Modeling black hole evaporative mass evolution via radiation from moving mirrors

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We investigate the evaporation of an uncharged and nonrotating black hole in vacuum by taking into account the effects given by the shrinking of the horizon area. These include the backreaction on the metric and other smaller contributions arising from quantum fields in curved spacetime. Our approach is facilitated by the use of an analog accelerating moving mirror. We study the consequences of this modified evaporation on the black hole entropy. Insights are provided on the amount of information obtained from a black hole by considering nonequilibrium thermodynamics and the nonthermal part of Hawking radiation.

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I. INTRODUCTION

Despite the impressive efforts spent on black hole (BH) thermodynamics [1–8], it is still a challenge to know how a BH's mass changes in time. In the latter scenario, BH evaporation may cause a backreaction on the underlying spacetime metric. Consequently, several attempts to describe BH metrics encompassing backreaction effects have been recently developed [9–11], leading to no unanimous consensus on how backreaction occurs.

Naively if a BH radiates, the horizon area shrinks and thermal Hawking radiative power would increase. However, even the opposite perspective may be plausible, see, e.g., [12], as quantum gravitational effects are not fully employed [13–16]. In addition, the modification of BH particle production is also associated with other effects, i.e., due to horizon shrinking [17], where, for instance, the evolution in time of the background spacetime provides a nonzero small particle count [18].

In these scenarios, analog systems mimicking BHs, namely BH mimickers [19–21], are helpful to overcome the mathematical difficulties related to time-dependent thermodynamic quantities, e.g., mass, temperature, entropy, and so forth. Among all possibilities, perfectly reflecting moving mirrors in $(1+1)$-dimensional flat spacetime, characterized by a given trajectory, see, e.g., [22–25], can reproduce thermal Hawking radiation.[1]

A net advantage of mirrors consists in studying BH radiation properties, e.g., Hawking radiation, thermodynamics, etc., without considering an underlying spacetime associated with the BH itself[2] as analog systems. As a consequence, by using mirrors, one can deal with BH radiation models without having a precise description of the (apparent) horizon area and/or of the BH surface gravity.[3]

In this work, we investigate the thermodynamic properties of a mass-varying BH adopting the BH analog provided by a thermal moving mirror. We focus in particular on the mass evolution of an evaporating BH in vacuum. The mathematical simplification of moving mirrors easily describes the mass evolution through a differential equation that can be numerically solved. We find corrections to

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[1]On the other hand, semitransparent moving mirrors may exhibit quite different energy emission and particle creation [22,25–35].

[2]Attempts towards investigating metrics considering the variation of its mass can be found in [36–39].

[3]For evaporating BHs, there is no Killing horizon and the concept of surface gravity is controversial. The definition of a surface gravity in these contexts is an ongoing subject of study (see, e.g., [40]).

Hawking radiation without postulating the horizon area and/or the surface gravity. Those corrections are related to the effects that the evaporation is expected to cause to the radiation, above all, mimicking the backreaction effect on the metric. We compare the results that we infer with those in Ref. [23], where qualitative arguments for the evaporation have been discussed in view of mirrors. We debate how the expected small corrections to Hawking radiation, obtained as the BH evaporates, are of primary importance to help understand BH information loss [41–43]. Indeed, if BH radiation is not precisely thermal, then it carries some information from inside to outside the event horizon. Hence, nonthermality of BH radiation represents a landscape for the information paradox.[4] We work out the hypothesis of quasistatic processes to approximate the first thermodynamics principle by means of an effective nonequilibrium temperature. In this respect, we show that the deviations from Hawking radiation is initially small, becoming larger as BHs evaporate. This causes a decrease of a BH's lifetime by a factor $\sim 3/8$. Thus, since the effects of BH evaporation drastically affects Hawking radiation, mirrors may confirm quantum tunneling models for Hawking radiation [13,14], showing the emitted radiation to be less entropic than the one predicted in the literature [2,49–53]. This may be interpreted by assuming part of the information can be transmitted by BH radiation. Furthermore, we emphasize in our treatment, it is possible to construct an argument for the BH age from its mass and Hawking radiation.

The paper is organized as follows. In Sec. II we explain how moving mirror radiation emulates BHs. In Sec. III we use this analogy to study BH radiation and its mass evolution from its creation to its complete evaporation. In Sec. IV we study the nonequilibrium thermodynamics of BH evaporation, adopting the quasistatic approximation. Finally, Sec. V is devoted to conclusions and perspectives of our scheme. Throughout the paper, we use Planck units $c = G = \hbar = k_B = 1$.

II. BLACK HOLES FROM MIRROR ANALOGY

Here we briefly review the radiation emitted by BHs and by moving mirrors. We confirm that a trajectory for a $(1+1)$D mirror exactly reproduces Hawking radiation emitted by a $(3+1)$D BH. We limit our analysis to the emission of scalar massless particles. The discussion is split into two subsections focusing on BHs first and then the moving mirror analog.

[4]In particular, by considering BH evaporation effects, quantum tunneling models for Hawking radiation provide a significant deviation from the thermal spectrum [13–16], which cause a reduction of the total BH entropy similar to the one predicted by quantum gravity [14,16,44–46]. Moreover, there exist a model-independent argument proving that the nonthermal part of Hawking radiation cannot be omitted [47,48].

A. Black hole radiation

By quantum field theory in curved spacetime, particle creation occurs whenever the background spacetime evolves in time [26]. This particle production is easy to quantify when a spacetime is flat in the infinite past and infinite future. Indeed, in this case, the normal modes of the scalar field in the infinite future (or output modes) $\{\phi^{\mathrm{out}}_{\omega}\}_{\omega}$ could be obtained from the ones in the infinite past (or input modes) $\{\phi^{\mathrm{in}}_{\omega}\}$ through the following Bogoliubov transformation:

$$\phi^{\mathrm{out}}_{\omega} = \int_0^{\infty} (\alpha_{\omega\omega'}\phi^{\mathrm{in}}_{\omega'} + \beta_{\omega\omega'}\phi^{in*}_{\omega'})d\omega'. \tag{1}$$

The nontrivial Bogoliubov coefficients $\beta_{\omega\omega'}$ are not zero, indicating that particle creation occurs from the vacuum[5] [26,56]. The spectrum of particles produced is given by

$$N_{\omega} = \int_0^{\infty} |\beta_{\omega\omega'}|^2 d\omega'. \tag{2}$$

Hawking calculated [1] the Bogoliubov coefficients relative to a spacetime where a star collapses into a black hole. In this context, Eq. (1) holds by considering the output modes $\{\phi^{\mathrm{out}}_{\omega}\}_{\omega}$ as the modes outgoing from the collapsing star and the input modes $\{\phi^{\mathrm{in}}_{\omega}\}_{\omega}$ as the modes ingoing towards it. The Bogoliubov coefficient $\beta^{\mathrm{BH}}_{\omega\omega'}$ arising from a star collapsing into a black hole, with mass M_{BH} reads

$$\beta^{\mathrm{BH}}_{\omega\omega'} = \sqrt{\frac{\omega'}{\omega}}\Gamma(1 - 4iM_{\mathrm{BH}}\omega)(i\omega')^{-1+4iM_{\mathrm{BH}}\omega}, \tag{3}$$

where Γ is the Euler gamma function. By applying the modulus square of Eq. (3) we obtain

$$|\beta^{\mathrm{BH}}_{\omega\omega'}|^2 = \frac{2M_{\mathrm{BH}}}{\pi\omega'}\frac{1}{e^{8\pi M_{\mathrm{BH}}\omega} - 1}, \tag{4}$$

leading to the known thermal spectrum with temperature

$$T = \frac{1}{8\pi M_{\mathrm{BH}}}. \tag{5}$$

The spectrum of particles radiated, obtained from Eq. (2), is divergent because BH mass evaporation is not considered, so that the BH continues to emit forever. This is the model for what is now called an eternal BH with exact thermal emission. Nevertheless, by making use of wave packets, we can localize the input and output modes in a finite range of time and frequencies. In this way, Hawking proved that [1], in a finite range of time, the collapsing star emits a finite number of particles, following a thermal spectrum with temperature $1/8\pi M_{\mathrm{BH}}$. The astonishing result is that this

[5]For relevant cosmological applications see, e.g., [54,55].

radiation is always constant in time, with exact Planck-distributed particles originating from the collapsed star.

The fact that a BH continues to emit even when the BH is created is justified by the presence of the horizon when considering vacuum fluctuations near it [1,2]. Finally, the renormalized stress energy tensor in presence of an emitting BH was also calculated [28,57]. From it, one can find the flux of energy (power) radiated by a BH as

$$F_{\rm BH} = \frac{1}{768\pi M_{\rm BH}^2} = \frac{\pi}{12}T^2. \tag{6}$$

The conclusion is that, following the first quantum BH model [1,2,28,57] an eternal BH emits as a $1D$ black body[6] [59]. It is worth specifying that the radiation studied until now is valid as long as the BH does not evaporate. As a consequence, this kind of BH (eternal BH) is an open system, as it continues emitting forever without losing mass.

Let us now consider the black hole evaporation. By applying the adiabatic approximation, one assumes that the flux emitted by the BH remains the thermal one, as in Eq. (6), despite the mass loss. In this case, if we impose energy conservation, then the flux of energy radiated by a BH should drain the BH mass, namely $\dot{M}_{\rm BH} = -F_{\rm BH}$. By considering the flux (6) we have

$$\dot{M}_{\rm BH} = -\frac{1}{768\pi M_{\rm BH}^2}, \tag{7}$$

providing

$$M_{\rm BH}(t) = \left(M_0^3 - \frac{t}{256\pi}\right)^{\frac{1}{3}}, \tag{8}$$

where $M_0 = M_{\rm BH}(t=0)$. Following this model, the BH evaporates completely in a time

$$t_{ev}^H = 256\pi M_0^3. \tag{9}$$

In this case, the system is closed, since the total energy emitted is the black hole mass.

B. Mirrors and black hole analogy

Another physical system providing particle production is given by an accelerating mirror. In particular, the radiation by perfectly reflecting accelerating mirrors comes from the acceleration of the boundary condition imposed by perfect reflection, providing the well-known dynamical Casimir effect [26–28,60].

[6]To model the BH as an n-dimensional black body, it is sufficient to modify the prefactor $\frac{1}{768\pi}$ from Eq. (6) according to the n-dimensional Stefan-Boltzmann constant [58] and appropriate temperature scaling.

Let us consider a $(1+1)$D mirror with mass M_m and a generic trajectory $z(t)$. The Lagrangian of the system including this mirror coupled with a scalar field ϕ is [61]

$$\mathcal{L} = -(M_m + \gamma\phi^2(z(t),t))\sqrt{1-\dot{z}(t)} + \frac{1}{2}\int\left(\dot{\phi}(x,t)^2 - \phi'^2(x,t)\right)dx, \tag{10}$$

where the dot indicates a time derivative and the $'$ the derivative with respect to x. The parameter γ indicates that the coupling between the moving mirror and the scalar field. If the mirror is at rest $[z(t)=0]$, then γ is related to the reflection coefficient of the mirror $R(\omega)$, which is dependent on the frequency of the incident particles ω, through

$$R(\omega) = -\frac{i\gamma}{\omega + i\gamma}. \tag{11}$$

As a consequence, the mirror is transparent for frequencies $\omega \gg \gamma$ and reflecting for frequencies $\omega \ll \gamma$. A perfectly reflecting mirror is obtained in the limit $\gamma \to \infty$.

For the mirror, we impose a classical trajectory $z(t)$. To do that, the indeterminacy on the mirror position, $z(t)$, must be negligible. This condition is reached as $\gamma \ll M_m$ (see Sec. VI of Ref. [61]). If the mirror is perfectly reflecting, then the latter condition is unreachable unless we consider a mirror with an infinite mass. Nevertheless, if the spectrum of particles produced by the dynamical Casimir effect vanishes for high frequencies, then the mirror behaves as transparent for those frequencies. In this occurrence, we can assign a finite γ even for a perfectly reflecting mirror, representing an ultraviolet frequency cutoff. Hence, it is even possible to assign to the mirror a finite mass $M_m \gg \gamma$ such that its trajectory can be considered classical.

From now on, for simplicity, we consider a perfectly reflecting mirror, evaluating later if we can assign to it a finite or infinite mass M_m. Each normal mode, reflected back by a mirror with frequency ω, i.e., $\phi_\omega^{\rm out}$, can be written as a combination of the normal modes incoming to the mirror $\{\phi_\omega^{\rm in}\}_\omega$ as Eq. (1).

The Bogoliubov coefficient $\beta_{\omega\omega'}$ is [24,62]

$$\beta_{\omega\omega'}^m = \frac{1}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{+\infty}\exp\left(-i(\omega+\omega')t + i(\omega-\omega')z(t)\right) \times ((\omega+\omega')\dot{z}(t) - \omega + \omega')dt. \tag{12}$$

Using the renormalized stress energy tensor [28] one can derive the flux of energy radiated by the mirror, say to its right, as

$$F_m = \frac{\dddot{z}(\dot{z}^2-1) - 3\dot{z}\ddot{z}^2}{12\pi(\dot{z}-1)^4(\dot{z}+1)^2}, \\ = \frac{1}{12\pi}\left(\frac{\dddot{z}}{(\dot{z}-1)^3(\dot{z}+1)} - 3\frac{\dot{z}\ddot{z}^2}{(\dot{z}-1)^4(\dot{z}+1)^2}\right). \quad (13)$$

In order to investigate whether the system turns out to be open or closed, we may consider the force strength, say $\mathcal{F}$, one requires to apply to the mirror itself to preserve its trajectory, $z(t)$. Thus, from Ref. [61], this force can be written by

$$\mathcal{F}(t) = \left(M_m + \frac{\gamma}{2}\phi^2(0)\right)\ddot{z}(t) + \frac{F_m}{\dot{z}(t)}. \quad (14)$$

The first term represents the mechanical force, while the second term is the force needed to compensate the recoil the mirror would undergo while emitting the flux of energy F_m. We remind the reader that the trajectory could be considered as classical only if $\gamma \ll M_m$. If the mirror is perfectly reflecting and no ultraviolet frequency cutoff could be imposed, then we have no choice but an infinite mass M_m. In this case, the external force $\mathcal{F}(t)$ diverges and the system is clearly open.

The Carlitz-Willey trajectory corresponds to a $(1+1)$D trajectory and represents a simple approach to model thermal mirror trajectories. It reads

$$z(t) = -t - \frac{1}{\kappa}W(e^{-2\kappa t}), \quad (15)$$

where W is the Lambert function and κ a free constant related to mirror acceleration [22]. If a mirror has this trajectory, then by Eq. (12) we get

$$\beta^m_{\omega\omega'} = -\frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega'}}e^{-\frac{\pi\omega}{2\kappa}}\Gamma\left(i\frac{\omega}{\kappa}\right)\left(\frac{\omega'}{\kappa}\right)^{-i\frac{\omega}{\kappa}}, \quad (16)$$

and its modulus square is

$$|\beta^m_{\omega\omega'}|^2 = \frac{1}{2\pi\kappa\omega'}\frac{1}{e^{2\pi\omega/\kappa}-1}. \quad (17)$$

The spectrum of particles produced by this mirror is divergent even for high frequencies. So, it is straightforward that an ultraviolet frequency cutoff cannot be imposed. The mass of this mirror M_m is then infinite and so is the magnitude of the external force needed to keep the mirror trajectory (15).

By computing from Eq. (13) the flux of energy that a mirror with trajectory (15) radiates to its right we obtain

$$F_m = \frac{\kappa^2}{48\pi}. \quad (18)$$

By comparing Eq. (4) with (17), we get $|\beta^{\mathrm{BH}}_{\omega\omega'}|^2 = |\beta^m_{\omega\omega'}|^2$ by putting $\kappa = \frac{1}{4M_{\mathrm{BH}}}$. In this case, from Eqs. (6) and (18), we get also $F_{\mathrm{BH}} = F_m$.

Hence, a $(1+1)$-dimensional mirror with a trajectory given by Eq. (15) exactly reproduces the Hawking radiation from a $(3+1)$-dimensional Schwarzschild BH with mass $\frac{1}{4\kappa}$: both in terms of particles produced and in terms of energy radiated. The analogy between a Carlitz-Willey accelerated mirror and an eternal black hole is confirmed also by the fact that both the systems are open.

Considering an appropriate modification of the Carlitz-Willey trajectory (15), it is possible to find an analog mirror emulating the particle production properties of a Kerr BH [32], a Reissman-Nordstrom BH [29] and a de Sitter/anti–de Sitter BH [34]. The exact eternal thermal emission of the Carlitz-Willey moving mirror is given by the late-time emission of the Schwarzschild mirror [25].

Since the modulus square of the Bogoliubov coefficients (4) and (17) are the same when $M_{\mathrm{BH}} = \frac{1}{4\kappa}$, the Bogoliubov coefficients (3) and (16) are the same up to a phase. In the mirror framework, a phase factor on the Bogoliubov coefficient is related to a translation of the trajectory, which does not change the particle production (17). As a consequence, a mirror can emulate all the BH properties related to its Bogoliubov coefficients, e.g., localized wave packets particle production [24], quantum communication properties [63], etc.

The Carlitz-Willey accelerated mirror has also a horizon representing the BH event horizon, i.e., from Eq. (15), we can see that the mirror approaches $z = -t$ as $t \to \infty$. This means that no particle can reach the mirror after $t = 0$ and be reflected back by it. As a consequence, the information on the input particle disappears, as does the information of a particle sent to a BH. In other words, at $t = 0$, the mirror creates its horizon, as it happens for the creation of the event horizon of the BH the mirror wants to emulate.

We can notice that the energy radiated, Eq. (6), does not depend upon time. Namely, the same flux arises even when the horizon is not created yet. This is due to an approximation performed in Ref. [1]. A more realistic model should involve radiation which turns on smoothly after the creation of the horizon. The simplest of these models arises by modeling the BH as a collapsing null shell, see, e.g., Ref. [64] for a review.

The mirror emulating its spectrum and its energy radiated is given by the Schwarzschild mirror trajectory [25]

$$z(t) = -t - \frac{1}{2\kappa}W(2e^{-2\kappa t}). \quad (19)$$

From Ref. [25], for $t < 0$, the spectrum of particles and the flux of energy radiated by the mirror drops to zero exponentially as t decreases. On the contrary, for $t > 0$, both those quantities go exponentially to the Hawking ones, Eqs. (3) and (6), respectively, as t increases. The deviation with respect to the particle spectrum and energy radiated, Eq. (6), predicted by Hawking drops as $\sim e^{-t/M_{\mathrm{BH}}}$. Hence, the radiation could be considered as completely "turned on" when $t \gtrsim M_{\mathrm{BH}}$.

In the next section, for simplicity, we consider a BH starting to evaporate only when $t \gtrsim M_{\mathrm{BH}}$, or fully turned on. We see that, if the initial mass of the BH is large enough, the turning on period occurs in a time negligible with respect to the evaporation period, justifying why this approximation may hold.

III. BLACK HOLE EVAPORATION

In the following, we generalize the Carlitz-Willey trajectory of Eq. (15) by taking κ time dependent. Thus, from the BH mirror analogy, $\kappa = \frac{1}{4M_{\mathrm{BH}}}$, a variation of κ induces a variation over the BH mass and also represents a class of trajectories quite different from the standard Carlitz-Willey trajectory.

By imposing energy conservation, we can thus find $M_{\mathrm{BH}}(t)$, with the corresponding flux deviating from Eq. (6) as due to the time dependence of κ.

This deviation can be easily related to the effects expected to slightly modify Hawking radiation during BH evaporation, such as the backreaction on the metric or the shrinking of the horizon modifying local boundary conditions. In all these situations, we expect departures from genuine equilibrium thermodynamics, in favor of nonequilibrium effects that we will discuss later in the text.

A. Modeling BH evaporation with mirrors

As discussed at the end of Sec. II B, the black hole radiation turns on smoothly once the black hole is created. For simplicity, we consider that the black hole starts to evaporate once the radiation is fully turned on. In this way, we can consider the analog mirror to follow a generalization of the standard Carlitz-Willey trajectory, Eq. (15), namely

$$z(t) = -t - 4M_{\mathrm{BH}}(t)W(e^{-\frac{t}{2M_{\mathrm{BH}}(t)}}). \tag{20}$$

The time $t = 0$ corresponds to the time at which the BH starts to evaporate—this makes $t \sim -4M_0$ the time at which the BH has been created (see the discussion at the end of Sec. II B).

We define $M_0 = M(t=0)$ as the initial mass hold by the underlying BH. To generalize the flux, we should plug Eq. (20) and its time derivatives into Eq. (13). In this way, we can obtain a general expression for the flux[7] $F_m = F_m(t, M_{\mathrm{BH}}, \dot{M}_{\mathrm{BH}}, \ddot{M}_{\mathrm{BH}}, \dddot{M}_{\mathrm{BH}})$ radiated by a mirror with trajectory (20). We now ansatz the black-hole-mirror analogy. In other words, since for M_{BH} constant (i.e., for an eternal black hole) we have seen that $F_m = F_{\mathrm{BH}}$, we impose the same equality also in the case of a time-dependent M_{BH}. In so doing, from the energy conservation $\dot{M}_{\mathrm{BH}} = -F_{\mathrm{BH}} = -F_m$, we can study the evaporation of the black hole. In particular, we get a third order ordinary differential equation:

$$\dot{M}_{\mathrm{BH}} = -F_m(t, M_{\mathrm{BH}}, \dot{M}_{\mathrm{BH}}, \ddot{M}_{\mathrm{BH}}, \dddot{M}_{\mathrm{BH}}), \tag{21}$$

giving the evolution of the mass $M_{\mathrm{BH}}(t)$ from $t = 0$ to the complete evaporation time t_{ev}, defined as the time in which $M_{\mathrm{BH}}(t_{ev}) = 0$. From now on, for the sake of simplicity we omit the label "BH" when referring to the mass $M_{\mathrm{BH}} \coloneqq M$ and the flux $F_{\mathrm{BH}} \coloneqq F$, which always refer to the black hole. The study of the mirror trajectory, mass and the forces needed to keep its trajectory (20) is provided in Sec. III E.

To simplify the flux expression $F(t, M, \dot{M}, \ddot{M}, \dddot{M})$, we employ two ranges of time:

(1) $0 \le t \le t_0$, where we consider negligible the deviations from the Hawking flux (6).

(2) $t_0 < t < t_{ev}$, where the corrections to the Hawking flux (6) given by the BH evaporation becomes nonnegligible.

The time t_0 is fixed as $t_0 \gg 2M_0$, implying $t \gg 2M(t)$ for $t_0 < t < t_{ev}$, since $M(t)$ decreases in time as the BH evaporates.

Hence, in the generalized Carlitz-Willey trajectory, we may approximate $W(\exp(-t/M(t))) \sim \exp(-t/M(t))$. Thus, for $t \le t_0$ we easily recover the Hawking flux of Eq. (6), whereas for $t > t_0$ the flux can be computed from Eq. (13), applying the approximation $t \gg 2M(t)$. Hence, the flux of energy radiated from $t = 0$ to $t = t_{ev}$ is

$$\begin{cases} F = \frac{1}{768\pi M^2} & \text{for } 0 < t \le t_0; \\ F = \frac{1}{192\pi(1-\frac{\dot{M}}{M}t)^2}\left(3\frac{\ddot{M}^2}{M^2}t^2 + \frac{1}{4M^2}\left(1-\frac{\dot{M}}{M}t\right)^4 + 2\frac{\dddot{M}}{M}t\left(1-\frac{\dot{M}}{M}t\right) - 12\frac{\ddot{M}\dot{M}}{M^2}t\right) & \text{for } t > t_0 \end{cases}. \tag{22}$$

To apply this simplification, we must ensure $\dot{M}$ to be negligible for $t \le t_0$, i.e., $-\dot{M}(t_0) \ll M(t_0)$.

[7] The expression is not explicitly reported because it is too cumbersome.

Since, for $t \leq t_0$, Eq. (8) is valid, then at $t = t_0$ we obtain

$$|\dot{M}(t_0)| = \frac{1}{768\pi M^2(t_0)} \ll M(t_0). \qquad (23)$$

From Eq. (8), which is valid for $t \leq t_0$, we can therefore evaluate $M(t_0)$. Thus, the condition $-\dot{M}(t_0) \ll M(t_0)$ becomes

$$t_0 \ll 256\pi M_0^3 - \frac{1}{3}. \qquad (24)$$

Summing up, we need to choose a time, t_0, such that $2M_0 \ll t_0 \ll 256\pi M_0^3 - \frac{1}{3}$. So, in order to have a t_0 satisfying this condition, we need an initial mass, M_0, large enough.[8]

B. Evaluating the mass evolution

Afterwards, to evaluate the function $M(t)$ we impose the energy condition, $\dot{M}(t) = -F$.

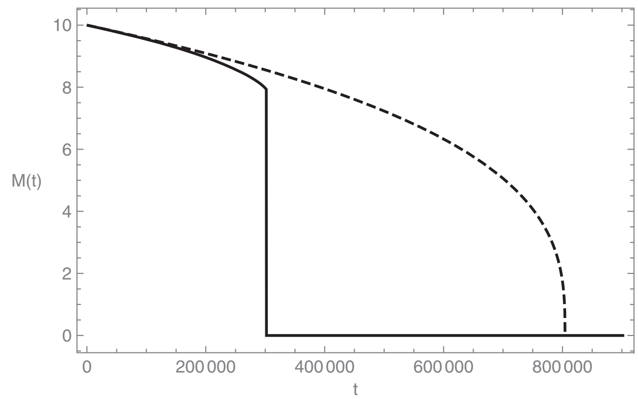

FIG. 1. Plot of the numerical solution of $M(t)$ from Eq. (25) (continuous line) and Hawking solution for $M(t)$ following Eq. (8) (dashed line) considering $M_0 = 10$ and $t_0 = 200M_0$.

In this way, Eq. (22) becomes a third order, nonlinear differential equation

$$\begin{cases} \dot{M} = \frac{1}{768\pi M^2} & \text{for } 0 < t < t_0 \\ \dot{M} = -\frac{1}{192\pi(1-\frac{\dot{M}}{M}t)^2}\left(3\frac{\dot{M}^2}{M^2}t^2 + \frac{1}{4M^2}\left(1-\frac{\dot{M}}{M}t\right)^4 + 2\frac{\ddot{M}}{M}t\left(1-\frac{\dot{M}}{M}t\right) - 12\frac{\ddot{M}\dot{M}}{M^2}t\right) & \text{for } t > t_0 \\ M(t=0) = M_0 \end{cases}. \qquad (25)$$

The numerical solution of Eq. (25) is drawn in Fig. 1, where $t_0 = 200M_0$ and $M_0 = 10$ were considered, having $t_0 \sim 2000$. The period of time $0 < t < t_0$, in which the evaporation effects on the radiation are neglected, is very small with respect to the overall evaporation period, i.e., one part over a thousand. This is what we wanted, since we want to study the deviations from the Hawking radiation (6) in a period of time as large as possible.

In Sec. II B, the "turning on period," namely the period in which the radiation turns on after BH creation, is $\sim 4M_0$, being very small than periods of time here considered. Consequently, we can ignore the turning on period, identifying the time $t = 0$ as the time at which the horizon of the BH originates as consequence of star collapse.

From the numerical solution in Fig. 1, a sudden drop of the mass occurs at $t = t_c$, namely at a critical time, unavoidable for any initial mass value. To analytically explain this sharp behavior, we now approximate Eq. (25) at the range of times $t_0 < t < t_c$.

To do so, we first estimate the magnitude of $\dot{M}$, $\ddot{M}$, and $\dddot{M}$ when there are no evaporation effects on the radiation. Using Eq. (7), we get

$$\begin{cases} \dot{M} \sim 10^{-3}/M_0^2 \\ \ddot{M} \sim 10^{-6}/M_0^5 \\ \dddot{M} \sim 10^{-9}/M_0^8 \end{cases}. \qquad (26)$$

Considering the evaporation effects on radiation, the derivatives of the mass (26) are expected to increase in magnitude. However, from Fig. 1, we see that such increase is relatively small. Bearing this in mind, we study the orders of the terms at the rhs of Eq. (25), as $t > t_0$, using Eqs. (26).

(1) The term proportional to $\frac{\dot{M}^2 t^2}{M^2}$ has order $\sim 10^{-8}/M_0^6$. The denominator $(1 - \frac{\dot{M}}{M}t)^2$ decreases the magnitude of this term as t increases.[9] The same thing is valid for the third and last term, respectively.
(2) The second term is $\sim 1/M_0^2$ and simplifies as $\frac{1}{M^2}(1 - \frac{\dot{M}}{M}t)^2$ with the denominator, increasing *de facto* with time.
(3) The third term has order $10^{-7}/M^7$, with magnitude decreasing as time increases.
(4) The last term has order $10^{-7}/M^6$, with magnitude decreasing as time increases.

[8]For instance, $M_0 \sim 5$ ensures the existence of a t_0 which is 100 times smaller than $256\pi M_0^3$ and 100 times larger than $2M_0$, making the approximation valid.

[9]Consider that $\dot{M}$ is forced to be negative, so $(1 - \frac{\dot{M}}{M}t)$ is always larger than 1, increasing as time increases.

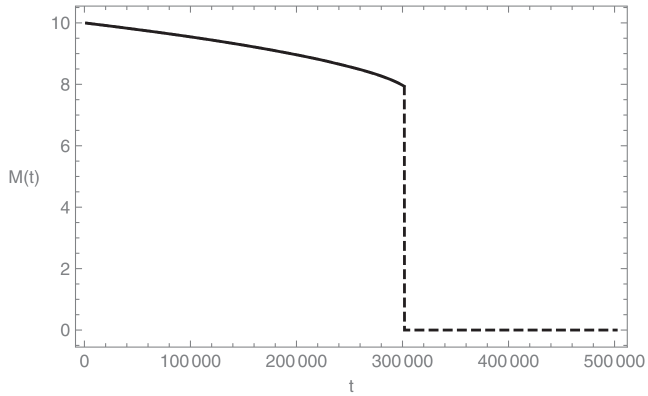

FIG. 2. Comparison between the solution of the approximated Eq. (27) (thick line) and the solution of Eq. (25) (dashed line).

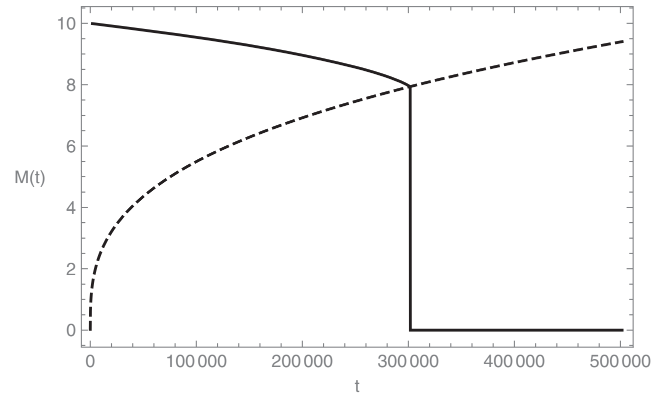

FIG. 3. This figure shows that the critical point of the solution of Eq. (25) for $M_0 = 10$ (thick line) satisfies the relation (29), since the critical point (t_c, M_c) lies on the curve $t = 192\pi M^3$ (dashed line).

From this analysis, since M_0 cannot be small, we conclude that the second term, i.e., $\frac{1}{4M}(1 - \frac{\dot{M}}{M}t)^2$ is dominant before t_c, leading to an approximation of Eq. (25) of the kind:

$$\dot{M} = -\frac{1}{768\pi M^2}\left(1 - \frac{\dot{M}}{M}t\right)^2. \tag{27}$$

The validity of Eq. (27) before t_c is confirmed in Fig. 2, where the solutions of Eqs. (25) and (27) effectively coincide before t_c.

C. Interpreting the critical time

The issue of inferring the physical consequences of the above-defined critical time is challenging but it helps to justify the existence of critical time via the use of the simplified differential Eq. (27). Indeed, Eq. (27) can be written as

$$\dot{M} = -\frac{384\pi M^4}{t^2} + \frac{M}{t} + \frac{384\pi M^4}{t^2}\sqrt{1 - \frac{t}{192\pi M^3}}, \tag{28}$$

whose solution exists if $t \le 192\pi M^3(t)$. So, as $t \to 192\pi M^3$, $\ddot{M}(t)$ diverges as $\sim(1 - \frac{t}{192\pi M^3})^{-1/2}$. As a consequence, also the third time derivative of the mass diverges. So, in a neighborhood of $t = 192\pi M^3(t)$, the first, third, and fourth terms at the right-hand side of the second of Eq. (25) suddenly increase, becoming dominant and making $M(t)$ to drop sharply as Fig. 1 shows.

At this point, we can associate t_c to the time at which the square root of Eq. (28) nullifies. Further, we define the critical mass as $M_c \coloneqq M(t_c)$, having this relation:

$$t_c = 192\pi M_c^3. \tag{29}$$

Figure 3 shows that the critical point lies on the curve $t = 192\pi M^3$ confirming the relation (29) between critical time t_c and critical mass M_c. Numerically, taking a sample of initial masses stepping by $\Delta M_0 = 1$ from $M_0 = 10$ to $M_0 = 50$, the critical mass becomes $M_c \sim 0.7937 M_0 \simeq 2^{-1/3} M_0$.

With this information, we can give a value to the critical time and mass:

$$M_c \simeq \frac{M_0}{\sqrt[3]{2}} \Rightarrow t_c \simeq 96\pi M_0^3 = \frac{3}{8} t_{ev}^H, \tag{30}$$

where t_{ev}^H is the evaporation time predicted by Hawking (9).

After the critical time t_c, the mass drops to zero in a finite but very short time, that we can neglect. Thus, the new evaporation time of the BH is modified as $t_{ev} \sim \frac{3}{8} t_{ev}^H$, demonstrating the BH evaporates faster than the standard Hawking case when accounting for the mass evaporation effects on the radiation. This fact reduces the evaporation time by a factor $\sim 3/8$.

The mass behavior after t_c requires a physical interpretation.

(1) For instance, a possible justification may include quantum gravity effects. Indeed, since $\dot{M}$ increases sharply after the critical time, $|\dot{M}|$ quickly reaches order M and likely, as this fact occurs, quantum gravity effects cannot be neglected. Hence, our modeling predicts that, when considering the mass evaporation of a black hole in vacuum and the effects the evaporation induces to the metric, quantum gravity effects have to be considered when the black hole mass reaches $2^{-1/3}$ of its initial value.

(2) Analytically, the sharp mass drop is due to the sudden increasing of $-\ddot{M}$, i.e., of the mass loss acceleration, analyzed after Eq. (28). It appears evident that the accelerated behavior resembles a jetlike form, similar to Fermi processes [65], where the mass loss acceleration does not smoothly behave, leading to uncontrolled astrophysical processes. This interpretation may be framed in more practical physical scenarios related to a compact

object, leaving open the possibility to model those objects by accelerated mirrors.

We give these explanations for the sake of completeness, however they lie beyond the main purposes of our work, however interesting they may be for future investigations.

D. Mass evolution at early times

In this subsection, we provide an approximation for the differential Eq. (28) valid when $t \gtrsim t_0$. In this way, we obtain an analytic expression for $M(t)$, providing an explanation for how the mass evaporation of the black hole modifies the Hawking radiation at early times.

From the condition (24), by considering $t \gtrsim t_0$ we expect also $t \ll 192\pi M^3(t)$. Hence, the square root in the right-hand side of Eq. (28) can be expanded up to third order as

$$\sqrt{1-\frac{t}{192\pi M^3}} \sim 1-\frac{t}{384\pi M^3}-\frac{2t^2}{9\cdot(256\pi)^2M^6} - \frac{4t^3}{27\cdot(256\pi)^3M^9}+\dots \tag{31}$$

Considering the first two terms of the expansion (31), Eq. (28) becomes $\dot{M}=0$, i.e., the case in which there is no evaporation. Considering the first three terms of (31) we get exactly the known differential equation (7) with the known solution (8). To provide a first correction on $M(t)$ given by the evaporation effects, we can consider also the fourth term of the expansion (31). In this case Eq. (28) becomes

$$\dot{M} = -\frac{1}{768\pi M^2}-\frac{2t}{9\cdot(256\pi)^2M^5}. \tag{32}$$

This can be rewritten in terms of $t(M)$:

$$\frac{dt}{dM} = -\frac{768\pi M^2}{\left(1+\frac{t}{384\pi M^3}\right)}. \tag{33}$$

By considering $(1+\frac{t}{384\pi M^3})^{-1} \sim 1$ then the solution (8) is restored. Hence, to study its first order deviation, we expand the latter to first order for $t \ll 256\pi M^3$, namely $(1+\frac{t}{384\pi M^3})^{-1} \sim 1-\frac{t}{384\pi M^3}$. In this way, Eq. (33) becomes the linear differential equation:

$$\frac{dt}{dM}-\frac{2t}{M} = -768\pi M^2. \tag{34}$$

From Eq. (25), for $0 < t \le t_0$, the mass evolution is given by Eq. (8). Since Eq. (34) is valid for $t \gtrsim t_0$, the initial condition for it is defined using Eq. (8) at t_0, i.e.,

$$t\left(\left(M_0^3-\frac{t_0}{256\pi}\right)^{1/3}\right) = t_0. \tag{35}$$

The solution of Eq. (34) with the condition (35) is

$$t(M) = \frac{1}{M^2}\left(\frac{2}{5}t_0+\frac{768\pi M_0^3}{5}\right)\left(M_0^3-\frac{t_0}{256\pi}\right)^{2/3}+\frac{768\pi M^3}{5}. \tag{36}$$

Here, t_0 is arbitrary, but since $t_0 \ll t_{ev}^H$ we expand the two factors in Eq. (36) depending on t_0:

$$\begin{aligned}&\left(\frac{2}{5}t_0+\frac{768\pi M_0^3}{5}\right)\left(M_0^3-\frac{t_0}{256\pi}\right)^{2/3}\\ &\quad=\frac{768\pi M_0^5}{5}\left(1+\frac{t_0}{384\pi M_0^3}\right)\left(1-\frac{t_0}{256\pi M_0^3}\right)^{2/3}\\ &\quad\sim\frac{768\pi M_0^5}{5}\left(1+\mathcal{O}\left(\left(\frac{t_0}{t_{ev}^H}\right)^2\right)\right),\end{aligned}$$

proving that $t_0 \neq 0$ provides a second order deviation from the Hawking case. However, since we have considered only first deviations from the Hawking mass evolution (8), we can neglect the contribution of $t_0 \neq 0$. In this way, the solution of (34) becomes

$$t(M) = \frac{768\pi}{5}\left(\frac{M_0^5-M^5}{M^2}\right). \tag{37}$$

In contrast, the one without evaporation effects (8) reads

$$t(M) = 256\pi(M_0^3-M^3). \tag{38}$$

Summarizing, Eq. (37) expresses the correct behavior of the mass in time when $t \ll t_{ev}$, i.e., when evaporation effects are small but different than 0. One can study further corrections of Eq. (8) by considering higher orders of the expansion (31).

E. Dynamical behavior of mirrors at intermediate stages

By virtue of the general mass loss behavior, Eq. (25), one can infer how the mirror trajectory evolves throughout the evolution of our dynamical system.

For our purposes, the trajectory of the mirror (20) has been defined only in the restricted range of times $0 < t < t_{ev}$ as a modification of the Carlitz-Willey trajectory [22], Eq. (20), with M being time dependent and following the differential Eq. (25). The modification of the Carlitz-Willey trajectory is shown in Fig. 4. In our trajectory, the mirror approaches the asymptote $z=-t$ faster than the normal Carlitz-Willey trajectory. However, the difference between the two is vanishingly small, namely, of an order $e^{-2t/M_0}-e^{-2t/M(t)}$. Now, we can argue which consequences occur to the mirror at the critical time t_c. To do so, we know that, near t_c, $\ddot{M}$ suddenly increases proportionally to $(1-t/t_c)^{-1/2}$. Thus, using Eq. (20) and $t \gg 2M(t)$, we can compute the mirror

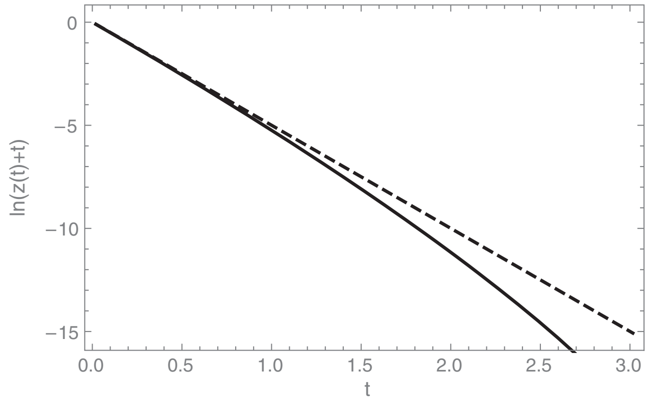

FIG. 4. Trajectory (20) of the mirror emulating an evaporating BH (thick line) and trajectory (15) emulating an eternal BH (dashed line). The numbers on the axes x and y are in powers of 10^5 and 10^3, respectively.

velocity and acceleration with respect to an external observer, respectively, as

$$\dot{z}(t) \sim -1 + 2\left(1 - \frac{\dot{M}}{M}t\right)e^{-\frac{t}{2M}}, \tag{39}$$

$$\ddot{z}(t) \sim -2\left(\frac{\ddot{M}}{M}t + \frac{1}{2M}\left(1 - \frac{\dot{M}}{M}t\right)^2\right)e^{-\frac{t}{2M}}. \tag{40}$$

Looking at Eq. (40), the acceleration drops always as an exponential $e^{-2\kappa t}$. As t approaches t_c, the first term of Eq. (40) dominates and the acceleration of the mirror becomes proportional to

$$\ddot{z}(t) \propto \frac{e^{-\frac{M}{2t}}}{\sqrt{1 - \frac{t}{t_c}}}. \tag{41}$$

For $t \lesssim t_c$, the acceleration of the mirror is vanishingly small. However, $e^{-t/2M}$ never goes precisely to 0, but the square root in the denominator does. As a consequence, when $t_c - t$ is really close to zero, the acceleration, from being vanishingly small, suddenly diverges. As this happens, the mirror reaches its asymptote $z = -t$ suddenly. In particular, this occurs when the BH completely evaporates, namely for $M \to 0$)—as it can be easily verified from Eq. (15). This is consistent with the fact that, at the moment of the evaporation t_{ev}, both the BH horizon and the mirror horizon disappear (see the discussion in Sec. II B on the horizon analogy).

Once the BH has evaporated, the mirror should be static in order to reproduce a flat spacetime where signals are not redshifted [23,30,66]. In particular, the trajectory of the mirror is given by Eq. (20) when $t < t_{ev}$, and $z(t) = -t_{ev}$ when $t > t_{ev}$. This implies that $\dot{z}(t_{ev})$ should be zero, but this means that we can assert the behavior of $\dot{M}$ when M approaches zero. In fact, taking the velocity of the mirror as Eq. (39) and imposing it to be null at $t = t_{ev}$, we obtain

$$\dot{M} = -\frac{M}{t_{ev}}\left(\frac{e^{\frac{t}{2M}}}{2} - 1\right). \tag{42}$$

In conclusion, in the limit $M \to 0$, $\dot{M}$ diverges to $-\infty$ asymptotically to

$$\dot{M} \sim -\frac{M}{t_{ev}}e^{\frac{t_{ev}}{2M}}. \tag{43}$$

Since this condition makes the mirror velocity $\dot{z}(t)$ continuous at $t = t_{ev}$, it ensures also that the mirror acceleration $\ddot{z}(t)$ is finite (but not continuous) at $t = t_{ev}$.

Comparing the behavior of our mirror with respect to Ref. [23], the deceleration of our mirror to become static is very fast. As a future perspective, one can try to use the formalism of Ref. [23] to slow down the mirror deceleration after t_c.

Finally, since the total energy radiated by the mirror is finite and equal to M_0, the system is expected to be closed. To confirm that, we study the force applied to the mirror to keep its trajectory. First of all, let us study if we can assign a finite mass M_m to the mirror. As we just said, since the mirror is static after the time t_{ev}, it emits a finite total energy $E_{\rm tot}$. The latter, in general, can expressed in terms of the spectrum of particles produced N_ω as

$$E_{\rm tot} = \int_0^\infty \omega N_\omega d\omega. \tag{44}$$

For the integral on the right-hand side of Eq. (44), in order to be convergent at $\omega \to \infty$, we need that $N_\omega \propto \omega^{-x}$, where $x > 2$, in the limit $\omega \to \infty$. As a consequence, it is possible to insert a finite ultraviolet cutoff γ for the frequency ω leaving the physics unchanged. Hence, the mirror can be considered as semitransparent with a reflectivity γ. Finally, the mass of the mirror M_m can be considered finite and still satisfying $M_m \gg \gamma$.

At this point, one can evaluate the total external force the mirror undergoes through Eq. (14). Despite the acceleration (40) is very high when t is close to t_c, the mirror becomes static before the latter reaches infinity. As a consequence, the acceleration is very high but finite, meaning that the force $\mathcal{F}$ that should be applied to the mirror is always finite. This means that the system involving this kind of mirror is closed, as well as a completely evaporating black hole. For the sake of completeness, we must say that in Eq. (14) we have not considered the loss of mass of the mirror due to the dynamical Casimir effect emission (13). However, to make this loss negligible in our context, it is sufficient to consider $M_m \gg M_{\rm BH}$, so that the mirror mass after t_{ev} becomes $M_m - M_{\rm BH} \sim M_m$.

IV. MIRROR THERMODYNAMICS AND BLACK HOLE ANALOGY

In this section, we would like to study the thermodynamics of the evaporating BH modeled in Sec. III. We aim to know if the entropy released by the BH during its evaporation is less than the one predicted by Bekenstein and Hawking [1,3].

Eternal BH Hawking radiation has a thermal spectrum, and the radiation of the BH modeled in Sec. III deviates from the thermal one. We expect the nonthermal part of the radiation to contain some information, unavailable otherwise, about the BH. It is worth pointing out that, with the mirror model, we cannot find exact results for the thermality of the spectrum of particles radiated.[10] However, the expressions obtained in the previous section allow a physically reliable assumption for the nonthermal part of the flux radiated. The price to be paid is that the final results are dependent on an unknown index. Nevertheless, we are able to restrict this parameter to a small range by putting ourselves in a quasistatic regime.

A. Nonthermality

From Sec. II, the flux of energy radiated by an evaporating black hole, i.e., its power, is given by Eq. (22). When $0 \leq t \leq t_0$, the power radiated is exactly the one predicted by Hawking, Eq. (6). For this reason, we can suppose the radiation to be completely thermal in this range of times. When $t_0 < t < t_{ev}$, the radiation deviates from the Hawking one (6), by

$$\Delta F = F - \frac{1}{768\pi M^2}. \tag{45}$$

By Eq. (22) we have in particular

$$\begin{cases} \Delta F = 0 & \text{for } 0 < t \leq t_0 \\ \Delta F = \frac{1}{192\pi(1-\frac{\dot{M}}{M}t)^2}\left(3\frac{\ddot{M}^2}{M^2}t^2 + 2\frac{\dddot{M}}{M}t\left(1-\frac{\dot{M}}{M}t\right) - 12\frac{\ddot{M}\dot{M}}{M^2}t\right) + \frac{1}{768\pi M^2}\left(\frac{\dot{M}^2}{M^2}t^2 - 2\frac{\dot{M}}{M}t\right) & \text{for } t > t_0 \end{cases}. \tag{46}$$

We notice that ΔF nullifies whenever $\dot{M} = 0$, whereas when $\Delta F = 0$ the spectrum turns out to be exactly thermal. So, the nonthermality of Hawking radiation is expected to be proportional to ΔF. In particular we state that the BH power is composed of a thermal contribution F_{th} and a nonthermal one $F_{\text{no-th}}$. We thus have

$$F = F_{\text{th}} + F_{\text{no-th}}. \tag{47}$$

We know that the Hawking flux $1/(768\pi M^2)$ gives an exact thermal contribute, so that this term is included in F_{th}. However, the possibility that part of ΔF gives a small thermal contribute cannot be excluded.[11] Since the nonthermality of the spectrum must be proportional to the deviation from Hawking radiation ΔF, we write

$$F_{\text{no-th}} = \alpha \Delta F, \tag{48}$$

where $\alpha \in [0, 1]$ is an unknown parameter that acts to quantify how much the deviations from Hawking power are nonthermal. The thermal part of the radiation (the one giving the thermal spectrum) is then

$$F_{\text{th}} = \frac{1}{768\pi M^2} + (1-\alpha)\Delta F. \tag{49}$$

Thus, without considering the Bogoliubov coefficients to study the spectra of particles radiated [24], the BH thermodynamics is then studied considering the above parameter α, restricted to be close to unity in order to fulfill the quasistatic regime, as we clarify in the subsection below.

B. Temperature and entropy of an evaporating BH

Consider the first law of thermodynamics, $dE = dQ - d\Theta$, where Θ is a generic quantity associated with a loss of energy and not given by heat exchange (indicated by Q). The rate of heat released in time by the BH can be associated with the thermal part of the power radiated, namely

$$dQ = -F_{\text{th}} dt. \tag{50}$$

Moreover, in the equilibrium thermodynamics context, the thermal part of the spectrum is associated with the BH temperature through the $(1+1)$-dimensional Stefan-Boltzmann law:

[10] Indeed, the Bogoliubov coefficient $\beta_{\omega\omega'}$ [Eq. (12)] gives the spectrum of the overall particles radiated during the evaporation, without time dependence. Moreover, Ref. [24] shows that, in the mirror context, the study of time-dependent particle production through localized wave packets may give controversies.

[11] We can imagine the overall spectrum of particles radiated as the superposition between the thermal one given by $1/(768\pi M^2)$ and an unknown contribution given by ΔF. However, the unknown contribution can modify the thermal spectrum created by $1/(768\pi M^2)$ such that another thermal spectrum, with a different temperature, arises.

$$F_{\mathrm{th}} = \frac{\pi}{12} T^2. \tag{51}$$

However, the Stefan-Boltzmann law, Eq. (51), and the temperature holds in the thermodynamics of equilibrium only, leaving unclear how to define the concept of temperature, i.e., of entropy when those quantities depend upon time.

To overcome this issue, we follow the standard procedure of defining a thermodynamic quasistatic approximation, imposing that, for a short time interval, the equilibrium is realized only locally. Obviously, the latter cannot be realized after the critical time, t_c, where the mass suddenly drops.

Hence, we restrict in the interval of times given by $0 < t < t_c$, where the second time derivatives of the mass can be neglected, see Fig. 2, and the flux can be approximated by

$$F = \frac{1}{768\pi M^2}\left(1 - 2\frac{\dot{M}}{M}t + \frac{\dot{M}^2}{M^2}t^2\right). \tag{52}$$

From Eq. (52), by using Eqs. (47) and (48), we have for the nonthermal and thermal counterparts, respectively:

$$F_{\text{no-th}} = \frac{\alpha}{768\pi M^2}\left(-2\frac{\dot{M}}{M}t + \frac{\dot{M}^2}{M^2}t^2\right), \tag{53a}$$

$$F_{\mathrm{th}} = \frac{1}{768\pi M^2} + \frac{1-\alpha}{768\pi M^2}\left(-2\frac{\dot{M}}{M}t + \frac{\dot{M}^2}{M^2}t^2\right). \tag{53b}$$

So, in this range of times, the temperature defined from the Stefan-Boltzmann law, Eq. (51), is

$$T = \frac{1}{8\pi M}\sqrt{1 + (1-\alpha)\left(-2\frac{\dot{M}}{M}t + \frac{\dot{M}^2}{M^2}t^2\right)}. \tag{54}$$

C. Effective temperature

To study nonequilibrium thermodynamics in a quasistatic regime, we may define an effective temperature, valid for quite short time intervals, by

$$T_{\mathrm{eff}}(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} T(t')dt', \tag{55}$$

where T is the temperature defined from the Stefan-Boltzmann law, Eq. (54).

A plot of T_{eff} is provided in Fig. 5 for different values of α, close to unity to guarantee the quasistatic regime. To check whether our quasistatic approximation is suitable, we compare the effective temperature, T_{eff}, with the equilibrium temperature, namely the Bekenstein temperature [1,3] given by $T_{\mathrm{HW}} = \frac{1}{8\pi M}$, as prompted in Fig. 5.

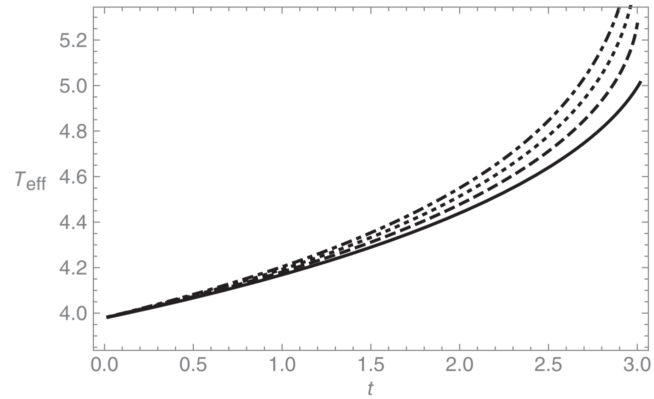

FIG. 5. Plot of the effective temperature of the BH in function of time [by Eq. (55)] for different values of α. The numbers in the axes x and y are in powers of 10^5 and 10^{-3}, respectively. In particular $\alpha = 1$ (thick line), $\alpha = 0.95$ (dashed line), $\alpha = 0.9$ (dotted line), and $\alpha = 0.85$ (dot-dashed line). The integration time Δt was chosen as $\Delta t = 1000$.

To certify the goodness of our hypothesis toward the quasistatic approximation, the effective temperature can be easily recast by

$$T_{\mathrm{eff}} = T_{\mathrm{HW}} + \delta T, \tag{56}$$

reproducing it in terms of a small deviation, globally vanishing, of the Hawking temperature, that is slightly significant for small intervals of time. A numerical study of $\delta T/T_{\mathrm{HW}}$ is performed in Table I for different times and for different α. Since with T_{eff} we want to approximate a thermodynamic equilibrium situation, T_{eff} should be close to the equilibrium temperature T_{HW}, i.e., $\delta T \ll T_{\mathrm{HW}}$. As we can see from Fig. 5 and Table I, this occurs when α is close to 1, as anticipated above.

Another relevant fact, evident from Fig. 5 and Table I, is that δT increases as t approaches t_c. From Table I, in particular, we can notice that the quasistatic approximation turns out to be still acceptable at $t = 0.95t_c$, as long as α is close to 1. After t_c, giving the sudden increasing of the power radiated, the quasistatic approximation breaks down.

D. Consequences on thermodynamics

Once defined as a quasistatic temperature, we rewrite the first law of thermodynamics through the following assumption

TABLE I. Values of $\delta T/T_{\mathrm{HW}}$, in percentage, for specific values of α (indicated in the first column) and for different times t (indicated in the first row).

α	$t = 0.25t_c$	$t = 0.5t_c$	$t = 0.75t_c$	$t = 0.95t_c$
0.95	0.1423%	0.4699%	1.023%	2.820%
0.90	0.3286%	0.9559%	2.096%	5.567%
0.85	0.5146%	1.440%	3.158%	8.245%

TABLE II. Values of β [indicating the entropy released during the evaporation from M_0 to M_c from Eq. (60)], for different values of α, indicating the nonthermality of the spectrum, by Eq. (53b).

α	β
0.95	1.098
0.90	1.107
0.85	1.112

$$dE = T_{\text{eff}} dS - d\Theta, \tag{57}$$

which resembles the usual version of first thermodynamics principle, but with T_{eff}, which replaces the equilibrium temperature, as given by Eq. (55) with a corresponding net entropy,[12] say S.

Using Eq. (50) and $dQ = T_{\text{eff}} dS$ we obtain an expression for the rate of entropy loss of the BH as it evaporates:

$$\frac{dS}{dt} = -\frac{F_{\text{th}}}{T_{\text{eff}}}. \tag{58}$$

Integrating the last over a period of time, we obtain the entropy that the BH loses during this period. In particular, since the quasistatic thermodynamics approach is not possible after t_c, we study the entropy released by the BH from its creation $t = 0$ (corresponding to the mass M_0) to t_c (corresponding to the mass M_c), namely

$$S_{\text{rel}}(M_0 \colon M_c) = -\int_0^{t_c} \frac{P_{\text{th}}}{T_{\text{eff}}}. \tag{59}$$

To compare the latter with the Bekenstein-Hawking entropy $4\pi M_0^2$, it is useful to write

$$S_{\text{rel}}(M_0 \colon M_c) = \beta\pi M_0^2. \tag{60}$$

Taking various values for α, the corresponding findings for β, obtained from Eq. (60), are shown in Table II. As we can see from this table, the more the spectrum is nonthermal (the more is α), the less is the entropy lost by the BH during the period $0 < t < t_c$ (the less is β).

We can make the reasonable assumption that the entropy of the particles radiated by the BH is proportional through a constant, say γ, to the one lost by the BH itself, i.e., $\frac{dS_{\text{rad}}}{dt} = -\gamma \frac{dS_{\text{rel}}}{dt}$ (see, e.g., [49–53] for more information about the value of γ). In this case, from Table II, we conclude that the more nonthermal the spectrum the less entropic the BH radiation.

This result seems to be consistent with the fact that part of the information swallowed by the BH is retrievable in the eventual nonthermal part of the radiation, slightly suggesting some resolution of BH information loss.

E. Consequences on entropy

Lastly, we compare the entropies we have computed in Table II $S_{\text{rel}}(M_0 \colon M_c) = \beta\pi M_0^2$ with the one released from an evaporating BH following the evaporation predicted by Hawking (8), i.e., without considering evaporation effects on the radiation, until the mass of the BH reaches M_c. The entropy of such a BH is given by the Bekenstein-Hawking entropy [3], $S = 4\pi M^2$. Using the indicative value of M_c provided in Eq. (30), we obtain

$$\beta = \frac{S_{\text{rel}}(M_0 \colon M_c)}{\pi M_0^2} \sim 1.48. \tag{61}$$

This means that, by considering the same mass evaporated $M_0 - M_c$, i.e., the same amount of radiation, the Hawking radiation is more entropic than our findings in Sec. II.

By looking at the expressions (53b) and (53a) for the thermal and nonthermal parts of the power radiated, respectively, we can explain qualitatively what is the further information retrievable from a BH in our case, with respect to the Hawking case. To do so, we summarize our steps here:

(1) First, as we stressed, radiation is not fully thermal. By considering, for instance, the photon evaporation [1,49,53], we expect that the radiated photons are no longer completely unpolarized. So, part of the information swallowed by the BH could be encoded in the polarization of the radiated photons. We confirm this fact by looking at Table II, in which we show that the more nonthermal the radiation, the less entropy radiated as stated above. Consequently, the more the photons are polarized.

(2) Suppose that we retrieve the radiated BH energy within a finite period of time while knowing its mass $M(t)$. In Hawking's case, the observed power radiated is $(768\pi M^2)^{-1}$. This contains information only about the mass $M(t)$, which we are observing. Hence, as expected, in the Hawking case we do not retrieve further information by observing Hawking radiation. Instead, by considering our model, the power radiated, in its explicit form, using Eq. (28), is given by

$$F = \frac{384\pi M^4}{t^2} - \frac{M}{t} - \frac{384\pi M^4}{t^2}\sqrt{1 - \frac{t}{192\pi M^3}}. \tag{62}$$

From this expression, by observing the mass of the BH and its energy radiated, we are able to retrieve the parameter t, giving the time passed since the BH started to evaporate. As a consequence, while without evaporation effects on the radiation only the mass of the BH is retrievable from Hawking radiation, by considering these effects, we are able to

[12] For the sake of clearness, one would require to add a subscript "eff" to the entropy also. However, we leave S without any subscripts in order to simplify the notation.

retrieve the history of the BH mass, i.e., $M(t)$. The latter implies also the information about $M(t=0)$, i.e., of the mass of the BH once it was created, and on t, the black hole age. This gives a decrease of the degrees of freedom of the microstates composing the BH mass, reducing the entropy as confirmed by comparing the values in Table II and Eq. (61).

V. OUTLOOKS

We have studied the analogy between moving mirrors and BHs, with particular attention devoted to the mass evolution of an evaporating BH in vacuum and to the corresponding nonequilibrium thermodynamics.

In particular, we adopted mirror analogs to BHs since these objects provide simplified descriptions of the time-evolving BH nature, indicating how BHs can evaporate. We described the mass evolution by means of numerical solutions obtained in the framework of Carlitz-Willey trajectory of mirrors. Consequently, we obtained suitable corrections to Hawking radiation without assuming a horizon area and/or surface gravity. We showed that these corrections are related to evaporation and we argued about possible deviation effects that appeared similar to those induced by backreaction on the metric, investigated in previous literature. We inferred (small) corrections to Hawking radiation, obtained as the BH evaporates, and we proposed a view of the BH information paradox in light of our findings. Moreover, in the case of not-fully thermal radiation, we studied the nonequilibrium thermodynamics associated with BHs, passing through mirror analogs and showing, again, how to relate these outcomes to the information paradox. To do so, we worked out the hypothesis of quasistatic processes, leading to an approximate version of the first principle of thermodynamics. Deviations from Hawking radiation were computed, showing at the same time a decrease of a BH's lifetime by a factor $\sim 3/8$. The entropy decrease was interpreted by assuming that part of information can be retrieved by BH radiation. Consequences about the role of an effective temperature, in view of revising the first principle of thermodynamics, has been discussed critically.

For future perspective, several aspects related to our work could be developed. One example is the role of Bogoliubov transformations in the context of mirrors, while another is the role of thermodynamics. Further applications of accelerating mirrors as BH analogs with different trajectory classes could also be pursued.

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[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] S. W. Hawking, Phys. Rev. D **13**, 191 (1976).
[3] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[4] J. D. Bekenstein, Phys. Rev. D **12**, 3077 (1975).
[5] R. Brout, S. Massar, R. Parentani, and P. Spindel, Phys. Rev. D **52**, 4559 (1995).
[6] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, Phys. Rev. Lett. **80**, 904 (1998).
[7] C. Barceló, S. Liberati, and M. Visser, Int. J. Mod. Phys. A **18**, 3735 (2003).
[8] J. R. M. de Nova, K. Golubkov, V. I. Kolobov, and J. Steinhauer, Nature (London) **569**, 688 (2019).
[9] R. Balbinot and A. Barletta, Classical Quantum Gravity **6**, 195 (1989).
[10] G. Vilkovisky, Phys. Lett. B **638**, 523 (2006).
[11] L. Mersini-Houghton, Phys. Lett. B **738**, 61 (2014).
[12] L. Susskind and L. Thorlacius, Nucl. Phys. B **B382**, 123 (1992).
[13] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).
[14] J. Zhang, Phys. Lett. B **668**, 353 (2008).
[15] B. Zhang, Q. yu Cai, L. You, and M. sheng Zhan, Phys. Lett. B **675**, 98 (2009).
[16] M. Arzano, A. J. M. Medved, and E. C. Vagenas, J. High Energy Phys. 09 (2005) 037.
[17] S. Bose, L. Parker, and Y. Peleg, Phys. Rev. D **52**, 3512 (1995).
[18] S. A. Fulling, J. Mod. Opt. **52**, 2207 (2005).
[19] J. P. S. Lemos and O. B. Zaslavskii, Phys. Rev. D **78** , 024040 (2008).
[20] B. Ilyas, J. Yang, D. Malafarina, and C. Bambi, Eur. Phys. J. C **77** (2017).
[21] A. B. Abdikamalov, A. A. Abdujabbarov, D. Ayzenberg, D. Malafarina, C. Bambi, and B. Ahmedov, Phys. Rev. D **100**, 024014 (2019).
[22] R. D. Carlitz and R. S. Willey, Phys. Rev. D **36**, 2327 (1987).
[23] R. D. Carlitz and R. S. Willey, Phys. Rev. D **36**, 2336 (1987).
[24] M. R. R. Good, P. R. Anderson, and C. R. Evans, Phys. Rev. D **88**, 025023 (2013).
[25] M. R. R. Good, P. R. Anderson, and C. R. Evans, Phys. Rev. D **94**, 065010 (2016).

[26] B. S. DeWitt, Phys. Rep. **19**, 295 (1975).
[27] S. A. Fulling and P. C. W. Davies, Proc. R. Soc. A **348**, 393 (1976).
[28] P. C. W. Davies and S. A. Fulling, Proc. R. Soc. A **356**, 237 (1977).
[29] M. R. R. Good and Y. C. Ong, Eur. Phys. J. C **80**, 1169 (2020).
[30] W. R. Walker and P. C. W. Davies, J. Phys. A **15**, L477 (1982).
[31] W. R. Walker, Classical Quantum Gravity **2**, L37 (1985).
[32] M. R. R. Good, J. Foo, and E. V. Linder, Classical Quantum Gravity **38**, 085011 (2021).
[33] W. R. Walker, Phys. Rev. D **31**, 767 (1985).
[34] M. R. R. Good, A. Zhakenuly, and E. V. Linder, Phys. Rev. D **102**, 045020 (2020).
[35] L. Ford and A. Vilenkin, Phys. Rev. D **25**, 2569 (1982).
[36] R. Balbinot, R. Bergamini, and B. Giorgini, Nuovo Cimento B **71**, 27 (1982).
[37] R. Balbinot and R. Bergamini, Nuovo Cimento B **68**, 104 (1982).
[38] S. A. Hayward, Phys. Rev. Lett. **96**, 031103 (2006).
[39] S. Abdolrahimi, D. N. Page, and C. Tzounis, Phys. Rev. D **100**, 124038 (2019).
[40] R. Li and J. Wang, Phys. Rev. D **104**, 026011 (2021).
[41] S. D. Mathur, Pramana **79**, 1059 (2012).
[42] V. Balasubramanian and B. Czech, Classical Quantum Gravity **28**, 163001 (2011).
[43] S. D. Mathur, Classical Quantum Gravity **26**, 224001 (2009).
[44] C. Rovelli, Phys. Rev. Lett. **77**, 3288 (1996).
[45] S. N. Solodukhin, Phys. Rev. D **57**, 2410 (1998).
[46] A. Ghosh and P. Mitra, Phys. Rev. D **71**, 027502 (2005).
[47] G. Dvali and C. Gomez, Phys. Lett. B **719**, 419 (2013).
[48] G. Dvali, Fortschr. Phys. **64**, 106 (2016).
[49] D. N. Page, Phys. Rev. D **13**, 198 (1976).
[50] D. N. Page, Phys. Rev. D **16**, 2402 (1977).
[51] D. N. Page, Phys. Rev. D **14**, 3260 (1976).
[52] D. N. Page, New J. Phys. **7**, 203 (2005).
[53] D. N. Page, J. Cosmol. Astropart. Phys. 09 (2013) 028.
[54] A. Belfiglio, O. Luongo, and S. Mancini, Phys. Rev. D **105**, 123523 (2022).
[55] A. Belfiglio, R. Giambò, and O. Luongo, Classical Quantum Gravity **40**, 105004 (2023).
[56] P. C. W. Davies, J. Phys. A **8**, 609 (1975).
[57] P. Davies, Proc. R. Soc. A **351**, 129 (1976).
[58] P. Landsberg and A. D. Vos, J. Phys. A **22**, 1073 (1989).
[59] J. D. Bekenstein and A. E. Mayo, Gen. Relativ. Gravit. **33**, 2095 (2001).
[60] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, Nature (London) **479**, 376 (2011).
[61] G. Barton and A. Calogeracos, Ann. Phys. (N.Y.) **238**, 227 (1995).
[62] S. A. Fulling, Phys. Rev. D **7**, 2850 (1973).
[63] M. R. R. Good, A. Lapponi, O. Luongo, and S. Mancini, Phys. Rev. D **104**, 105020 (2021).
[64] S. Massar and R. Parentani, Phys. Rev. D **54**, 7444 (1996).
[65] O. Luongo and M. Muccino, Galaxies **9**, 77 (2021).
[66] M. R. Good and E. V. Linder, Phys. Rev. D **97**, 065006 (2018).