Physical tuning and naturalness

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We present a radically new proposal for the solution of the naturalness/hierarchy problem, where the fine-tuning of the Higgs mass finds its physical explanation and the well-known multiplicative renormalization of the usual perturbative approach emerges as an IR property of the nonperturbative running of the mass.

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I. INTRODUCTION

The Standard Model (SM) of particle physics is an effective theory, i.e., a quantum field theory valid up to a certain scale $(M_P, M_{\rm GUT}, \ldots)$, above which it has to be replaced by its ultraviolet (UV) completion. This scale, which we generically indicate with Λ, is the *physical cutoff* of the theory: the SM effective Lagrangian $\mathcal{L}_{\rm SM}^{(\Lambda)}$ allows to describe processes at momenta $p \lesssim \Lambda$.

Because of unsuppressed quantum fluctuations, the square of the Higgs boson mass m_H^2 receives contributions proportional to Λ^2. In this respect, we stress that the result $m_H^2 \sim \Lambda^2$ from the physical point of view indicates a "quadratic sensitivity" of m_H^2 to the ultimate scale of the theory, rather than a "quadratic divergence." Moreover, this value of m_H^2 is nothing but the square of the running $m_H^2(\mu)$ at the scale $\mu = \Lambda$. If Λ is too large, $m_H^2(\Lambda)$ is "unnaturally" large, and this poses a problem of "hierarchy" with the Fermi scale μ_F, where $m_H(\mu_F) \sim 125$ GeV. Several attempts have been made to find the "solution" to this naturalness/hierarchy (NH) problem. Here we focus on three of them, as they will help to introduce our proposal.

(1) A popular approach is based on the assumption that the UV completion of the SM could provide the condition

$$m_H^2(\Lambda) \ll \Lambda^2 \tag{1}$$

at the scale Λ [1–3]. Sometimes (1) is viewed as a "quantum gravity miracle" [1], which could result from a conspiracy among the SM couplings at the scale Λ. This is for instance the case of the so-called Veltman condition[1] [4]. In such a scenario: (i) the naturalness problem is solved from physics "outside" the SM realm, since (1) is considered as a leftover of its UV completion (or extensions of it) and (ii) the hierarchy problem is solved "inside" the SM, as for the running Higgs mass $m_H^2(\mu)$ the perturbative renormalization group (RG) equation ($\gamma \ll 1$ is the perturbative anomalous mass dimension)

$$\mu \frac{d}{d\mu} m_H^2(\mu) = \gamma m_H^2(\mu) \tag{2}$$

is considered. In fact, from (1) and (2) it turns out that $m_H^2(\mu_F)$ and $m_H^2(\Lambda)$ are of the same order. Therefore, the combined use of these two equations leads one to conclude there is no problem of hierarchy.

(2) A somehow complementary approach consists in considering again Eq. (2) for $m_H^2(\mu)$, but assuming this time that gravity could provide a nonperturbative value for γ (~ 2) [5–12]. In this case, the large hierarchy between the Fermi scale μ_F and the UV scale Λ could be easily accommodated, and again no NH problem would arise.

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[1]If not directly to the SM itself, this condition can be applied to some of its extensions [3].

(3) Finally, some authors suggest that dimensional regularization (DR) could be endowed with special physical properties that make it the correct "physical" way to calculate the radiative corrections in quantum field theory (QFT). If no new heavy particles are coupled to the Higgs boson, once again the NH problem would seem to be absent from the beginning [13–36].

However, as shown in a recent paper [37], none of these approaches can really provide a solution to the problem. The reason is that any effective field theory (EFT), including the SM and extensions of it, is *necessarily* defined and interpreted in a Wilsonian framework. The meaning of this statement is twofold: (i) the parameters (masses and couplings) $g_i(\Lambda)$ in the effective Lagrangian $\mathcal{L}_{\rm SM}^{(\Lambda)}$ result from integrating out the higher energy modes $k > \Lambda$ related to the UV completion of the SM and (ii) the same parameters $g_i(\mu)$ at a lower scale $\mu < \Lambda$ result from integrating out the modes of the fields that appear in $\mathcal{L}_{\rm SM}^{(\Lambda)}$ in the range $[\mu, \Lambda]$.

It is shown in [37] that DR provides a specific implementation of the Wilsonian strategy, where the fine-tuning is *automatically* encoded in the calculations, although in a *hidden* manner. As a consequence, DR cannot provide a solution to the problem. Moreover, it is shown that the RG equation (2) is obtained when the "critical value" of the scalar (Higgs) mass $m^2_{\rm cr}(\mu)$ is subtracted to the Wilsonian running mass $m^2(\mu)$. In other words, $m^2_H(\mu)$ in (2) is not the Wilsonian mass $m^2(\mu)$ but rather the difference $m^2_H(\mu) \equiv m^2(\mu) - m^2_{\rm cr}(\mu)$. Equation (2) then incorporates the fine-tuning, and cannot be invoked to solve the NH problem.

In this work we make a radically different proposal, rooted in simple and (in our opinion) indisputable "facts": (i) the SM is an EFT valid up to an ultimate UV scale Λ; (ii) the Wilsonian integration of modes is the only *physically consistent* way of including the quantum fluctuations in an EFT.

II. RUNNING SCALAR MASS

To introduce our proposal, we begin by considering the Wilsonian RG equations for the scalar ϕ^4 theory, whose (Euclidean) Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2_\Lambda\phi^2 + \frac{\lambda_\Lambda}{4!}\phi^4, \tag{3}$$

where $m^2_\Lambda \equiv m^2(\Lambda)$ and $\lambda_\Lambda \equiv \lambda(\Lambda)$ are the mass and coupling constant at the (physical) scale Λ. By considering the corresponding Wilsonian action within the so-called "local potential approximation," $S_k[\phi] = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + U_k(\phi))$, and truncating the potential to the first two terms, $U_k(\phi) = \frac{1}{2}m^2_k\phi^2 + \frac{1}{4!}\lambda_k\phi^4$, the RG equations for m^2_k and λ_k are (see Refs. [37,38])

$$k\frac{dm^2_k}{dk} = -\frac{k^4}{16\pi^2}\frac{\lambda_k}{k^2 + m^2_k}, \tag{4}$$

$$k\frac{d\lambda_k}{dk} = \frac{k^4}{16\pi^2}\frac{3\lambda^2_k}{(k^2 + m^2_k)^2}. \tag{5}$$

When the UV boundaries for (4) and (5) are such that the condition $m^2_k \ll k^2$ is satisfied in the whole range of integration, this system is well approximated by

$$k\frac{dm^2_k}{dk} = -\frac{\lambda_k}{16\pi^2}k^2 + \frac{\lambda_k}{16\pi^2}m^2_k \tag{6}$$

$$k\frac{d\lambda_k}{dk} = \frac{3\lambda^2_k}{16\pi^2}. \tag{7}$$

Taking for instance "SM-like" IR boundaries, $m(\mu_F) = 125.7$ GeV and $\lambda(\mu_F) = 0.1272$, and solving both systems numerically, we find that the solutions to (4) and (5), and (6) and (7), coincide with great accuracy.

The flow equations (6) and (7) can be solved analytically. The solution to (7) is the well-known one-loop-improved running quartic coupling

$$\lambda(\mu) = \frac{\lambda_\Lambda}{1 - \frac{3}{16\pi^2}\lambda_\Lambda \log(\frac{\mu}{\Lambda})}, \tag{8}$$

while the exact solution of (6) is (for notational simplicity from now on we replace $\lambda_\Lambda \to \lambda$)

$$\begin{aligned} m^2(\mu) = {} & \frac{1}{3\cdot 2^{2/3}(3\lambda\log\frac{\mu}{\Lambda} - 16\pi^2)}\Bigg(2^{2/3}\Lambda^2 e^{\frac{32\pi^2}{3\lambda}} \\ & \times\left(16\pi^2 - 3\lambda\log\frac{\mu}{\Lambda}\right)E_{\frac{2}{3}}\left(\frac{32\pi^2}{3\lambda} - 2\log\frac{\mu}{\Lambda}\right) \\ & + 4\lambda\sqrt[3]{-\frac{1}{\lambda}}\left(\Lambda^2 e^{\frac{32\pi^2}{3\lambda}}E_{\frac{2}{3}}\left(\frac{32\pi^2}{3\lambda}\right) + 3m^2_\Lambda\right) \\ & \times\left(3\pi\log\frac{\mu}{\Lambda} - \frac{16\pi^3}{\lambda}\right)^{2/3}\Bigg), \end{aligned} \tag{9}$$

where $E_{\frac{2}{3}}(x)$ is the generalized exponential integral function $E_p(x)$ for $p = \frac{2}{3}$. To get closer to the notation typically used in phenomenological applications, we indicate the running scale with μ rather than with k. Moreover, according to notational convenience, we will equivalently write the running mass as $m^2(\mu)$ or m^2_μ.

Let us focus now on the *nonperturbative* evolution equation (9) that we have just found. First of all we note that, expanding it for $\lambda \ll 1$ (and $\mu^2 \ll \Lambda^2$), we obtain the well-known perturbative result

$$m^2_\mu = m^2_\Lambda + \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2_\Lambda\log\frac{\Lambda^2}{\mu^2}\right). \tag{10}$$

More important for our scopes, however, is to note that the flow equation (9) has a very interesting *nonperturbative* approximation, which can be obtained replacing in the right-hand side of (6) λ_k with λ. Solving the resulting equation for the running mass m_μ^2 we have

$$m_\mu^2 = \left(\frac{\mu}{\Lambda}\right)^{\frac{\lambda}{16\pi^2}}\left(m_\Lambda^2 + \frac{\lambda\Lambda^2}{32\pi^2 - \lambda}\right) - \frac{\lambda\mu^2}{32\pi^2 - \lambda}. \tag{11}$$

This solution can be improved if in $(\frac{\mu}{\Lambda})^{\frac{\lambda}{16\pi^2}}$ and in $\frac{\lambda\mu^2}{32\pi^2-\lambda}$ the (UV value of the) coupling λ is replaced with the running $\lambda(\mu)$ in (8). Note that the coefficient of $(\frac{\mu}{\Lambda})^{\frac{\lambda}{16\pi^2}}$ in the round bracket is a UV boundary value (the integration constant), and this is why we should not make the replacement $\lambda \to \lambda(\mu)$ in it. Using for instance the same boundary values considered before [see below (7)], we easily check that this improved version of (11) provides a very good approximation to the flow governed by (4).

Equation (11) is a crucial result of the present work (below it will be extended to the SM) and contains several important lessons. First it shows how the fine-tuning usually realized in perturbative QFT operates in the Wilsonian framework: It simply fixes the boundary at the UV scale $\mu = \Lambda$ for the running of the mass m_μ^2. This is given by the terms in the round brackets in the right-hand side of (11), where m_Λ^2 and $\frac{\lambda\Lambda^2}{32\pi^2-\lambda}$ need to be *enormously fine-tuned* if at the IR scale $\mu_{\rm low}$ we want $m_{\mu_{\rm low}} \sim \mathcal{O}(100)$ GeV.

There is another important lesson in (11). By simple inspection, we see that the combination

$$m_{\mu,r}^2 \equiv m_\mu^2 + \frac{\lambda\mu^2}{32\pi^2 - \lambda} \tag{12}$$

obeys the RG equation

$$\mu\frac{d}{d\mu}m_{\mu,r}^2 = \gamma m_{\mu,r}^2, \tag{13}$$

where $\gamma = \frac{\lambda}{16\pi^2}$ is the mass anomalous dimension for the ϕ^4 theory at one-loop order. Equation (13) coincides with the well-known one-loop improved flow equation for the renormalized running mass. Therefore, $m_r^2(\mu)$ defined in (12) *is* the renormalized running mass. At the same time we note that

$$m_{\mu,\rm cr}^2 \equiv -\frac{\lambda\mu^2}{32\pi^2 - \lambda} \tag{14}$$

is the "critical mass" defined at each value of the running scale μ, and that the subtraction in (13) drives the RG flow close to the critical surface of the Gaussian fixed point. The simple integration of (13) gives ($\mu_0 > \mu$)

$$m_{\mu,r}^2 = \left(\frac{\mu}{\mu_0}\right)^{\frac{\lambda}{16\pi^2}} m_{\mu_0,r}^2. \tag{15}$$

For the purposes of our analysis, it is important to note that we derived Eq. (13) in the Wilsonian framework, namely from the RG flow (11), whereas usually it is derived in the context of "technical schemes," as dimensional, heat kernel, or zeta function regularization. In this respect, we stress that, when the quantum fluctuations are calculated in the framework of a technical scheme, we only have access to (13) [and then to its solution (15)], but we are blind to the fact that the renormalized running mass is obtained only after operating at each scale μ the subtraction in (12). When, on the contrary, the quantum fluctuations are calculated within the Wilsonian "physical scheme," we clearly see how the renormalized mass emerges.

There is a third important lesson contained in (11), which is related to the following question. Should we identify the physical running mass with (11) or with (15), i.e. with the original Wilsonian mass m_μ^2, or with the subtracted mass $m_{\mu,r}^2$? In QFT the running mass is usually identified with (15). On the other hand, according to the definition of Wilsonian action, the running couplings $g_i(\mu)$ at the scale μ result from the integration over the quantum fluctuations in the range $[\mu, \Lambda]$, and are the effective couplings at this scale. This is true, in particular, for the mass. Therefore, it is the original Wilsonian mass m_μ^2, not the subtracted $m_{\mu,r}^2$ that has to be identified with the physical mass at the scale μ.

Being this the case, how can we justify the traditional (textbook) approach to QFT, where it is $m_{\mu,r}^2$ that is identified with the physical running mass at the scale μ?

The answer to this question comes from the comparison between our result (11) and (the textbook) Eq. (15). As long as we confine ourselves to sufficiently low values of μ (IR regime), the flow governed by (15) practically coincides with the flow (11). The overlap region is defined by the condition

$$\frac{\lambda\mu^2}{32\pi^2 - \lambda} \ll \left(\frac{\mu}{\Lambda}\right)^{\frac{\lambda}{16\pi^2}}\left(m_\Lambda^2 + \frac{\lambda\Lambda^2}{32\pi^2 - \lambda}\right), \tag{16}$$

where (we stress again) the term inside the parentheses contains the fine-tuning necessary to obtain the IR (measured) value of the physical mass. Moreover, our equation (11) allows to find the energy range to which Eq. (15) is limited. Clearly, if we are interested in energy scales μ above the region determined by (16), we must go back to the original flow (11), which has a much wider range of validity.

III. THE PROPOSAL

We are now ready to move to the SM (similar considerations hold even for an extended version of the SM) and to present our proposal. Following steps similar to those that led to (6), for the running Higgs mass we get the Wilsonian RG equation

$$\mu \frac{d}{d\mu} m_H^2 = \alpha(\mu)\mu^2 + \gamma(\mu) m_H^2, \tag{17}$$

where $\alpha(\mu)$ is a combination of SM running couplings (gauge, Yukawa, scalar), which at one-loop level reads (for the purposes of the present work it is sufficient to restrict ourselves to the one-loop order)

$$16\pi^2 \alpha(\mu) = 12 y_t^2 - 12\lambda - \frac{3}{2} g_1^2 - \frac{9}{2} g_2^2, \tag{18}$$

and $\gamma(\mu)$ is the mass anomalous dimension

$$16\pi^2 \gamma(\mu) = 6 y_t^2 + 12\lambda - \frac{3}{2} g_1^2 - \frac{9}{2} g_2^2. \tag{19}$$

Considering constant values for the couplings, from (17) we obtain

$$m_H^2(\mu) = \left(\frac{\mu}{\Lambda}\right)^\gamma \left(m_H^2(\Lambda) - \frac{\alpha\Lambda^2}{2-\gamma}\right) + \frac{\alpha\mu^2}{2-\gamma}, \tag{20}$$

which, as it is easy to check, provides a very good approximation to the flow governed by (17). As for the scalar theory previously considered [see Eq. (11) and comments below it], an improvement to (20) is obtained if α and γ outside the round brackets are replaced with $\alpha(\mu)$ and $\gamma(\mu)$ [the term in the round brackets is an integration constant, where $\alpha = \alpha(\Lambda)$ and $\gamma = \gamma(\Lambda)$]. In Fig. 1 both the numerical solution to (17) and the approximate analytical solution (20) (with the just mentioned improvement) are plotted. They are practically indistinguishable.

Equation (20) is one of the most important results of the present work, and deserves several comments. Before doing that, however, it is worth deriving a few other related results. Let us define [as in (14)] the critical mass,

$$m_{H,\mathrm{cr}}^2(\mu) \equiv \frac{\alpha\mu^2}{2-\gamma}, \tag{21}$$

and [as in (12)] the subtracted mass

$$m_{H,r}^2(\mu) \equiv m_H^2(\mu) - m_{H,\mathrm{cr}}^2(\mu). \tag{22}$$

From (20) we derive the equation

$$\mu \frac{d}{d\mu} m_{H,r}^2(\mu) = \gamma m_{H,r}^2(\mu), \tag{23}$$

which once solved gives ($\mu_0 > \mu$)

$$m_{H,r}^2(\mu) = \left(\frac{\mu}{\mu_0}\right)^\gamma m_{H,r}^2(\mu_0). \tag{24}$$

Equation (23) coincides with the well-known (textbook) one-loop improved RG equation for the renormalized running Higgs mass, and is nothing but Eq. (2). We then conclude that $m_{H,r}^2(\mu)$ defined in (22) *is* the usual renormalized running Higgs mass. However, we observe that (as explained in the section devoted to the scalar theory) it is the Wilsonian mass parameter $m_H^2(\mu)$ in (20) that has to be identified with the running Higgs mass. In this respect, let us consider the two following points.

(i) Requiring that $m_H^2(\mu)$ at the Fermi scale μ_F is the measured $m_{H,\mathrm{exp}}^2 \sim (125.7)^2$ GeV^2, from (20) we see that $m_H^2(\Lambda)$ needs to be enormously fine-tuned.

(ii) Turning to the RG flow (24) for $m_{H,r}^2(\mu)$, and requiring this time that it is $m_{H,r}^2(\mu_F)$ that takes the experimental value $\sim (125.7)^2$ GeV^2, we see that the two flows $m_H^2(\mu)$ and $m_{H,r}^2(\mu)$ *coincide* for all the values of μ that satisfy the condition

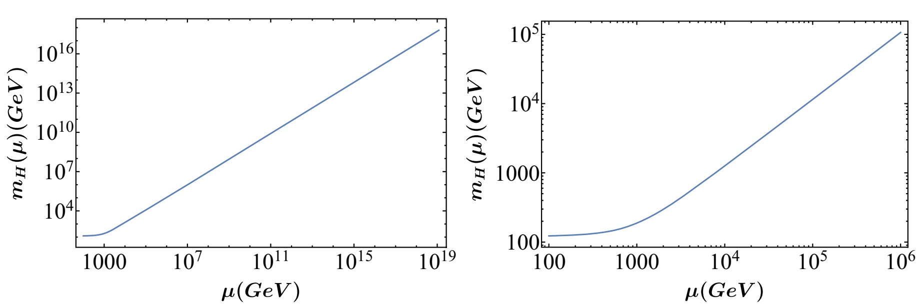

FIG. 1. Left panel: log-log plot of $m_H(\mu)$, with UV boundary $m_H(M_P) \sim 6.347 \times 10^{17}$ GeV, see Eq. (17). The latter is coupled to the RG equations for the SM couplings, λ, y_t, g_1, g_2 and g_3, solved numerically using one-loop beta functions, and IR (experimental) boundary values: $\lambda(m_t) = 0.1272$, $y_t(m_t) = 0.9369$, $g_1(m_t) = 0.3587$, $g_2(m_t) = 0.6483$, $g_3(m_t) = 1.1671$ (m_t is the top quark mass). Equation (20) [analytical approximation to the solution of (17)] is also plotted, but the two curves are indistinguishable. Right panel: Zoom in the region 10^2–10^6 GeV of the running shown in the left panel. The "elbow" around $\mu \sim 10^3$ GeV signals that the IR flow is entering the region where $m_H(\mu)$ is very well approximated by $m_{H,r}(\mu)$ [Eq. (24)].

$$\frac{\alpha\mu^2}{2-\gamma} \ll \left(\frac{\mu}{\Lambda}\right)^{\gamma}\left(m_H^2(\Lambda) - \frac{\alpha\Lambda^2}{2-\gamma}\right). \qquad (25)$$

These results contain crucial physical lessons. First of all, we learn that the "fine-tuning" of $m_H^2(\Lambda)$, which in the traditional approach to QFT is *formally* realized through the introduction of counterterms, has a profound *physical* meaning. It provides the boundary for the RG flow of the running mass $m_H^2(\mu)$ at the UV scale Λ. A very large value of m_H^2 at Λ is then *physically necessary* and welcome, not an unwanted result to get rid of.

Moreover, from Eq. (20) and from Fig. 1, we see that through a *quadratic running* that lasts for most of the $m_H^2(\mu)$ flow towards the IR, this finely tuned value of $m_H^2(\Lambda)$ allows to reach the experimental value of the Higgs mass at the Fermi scale. What is crucial to realize is that, proceeding towards the IR, the initial "quadratic running" $m_H^2(\mu) \sim \mu^2$ sooner or later gives way to a lower energy running, where the "multiplicative renormalization" [see Eq. (24)] emerges. In schemes as DR we only have access to (24), but from a truly physical perspective the latter is an "emergent property" of the running, which rises when the flow approaches the IR. The region, say the scale $\bar{\mu}$, where the transition from the quadratic to the multiplicative running occurs, and the value of the squared mass $m_H^2(\bar{\mu})$, are determined by the fine-tuning, i.e. by the quantity inside the parentheses in (20) [(11) for the ϕ^4 theory]. For $\mu \lesssim \bar{\mu}$, the running is extremely slow, and we then understand the profound role of the boundary condition $m_H^2(\Lambda)$, and thus of the fine-tuning: it determines the scale $\bar{\mu}$ below which the running appears approximately frozen and, up to small corrections, the value $m_H^2(\mu)$ that the mass necessarily assumes for all scales[2] $\mu \lesssim \bar{\mu}$.

This is a great change in the usual paradigm. In the physically unavoidable top-down Wilsonian approach, a large hierarchy between the UV and the IR values of m_H^2, together with the fine-tuning of $m_H^2(\Lambda)$, are *physically mandatory* and the typical multiplicative renormalization (24) emerges as an IR property of the complete physical running (20). It is also worth observing that, while in traditional approaches (Veltman condition, supersymmetric models, Higgs composite models) to the naturalness problem the central idea is to avoid/cancel terms quadratically sensitive to the scale, in our Wilsonian approach their presence is necessary to drive the mass towards the IR measured value. Moreover, we stress that this physical picture holds not only for the SM but also for beyond Standard Model (BSM) models as long as γ stays perturbative. The change in the parameter α only determines a change in the boundary condition $m_H^2(\Lambda)$.

Moreover, Eq. (20) shows the limitations of (23), or equivalently of its solution (24): they can be used only at sufficiently low energies [where (25) is satisfied], which in most of the cases are the only experimentally reachable energies. In the right panel of Fig. 1, a zoom of the $m_H(\mu)$ running is shown. The presence of an elbow near $\mu \sim 10^3$ GeV and the almost "freezing" of the $m_H(\mu)$ flow at lower scales signal that the first term in the right-hand side of (20) takes over the second one. This realizes the "transition" from the additive to the multiplicative renormalization of the mass, which is the transition from (20) to (24). If one were to extend (24) outside its realm of validity, the experimental value $m_H = 125.7$ GeV at the Fermi scale would be reached starting with the UV boundary $m_H(M_P) \sim 132.4$ GeV. This is connected with one of the popular (but, as shown above, incorrect) approaches to the NH problem [see Eq. (1) and the related discussion].

The fine-tuning manifests itself through the term $(m_H^2(\Lambda) - \alpha/(2-\gamma)\Lambda^2)$ in (20), and the choice of this combination in the UV determines the measured (IR) value of m_H^2. Therefore, taking into account the experimental uncertainties, we conclude that there exists a region of "tiny size" in the SM parameter space from which very large UV boundary values of m_H^2 give rise, through the RG flow, to the measured (within errors) value of the Higgs mass. Such a region can only be inherited from the ultimate UV completion of the SM (or of the yet unknown BSM), namely the theory of everything. In string theory, for instance, an enormous variety of theories/vacua has to be considered, and the conditions for the existence of such a region are certainly met. Moreover, in connection with the quadratic dependence of the Higgs mass in the UV, we note that in the string framework the common expectation is that the Higgs mass at the string scale M_S is $m_H^2(M_S) \sim M_S^2$ [39].

IV. IR RUNNING AND PHYSICAL MASS

Going back to the running (24) (multiplicative renormalization), we observe that it can be obtained within different schemes (DR, heat kernel, …), and that no physical content can ever be related to the choice of a specific scheme. However, our analysis has shown that this behavior is related *only* to the IR sector of the flow. The whole UV $\to$ IR running is given by the Wilsonian flow (20), while (24) is confined to the IR regime alone.

In light of these findings, an interesting question arises that might be subject to experimental investigation in the (hopefully not too far) future. Although no one has observed up to now the running of the Higgs mass, we can

[2]Note that at very low scales the expansion of the kind (6) no longer holds (in the present case this happens for scales much lower than m_t). The full Wilsonian equations predict a complete freezing of the running of $m_H^2(\mu)$, so that the fact that in our approximation the solutions (20) and (11) seem to flow towards zero in a small region of μ very close to $\mu = 0$ should not be taken as a physical prediction: when we refer to freezing we mean that this really extends down to $\mu \to 0$.

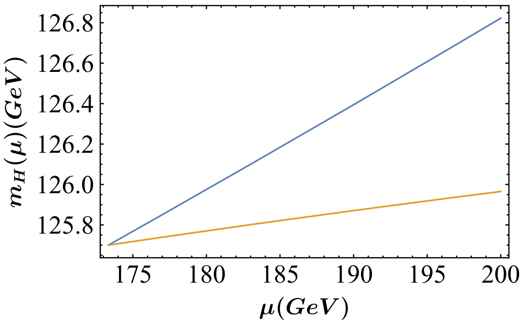

FIG. 2. This figure shows a focus of the $m_H(\mu)$ flow of Fig. 1, Eq. (20), in the IR region between m_t [where $m_H(m_t) \sim 125.7$ GeV] and 200 GeV (blue line). The yellow curve is the flow given by (24) again with $m_H(m_t) \sim 125.7$ GeV. Future experiments should allow to evidentiate the difference between these two flows.

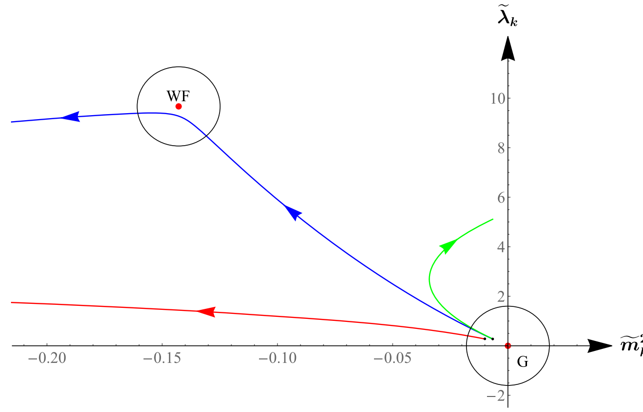

FIG. 3. RG flows [Eqs. (26) and (27)] in the parameter space $(\tilde{m}_k^2, \tilde{\lambda}_k)$ of a ϕ^4 theory in $d = 3$ dimensions. The blue and red flows emanate from the UV region close to the Gaussian fixed point G (different boundary values). The green line is obtained linearizing (26) and (27) around G, with the same boundary as the blue one.

consider physical processes that should allow to test the $m_H^2(\mu)$ flow (much in the same spirit of what is done with the running bottom quark mass [40]): think, for instance, of ongoing work on precision measurements of the trilinear coupling [41]. If future experiments will be able to enter the energy regime where the complete flow (20) and the approximate IR flow (24) start to be significantly different, and experimentally distinguishable, it should become possible to discriminate between these two alternatives (see Fig. 2).

In this respect, we observe that the connection between quantum field theory and statistical physics is usually done by establishing a one-to-one correspondence between the request $\xi \gg a$ in the theory of critical phenomena (a is the lattice spacing, ξ the correlation length) and the request $m^2 \ll \Lambda^2$ in the QFT framework (m is the particle mass, Λ the ultimate UV scale of the theory). When phrased in RG language, this corresponds to the tuning towards the "critical surface," achieved through the subtraction of the critical mass: $m^2_{\rm ren}(\mu) = m^2(\mu) - m^2_{\rm cr}(\mu)$. However, we have seen that $m^2_{\rm ren}(\mu)$ only captures the *final part* of the running of the physical mass. Actually, the RG flow is physically meaningful even far from the critical surface and from fixed points. This is in fact what happens to our flow (20) that approaches the critical surface of the Gaussian fixed point in the IR, giving eventually rise to the flow (24).

These points can be well illustrated if we go for a moment to $d = 3$ dimensions, and consider the Ginzburg-Landau free energy (used to describe the ferromagnetic transition), $F[\phi] = \int d^3x(\frac{1}{2}(\vec{\nabla}\phi)^2 + V_k(\phi))$, with potential $V_k(\phi) = \frac{1}{2}m_k^2\phi^2 + \frac{\lambda_k}{4!}\phi^4$. The RG equations for the dimensionless couplings $\tilde{m}_k^2 \equiv k^{-2}m_k^2$ and $\tilde{\lambda}_k \equiv k^{-1}\lambda_k$ ($t \equiv \log k/k_0$, with k_0 a reference scale) are

$$\frac{d\tilde{m}_k^2}{dt} = -2\tilde{m}_k^2 - \frac{\tilde{\lambda}_k}{4\pi^2(1+\tilde{m}_k^2)} \tag{26}$$

$$\frac{d\tilde{\lambda}_k}{dt} = -\tilde{\lambda}_k + \frac{3\tilde{\lambda}_k^2}{4\pi^2(1+\tilde{m}_k^2)^2}. \tag{27}$$

It is immediate to see that these equations have a Gaussian and a Wilson-Fisher fixed point, G and WF in Fig. 3, and that G is an IR repulsive fixed point.

Figure 3 conveys two messages: (a) Let us consider the UV → IR flow given by the blue line. In the region around G, where (26) and (27) can be linearized, this flow is well approximated by the "subtracted flow" (green line), the analog of (24) in this case. Beyond this region, however, the green flow deviates from the true physical flow (blue line). The very existence of the ferromagnetic transition shows that the green flow *cannot* be the true one. (b) The blue and red flows have slightly different UV boundaries. Thanks to the fine-tuning operated in the UV, the blue flow is driven towards WF (i.e. towards the ferromagnetic transition). This example clearly shows that, if (as it is certainly the case) the IR physics is dictated by WF, the fine-tuning in the UV is *physical* and *unavoidable*.

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