Construction of non-Abelian electric strings

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We detail the construction of electric string solutions in $SU(2)$ Yang-Mills-Higgs theory with a scalar in the fundamental representation and discuss the properties of the solution. We show that Schwinger gluon pair production in the electric string background is absent. A similar construction in other models, such as with an adjoint scalar field and the electroweak model, does not yield solutions.

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I. INTRODUCTION

A homogeneous electric field in Maxwell electrodynamics corresponds to the familiar gauge potential

$$A^{\mu} = (-Ez, 0, 0, 0), \tag{1}$$

where E is the electric field strength. When coupled to external charges, the electric field is known to decay by Schwinger pair production [1]. Similarly, an $SU(2)$ non-Abelian electric field[1] can be derived from the gauge potential

$$A^{\mu a} = (-Ez, 0, 0, 0)\delta^{a3}, \tag{2}$$

where $a = 1, 2, 3$ is the group index. Unlike in the Maxwell case, it is not necessary to introduce "external" charges as even the pure non-Abelian gauge theory includes charged quanta ("gluons"). Schwinger pair production of gluons will cause the non-Abelian electric field to decay rapidly [2–16]. However, the story for non-Abelian electric fields is more subtle, as embedding the Maxwell gauge potential into the non-Abelian theory is not the only way to obtain a non-Abelian electric field. As shown by Brown and Weisberger (BW) [3], and described in Sec. II, there is a one parameter set of gauge inequivalent gauge fields that all lead to the same homogeneous electric field. An analysis of Schwinger pair production in such gauge field backgrounds shows that Schwinger gluon production is absent [17].

[1]We will explicitly only consider $SU(2)$ non-Abelian gauge theory. The solutions may be embedded in theories with larger gauge groups.

An issue with BW gauge fields is that, unlike Eq. (2), they are not classical solutions of the vacuum equations of non-Abelian gauge theory; instead, they require sources. A possibility is that quantum backreaction on the classical dynamics effectively provides such sources, but this is difficult to show. A second possibility, one that has been reported in Ref. [18] and that we will detail in this paper, is that suitable sources can be provided by an external classical field, such as a scalar field. Then the BW gauge fields are solutions of the Yang-Mills-Higgs classical equations of motion, much like other classical solutions such as strings and magnetic monopoles [19], but a key difference is that the solution contains a flux of electric field instead of a magnetic field. Such solutions are called "electric strings."

The electric string solution presented in Ref. [18] was arrived at by using a certain amount of guesswork. In the present paper, we present some rationale for the guesses, in addition to exploring certain other issues. The starting point for our discussions is to consider a homogeneous BW electric field, discussed in Sec. II. In Sec. III, we show that a homogeneous BW electric field can indeed be sourced by a scalar field that transforms in the fundamental representation of $SU(2)$. The necessary scalar field configuration also solves its own equation of motion but only for certain parameters.

Encouraged by the case of the homogeneous electric field, we turn our attention to "electric string" solutions in which the electric field is localized to a tubular region. In Sec. IV, we construct such a solution. We find that the tubular electric field is wrapped by magnetic fields in an oscillating pattern with a slow asymptotic falloff.

The question of Schwinger gluon production in the background of an electric string is considered in Sec. V, and as in the homogeneous electric field case, this process is absent. A classical stability analysis of the electric string solution is postponed for future work.

The existence of an electric string solution in a Yang-Mills with a fundamental Higgs naturally raises the question if such strings can arise in other theories. In Appendix A, we examine the case when the scalar field is in the adjoint representation of $SU(2)$ and show that the solution does not exist. Similarly in Appendix B, we examine if the solution can be found in the electroweak model that has an additional $U(1)$ hypercharge gauge field, and there too, we find that a solution does not exist. These no-go results though are based on certain assumptions about the structure of the solutions and it is possible that a more general analysis might successfully find solutions.

Our conclusions are summarized in Sec. VI, and some helpful formulas are listed in Appendix C.

II. GAUGE FIELDS

A. BW homogeneous gauge fields

The nonvanishing $SU(2)$ gauge fields that give a homogeneous electric field are [3]

$$W^1_\mu = \frac{\Omega}{g}\partial_\mu t, \qquad W^2_\mu = -\frac{E}{\Omega}\partial_\mu z, \qquad W^3_\mu = 0. \tag{3}$$

where Ω is a parameter. We will assume without loss of generality that $\Omega > 0$. The field strength is found from

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu, \tag{4}$$

and the nonvanishing field strength is

$$W^3_{\mu\nu} = -E(\partial_\mu t \partial_\nu z - \partial_\nu t \partial_\mu z), \tag{5}$$

and $W^1_{\mu\nu} = 0 = W^2_{\mu\nu}$.

The currents are found from the classical equations of motion for the gauge fields,

$$j^{\mu a} = \partial_\nu W^{\mu\nu a} + g\epsilon^{abc} W^b_\nu W^{\mu\nu c}, \tag{6}$$

and (in the BW gauge) are given by

$$j^{\mu 1} = -g\frac{E^2}{\Omega}\partial^\mu t, \qquad j^{\mu 2} = -\Omega E \partial^\mu z, \qquad j^{\mu 3} = 0. \tag{7}$$

B. Temporal gauge

To bring the BW gauge fields to temporal gauge, we perform the gauge transformation,

$$U = e^{i\sigma^2\pi/4} e^{i\sigma^1\pi/4} e^{-i\sigma^1\Omega t/2}, \tag{8}$$

where σ^a are the Pauli spin matrices. Then,

$$W_\mu \to W'_\mu = U W_\mu U^\dagger + \frac{2i}{g} U \partial_\mu U^\dagger, \tag{9}$$

where $W_\mu \equiv W^a_\mu \sigma^a$. This gives the gauge fields for a homogeneous electric field in the form,

$$W^\pm_\mu = -\frac{\epsilon}{g} e^{\pm i\Omega t}\partial_\mu z, \qquad W^3_\mu = 0, \tag{10}$$

where

$$\epsilon \equiv gE/\Omega. \tag{11}$$

Next we also include a string profile function, $f(r)$, since we eventually want to discuss electric string configurations. Then the gauge fields we will consider are

$$W^\pm_\mu = -\frac{\epsilon}{g} e^{\pm i\Omega t} f(r)\partial_\mu z, \qquad W^3_\mu = 0, \tag{12}$$

where $W^\pm_\mu \equiv W^1_\mu \pm i W^2_\mu$, r refers to the cylindrical radial coordinate. Alternately, we can write

$$W^1_\mu = -\frac{\epsilon}{g}\cos(\Omega t) f(r)\partial_\mu z, \tag{13}$$

$$W^2_\mu = -\frac{\epsilon}{g}\sin(\Omega t) f(r)\partial_\mu z. \tag{14}$$

The field strengths are

$$\begin{aligned} W^1_{\mu\nu} = -\frac{\epsilon}{g}[&-\Omega\sin(\Omega t) f(r)(\partial_\mu t\partial_\nu z - \partial_\nu t\partial_\mu z) \\ &+ \cos(\Omega t) f'(r)(\partial_\mu r\partial_\nu z - \partial_\nu r\partial_\mu z)], \end{aligned} \tag{15}$$

$$\begin{aligned} W^2_{\mu\nu} = -\frac{\epsilon}{g}[&\Omega\cos(\Omega t) f(r)(\partial_\mu t\partial_\nu z - \partial_\nu t\partial_\mu z) \\ &+ \sin(\Omega t) f'(r)(\partial_\mu r\partial_\nu z - \partial_\nu r\partial_\mu z)], \end{aligned} \tag{16}$$

$$W^3_{\mu\nu} = 0, \tag{17}$$

where prime denotes differentiation with respect to the argument. Note that the electric field is accompanied by a magnetic field in the azimuthal direction. To decide if the field configuration is electric or magnetic, we will calculate the gauge and Lorentz invariant Lagrangian density,

$$\mathcal{L}_g = -\frac{1}{4}W^a_{\mu\nu}W^{\mu\nu a} = \frac{\epsilon^2}{2g^2}(\Omega^2 f^2 - f'^2). \tag{18}$$

Positive $\mathcal{L}_g$ implies that the field is electriclike while negative values imply a magneticlike field.

The $SU(2)$ currents are obtained from (6),

$$j^1_\mu = -\frac{\epsilon}{g}\cos(\Omega t)\left[f'' + \frac{f'}{r} + \Omega^2 f\right]\partial_\mu z, \tag{19}$$

$$j^2_\mu = -\frac{\epsilon}{g}\sin(\Omega t)\left[f'' + \frac{f'}{r} + \Omega^2 f\right]\partial_\mu z, \tag{20}$$

$$j^3_\mu = -\frac{\epsilon^2}{g}\Omega f^2 \partial_\mu t. \tag{21}$$

Note that the $a = 1$, 2 equations may also be written as

$$j^a_\mu = \frac{1}{f}\left[f'' + \frac{f'}{r} + \Omega^2 f\right] W^a_\mu, \quad (a = 1, 2). \quad (22)$$

We are interested in finding if these currents can be sourced by scalar fields. We will first consider the simpler case of a homogeneous electric field ($f(r) = 1$) and show that it can be sourced by a scalar field in the fundamental representation. The case of a scalar field in the adjoint representation is discussed in Appendix A with the conclusion that it *cannot* source the electric field.

III. HOMOGENEOUS ELECTRIC FIELD

In this section, $f(r) = 1$.

A. Fundamental scalar

The model now is similar to the electroweak model where the symmetry is $SU(2) \times U(1)$, except that the $U(1)$ charge, commonly denoted by g', is set to zero. In other words, the $SU(2)$ is a gauged symmetry while the $U(1)$ is a global symmetry.

We denote the fundamental scalar field by Φ. The Lagrangian for the model is

$$L = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} + |D_\mu \Phi|^2 - V(\Phi), \quad (23)$$

where

$$D_\mu \Phi = \partial_\mu \Phi - i\frac{g}{2} W^a_\mu \sigma^a \Phi, \quad (24)$$

$$V(\Phi) = m^2 |\Phi|^2 + \lambda |\Phi|^4, \quad (25)$$

where m^2 may be negative or positive but $\lambda > 0$.

The gauge field equations of motion are

$$D_\nu W^{\mu\nu a} = j^a_\mu = i\frac{g}{2}[\Phi^\dagger \sigma^a D_\mu \Phi - \text{H.c.}], \quad (26)$$

where H.c. stands for Hermitian conjugate.

Using $\sigma^a \sigma^b = \delta^{ab} + i\epsilon^{abc}\sigma^c$, we find

$$\Phi^\dagger \sigma^a D_\mu \Phi = \Phi^\dagger \sigma^a \partial_\mu \Phi - \frac{ig}{2}|\Phi|^2 W^a_\mu + \frac{g}{2}|\Phi|^2 (\vec{W}_\mu \times \hat{n})^a, \quad (27)$$

where the unit vector $\hat{n}^a$ is given by

$$\hat{n}^a \equiv \frac{\Phi^\dagger \sigma^a \Phi}{\Phi^\dagger \Phi}. \quad (28)$$

Inserting (27) in the expression for the current, we get

$$j^a_\mu = i\frac{g}{2}[\Phi^\dagger \sigma^a \partial_\mu \Phi - \text{H.c.}] + \frac{g^2}{2}|\Phi|^2 W^a_\mu. \quad (29)$$

We will start by solving (29) for Φ with j^a_μ given in (19)–(21) with $f = 1$. We can write Φ in the Hopf parametrization,

$$\Phi = \eta \begin{pmatrix} \cos\alpha e^{i\beta} \\ \sin\alpha e^{i\gamma} \end{pmatrix}. \quad (30)$$

We will assume that Φ is homogeneous, so α, β, and γ are only functions of time, and η is a constant.

The $\mu = 3$ component of (29) is nontrivial only for $a = 1$, 2, giving

$$j^a_z = \frac{g^2}{2}|\Phi|^2 W^a_z, \quad (a = 1, 2). \quad (31)$$

Comparison with (22) gives

$$\Omega^2 = \frac{1}{2} g^2 \eta^2, \quad \text{or}, \quad \eta = \frac{\sqrt{2}\Omega}{g}. \quad (32)$$

(Recall that we are considering $f = 1$.) Next we turn to the $\mu = 0$ components. Some algebra (see Appendix C) leads to

$$2\dot{\alpha}\sin(\gamma - \beta) + \sin(2\alpha)\cos(\gamma - \beta)(\dot{\beta} + \dot{\gamma}) = 0, \quad (33)$$

$$2\dot{\alpha}\cos(\gamma - \beta) - \sin(2\alpha)\sin(\gamma - \beta)(\dot{\beta} + \dot{\gamma}) = 0, \quad (34)$$

$$g\eta^2(\cos^2\alpha\dot{\beta} - \sin^2\alpha\dot{\gamma}) - \frac{\epsilon^2 \Omega}{g} = 0. \quad (35)$$

The solution to these three equations leads to

$$\Phi = \eta \begin{pmatrix} z_1 e^{+i\omega t} \\ z_2 e^{-i\omega t} \end{pmatrix}, \quad (36)$$

where $z_1, z_2 \in \mathbb{C}$ are constants with $|z_1|^2 + |z_2|^2 = 1$, and

$$\omega = \frac{\epsilon^2}{2\Omega}. \quad (37)$$

Now we have to make sure that Φ solves its own equation of motion,

$$D_\mu D^\mu \Phi + V'(\Phi) = 0, \quad (38)$$

where the prime denotes derivative with respect to $\Phi^\dagger$. For V in (25) and Φ in (36), we can write

$$V'(\Phi) = (m^2 + 2\lambda\eta^2)\Phi. \quad (39)$$

We also evaluate

$$D_\mu D^\mu \Phi = -\left(\omega^2 - \frac{\epsilon^2}{4}\right)\Phi. \quad (40)$$

Therefore (38) leads to the equation,

$$-\omega^2 + \frac{\epsilon^2}{4} + m^2 + 2\lambda\eta^2 = 0. \tag{41}$$

Consistency with (37) implies

$$\omega^2 - \frac{\omega\Omega}{2} - m^2 - 2\lambda\eta^2 = 0. \tag{42}$$

Therefore,

$$\omega = \frac{1}{2}\left[\frac{\Omega}{2} \pm \sqrt{\frac{\Omega^2}{4} + 4\left(m^2 + \frac{4\lambda}{g^2}\Omega^2\right)}\right]. \tag{43}$$

Note that (37) implies that $\omega/\Omega > 0$. Depending on the signs and magnitude of the parameters m^2, λ/g^2 and Ω^2, only one or both roots in (43) are valid.

If we choose potential parameters such that $m^2 + 2\lambda\eta^2 = 0$, we obtain simpler expressions. With this choice of parameters, Eq. (39) gives $V'(\Phi) = 0$; i.e., Φ is at an extrema of its potential. Then (43) gives two solutions: $\omega = 0$ or $\omega = \Omega/2$. The solution with $\omega = 0$ is trivial for then $\epsilon = 0$ because of (37) and the electric field vanishes. For the nontrivial solution $\omega = \Omega/2$, (37) gives

$$\epsilon = \Omega, \qquad (V' = 0). \tag{44}$$

To summarize these results, we have found a solution of the classical equations of motion that corresponds to a homogeneous electric field,

$$\Phi = \frac{\sqrt{2}\Omega}{g}\begin{pmatrix} z_1 e^{+i\omega t} \\ z_2 e^{-i\omega t}\end{pmatrix} \tag{45}$$

$$\vec{W}_\mu = -\frac{\sqrt{2\omega\Omega}}{g}(\cos(\Omega t), \sin(\Omega t), 0)\partial_\mu z, \tag{46}$$

where $z_1, z_2 \in \mathbb{C}$ are constants with $|z_1|^2 + |z_2|^2 = 1$, and ω is given in terms of Ω and the parameters in the scalar potential by (43). The electric field is found from (11) and (37),

$$E = \frac{\Omega\sqrt{2\omega\Omega}}{g}. \tag{47}$$

IV. ELECTRIC STRING ($f(r) \neq 1$)

We now move on from the homogeneous electric field to electric string solutions. The gauge fields are given by (12), the required currents by (19)–(21), and the currents that Φ can source by (29). Hence, we must now find Φ by solving

$$\frac{i}{2}g(\Phi^\dagger\sigma^1 D_\mu\Phi - (D_\mu\Phi)^\dagger\sigma^1\Phi) = -\frac{\epsilon}{g}\cos(\Omega t)\left(f'' + \frac{f'}{r} + \Omega^2 f\right)\partial_\mu z, \tag{48}$$

$$\frac{i}{2}g(\Phi^\dagger\sigma^2 D_\mu\Phi - (D_\mu\Phi)^\dagger\sigma^2\Phi) = -\frac{\epsilon}{g}\sin(\Omega t)\left(f'' + \frac{f'}{r} + \Omega^2 f\right)\partial_\mu z, \tag{49}$$

$$\frac{i}{2}g(\Phi^\dagger\sigma^3 D_\mu\Phi - (D_\mu\Phi)^\dagger\sigma^3\Phi) = -\frac{\epsilon^2}{g}\Omega f^2\partial_\mu t. \tag{50}$$

Guided by the homogeneous electric field case of Sec. III, we try

$$\Phi = \eta h(r)\begin{pmatrix} z_1 e^{+i\omega t} \\ z_2 e^{-i\omega t}\end{pmatrix}, \tag{51}$$

where $h(r)$ is a real profile function that is to be determined.

We first consider $\mu = 0$ in (48)–(50). We find that (48) and (49) are trivially satisfied, while Eq. (50) gives

$$-g\eta^2 h^2\omega = -\frac{\epsilon^2}{g}\Omega f^2, \tag{52}$$

which implies

$$h(r) = \frac{\epsilon}{g\eta}\sqrt{\frac{\Omega}{\omega}}f(r), \tag{53}$$

and we should have $\omega/\Omega > 0$ since h is real.

The $\mu = 1, 2$ equations are trivially satisfied, so we now consider $\mu = 3$; i.e., the $\mu = z$ equations. Equations (48) and (49) then give

$$f'' + \frac{f'}{r} + \left(\Omega^2 - \frac{g^2\eta^2}{2}h^2\right)f = 0 \tag{54}$$

and (50) is trivially satisfied.

Inserting (53) in (54) gives

$$f'' + \frac{f'}{r} + \Omega^2\left(1 - \frac{\epsilon^2}{2\omega\Omega}f^2\right)f = 0. \tag{55}$$

Next we consider the Φ equation of motion in (38) together with (53) to get

$$f'' + \frac{f'}{r} + \left[(\omega^2 - m^2) - \left(\frac{\epsilon^2}{4} + \frac{2\lambda\epsilon^2\Omega}{g^2\omega}\right)f^2\right]f = 0. \tag{56}$$

Consistency with (54) requires

$$\Omega^2 = \omega^2 - m^2 \tag{57}$$

$$\frac{\Omega}{2\omega} = \frac{1}{4} + \frac{2\lambda\Omega}{g^2\omega}. \tag{58}$$

Equation (58) can be written as

$$\omega = 2\left(1 - \frac{4\lambda}{g^2}\right)\Omega, \tag{59}$$

and since we have already assumed $\omega/\Omega > 0$ [see Eq. (53)], we should restrict to $\lambda \leq g^2/4$. Also, since the potential V should be bounded from below, we have $0 \leq \lambda \leq g^2/4$.

For fixed parameters m^2, λ, Eqs. (57) and (58) can be solved to obtain Ω and ω,

$$\Omega^2 = \frac{m^2}{4(1-4\lambda/g^2)^2 - 1}, \tag{60}$$

$$\omega^2 = \frac{4(1-4\lambda/g^2)^2 m^2}{4(1-4\lambda/g^2)^2 - 1}. \tag{61}$$

This is a valid solution provided

$$\frac{m^2}{4(1-4\lambda/g^2)^2 - 1} > 0, \tag{62}$$

which gives the conditions $0 < \lambda < g^2/8$ or $\lambda > 3g^2/8$ if $m^2 > 0$, and $g^2/8 < \lambda < 3g^2/8$ if $m^2 < 0$. Taking into account the tighter restriction discussed below (59) that $\omega > 0$ for $\Omega > 0$, the range of λ for $m^2 < 0$ is $g^2/8 < \lambda < g^2/4$. These constraints are shown in Fig. 1.

Hence, there is a range of values of the parameters m^2 and λ of the potential for which (55) and (56) are identical, provided (60) and (61) hold. In this case, (55) can be solved numerically. It is preferable to rescale variables,

$$R = \Omega r, \qquad F = \frac{\epsilon}{\sqrt{2\omega\Omega}} f, \tag{63}$$

and then

$$F'' + \frac{F'}{R} + (1 - F^2)F = 0. \tag{64}$$

Only the combination $\epsilon f(r)$ appears in the gauge field [see Eq. (12)], and we are free to take $f(0) = 1$ or

$$F(0) = \frac{\epsilon}{\sqrt{2\omega\Omega}}. \tag{65}$$

Since the electric field strength is proportional to F, we would like to choose boundary conditions such that $F(0) = F_0 \neq 0$ and $F(\infty) \to 0$. Different values of F_0 correspond to different external charges placed at $z \to \pm\infty$ that produce the electric field. Smoothness at $R = 0$ requires $F'(0) = 0$.

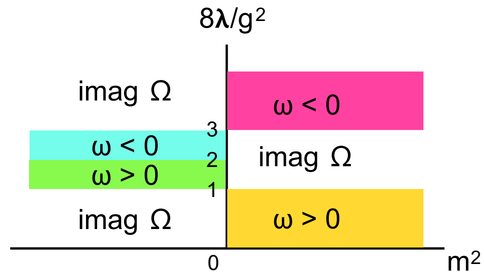

FIG. 1. Constraints on the parameters in the m^2-$8\lambda/g^2$ plane. The unshaded regions give imaginary Ω and are not allowed. The solution is only valid in the regions of parameter space where $\omega > 0$.

There are no solutions that fall off asymptotically for $F_0 > 1$. For $F_0 \ll 1$, the nonlinear term with F^2 is subdominant, and the solutions are well approximated by a Bessel function of zero order,

$$F(R) \approx F_0 J_0(R). \tag{66}$$

A plot of the numerically evaluated $F(R)$ is shown in Fig. 2. The asymptotic behavior of $F(R)$ is therefore,

$$F(R) \sim F_0 \sqrt{\frac{2}{\pi R}} \cos\left(R - \frac{\pi}{4}\right). \tag{67}$$

The energy density in all the fields is given by the expression,

$$\mathcal{E} = \frac{1}{2}(W^a_{0i})^2 + \frac{1}{4}(W^a_{ij})^2 + |D_t\Phi|^2 + |D_i\Phi|^2 + V(\Phi). \tag{68}$$

Inserting the expressions for the solution, we get

$$\mathcal{E} = \frac{\epsilon^2\Omega^2}{2g^2} f^2 + \frac{\epsilon^2}{2g^2} f'^2 + \frac{\epsilon^2\omega\Omega}{g^2} f^2 + \frac{\epsilon^2\Omega}{g^2\omega} f'^2 + \frac{\epsilon^4\Omega}{4g^2\omega} f^4 + m^2 \frac{\epsilon^2\Omega}{g^2\omega} f^2 + \lambda \frac{\epsilon^4\Omega^2}{g^4\omega^2} f^4, \tag{69}$$

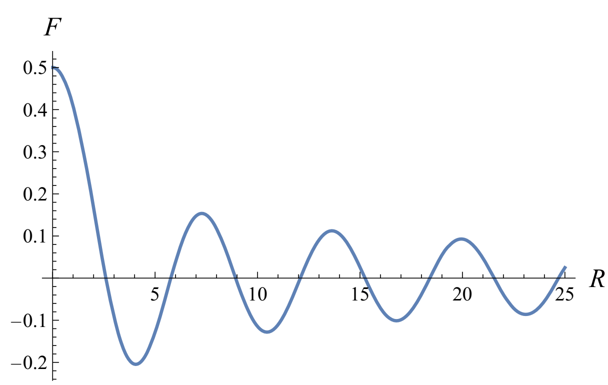

FIG. 2. $F(R)$ vs R for $F_0 = 0.5$.

or in terms of rescaled variables,

$$\mathcal{E}' \equiv \frac{g^2}{2\omega\Omega^3}\mathcal{E} = \left(\frac{1}{2}+\frac{1}{\kappa}\right)F'^2 + \left(\frac{1}{2}+2\kappa-\frac{1}{\kappa}\right)F^2 + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{\kappa}\right)F^4, \tag{70}$$

where [see Eq. (59)],

$$\kappa \equiv \frac{\omega}{\Omega} = 2\left(1-\frac{4\lambda}{g^2}\right). \tag{71}$$

Due to the constraints on λ/g^2, we find $1<\kappa<2$ for $m^2>0$ and $0<\kappa<1$ for $m^2<0$. In Fig. 3, we show $\mathcal{E}'$ for a sample value of κ and F_0.

The energy per unit length along the z direction is defined to be the tension of the string. Therefore, the rescaled tension with a radial cutoff at R_c is

$$\mu(R_c) = 2\pi\int_0^{R_c} dR\, R\mathcal{E}' \approx \left[\left(\frac{1}{2}+\frac{1}{\kappa}\right)0.54 + \left(\frac{1}{2}+2\kappa-\frac{1}{\kappa}\right)0.54\right]R_c + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{\kappa}\right)0.09\ln(R_c). \tag{72}$$

Therefore, the tension diverges linearly with the radial cutoff. Thus, while the electric field is axisymmetric and concentrated along the central axis, it is not as sharply localized as in a magnetic Nielsen-Olesen string [20], though it is more localized than the homogeneous configuration. The situation is similar to that of a global string for which the energy diverges logarithmically and to global monopoles for which the energy diverges linearly [19].

For $m^2<0$, the potential $V(\Phi)$ has a minimum at $\Phi \neq 0$. However, the solution of (64) implies that $\Phi \to 0$ as $r\to\infty$.

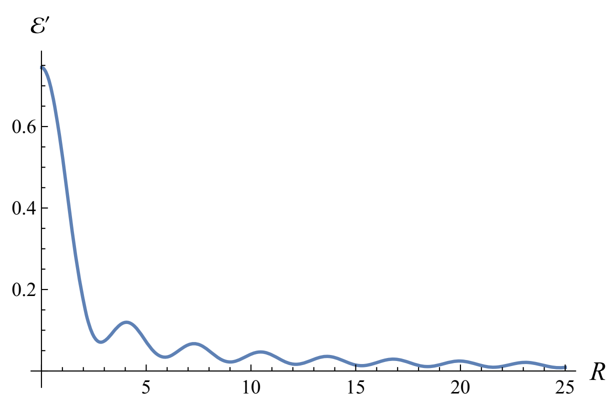

FIG. 3. $\mathcal{E}'$ vs R for $F_0 = 0.5$ and $\kappa = 1.5$.

Therefore, Φ is not in its true vacuum asymptotically, and there is nonvanishing vacuum energy at spatial infinity. In this case, the electric string is a cylindrical “bubble” of the true vacuum (with nonvanishing Φ) in a background of the false vacuum phase with $\Phi = 0$. This is different from the case of the electric string with $m^2>0$ for then, the true vacuum is at $\Phi=0$, and the potential energy of Φ goes to zero in the asymptotic region.

Finally, we return to the question of whether the solution corresponds to an *electric* string as azimuthal magnetic fields are also present [see Eqs. (15)–(17)]. Hence, we calculate $\mathcal{L}_g$ using (18) and plot the quantity $F^2-(F')^2$ in Fig. 4. The behavior at large R can be seen from the properties of the Bessel functions,

$$F^2-(F')^2 \sim F_0^2[J_0^2(R)-J_1^2(R)] \to F_0^2\frac{2\sin(2R)}{\pi R}, \tag{73}$$

where we have used $J_1(R) = -J_0'(R)$ and (67). The gauge field strength is electriclike where $F^2-(F')^2>0$; otherwise, it is magneticlike. This shows that the solution has alternating electric and magnetic fields where the electric field is along the z direction and the magnetic field is along the azimuthal direction. So, a caricature of the electric string configuration is a tube of electric field along the z direction, wrapped by weak azimuthal magnetic fields, that are again contained in a sheath of weaker electric field, ad infinitum (see the sketch in Fig. 5).

To summarize the main result of this section, we have found the electric string solution,

$$\Phi = \frac{\epsilon}{g}\sqrt{\frac{\Omega}{\omega}}f(r)\begin{pmatrix} z_1 e^{+i\omega t} \\ z_2 e^{-i\omega t}\end{pmatrix} \tag{74}$$

$$W_\mu^\pm = -\frac{\epsilon}{g}e^{\pm i\Omega t}f(r)\partial_\mu z, \qquad W_\mu^3 = 0, \tag{75}$$

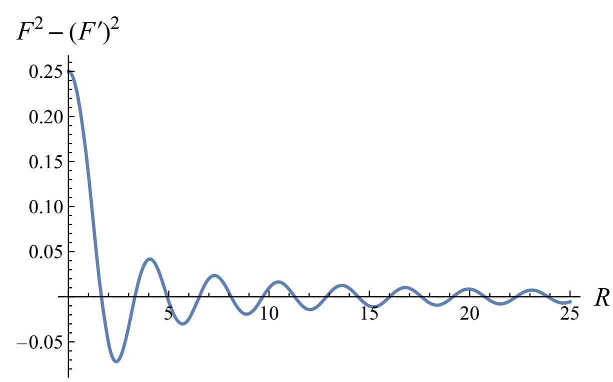

FIG. 4. $F(R)^2 - F'(R)^2$ vs R for $F_0 = 0.5$. The field strength is electriclike where $F(R)^2 - F'(R)^2$ is positive and magneticlike where $F(R)^2 - F'(R)^2$ is negative.

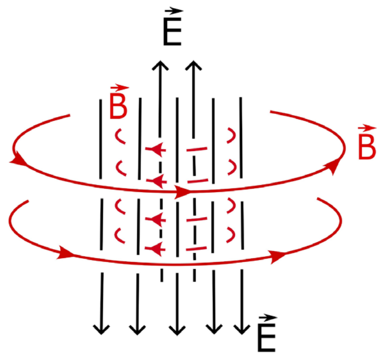

FIG. 5. A sketch of the electric and magnetic fields in an electric string.

where $|z_1|^2 + |z_2|^2 = 1$, and Ω and ω are given by

$$\Omega = \left[\frac{m^2}{4(1-4\lambda/g^2)^2 - 1}\right]^{1/2}, \tag{76}$$

$$\omega = \left[\frac{4(1-4\lambda/g^2)^2 m^2}{4(1-4\lambda/g^2)^2 - 1}\right]^{1/2}, \tag{77}$$

and m^2 and λ are parameters of the scalar potential [see Eq. (25)]. The solution is valid for $0 < \lambda < g^2/8$ for $m^2 > 0$ and for $g^2/8 < \lambda < g^2/4$ for $m^2 < 0$. The profile function, $f(r)$, is common to both the gauge field and the scalar field and satisfies (55) with boundary conditions $f(0) = 1$, $f'(0) = 0$. The solution for the profile is closely approximated by the zeroth order Bessel function, $J_0(\Omega r)$, up to a multiplicative constant that fixes the strength of the electric field at the origin.

V. SCHWINGER GLUON PRODUCTION?

The gauge particles ("gluons") are charged under $SU(2)$, as are excitations of the scalar field Φ, and there can be Schwinger pair production of both kinds of excitations. Here, we are interested in whether the electric field is protected from pair production of gluons. Pair production of scalar quanta is similar to pair production of quarks in QCD that results in string breaking. The effect can be suppressed by considering large masses of the scalar quanta, i.e., large positive values of the parameter m^2.

To see that the electric field is stable to Schwinger gluon production, we perturb the gauge field,

$$W^{\pm}_{\mu} = A^{\pm}_{\mu} + e^{\pm i\Omega t} Q^{\pm}_{\mu}, \qquad W^3_{\mu} = A^3_{\mu} + Q^3_{\mu}, \tag{78}$$

where A^a_μ is the background [see Eq. (12)]

$$A^{\pm}_{\mu} = -\frac{\epsilon}{g} e^{\pm i\Omega t} f(r)\partial_\mu z, \qquad A^3_\mu = 0, \tag{79}$$

and Q^a_μ are the perturbations. The scalar field is left unperturbed,

$$\Phi = \eta h(r) \begin{pmatrix} z_1 e^{+i\omega t} \\ z_2 e^{-i\omega t} \end{pmatrix}, \tag{80}$$

since we are interested in gluon production, and the mass of the Φ field can be taken to be large.

As in Ref. [17], the expressions in (78) are inserted in the Lagrangian density for the gauge fields

$$\mathcal{L}_g = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a}, \tag{81}$$

to obtain the Lagrangian density for the perturbations Q^a_μ. The expressions are lengthy, but the important point is that there is no explicit time dependence in the Lagrangian even though the background in (79) is time dependent. For example, the Lagrangian density to second order in the perturbations is

$$\begin{aligned} \mathcal{L}^{(2)}_g = {} & \frac{1}{2}\left(\dot{Q}^{(1)}_i - \Omega Q^{(2)}_i\right)^2 + \frac{1}{2}\left(\dot{Q}^{(2)}_i + \Omega Q^{(1)}_i\right)^2 \\ & + \frac{1}{2}\left(\dot{Q}^{(3)}_i\right)^2 - \frac{1}{4}\left(\partial_i Q^{(1)}_j - \partial_j Q^{(1)}_i\right)^2 \\ & - \frac{1}{4}\left(\partial_i Q^{(2)}_j - \partial_j Q^{(2)}_i + \epsilon f\left(\hat{z}_i Q^{(3)}_j - \hat{z}_j Q^{(3)}_i\right)\right)^2 \\ & - \frac{1}{4}\left(\partial_i Q^{(3)}_j - \partial_j Q^{(3)}_i - \epsilon f\left(\hat{z}_i Q^{(2)}_j - \hat{z}_j Q^{(2)}_i\right)\right)^2 \\ & + \epsilon f'(\hat{z}_i \hat{r}_j - \hat{z}_j \hat{r}_i) Q^{(2)}_i Q^{(3)}_j, \end{aligned} \tag{82}$$

where $\hat{z}_i$, $\hat{r}_i$ are unit vectors in the z and r directions, and the contraction of spatial indices is with the Kronecker delta, e.g., $(\dot{Q}^{(3)}_i)^2 = \dot{Q}^{(3)}_i \dot{Q}^{(3)}_i$. Similar expressions are obtained at all orders in perturbations, and there is no explicit time dependence in any of them. If we expand the perturbations in modes in $\mathcal{L}^{(2)}_g$, the mode coefficients correspond to simple harmonic oscillators with time-independent frequencies, which implies that there is no particle production.

In the present analysis, since we include the scalar field Φ, there is an extra term in Q^a_μ coming from the covariant gradient term in the Lagrangian,

$$L_\Phi = |D_\mu \Phi|^2 - V(\Phi) \to \ldots + \frac{g^2}{4}|\Phi|^2 Q^a_\mu Q^{\mu a}, \tag{83}$$

where the … include terms that are zeroth and first order in Q^a_μ. Since $|\Phi|^2 = \eta^2 h^2$ is independent of time, the last term in (83) is simply a mass term for Q^a_μ with a mass that is independent of time. Once again, the quantum state of the modes of Q^a_μ will correspond to simple harmonic oscillators with time-independent frequencies. Thus the time dependence of the background gauge field does not lead to any Schwinger particle production of the gauge excitations.

VI. CONCLUSIONS

We have first constructed homogeneous electric field solutions in non-Abelian gauge theories with a scalar field that transforms in the fundamental representation. This construction paved the way for the construction of *electric string* solutions that are summarized in Eqs. (74), (75), (76), and (77). The solutions describe a flux tube of electric field, wrapped by azimuthal magnetic fields, followed by a sheath of electric field, which is again wrapped by azimuthal magnetic field, ad infinitum. The strength of the electric and magnetic fields falls off with distance as $1/\sqrt{r}$. The slow falloff implies a linear divergence in the energy per unit length of the electric string for $m^2 > 0$. In the case where $m^2 < 0$, the electric solution has $\Phi = 0$ asymptotically while the true vacuum has $\Phi \neq 0$. Hence, the electric string for $m^2 < 0$ is like a cylindrical bubble solution that contains gauge fields and $\Phi \neq 0$ that is immersed in a false vacuum region with $\Phi = 0$.

The electric string solution could have some relevance in QCD because quantum fluctuations about the electric string background might effectively provide the required sources for the electric field. However, to actually show this is a difficult task. Electric string solutions with vanishing hypercharge gauge field do not exist, but a more reasonable requirement might be to set the electromagnetic gauge field to vanish. Then the electric string would be composed of $W^\pm$ and Z gauge fields. Another interesting direction to explore is whether fermions can provide suitable sources for electric string solutions.

In Sec. V, we have shown that Schwinger gluon production is absent in the electric string background. This still leaves room for classical instabilities of the electric string solution, especially since unstable modes are known to exist in the *homogeneous* BW gauge field background in pure gauge theory [21]. With the Φ field included, the main difference is that there is now an interaction term $|\Phi|^2(W_i^a)^2$ in the energy functional. Since Φ is nonzero in the electric string solution, the gauge field excitations above the background are massive. This should suppress instabilities, but it is difficult to say if the suppression is sufficient to eliminate the instabilities. We plan to perform a classical stability analysis of the electric string solution in future work.

The electric string solution also contains azimuthal magnetic fields, and there is danger of an Ambjorn-Olesen (or W-condensation) instability [22]. The instability was found for a uniform non-Abelian magnetic field and can be understood in terms of the gauge particle's magnetic dipole coupling to the magnetic field. Since gauge particles are spin 1 particles, the coupling can be large and negative, enough to overcome any energy costs in producing the particles from the vacuum. The leads to an instability toward condensation of the gauge particles. In our case, the magnetic field is in the azimuthal direction (see Fig. 5) and is not homogeneous. It remains to be seen if the Ambjorn-Olesen instability applies to the magnetic field in the electric string solution.

There is a large body of work on the quantization of classical solutions [23], The procedure is to consider fluctuations around the background solution. In our case, the gauge field with fluctuations can be written as in Eq. (78) and similarly the scalar field is

$$\Phi = \Phi_0 + \hat{\Phi}, \tag{84}$$

where Φ_0 is the classical solution and $\hat{\Phi}$ represent fluctuations. Assuming weak coupling and that there are no classical instabilities, the fluctuations can be treated to lowest quadratic order in the action, and their eigenmodes are simple harmonic oscillators that can be quantized in the standard way. The backreaction of these quantum fluctuations on the classical background will be small. However, this straightforward quantization does not hold at strong coupling. In that case, the action cannot be truncated to quadratic order in the fluctuations and the backreaction may change the classical solution in a significant way. Then lattice methods seem to be the only recourse. It would be very interesting if strong coupling effects could control the asymptotic behavior of the electric string so as to give a finite string tension.

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APPENDIX A: HOMOGENEOUS ELECTRIC FIELD AND ADJOINT SCALAR FIELD

The $SU(2)$ gauge fields [see Eqs. (13) and (14)] will be written as

$$\vec{W}_\mu = -\frac{\epsilon}{g}(\cos(\Omega t), \sin(\Omega t), 0)\partial_\mu z, \tag{A1}$$

where the vector sign denotes a vector in internal space. From (19)–(21), the current is

$$\vec{j}_\mu = -\frac{\epsilon\Omega}{g}(\Omega\cos(\Omega t)\partial_\mu z, \Omega\sin(\Omega t)\partial_\mu z, \epsilon\partial_\mu t). \tag{A2}$$

The adjoint scalar will be denoted by $\vec{\phi}$. In terms of $\vec{\phi}$, the current is

$$\vec{j}_\mu = g\vec{\phi}\times D_\mu\vec{\phi}, \tag{A3}$$

where

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} + g\vec{W}_\mu \times \vec{\phi}. \tag{A4}$$

Therefore, given $\vec{j}_\mu$, $\vec{\phi}$ must satisfy the constraint

$$\vec{\phi} \cdot \vec{j}_\mu = 0 \tag{A5}$$

for every μ. Setting $\mu = 0$, we obtain the requirement $\phi^3 = 0$. And, setting $\mu = 3$ gives

$$\cos(\Omega t)\phi^1 + \sin(\Omega t)\phi^2 = 0. \tag{A6}$$

Therefore,

$$\vec{\phi} = \eta(-\sin(\Omega t), \cos(\Omega t), 0), \tag{A7}$$

where η is some unspecified vacuum expectation value of the scalar. Then

$$\partial_\mu \vec{\phi} = -\eta\Omega(\cos(\Omega t), \sin(\Omega t), 0)\partial_\mu t, \tag{A8}$$

and

$$\vec{\phi} \times \partial_\mu \vec{\phi} = \eta^2 \Omega(0,0,1)\partial_\mu t, \tag{A9}$$

$$\vec{\phi} \times (\vec{W}_\mu \times \vec{\phi}) = (\vec{\phi}\cdot\vec{\phi})\vec{W}_\mu - (\vec{\phi}\cdot\vec{W}_\mu)\vec{\phi} = \eta^2 \vec{W}_\mu, \tag{A10}$$

since $\vec{\phi}\cdot\vec{W}_\mu = 0$.

Equation (A3) gives

$$\begin{aligned}\vec{j}_\mu &= g\eta^2[\Omega \hat{e}_3 \partial_\mu t + g\vec{W}_\mu] \\ &= g\eta^2(-\epsilon\cos(\Omega t)\partial_\mu z, -\epsilon \sin(\Omega t)\partial_\mu z, \Omega \partial_\mu t)\end{aligned} \tag{A11}$$

Comparison of the $\mu = 3$ expressions with the desired currents (A2) gives

$$\Omega = g\eta, \tag{A12}$$

The trouble arises in matching the $\mu = 0$ expressions, for then,

$$\epsilon^2 = -g^2\eta^2, \tag{A13}$$

and a real solution does not exist.

We conclude that the adjoint scalar $\vec{\phi}$ cannot source the initial gauge field in (A1).

APPENDIX B: HOMOGENOUS ELECTRIC FIELD AND ELECTROWEAK MODEL

The electroweak model has the same ingredients as our model with an electric string solution, except that the $U(1)$ symmetry is gauged with gauge coupling g', and we have an extra gauge field, Y_μ, called the hypercharge gauge field. We will look for an electric string solution of the same form as in (12) and with $Y_\mu = 0$.

The W currents are unchanged, and we still need to satisfy (48)–(50). In addition, since $Y_\mu = 0$, the hypercharge current must also vanish

$$\partial_\nu Y^{\mu\nu} = j^Y_\mu = i\frac{g'}{2}(\Phi^\dagger D_\mu \Phi - \text{H.c.}) = 0 \tag{B1}$$

The form of Φ and W^a_μ is fixed and given in (74) and (75). Inserting these in (B1) gives

$$\begin{aligned}0 = &-\frac{g'\epsilon^2}{g^2}\Omega f^2(|z_1|^2 - |z_2|^2)\partial_\mu t \\ &-\frac{g'\epsilon^3}{2g^2}\frac{\Omega}{\omega} f^3[z_1 z_2^* e^{i(\Omega+2\omega)t} + \text{H.c.}]\partial_\mu z,\end{aligned} \tag{B2}$$

The $\mu = 0$ component requires $|z_1|^2 = |z_2|^2 = 1/2$, while the $\mu = z$ component requires

$$z_1 z_2^* e^{i(\Omega+2\omega)t} + \text{H.c.} = 0. \tag{B3}$$

This forces $\omega/\Omega = -1/2$, but this is in conflict with the requirement that $\omega/\Omega > 0$ discussed below (53). Hence, the electric string solution with $Y_\mu = 0$ does not exist in the electroweak model. This does not exclude the possibility of an electric string solution with $Y_\mu \neq 0$.

APPENDIX C: SOME HELPFUL FORMULAS

If we use the Hopf parametrization to write Φ,

$$\Phi = \eta\begin{pmatrix}\cos\alpha e^{i\beta} \\ \sin\alpha e^{i\gamma}\end{pmatrix}, \tag{C1}$$

where α, β, γ only depend on time. Then

$$\vec{n} \equiv \Phi^\dagger \vec{\sigma}\Phi = \eta^2(\sin(2\alpha)\cos\theta, \sin(2\alpha)\sin\theta, \cos(2\alpha)), \tag{C2}$$

where $\theta \equiv \gamma - \beta$. And,

$$\Phi^\dagger\sigma^1\dot{\Phi} - \text{H.c.} = i\eta^2[2\sin\theta\dot{\alpha} + \sin 2\alpha\cos\theta(\dot{\beta}+\dot{\gamma})], \tag{C3}$$

$$\Phi^\dagger\sigma^2\dot{\Phi} - \text{H.c.} = i\eta^2[-2\cos\theta\dot{\alpha} + \sin 2\alpha\sin\theta(\dot{\beta}+\dot{\gamma})], \tag{C4}$$

$$\Phi^\dagger\sigma^3\dot{\Phi} - \text{H.c.} = i\eta^2 2(\cos^2\alpha\dot{\beta} - \sin^2\alpha\dot{\gamma}), \tag{C5}$$

$$\Phi^\dagger\dot{\Phi} - \text{H.c.} = i\eta^2 2(\cos^2\alpha\dot{\beta} + \sin^2\alpha\dot{\gamma}). \tag{C6}$$

[1] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).
[2] S. G. Matinyan and G. K. Savvidy, Nucl. Phys. **B134**, 539 (1978).
[3] L. S. Brown and W. I. Weisberger, Nucl. Phys. **B157**, 285 (1979); **B172**, 544(E) (1980).
[4] A. Yildiz and P. H. Cox, Phys. Rev. D **21**, 1095 (1980).
[5] J. Ambjorn and R. J. Hughes, Nucl. Phys. **B197**, 113 (1982).
[6] J. Ambjorn and R. J. Hughes, Phys. Lett. **113B**, 305 (1982).
[7] G. C. Nayak and P. van Nieuwenhuizen, Phys. Rev. D **71**, 125001 (2005).
[8] F. Cooper and G. C. Nayak, Phys. Rev. D **73**, 065005 (2006).
[9] F. Cooper, J. F. Dawson, and B. Mihaila, in *Proceedings of the Conference on Nonequilibrium Phenomena in Cosmology and Particle Physics* (2008), arXiv:0806.1249.
[10] V. P. Nair and A. Yelnikov, Phys. Rev. D **82**, 125005 (2010).
[11] S. P. Kim and C. Schubert, Phys. Rev. D **84**, 125028 (2011).
[12] A. Ilderton, Phys. Rev. D **105**, 016021 (2022).
[13] A. Huet, S. P. Kim, and C. Schubert, Phys. Rev. D **90**, 125033 (2014).
[14] M. Ragsdale and D. Singleton, J. Phys. Conf. Ser. **883**, 012014 (2017).
[15] D. Karabali, S. Kurkcuoglu, and V. P. Nair, Phys. Rev. D **100**, 065006 (2019).
[16] C. Cardona and T. Vachaspati, Phys. Rev. D **104**, 045009 (2021).
[17] T. Vachaspati, Phys. Rev. D **105**, 105011 (2022).
[18] T. Vachaspati, Phys. Rev. D **107**, L031903 (2023).
[19] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 2000), ISBN 978-0-521-65476-0.
[20] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[21] J. Pereira and T. Vachaspati, Phys. Rev. D **106**, 096019 (2022).
[22] J. Ambjorn and P. Olesen, Phys. Lett. B **214**, 565 (1988).
[23] R. Rajaraman, *Solitons and Instantons. An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982).