)p, \quad (10)$$

with the Ricci scalar equal

$$\mathcal{R}(r) = 4\Lambda e^{-k\int \frac{dr}{rq(r)}}. \quad (11)$$

The most crucial element of the metric is the new (Buchdahl) parameter k, which allows the metric to be non-Schwarzschild and enables the Ricci scalar to *vary* on the manifold. At largest distances, the Ricci scalar converges to 4Λ. Metric (8)–(11) is a bona fide enlargement of the Schwarzschild–de Sitter (SdS) metric and duly recovers the SdS metric when $k = 0$ (see the subsection "The small k limit" right below). Our investigation on the phase-space $\{p(r), q(r)\}$ of the evolution rules (9)–(10) points towards very interesting new physics, e.g., the existence of horizonless objects; our findings shall be presented in a separate report.

To alley any lingering doubt, in [12,13], the present author and Shurtleff successfully checked the solution (8)–(11) against the pure $\mathcal{R}^2$ vacuo field equation, thereby affirming

its validity. The existence of vacua with *nonconstant* scalar curvature in a quadratic theory of gravity is a direct counterexample against the generalized Lichnerowicz theorem stated in Eq. (5).

Moreover, to our surprise, despite being nonlinear, the evolution rules (9)–(10) are *fully soluble* for $\Lambda = 0$. In a companion paper of this "Beyond Schwarzschild–de Sitter spacetimes" series [14], we exploited this advantage to derive an *exact closed analytical* form for a new metric, which we called the *special* Buchdahl-inspired metric that describes an asymptotically flat non-Schwarzschild $\mathcal{R}^2$ spacetime. The Kretschmann invariant of this metric exhibits curvature singularities on the interior-exterior boundary. Novel anomalous behaviors in the interior-exterior boundary and in the Kruskal-Szekeres diagram of pure $\mathcal{R}^2$ spacetime structures are discovered and reported in our companion paper [14].

A. The small k limit

For a nonzero Λ but with a small value of k, the Buchdahl-inspired metric (8)–(11) admits a perturbative form which we derived in [12] and shall briefly reproduce here for the reader's convenience. The crux of the argument is that since the evolution rules (9)–(10) depend on k^2 instead of k, they admit the perturbative solution

$$p(r) = 1 + \mathcal{O}(k^2), \quad (12)$$

$$q(r) = r - r_s - \frac{\Lambda}{3}r^3 + \mathcal{O}(k^2), \quad (13)$$

with r_s being a constant. Note that the conformal factor $e^{k\int \frac{dr}{rq(r)}}$ depends directly on k, however. Plugging (12)–(13) into (8) we obtain

$$ds^2 = e^{k\int \frac{dr}{r^2\Psi(r)}}\left[-\Psi(r)dt^2 + \frac{dr^2}{\Psi(r)} + r^2 d\Omega^2\right] + \mathcal{O}(k^2), \quad (14)$$

with $\Psi(r) := 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2$ and the Ricci scalar given by

$$\mathcal{R}(r) = 4\Lambda e^{-k\int \frac{dr}{r^2\Psi(r)}} + \mathcal{O}(k^2). \quad (15)$$

Metric (14)–(15) is applicable for pure $\mathcal{R}^2$ gravity up to $\mathcal{O}(k^2)$, with the Buchdahl parameter k measuring the amount of deviation from being Schwarzschild–de Sitter for the said metric. At $k = 0$, metric (14)–(15) is nothing but an SdS metric with a constant scalar curvature of 4Λ.

B. The purpose of this paper

Hereafter we shall concern with the following quadratic action

$$\mathcal{R}^2 + \gamma(\mathcal{R} - 2\Lambda). \quad (16)$$

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[1]Buchdahl's paper has gathered a paltry sum of 40+ citations since its inception in 1962, according to the NASA ADS and InpireHEP databases. Yet, none of these citations attempted to solve Buchdahl's ODE.

We aim to show that the perturbative metric specified in Eq. (14) for pure $\mathcal{R}^2$ gravity is *extendible* to action (16) upon a *minor* modification. Inspired by the expression (14), we shall seek a metric in the following form:

$$ds^2 = e^{k\varphi(r)}\left[-\Psi(r)dt^2 + \frac{dr^2}{\Psi(r)} + r^2 d\Omega^2\right] + \mathcal{O}(k^2), \quad (17)$$

with $\Psi(r)$ still given by $1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2$, while $\varphi(r)$ *is to be determined*. The case with $\varphi(r) \equiv 0$ is obviously the classic SdS metric.

The rest of our paper is organized as follows. In Sec. III we shall derive the perturbative vacuo (17) for action (16) which is valid up to $\mathcal{O}(k^2)$, with k being a Buchdahl-like parameter reflecting the higher-derivative nature of the action. Lemma 1 is the central result of our paper. Next, we find the asymptotic limits for the new metric at spatial infinity, in Sec. IV. We then embed the new metric in the larger context of full quadratic-gravity theory, viz. action (1), with emphasis on its connection to the Lü-Perkins-Pope-Stelle solution in Einstein-Weyl gravity, in Sec. V.

III. CONSTRUCTING A PERTURBATIVE VACUO FOR THE $R^2 + R + \Lambda$ ACTION

The quadratic action in (16) has the vacuo field equation

$$2\left[\mathcal{R}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathcal{R}\right]$$
$$+ \gamma\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu}\right) = 0. \quad (18)$$

Upon taking the trace

$$6\Box\mathcal{R} - \gamma(\mathcal{R} - 4\Lambda) = 0 \quad (19)$$

to get rid of the cumbersome $\Box\mathcal{R}$ term, Eq. (18) is transformed into

$$\mathcal{R}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) - \nabla_\mu\nabla_\nu\mathcal{R}$$
$$+ \frac{\gamma}{2}\left(\mathcal{R}_{\mu\nu} - \frac{1}{6}g_{\mu\nu}\mathcal{R} - \frac{\Lambda}{3}g_{\mu\nu}\right) = 0. \quad (20)$$

With the metric components dependent on r, the field equation (20) has three remaining independent components against one unknown function $\varphi(r)$ [while $\Phi(r)$ has been fixed to be $1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2$]. At this stage, the problem appears to be overdetermined; this mirage will resolve itself, as we shall see. The relevant components of the fourth-order derivative term are[2]

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[2]Recall that for a scalar field ϕ: $\nabla_\mu\nabla_\nu\phi = \partial_\mu\partial_\nu\phi - \Gamma^\lambda_{\mu\nu}\partial_\lambda\phi$.

$$\nabla_0\nabla_0\mathcal{R} = -\Gamma^1_{00}\mathcal{R}'(r), \quad (21)$$

$$\nabla_1\nabla_1\mathcal{R} = -\Gamma^1_{11}\mathcal{R}'(r) + \mathcal{R}''(r), \quad (22)$$

$$\nabla_2\nabla_2\mathcal{R} = -\Gamma^1_{22}\mathcal{R}'(r). \quad (23)$$

The rr component involves the second derivative of $\mathcal{R}$ with respect to r and is thus quite cumbersome to deal with. Therefore, in place of the rr component of the field equation, we shall use the trace equation as a surrogate for it, together with the tt and $\theta\theta$ components in our calculations below.

The key result is summarized in Lemma 1.

Lemma 1.—The line element

$$ds^2 = e^{k\varphi(r)}\left[-\Psi(r)dt^2 + \frac{dr^2}{\Psi(r)} + r^2 d\Omega^2\right] \quad (24)$$

with $\Psi(r) := 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2$ and $\varphi(r)$ obeying

$$6(r^2\Psi(r)\varphi'(r))' = \gamma r^2\varphi(r) \quad (25)$$

satisfies the field equation (18) up to $\mathcal{O}(k^2)$.

Proof.—We shall use Eq. (20) in place of Eq. (18). The $tt-$ and $\theta\theta-$ components of Eq. (20) can be cast as

$$\frac{\mathcal{R}_{00}}{\Psi} - \frac{1}{4}\frac{g_{00}e^{-\varphi}}{\Psi}(\mathcal{R}e^\varphi) + \frac{\Gamma^1_{00}}{\Psi}\frac{\mathcal{R}'}{\mathcal{R}}$$
$$= -\frac{\gamma}{2\mathcal{R}}\left[\frac{\mathcal{R}_{00}}{\Psi} - \frac{1}{6}\frac{g_{00}e^{-\varphi}}{\Psi}(\mathcal{R}e^\varphi) - \frac{\Lambda}{3}\frac{g_{00}e^{-\varphi}}{\Psi}e^\varphi\right], \quad (26)$$

$$\frac{\mathcal{R}_{22}}{r^2} - \frac{1}{4}\frac{g_{22}e^{-\varphi}}{r^2}(\mathcal{R}e^\varphi) + \frac{\Gamma^1_{22}}{r^2}\frac{\mathcal{R}'}{\mathcal{R}}$$
$$= -\frac{\gamma}{2\mathcal{R}}\left[\frac{\mathcal{R}_{22}}{r^2} - \frac{1}{6}\frac{g_{22}e^{-\varphi}}{r^2}(\mathcal{R}e^\varphi) - \frac{\Lambda}{3}\frac{g_{22}e^{-\varphi}}{r^2}e^\varphi\right]. \quad (27)$$

The trace equation (19) is recast as

$$6(e^\varphi r^2\Psi\mathcal{R}')' = \gamma r^2 e^{2\varphi}(\mathcal{R} - 4\Lambda). \quad (28)$$

In all calculations below, the $\approx$ sign when used means that we keep only up to the first order terms in k. The relevant tensor components are

$$\frac{\mathcal{R}_{00}}{\Psi} = \left(\frac{k\varphi''}{2} + \frac{k^2\varphi'^2}{2} + \frac{k\varphi'}{r}\right)\Psi + \frac{\Psi''}{2} + \left(k\varphi' + \frac{1}{r}\right)\Psi' \quad (29)$$

$$\approx \left(\frac{\Psi''}{2} + \frac{\Psi'}{r}\right) + k\left(\frac{\varphi''\Psi}{2} + \frac{\varphi'\Psi}{r} + \varphi'\Psi'\right) \quad (30)$$

$$= -\Lambda + k\left(\frac{\varphi''\Psi}{2} + \frac{\varphi'\Psi}{r} + \varphi'\Psi'\right), \quad (31)$$

$$\frac{\mathcal{R}_{22}}{r^2} = \frac{1}{r^2} - \left(\frac{k\varphi''}{2} + \frac{k^2\varphi'^2}{2} + \frac{2k\varphi'}{r} + \frac{1}{r^2}\right)\Psi$$
$$- \left(\frac{k\varphi'}{2} + \frac{1}{r}\right)\Psi' \tag{32}$$

$$\approx \left(\frac{1}{r^2} - \frac{\Psi}{r^2} - \frac{\Psi'}{r}\right) - k\left(\frac{\varphi''\Psi}{2} + \frac{2\varphi'\Psi}{r} + \frac{\varphi'\Psi'}{2}\right) \tag{33}$$

$$= \Lambda - k\left(\frac{\varphi''\Psi}{2} + \frac{2\varphi'\Psi}{r} + \frac{\varphi'\Psi'}{2}\right), \tag{34}$$

$$\mathcal{R}e^{k\varphi} = \frac{2}{r^2} - \left(3k\varphi'' + \frac{3k^2\varphi'^2}{2} + \frac{6k\varphi'}{r} + \frac{2}{r^2}\right)\Psi$$
$$- \left(3k\varphi' + \frac{4}{r}\right)\Psi' - \Psi'' \tag{35}$$

$$\approx \left(\frac{2}{r^2} - \frac{2\Psi}{r^2} - \frac{4\Psi'}{r} - \Psi''\right)$$
$$- k\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right) \tag{36}$$

$$= 4\Lambda - k\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right), \tag{37}$$

$$\mathcal{R} = (\mathcal{R}e^{k\varphi})e^{-k\varphi} \tag{38}$$

$$\approx 4\Lambda - 4\Lambda k\varphi - k\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right), \tag{39}$$

$$\mathcal{R}'e^{k\varphi} = (\mathcal{R}e^{k\varphi})' - k\varphi'(\mathcal{R}e^{k\varphi}) \tag{40}$$

$$\approx -k\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right)' - 4\Lambda k\varphi' \tag{41}$$

$$\frac{\mathcal{R}'}{\mathcal{R}} \approx -k\varphi' - \frac{k}{4\Lambda}\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right)', \tag{42}$$

and

$$\frac{g_{00}e^{-k\varphi}}{\Psi} = -1 \tag{43}$$

$$\frac{g_{22}e^{-k\varphi}}{r^2} = 1 \tag{44}$$

$$\frac{\Gamma_{00}^1}{\Psi} = k\frac{\varphi'\Psi}{2} + \frac{\Psi'}{2} \tag{45}$$

$$\frac{\Gamma_{22}^1}{r^2} = -\left(k\frac{\varphi'}{2} + \frac{1}{r}\right)\Psi. \tag{46}$$

The left-hand side of Eqs. (26) and (27) are

$$\frac{\mathcal{R}_{00}}{\Psi} - \frac{1}{4}\frac{g_{00}e^{-k\varphi}}{\Psi}(\mathcal{R}e^{k\varphi}) + \frac{\Gamma_{00}^1}{\Psi}\frac{\mathcal{R}'}{\mathcal{R}}$$
$$\approx -\frac{k}{4}\left(\varphi''\Psi + 2\frac{\varphi'\Psi}{r} + \varphi'\Psi'\right)$$
$$- \frac{\Psi'}{8\Lambda}k\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right)' \tag{47}$$

$$= -\frac{k}{4}\frac{(\varphi'r^2\Psi)'}{r^2} - \frac{3\Psi'}{8\Lambda}k\left(\frac{(\varphi'r^2\Psi)'}{r^2}\right)', \tag{48}$$

$$\frac{\mathcal{R}_{22}}{r^2} - \frac{1}{4}\frac{g_{22}e^{-k\varphi}}{r^2}(\mathcal{R}e^{k\varphi}) + \frac{\Gamma_{22}^1}{r^2}\frac{\mathcal{R}'}{\mathcal{R}}$$
$$\approx \frac{k}{4}\left(\varphi''\Psi + 2\frac{\varphi'\Psi}{r} + \varphi'\Psi'\right)$$
$$+ \frac{\Psi}{4\Lambda r}k\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right)' \tag{49}$$

$$= \frac{k}{4}\frac{(\varphi'r^2\Psi)'}{r^2} + \frac{3\Psi}{4\Lambda r}k\left(\frac{(\varphi'r^2\Psi)'}{r^2}\right)'. \tag{50}$$

The bracketed terms in the right-hand side of Eqs. (26) and (27) are

$$\frac{\mathcal{R}_{00}}{\Psi} - \frac{1}{6}\frac{g_{00}e^{-k\varphi}}{\Psi}(\mathcal{R}e^{k\varphi}) - \frac{\Lambda}{3}\frac{g_{00}e^{-k\varphi}}{\Psi}e^{k\varphi}$$
$$\approx k\frac{\varphi'\Psi'}{2} + \frac{\Lambda}{3}k\varphi \tag{51}$$

$$\frac{\mathcal{R}_{22}}{r^2} - \frac{1}{6}\frac{g_{22}e^{-k\varphi}}{r^2}(\mathcal{R}e^{k\varphi}) - \frac{\Lambda}{3}\frac{g_{22}e^{-k\varphi}}{r^2}e^{k\varphi}$$
$$\approx -k\frac{\varphi'\Psi}{r} - \frac{\Lambda}{3}k\varphi. \tag{52}$$

The terms in the trace equation (28) are

$$e^{k\varphi}r^2\Psi\mathcal{R}' = (\mathcal{R}e^{k\varphi})r^2\Psi\frac{\mathcal{R}'}{\mathcal{R}} \tag{53}$$

$$\approx 4\Lambda kr^2\Psi\left[-\varphi' - \frac{1}{4\Lambda}\left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right)'\right] \tag{54}$$

$$= -4\Lambda k\varphi'r^2\Psi - 3kr^2\Psi\left(\frac{(\varphi'r^2\Psi)'}{r^2}\right)' \tag{55}$$

$$r^2e^{2k\varphi}(\mathcal{R} - 4\Lambda) \approx kr^2\left[-4\Lambda\varphi - \left(3\varphi''\Psi + \frac{6\varphi'\Psi}{r} + 3\varphi'\Psi'\right)\right] \tag{56}$$

$$= -4\Lambda kr^2\varphi - 3k(\varphi'r^2\Psi)'. \tag{57}$$

Up to the first order in k, Eqs. (26), (27), and (28) are

$$\frac{1}{4}\frac{(\varphi' r^2\Psi)'}{r^2}+\frac{3\Psi'}{8\Lambda}\left(\frac{(\varphi' r^2\Psi)'}{r^2}\right)'=\frac{\gamma}{8\Lambda}\left(\frac{\varphi'\Psi'}{2}+\frac{\Lambda}{3}\varphi\right), \quad (58)$$

$$\frac{1}{4}\frac{(\varphi' r^2\Psi)'}{r^2}+\frac{3\Psi}{4\Lambda r}\left(\frac{(\varphi' r^2\Psi)'}{r^2}\right)'=\frac{\gamma}{8\Lambda}\left(\frac{\varphi'\Psi}{r}+\frac{\Lambda}{3}\varphi\right), \quad (59)$$

$$6\left(4\Lambda\varphi' r^2\Psi 3r^2\Psi\left(\frac{(\varphi' r^2\Psi)'}{r^2}\right)'\right)'=\gamma[4\Lambda r^2\varphi+3(\varphi' r^2\Psi)'], \tag{60}$$

which, upon rearranging, become

$$6\frac{(\varphi' r^2\Psi)'}{r^2}-\gamma\varphi=-\frac{3\Psi'}{2\Lambda}\left(6\frac{(\varphi' r^2\Psi)'}{r^2}-\gamma\varphi\right)', \quad (61)$$

$$6\frac{(\varphi' r^2\Psi)'}{r^2}-\gamma\varphi=-\frac{3\Psi}{\Lambda r}\left(6\frac{(\varphi' r^2\Psi)'}{r^2}-\gamma\varphi\right)', \quad (62)$$

$$6\frac{(\varphi' r^2\Psi)'}{r^2}-\gamma\varphi=-\frac{3}{4\Lambda r^2}\left(r^2\Psi\left(6\frac{(\varphi' r^2\Psi)'}{r^2}-\gamma\varphi\right)'\right)'. \tag{63}$$

Remarkably, all three equations (61)–(63) are automatically satisfied *if and only if* $\varphi(r)$ obeys the following linear second-order ODE

$$6(r^2\Psi\varphi')'=\gamma r^2\varphi. \tag{64}$$

Metric (24)–(25) is thus established.   ∎

Remark 1.—Let us ignore the conformal factor in expression (24) for the moment. The terms in the square bracket of (24) is an SdS metric—a "seed" metric—with a constant scalar curvature of 4Λ. The Ricci scalar of metric (24) can be obtained via a conformal transformation from the seed SdS metric [15], per

$$\mathcal{R}=e^{-k\varphi}(4\Lambda-3k\widetilde{\Box}\varphi)+\mathcal{O}(k^2), \tag{65}$$

with the tilde denoting derivatives using the seed SdS metric. By virtue of (25)

$$\widetilde{\Box}\varphi=\frac{1}{\sqrt{-\tilde{g}}}\partial_r\left(\sqrt{-\tilde{g}}\tilde{g}^{rr}\partial_r\varphi\right) \tag{66}$$

$$=\frac{1}{r^2\sin\theta}(r^2\sin\theta\Psi\varphi')' \tag{67}$$

$$=\frac{\gamma}{6}\varphi. \tag{68}$$

The Ricci scalar (65) is thus

$$\mathcal{R}(r)=(1-k\varphi)\left(4\Lambda-\frac{\gamma}{2}k\varphi\right)+\mathcal{O}(k^2) \tag{69}$$

$$=4\Lambda-k\left(4\Lambda+\frac{\gamma}{2}\right)\varphi(r)+\mathcal{O}(k^2). \tag{70}$$

Remark 2.—Using the symbolic manipulator MAXIMA ONLINE, we were able to verify the result in (70) concerning the Ricci scalar. We also verified that the trace equation (19) and the $tt-$, $rr-$, $\theta\theta-$ components of the field equation (20) vanish *up to* $\mathcal{O}(k^2)$.

Remark 3.—Per (70), the Ricci scalar is *nonconstant, including* the case with $\Lambda=0$, as long as $k\neq 0$ and $\gamma\neq 0$. The parameter k therefore acts as an equivalent to the Buchdahl parameter used in metric (8)–(11) and metric (14)–(15) in the pure $\mathcal{R}^2$ action.

Remark 4.—The case of $\gamma=0$ for action (16) amounts to pure $\mathcal{R}^2$ gravity. From (25), we have $r^2\Psi\varphi'=$ const, yielding $\varphi=\int\frac{dr}{r^2\Psi}$ in perfect agreement with the small-k expansion of the Buchdahl-inspired metric, per Eq. (14). The case of $\gamma=\infty$ for action (16) amounts to the Einstein-Hilbert action augmented with a cosmological constant. Eq. (25) then enforces that $\varphi=0$, thence reproducing the SdS metric as expected.

Remark 5.—The ODE (25) is of second differential order. It entails two boundary conditions. For the case of $\Lambda\leq 0$, in which the radial coordinate r is in the range $(0,\infty)$, we are at liberty to set $\varphi(r\to\infty)=0$. Since the ODE is linear, the magnitude of φ can be absorbed into the Buchdahl parameter k in the conformal factor of Eq. (24).

IV. LARGE-DISTANCE ASYMPTOTIC BEHAVIOR

A. The case of $\Lambda<0$

As $r\to\infty$, Eq. (25) asymptotically is

$$2|\Lambda|(r^4\varphi')'\simeq\gamma r^2\varphi, \tag{71}$$

which is soluble. Substituting $x=\ln r$, the equation above becomes

$$\frac{d^2\varphi}{dx^2}+3\frac{d\varphi}{dx}-\frac{\gamma}{2|\Lambda|}\varphi=0, \tag{72}$$

which has the solution

$$\varphi(x)=\xi_+ e^{\left(-\frac{3}{2}+\sqrt{\frac{9}{4}+\frac{\gamma}{2|\Lambda|}}\right)x}+\xi_- e^{\left(-\frac{3}{2}-\sqrt{\frac{9}{4}+\frac{\gamma}{2|\Lambda|}}\right)x}. \tag{73}$$

Since $\gamma>0$, the first term in (73) has a growing exponent and should be discarded. The asymptotic is left with

$$\varphi(r) \simeq r^{-\frac{3}{2}(1+\sqrt{1+\frac{2\gamma}{9|\Lambda|}})}, \qquad (74)$$

which vanishes at spatial infinity. The metric is asymptotically anti-de Sitter.

B. The case of $\Lambda = 0$

As $r \to \infty$, Eq. (25) asymptotically is

$$6(r^2\varphi')' \simeq \gamma r^2 \varphi. \qquad (75)$$

Using $\varphi(r) := \chi(r)/r$, the equation for $\chi(r)$ is soluble, giving the asymptotic

$$\varphi(x) \simeq \frac{e^{-\sqrt{\frac{\gamma}{6}}r}}{r}, \qquad (76)$$

which vanishes at spatial infinity. The metric is *asymptotically flat*.

This case is particularly interesting. The metric is asymptotically flat; yet, *in the bulk*, it develops nonconstant scalar curvature. The generalized Lichnerowicz theorem is evaded in its entirety.

V. IMPLICATIONS FOR THE LÜ-PERKINS-POPE-STELLE SOLUTION

In [2,3], influenced by the generalized Lichnerowicz theorem, Lü, Perkins, Pope, and Stelle suppressed the $\mathcal{R}^2$ term in their exploration of black hole configurations for quadratic gravity, viz. action (1). Their reason was that, *provided that* the generalized Lichnerowicz theorem were valid, a vanishing $\mathcal{R}$ would automatically kill off the terms in the square bracket—which are associated with β—in the second line of the field equation (2). The terms in question are the contributions of the $\mathcal{R}^2$ term of action (1) to the field equation. Accordingly, *solely* for the purpose of finding *static* vacua, in *assuming* the validity of the generalized Lichnerowicz theorem, it would have been legitimate to set β equal to zero.[3] This was indeed what Lü *et al.* did. They went on to discover the Lü-Perkins-Pope-Stelle numerical solution for the leftover Einstein-Weyl gravity. In [4] Podolský *et al.* followed up with an exact infinite-series solution in place of the numerical solution.

However, as established in our previous work [12] and in Lemma 1 herein, the existence of the class of Buchdahl-inspired metrics in pure $\mathcal{R}^2$ gravity alongside with the perturbative vacuo in the $\mathcal{R}^2 + \gamma(\mathcal{R} - 2\Lambda)$ action *denies* the generalized Lichnerowicz theorem in *its entirety*. These spacetimes project *nonconstant* scalar curvature. The $\mathcal{R}^2$ term must be restored into action (1), viz. $\gamma\mathcal{R} + \beta\mathcal{R}^2 - \alpha\mathcal{C}^{\mu\nu\rho\sigma}\mathcal{C}_{\mu\nu\rho\sigma}$, for the purpose of finding static

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[3]Note that for *other* purposes, such as finding *nonstatic* vacua, β must be restored into the investigation, however.

vacuo configurations. That is to say, unless one can suppress β by some other theoretical or observational reason, the generalized Lichnerowicz theorem—having lost its legitimacy—is *not* a justifiable cause to kill off β.

With β being reinstated, an immediate consequence would be to extend the Lü-Perkins-Pope-Stelle ansatz in [2,3] to the *full* quadratic action. Equivalently, the infinite-series approach pursued by Podolský *et al.* in [4] could be suitable for an extension with $\beta \neq 0$.

A more modest setup would be to rework the Lü-Perkins-Pope-Stelle ansatz (or that of Podolský *et al.*) for the $\gamma\mathcal{R} + \beta\mathcal{R}^2$ action, i.e., by excluding the Weyl term. This theory is ghost-free and is equivalent to the standard Einstein gravity with one additional scalar degree of freedom [16,17]. Regarding the Lü-Perkins-Pope-Stelle ansatz, for the $\gamma\mathcal{R} + \beta\mathcal{R}^2$ action, the mass $m_0 := \sqrt{\gamma/(6\beta)}$ of the massive spin-0 mode would stand in place for the mass $m_2 := \sqrt{\gamma/(2\alpha)}$ of the massive spin-2 mode which is now absent. Advantages in exploring the $\mathcal{R} + \mathcal{R}^2$ action would be to deal with a considerably simpler field equation, and that the issues with ghosts would stay silent.

Despite the absence of the Bach tensor in its field equation, the $\gamma\mathcal{R} + \beta\mathcal{R}^2$ action should already project very rich phenomenology. The reason is that the vacua of this theory should inherit some properties of the Lü-Perkins-Pope-Stelle solution *and* those of the Buchdahl-inspired solution, the latter being able to participate owing to the $\beta\mathcal{R}^2$ component in the action. Note that the two said solutions are of complementary nature. Whereas the Lü-Perkins-Pope-Stelle solution represents a second branch of static, spherically symmetric, and asymptotically flat spacetimes *separate* from the Schwarzschild branch, the Buchdahl-inspired solution *supersedes* the Schwarzschild branch.

For the $\gamma\mathcal{R} + \beta\mathcal{R}^2$ action, the $\mathcal{O}(k^2)$ perturbative result obtained in Lemma 1 would already provide a useful guidepost. The full solution—yet to be determined—needs to recover metric (24)–(25) in the limit of small k. An important question to find out is how the Buchdahl parameter k in (24)–(25) is translated into the built-in degree of "non-Schwarzschildness" in the Lü-Perkins-Pope-Stelle ansatz.

There is one serious caveat. As we briefly alluded to in Sec. II, in the asymptotic flatness limit, the Buchdahl-inspired metric given by Eqs. (8)–(11) admits an *exact closed analytical* form, which we called the *special* Buchdahl-inspired metric. Our detailed derivation is presented in our companion paper [14]. For the reader's convenience, we reproduce the *special* Buchdahl-inspired metric below:

$$ds^2 = \left|1 - \frac{r_s}{\rho}\right|^{\frac{k}{r_s}}\left\{-\left(1 - \frac{r_s}{\rho}\right)dt^2 + \frac{r^4(\rho)d\rho^2}{\rho^4(1 - \frac{r_s}{\rho})} + r^2(\rho)d\Omega^2\right\}$$

$$(77)$$

in which ρ is the radial coordinate and the *areal* coordinate r is given by

$$r(\rho) = \frac{\zeta r_s |1 - \frac{r_s}{\rho}|^{\frac{1}{2}(\zeta-1)}}{1 - \mathrm{sgn}(1 - \frac{r_s}{\rho})|1 - \frac{r_s}{\rho}|^\zeta}; \qquad \zeta := \sqrt{1 + 3k^2/r_s^2}.$$

(78)

It describes a static spherically symmetric $\mathcal{R}^2$ structure that lives on an asymptotically flat spacetime. The structure is found to possess *singular* (i.e., nonanalytic and anomalous) behaviors across its interior-exterior boundary as well as in its Kruskal-Szekeres diagram, with the Buchdahl parameter k being the root cause of all these anomalies; see our companion paper for a detailed exposition [14].

Back to the $\mathcal{R} + \mathcal{R}^2$ action at hand. The full vacuo solution to the $\gamma \mathcal{R} + \beta \mathcal{R}^2$ action must approach the *special* Buchdahl-inspired metric (77)–(78) in the limit of $\gamma = 0$ and $\Lambda = 0$ (i.e., asymptotic flatness). The Lü-Perkins-Pope-Stelle ansatz *in its current form* lacks a degree of nonanalyticity necessary to recover the *special* Buchdahl-inspired metric. In order to succeed, the Lü-Perkins-Pope-Stelle ansatz would need to have some nonanalytic built-in ingredients to be able to accommodate the powers of $|1 - \frac{r_s}{\rho}|$ in Eqs. (77) and (78).

Put another way, on the one hand, in the regime of $m_0 \to 0$ (i.e., $\gamma \to 0$ while β is fixed), the singular footprints of the *special* Buchdahl-inspired metric by way of $|1 - \frac{r_s}{\rho}|^\zeta$ and similar terms in (77)–(78) should manifest in the full solution–yet to be identified–for regions close to the interior-exterior boundary. Note that the auxiliary parameter ζ is not necessarily a rational number. On the other hand, in the regime of $m_0 \to \infty$ (i.e., $\beta \to 0$ while γ stays put), the classic Schwarzschild solution should become dominant in the full solution.

A tantalizing question arises: *What is the nature of the transition point between the two regimes, as m_0 is tuned from zero to infinity?*

VI. CONCLUSIONS

Lemma 1 in Sec. III is the central result of our paper. Inspired by the Buchdahl-inspired metric and the Buchdahl parameter k associated with it in pure $\mathcal{R}^2$ gravity [11,12], we carried the concept over to the quadratic action. In a broader context, the Buchdahl parameter k should be a generic universal characteristic of a higher-derivative theory of gravity. For the specific action $\mathcal{R}^2 + \gamma(\mathcal{R} - 2\Lambda)$, we obtained a perturbative solution valid up to $\mathcal{O}(k^2)$. The result is expressed by metric (24)–(25). This metric possesses nonconstant scalar curvature induced by a Buchdahl parameter k, in confirmation of our guiding intuition.

To our knowledge, considerations of metrics with non-constant scalar curvature have been exclusively in higher dimensions [18,19], or in a generic $f(\mathcal{R})$ theory [20–24].

In either situation, the generalized Lichnerowicz theorem as advocated in [1–3,10] is not applicable *per se*. The theorem, stated in Eq. (5), was "proved" strictly for (i) the quadratic action *and* (ii) in $3 + 1$ dimensions. Our result, summed up in Lemma 1, is thus *novel*. It defeats the generalized Lichnerowicz theorem which relied on overly strong restrictions assumed in the "proofs"; see Sec. I.

Considering the breakdown of the generalized Lichnerowicz theorem, the $\mathcal{R}^2$ term should be reinstated in the full quadratic action $\gamma(\mathcal{R} - 2\Lambda) + \beta \mathcal{R}^2 - \alpha \mathcal{C}^{\mu\nu\rho\sigma} \mathcal{C}_{\mu\nu\rho\sigma}$, in counter to the practice adopted in [2,3]; see our Sec. V herein for discussions. It remains to be seen whether a perturbative metric, akin to metric (24)–(25), can be found for the full quadratic action, $\gamma(\mathcal{R} - 2\Lambda) + \beta \mathcal{R}^2 - \alpha \mathcal{C}^{\mu\nu\rho\sigma} \mathcal{C}_{\mu\nu\rho\sigma}$. This would be an interesting possibility for future research.

The solution (24)–(25) obtained herein should be applicable as long as $e^{k\varphi(r)} \approx 1$, which means at large distances. Close to the interior-exterior boundary $r \approx r_s$, the solution should break down. The Lü-Perkins-Pope-Stelle ansatz could be suitable for the full quadratic action in the regions close to the interior-exterior boundary. We touched upon the aspects–advantages and caveats–of this direction in Sec. V.

On the no-hair theorem.—As a member of the $f(\mathcal{R})$ family, the $\mathcal{R} + \mathcal{R}^2$ action is equivalent to a scalar-tensor theory. Within scalar-tensor theories, Sotiriou and Faraoni [25] generalized Hawking's proof [26] that outside of a horizon—*provided that* there is one—the scalar field must be constant (hence making the Kerr-Newman metric an inevitable outcome of gravitational collapse). In support of this proof, Agnese and La Camera [27] illustrated that the Campanelli-Lousto solution [28] of Brans-Dicke gravity lacks a horizon; instead, it represents a naked singularity or a wormhole, depending on whether $\gamma < 1$ or $\gamma > 1$. These results indicate that a higher-derivative Buchdahl parameter, facilitated by a relaxed boundary condition in the quadratic field equation, could—in qualified circumstances—turn an $\mathcal{R} + \mathcal{R}^2$ spacetime structure into a naked singularity or a wormhole. Whether this conclusion is applicable for the full quadratic action, $\gamma(\mathcal{R} - 2\Lambda) + \beta \mathcal{R}^2 - \alpha \mathcal{C}^{\mu\nu\rho\sigma} \mathcal{C}_{\mu\nu\rho\sigma}$, is an open question.

In closing, this paper is the third and final installment of our "Beyond Schwarzschild–de Sitter spacetimes" series. The series started by advancing the obscure six-decades-old Buchdahl program to attain a novel exhaustive class of Buchdahl-inspired vacua for pure $\mathcal{R}^2$ gravity [12]. It then progressed to a *closed analytical* metric describing an asymptotically flat non-Schwarzschild spacetime with novel surprising properties [14]. The series closes with a perturbative metric for the $\mathcal{R}^2 + \mathcal{R} + \Lambda$ action, presented in this paper. The three sets of Buchdahl-inspired space-times, uncovered in our three papers in sequel, reveal a host of new interesting phenomenology that transcends the Einstein-Hilbert paradigm.

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