Study of the ultraviolet behavior of an O(N) $|\vec{\phi}|^6$ theory in $d=3$ dimensions

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We study the ultraviolet (UV) behavior of an O(N) $|\vec{\phi}|^6$ theory in $d=3$ spacetime dimensions, focusing on the question of the range in N over which the perturbative beta function exhibits robust evidence of a UV zero in the $|\vec{\phi}|^6$ coupling, g. The four-loop (4ℓ) beta function is known to have a (scheme-independent) UV zero at $g=g_{\mathrm{UV},4\ell}$, which is reliably calculable for large N. For our analysis we use the six-loop beta function calculated in the minimal subtraction scheme. We find that this six-loop beta function has a UV zero, $g_{\mathrm{UV},6\ell}$, if $N>N_c$, where $N_c\simeq 796$, and we calculate $g_{\mathrm{UV},6\ell}$. To investigate the reliability of the result in the region of $N\gtrsim N_c$, we apply three methods: (i) calculation of the fractional difference between $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},6\ell}$, (ii) a Padé approximant, and (iii) an assessment of scheme dependence. Our results provide quantitative measures of the range of N over which the six-loop beta function has a UV zero and of the $1/N$ corrections to the value of g at the UV zero for large but finite N. If one imposes a benchmark requirement that the fractional difference between $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},6\ell}$ must be less than 15%, then our results show that this requirement is satisfied for $N\gtrsim 2\times 10^3$. The possible role of nonperturbative effects is also noted.

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I. INTRODUCTION

In this paper we study the ultraviolet (UV) behavior of an O(N) $|\vec{\phi}|^6$ quantum field theory in $d=3$ spacetime dimensions. This theory, commonly denoted $|\vec{\phi}|^6_3$, involves an N-component real scalar field $\vec{\phi}=(\phi_1,\ldots,\phi_N)^T$ and is defined by the path integral $Z=\int\prod_x[d\phi_i(x)]e^{iS}$ with $S=\int d^3x\mathcal{L}$, and the bare Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial_\nu\vec{\phi})\cdot(\partial^\nu\vec{\phi})-\frac{1}{2}m^2|\vec{\phi}|^2-\frac{\lambda}{4N}|\vec{\phi}|^4-\frac{g}{6N^2}|\vec{\phi}|^6, \qquad (1.1)$$

where $|\vec{\phi}|=(\sum_{i=1}^N\phi_i^2)^{1/2}$. In $d=3-\epsilon$ (Euclidean) dimensions, the O(N) $|\vec{\phi}|^6$ theory has been extensively analyzed to obtain expressions for critical exponents describing tricritical points in condensed matter physics [1–4]. Early studies of the theory as a relativistic quantum field theory in $d=3$ spacetime dimensions include [5–10].

Because of quantum corrections, the physical coupling $g=g(\mu)$ depends on the Euclidean energy/momentum scale, μ, where it is measured. This dependence is described by the renormalization group (RG) beta function [11–13] of the theory, $\beta_g=dg/d\ln\mu$. The lowest-order [two-loop, $O(g^2)$] term in β_g is positive [1], so this theory is infrared (IR)-free, i.e., $g(\mu)\to 0$ as $\mu\to 0$. An important question is whether, for a given N, the theory has a UV zero in β_g at some value g_{UV}. If this is the case and if a perturbative analysis is adequate to describe the physics, then, as the reference energy/momentum scale μ increases from 0 to ∞, $g(\mu)$ increases from 0 and approaches g_{UV} from below. Since the coefficients of the quadratic and quartic terms in the Lagrangian (1.1) are both dimensionful, and since $\lim_{\mu\to\infty}m^2/\mu^2=0$ and $\lim_{\mu\to\infty}\lambda/\mu=0$, they are expected to play a negligible role in the ultraviolet limit $\mu\to\infty$. We denote the UV zero (presuming that it exists) of the n-loop ($n\ell$) beta function as $g_{\mathrm{UV},n\ell}$. The term of order g^p in β_g arises from graphs with a maximum number of loops n given by $n=2(p-1)$. The $O(g^3)$ term in the beta function is negative, so that at this order, this four-loop (4ℓ) beta function, $\beta_{g,4\ell}$, has a UV zero [6,7,9]. In the large-N limit, with the normalization in Eq. (1.1), this occurs at the value of the coupling $g_{\mathrm{UV},4\ell}=192$. It was noted in [6,7] that the N-dependence of higher-loop terms in β_g is such that for large N, the inclusion of these higher-loop terms would produce only a small fractional shift $\propto 1/N$ in the value of the coupling at the UV zero, and therefore the calculation of the value should be reliable in the large-N limit. Such a UV zero in the beta function is a UV fixed point (UVFP) of the renormalization group. The existence of a UVFP in an IR-free theory is of considerable interest, since it means that one has perturbative control of the theory in both the IR and

UV limits. A previous example of an IR-free theory with a UVFP is the nonlinear $O(N)$ σ model in $d = 2 + \epsilon$ dimensions [14–17]. The early studies [6,7] also cautioned that at small and moderate N, this formal UV zero at $g = g_{\mathrm{UV},4\ell}$ might be an artifact of the perturbative calculation.

We briefly review some further relevant work on this theory. After the studies in [6–9], a variational calculation was carried out in the $N \to \infty$ limit by Bardeen, Moshe, and Bander (BMB) in [10], who found that for $g > g_{cr}$, where $g_{cr} = (4\pi)^2 \simeq 158$, the theory undergoes a transition to a strongly coupled phase involving dynamical mass generation for the scalar field and spontaneous breaking of scale invariance, with the resultant appearance of a massless Nambu-Goldstone boson (NGB), namely a dilaton [10]. Since $g_{cr} < g_{\mathrm{UV},4\ell}$, it was concluded in [10] that in the large-N limit where the BMB calculations were performed, the physics is described by the properties of this strong-coupling phase rather than by a UV zero in the (perturbative) beta function. The properties in the $N \to \infty$ limit were further studied in [18–20]. Exploratory lattice studies to probe the BMB phase were performed in [21,22]. References [23,24] argued that at finite N, the BMB phase is unstable. More recently, Ref. [25] investigated the effect of higher-order corrections in $1/N$ on the BMB dilaton and found that it becomes a tachyon when one takes account of these $1/N$ corrections. On this basis, the authors of Ref. [25] concluded that at finite N, the BMB phase with spontaneously broken approximate scale invariance is unstable. Some recent related studies of this theory include [26–30].

In parallel with these continuing studies of the role of possible nonperturbative effects at finite N in the $|\vec{\phi}|^6_3$ theory, it seems worthwhile to investigate the UV properties of the (perturbative) beta function further. At the time of the early studies in [6–10], β_g had been calculated only up to $O(g^3)$. Subsequently, it was calculated to $O(g^4)$ in [4], and this remains the highest order to which it has been computed. A very basic question that, to our knowledge, has not been studied yet is whether, for a given N, this $O(g^4)$ beta function exhibits evidence for a (reliably calculable) UV zero, denoted $g_{\mathrm{UV},6\ell}$. We address this question in this paper.

A necessary condition for such evidence is that the values of the coupling at this UV zero obtained from calculations of the beta function to successive orders in g should be close to each other. To determine the region in N over which this condition is satisfied, we will compare the values of $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},6\ell}$ (for N values where the latter exists). Furthermore, the terms in the beta function at order g^p with $p \geq 4$ depend on the scheme used for regularization and renormalization. By itself, this property does not render these higher-order terms unphysical; for example, higher-order calculations of quark and gluon scattering in quantum chromodynamics (QCD) are also scheme-dependent but still play a crucial role in the analysis of experimental data, and work continues on the construction and application of optimal schemes for QCD calculations (see, e.g., [31] and references therein). However, this does mean that one must assess the effect of this scheme dependence, and we shall do this as part of our study. In carrying out this study, it should be stressed that nonperturbative effects may be important, and we refer the reader to the continuing analysis of this topic, e.g., in [25,30], as well as earlier works including [10]. However, bearing this caveat in mind, one should at least elucidate the predictions from the beta function calculated to the highest order to which it is known for general N, and that is the purpose of our present study.

In passing, it should be mentioned that theories of ϕ^6_3 type with various global symmetries, representations of the scalar fields, and sixth-degree interaction terms have also been of recent interest in the context of large-charge expansions and conformal field theory, e.g., [32–36]. Here we will confine our analysis to the simple realization of this theory in the Lagrangian (1.1).

This paper is organized as follows. In Sec. II we present the results of an analysis of the evidence, for a given N, of a UV zero in the six-loop beta function. In Sec. III we apply the method of Padé approximants to study this question further. Section IV contains an assessment of the effects of scheme dependence. Our conclusions are presented in Sec. V. Some auxiliary relations are given in the Appendix.

II. BETA FUNCTION AND UV ZERO

In this section we analyze the beta function of the $O(N)$ $|\vec{\phi}|^6_3$ theory. The beta function $\beta_g = dg/d\ln\mu$ has a series expansion in powers of the interaction coupling g, starting with a term of $O(g^2)$,

$$\beta_g = g\sum_{j=1}^{\infty} b_j g^j. \tag{2.1}$$

In the literature there are several different normalization conventions for the $|\vec{\phi}|^6$ coupling; for the reader's convenience, in the Appendix we list conversion formulas relating some of these. As noted above, the term in β_g of $O(g^p)$ arises from graphs with a maximal number of loops equal to $n = 2(p-1)$. We denote the truncation of the infinite series (2.1) to $O(g^p)$, as $\beta_{g,n\ell}$, where $n\ell$ is short for n-loops. As is the case with other scalar field theories, although Eq. (2.1) is an asymptotic expansion [37,38], it can still yield useful information about the properties of the theory.

With the normalization in Eq. (1.1), the first two coefficients, b_j, $j = 1, 2$, are [6,7,9] (see also [1,2])

$$b_1 = \frac{3N+22}{2\pi^2 N^2} \tag{2.2}$$

and

$$b_2 = -\frac{1}{2^7\pi^2N^4}\left[(N^3+34N^2+620N+2720)+\frac{8}{\pi^2}(53N^2+858N+3304)\right]$$
$$= -\frac{1}{N^4}[(0.791572\times10^{-3})N^3+0.0609195N^2+1.04129N+4.27300], \tag{2.3}$$

where floating-point numbers are given to the indicated accuracy. As mentioned before, these first two terms of $O(g^2)$ and $O(g^3)$ in β_g contain the maximal scheme-independent information in this function. The UV zero in the four-loop beta function occurs at $g_{\mathrm{UV},4\ell} = -b_1/b_2$, namely

$$g_{\mathrm{UV},4\ell} = \frac{64N^2(3N+22)}{(N^3+34N^2+620N+2720)+(8/\pi^2)(53N^2+858N+3304)}. \tag{2.4}$$

This is a monotonically increasing function of N. For large N,

$$g_{\mathrm{UV},4\ell} = 192\left[1-\frac{8(159+10\pi^2)}{3\pi^2N}+\frac{4(134832+14144\pi^2+215\pi^4)}{3\pi^4N}+O\left(\frac{1}{N^3}\right)\right]$$
$$= 192\left[1-\frac{69.62685}{N}+\frac{4043.0263}{N^2}+O\left(\frac{1}{N^3}\right)\right]. \tag{2.5}$$

The coefficient b_3 of the g^4 term in β_g has been calculated in [4] in the minimal subtraction scheme [39,40]. With the normalization in Eq. (1.1), it is

$$b_3 = \frac{1}{2^7\pi^6N^6}(1857N^3+45976N^2+367716N+950576)+\frac{1}{2^9\pi^4N^6}(36N^4+1607N^3+33568N^2+273772N+735392)$$
$$-\frac{3\ln(2)}{2^8\pi^4N^6}(N^4+N^3-700N^2-8236N-24816)+\frac{5}{2^{11}\pi^2N^6}(N^4+64N^3+1352N^2+12248N+36960)$$
$$-\frac{105\zeta(3)}{2^8\pi^6N^6}(11N^3+428N^2+4228N+12208)+\frac{1}{2^6\pi^6N^6}(24\beta(4)+\pi^2G)(31N^3+1126N^2+11876N+37592)$$
$$=\frac{1}{N^6}[(0.885804\times10^{-3})N^4+0.0739318N^3+1.81979N^2+16.3518N+47.4455], \tag{2.6}$$

where the Dirichlet beta function $\beta(s)$ is defined as

$$\beta(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}, \tag{2.7}$$

and $G=\beta(2)$ is the Catalan constant, with the values $G = 0.915965594$ and $\beta(4) = 0.98894455$ to the given floating-point accuracy. As is evident from the numerical evaluation of b_3 in Eq. (2.6), it is positive for all physical N and is a monotonically increasing function of N. With the normalization convention in Eq. (1.1), these coefficients have the large-N behavior $b_1, b_2 \sim 1/N$, $b_3, b_4 \sim 1/N^2$, with higher-order beta function coefficients having the large-N behavior $\sim 1/N^k$, $k \geq 3$, so that in the $N \to \infty$ limit, only the b_1 and b_2 terms are relevant, yielding the result $g_{\mathrm{UV}} = 192$ [7].

We now address the question of whether, for a given N, the six-loop beta function, $\beta_{g,6\ell}$, with b_3 computed in the minimal subtraction scheme, has a (perturbatively reliably calculable) UV zero. Aside from the double IR zero at $g=0$, the zeros of the n-loop beta function $\beta_{g,n\ell}$, i.e., the beta function calculated to $O(g^p)$, where $p=(n/2)+1$, are the solutions of the algebraic equation $\sum_{j=1}^{n/2} b_j g^{j-1} = 0$. For $\beta_{g,6\ell}$, this is the equation $b_1 + b_2 g + b_3 g^2 = 0$, with solutions $g_{\pm} = (2b_3)^{-1}(-b_2 \pm \sqrt{b_2^2 - 4b_1b_3})$. With N formally generalized from positive integers to positive real numbers, $b_2^2 - 4b_1b_3$ is positive if $N > N_c = 796.111$ and negative for the rest of the physical range $1 \leq N < N_c$. Hence, for $N > N_c$, the six-loop beta function $\beta_{g,6\ell}$ has two zeros on the positive g axis, and the one nearer to the origin is the UV zero, $g_{\mathrm{UV},6\ell} = g_- = (2b_3)^{-1}(-b_2 - \sqrt{b_2^2 - 4b_1b_3})$. This has the large-$N$ series expansion

$$g_{\mathrm{UV},6\ell} = 192\left[1 + \frac{4(978+25\pi^2-216\ln 2)}{3\pi^2 N} + \frac{4}{3\pi^4 N^2}(3232704 + 212612\pi^2 + 4565\pi^4 + 8928\pi^2 G - 1218249\ln 2 + 124416(\ln 2)^2 - 33192\pi^2\ln 2 - 83160\zeta(3)) + O\left(\frac{1}{N^3}\right)\right]$$
$$= 192\left[1 + \frac{145.230}{N} + \frac{67847.343}{N^2} + O\left(\frac{1}{N^3}\right)\right]. \quad (2.8)$$

It follows that

$$\lim_{N\to\infty} g_{\mathrm{UV},6\ell} = \lim_{N\to\infty} g_{\mathrm{UV},4\ell} = 192. \quad (2.9)$$

Note that the $1/N$ correction to $g_{\mathrm{UV},4\ell}$ is negative, while the $1/N$ correction to $g_{\mathrm{UV},6\ell}$ is positive. Given the N-dependence of still higher-order terms $O(g^p)$ with $p \geq 5$ in β_g, as discussed in [7], the result (2.9) can be generalized to

$$\lim_{N\to\infty} g_{\mathrm{UV},n\ell} = \lim_{N\to\infty} g_{\mathrm{UV},4\ell} = 192, \quad (2.10)$$

where $n = 2(p-1)$ with $p \geq 4$.

A necessary condition for a credible zero of a beta function is that when one calculates it to successive orders of perturbation theory, one obtains values that are close to each other, i.e., values with small fractional differences. We define the fractional difference as

$$\Delta(g_{\mathrm{UV},n\ell}, g_{\mathrm{UV},n'\ell}) \equiv \frac{g_{\mathrm{UV},n'\ell} - g_{\mathrm{UV},n\ell}}{g_{\mathrm{UV},n\ell}}. \quad (2.11)$$

In the large-N limit, the fractional difference between $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},6\ell}$ is

$$\Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},6\ell}) = \frac{12(144+5\pi^2-24\ln 2)}{\pi^2 N} + \frac{4}{\pi^4 N^2}(1215792 + 84036\pi^2 + 1850\pi^4 + 2976\pi^2 G - 436608\ln 2 + 41472(\ln 2)^2 - 12984\pi^2\ln 2 - 27729\zeta(3) + 71424\beta(4)) + O\left(\frac{1}{N^3}\right)$$
$$= \frac{214.8566}{N} + \frac{78764.106}{N^2} + O\left(\frac{1}{N^3}\right). \quad (2.12)$$

This fractional difference vanishes as $N \to \infty$, in agreement with the conclusion reached in [6,7,9] (before b_3 had been calculated). The new information obtained here is the calculation of the series expansion in powers of $1/N$ in Eq. (2.12), which provides a quantitative measure of the accuracy and reliability of the perturbative calculation of the value of the coupling at the UV zero for a given large N. In Table I we list $g_{\mathrm{UV},4\ell}$, $g_{\mathrm{UV},6\ell}$, and the fractional difference $\Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},6\ell})$ for some illustrative values of N. As N increases well beyond N_c, this fractional difference decreases reasonably quickly. For example, for $N = 2\times 10^3$, $N = 4\times 10^3$, and $N = 10^4$, $\Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},6\ell})$ has the approximate values 13%, 6%, and 2%, respectively. Thus, if one imposes a requirement that the fractional difference $\Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},6\ell})$ must be less than, say, 15%, in order for the calculation of the value of the UV zero to be reasonably reliable, then our results show that this criterion is satisfied for $N \gtrsim 2\times 10^3$.

TABLE I. From left to right, the columns of this table list (i) N; (ii) the UV zero, $g_{\mathrm{UV},4\ell}$, of the four-loop beta function, $\beta_{g,4\ell}$; (iii) the UV zero, $g_{\mathrm{UV},6\ell}$, of the six-loop beta function, $\beta_{g,6\ell}$; (iv) the UV zero, $g_{\mathrm{UV},[1,1]_{6\ell}}$, of the [1,1] Padé approximant to the reduced six-loop beta function, $\beta_{g,\mathrm{red},6\ell}$; and the fractional differences, denoted for short as (v) $\Delta_{4\ell,6\ell} \equiv \Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},6\ell})$; (vi) $\Delta_{4\ell,[1,1]_{6\ell}} \equiv \Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},[1,1]_{6\ell}})$; and (vii) $\Delta_{6\ell,[1,1]_{6\ell}} \equiv \Delta(g_{\mathrm{UV},6\ell}, g_{\mathrm{UV},[1,1]_{6\ell}})$. The last row lists the limiting values as $N \to \infty$. We use the standard notation −0.331e-2 for $-(0.331\times 10^{-2})$, etc. The symbol "n" means that the entry is unphysical or not relevant.

N	$g_{\mathrm{UV},4\ell}$	$g_{\mathrm{UV},6\ell}$	$g_{\mathrm{UV},[1,1]_{6\ell}}$	$\Delta_{4\ell,6\ell}$	$\Delta_{4\ell,[1,1]_{6\ell}}$	$\Delta_{6\ell,[1,1]_{6\ell}}$
1	0.2356	n	n	n	n	n
10	12.21	n	n	n	n	n
100	108.09	n	n	n	n	n
300	154.71	n	378.17	n	1.44	n
900	178.05	268.07	229.16	0.506	0.287	−0.145
1.0e3	179.37	249.49	224.79	0.391	0.253	−0.0990
2.0e3	185.505	210.34	207.07	0.134	0.116	−0.0156
4.0e3	188.71	199.90	199.24	0.0593	0.0558	−0.331e-2
1.0e4	190.67	194.93	194.83	0.0223	0.0218	−0.513e-3
∞	192	192	192	0	0	0

III. ANALYSIS WITH PADÉ APPROXIMANTS

One can gain further insight into the behavior of the beta function by the use of Padé approximants. Given a series expansion $f(z) = \sum_{n=0}^{n_{\rm max}} a_n z^n$, the $[r,s]$ Padé approximant (PA) is the rational function with numerator and denominator polynomials in z of degree r and s, respectively, where $r+s=n_{\rm max}$, such that the Taylor series expansion of this rational function matches the series expansion for $f(z)$ to its highest order, $n_{\rm max}$. The Padé method can be considered to be semiperturbative, since it uses as input a perturbative series expansion but produces a closed-form rational function, whose higher-order terms of order z^n with $n > n_{\rm max}$ are thus determined. Since the double IR zero in β_g at $g=0$ is not relevant here, it will be convenient to consider the reduced (red.) beta function normalized so that it is equal to 1 for $g=0$:

$$\beta_{g,{\rm red}} \equiv \frac{\beta_g}{\beta_{g,2\ell}} = \frac{\beta_g}{b_1 g^2} = 1 + \frac{1}{b_1}\sum_{j=2}^{\infty} b_j g^{j-1}. \tag{3.1}$$

From the beta function calculated to $O(g^p)$, one thus obtains the reduced beta function of degree $(p-2)$ in g. In particular, from $\beta_{g,6\ell}$, we have $\beta_{g,{\rm red},6\ell} = 1 + (b_2/b1)g + (b_3/b_1)g^2$. We denote the Padé approximants to $\beta_{g,{\rm red},n\ell}$ simply as $[r,s]_{n\ell}$. The PA $[2,0]_{6\ell}$ is this function itself, which we have already analyzed; the PA $[0,2]_{6\ell}$ has no zero, so we study the $[1,1]_{6\ell}$ Padé approximant. In terms of the coefficients b_j, $j=1,2,3$, this is

$$[1,1]_{6\ell} = \frac{1 + (\frac{b_2^2 - b_1 b_3}{b_1 b_2})g}{1 - (\frac{b_3}{b_2})g}. \tag{3.2}$$

We label the zero of this $[1,1]_{6\ell}$ PA as $g_{{\rm UV},[1,1]_{6\ell}}$. This is

$$g_{{\rm UV},[1,1]_{6\ell}} = \frac{b_1 b_2}{b_1 b_3 - b_2^2} = \frac{-(\frac{b_1}{b_2})}{1 - (\frac{b_1 b_3}{b_2^2})}. \tag{3.3}$$

We list illustrative values of $g_{{\rm UV},[1,1]_{6\ell}}$ in Table I. As is evident from the last term in Eq. (3.3), $g_{{\rm UV},[1,1]_{6\ell}}$ is related to the value of the UV zero of the four-loop beta function, $g_{{\rm UV},4\ell} = -b_1/b_2$, via division by the factor $1 - (b_1 b_3/b_2^2)$. Now $(b_1 b_3/b_2^2) > 0$, so if $(b_1 b_3/b_2^2) < 1$, then $g_{{\rm UV},[1,1]_{6\ell}} > g_{{\rm UV},4\ell}$. Since $b_1 b_3/b_2^2 \sim O(N^{-1})$ as $N \to \infty$, it follows that

$$\lim_{N\to\infty} g_{{\rm UV},[1,1]_{6\ell}} = \lim_{N\to\infty} g_{{\rm UV},4\ell}. \tag{3.4}$$

For $N \gg 1$, $g_{{\rm UV},[1,1]_{6\ell}}$ has the expansion

$$\begin{aligned} g_{{\rm UV},[1,1]_{6\ell}} &= 192\bigg[1 + \frac{4(978 + 25\pi^2 - 216\ln 2)}{3\pi^2 N} + \frac{4}{3\pi^4 N^2}(993216 + 57092\pi^2 + 1865\pi^4 + 8928\pi^2 G - 471744\ln 2 \\ &\quad + 62208(\ln 2)^2 - 83160\zeta(3) + 214272\beta(4)) + O\left(\frac{1}{N^3}\right)\bigg] \\ &= 192\left[1 + \frac{145.230}{N} + \frac{21683.974}{N^2} + O\left(\frac{1}{N^3}\right)\right]. \end{aligned} \tag{3.5}$$

As is evident from Eqs. (2.8) and (3.5), $g_{{\rm UV},6\ell}$ and $g_{{\rm UV},[1,1]_{6\ell}}$ have the same leading $1/N$ correction terms. This can be understood as a consequence of the fact that the $[1,1]_{6\ell}$ Padé approximant incorporates information from the $b_3 g^4$ term in $\beta_{g,6\ell}$, or equivalently, the $(b_3/b_1)g^2$ term in $\beta_{g,{\rm red},6\ell}$.

For large N, the fractional differences $\Delta(g_{{\rm UV},4\ell}, g_{{\rm UV},[1,1]_{6\ell}})$ and $\Delta(g_{{\rm UV},4\ell}, g_{{\rm UV},[1,1]_{6\ell}})$ are

$$\begin{aligned} \Delta(g_{{\rm UV},4\ell}, g_{{\rm UV},[1,1]_{6\ell}}) &= \frac{12(144 + 5\pi^2 - 24\ln 2)}{\pi^2 N} + \frac{4}{\pi^4 N^2}(469296 + 32196\pi^2 + 950\pi^4 + 2976\pi^2 G - 187776\ln 2 \\ &\quad + 20736(\ln 2)^2 - 4344\pi^2\ln 2 - 27729\zeta(3) + 71424\beta(4)) + O\left(\frac{1}{N^3}\right) \\ &= \frac{214.857}{N} + \frac{32600.74}{N^2} + O\left(\frac{1}{N^3}\right) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \Delta(g_{{\rm UV},6\ell}, g_{{\rm UV},[1,1]_{6\ell}}) &= -\frac{144(144 + 5\pi^2 - 24\ln 2)^2}{\pi^4 N^2} + O\left(\frac{1}{N^3}\right) \\ &= -\frac{46163.369}{N^2} + O\left(\frac{1}{N^3}\right). \end{aligned} \tag{3.7}$$

Since $g_{{\rm UV},6\ell}$ and $g_{{\rm UV},[1,1]_{6\ell}}$ have the same $1/N$ correction terms in the large-N limit, as observed above, it follows that $\Delta(g_{{\rm UV},6\ell}, g_{{\rm UV},[1,1]_{6\ell}})$ vanishes like $1/N^2$ in this limit. This is in contrast to $\Delta(g_{{\rm UV},4\ell}, g_{{\rm UV},6\ell})$ and $\Delta(g_{{\rm UV},4\ell}, g_{{\rm UV},[1,1]_{6\ell}})$, which both vanish like $1/N$ for large N.

In order for $g_{{\rm UV},[1,1]_{6\ell}}$ to be acceptable as a UV zero, it is necessary that there should not be a pole in the $[1,1]_{6\ell}$ PA on the positive real axis closer to the origin. The pole in this approximant occurs at

$$g_{[1,1]_{6\ell},\mathrm{pole}} = \frac{b_2}{b_3}. \tag{3.8}$$

Since $b_2 < 0$ and $b_3 > 0$ for all physical N, this pole occurs on the negative real axis, thereby fulfilling the above necessary condition. As $N \to \infty$, the value of g at this pole behaves as

$$\begin{aligned} g_{[1,1]_{6\ell},\mathrm{pole}} &= -\frac{16\pi^2 N}{144 + 5\pi^2 - 24\ln 2} + O(1) \\ &= -0.8936N + O(1). \end{aligned} \tag{3.9}$$

In general, in the large-N limit, the $1/N$ expansions given above show that

$$g_{\mathrm{UV},4\ell} \le g_{\mathrm{UV},[1,1]_{6\ell}} \le g_{\mathrm{UV},6\ell} \tag{3.10}$$

and

$$|\Delta(g_{\mathrm{UV},6\ell}, g_{\mathrm{UV},[1,1]_{6\ell}})| \le \Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},[1,1]_{6\ell}}) \tag{3.11}$$

with equality at $N = \infty$.

As N decreases from large values, $b_1 b_3 / b_2^2$ increases, and as N decreases below a value $N_d \simeq 150.799$, this ratio increases through 1, producing a pole in $g_{\mathrm{UV},[1,1]_{6\ell}}$. Clearly, this method of obtaining an estimate of a UV zero in $\beta_{g,6\ell}$ via a zero in the $[1,1]_{6\ell}$ approximant at $g_{\mathrm{UV},[1,1]_{6\ell}}$ is only reliable for values of N well above N_d.

Thus, the $[1,1]_{6\ell}$ Padé approximant to $\beta_{g,\mathrm{red},6\ell}$ yields a UV zero over a larger range of N than the beta function itself, extending below $N_c \simeq 796$ to the vicinity of $N_d \simeq 151$. However, as noted above, as N approaches the vicinity of N_d from above, the value of $g_{\mathrm{UV},[1,1]_{6\ell}}$ deviates substantially from the scheme-independent value, $g_{\mathrm{UV},4\ell}$. For example, at an illustrative value below N_c but above N_d, namely $N = 300$, although the [1,1] Padé approximant has a UV zero, $g_{\mathrm{UV},[1,1]_{6\ell}} = 378.17$, this is not close to $g_{\mathrm{UV},4\ell} = 154.71$. Consequently, in this vicinity, the method does not satisfy the requirement that different perturbative or semiperturbative methods of calculating this UV zero should yield values in approximate agreement with each other. Among the entries in Table I, in addition to the values of $g_{\mathrm{UV},[1,1]_{6\ell}}$ themselves, we list the fractional difference between $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},[1,1]_{6\ell}}$, denoted $\Delta(g_{\mathrm{UV},4\ell}, g_{\mathrm{UV},[1,1]_{6\ell}})$, and the fractional difference between $g_{\mathrm{UV},6\ell}$ and $g_{\mathrm{UV},[1,1]_{6\ell}}$, denoted $\Delta(g_{\mathrm{UV},6\ell}, g_{\mathrm{UV},[1,1]_{6\ell}})$.

IV. ASSESSMENT OF SCHEME TRANSFORMATIONS

Since b_3 and $g_{\mathrm{UV},6\ell}$ are scheme-dependent, one should assess the effect of this scheme dependence in a study of a UV zero of the beta function for this theory. A scheme transformation can be expressed as a mapping between g and g', which we write as $g = g' f(g')$, where $f(g')$ is the scheme transformation function, satisfying $f(0) = 1$. We will consider functions $f(g')$ that have a Taylor series expansion

$$f(g') = 1 + \sum_{j=1}^{j,\mathrm{max}} k_j g'^j, \tag{4.1}$$

where the k_s are constants, where j_{max} may be finite or infinite. The Jacobian of the transformation is $J = dg/dg'$,

$$J = 1 + \sum_{j=1}^{j_{\mathrm{max}}} (j+1) k_j g'^j. \tag{4.2}$$

After the scheme transformation is applied, the beta function in the new scheme has the form (2.1) with g replaced by g' and b_j replaced by b'_j. Expressions for the b'_j in terms of the b_j and k_s were derived in [41,42]. Aside from $b'_1 = b_1$ and $b'_2 = b_2$, these relations include

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \tag{4.3}$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1, \tag{4.4}$$

and so forth for b'_j with higher j.

As was discussed in [41,42] and studied further in [43–46], in order to be physically acceptable, a scheme transformation must satisfy several necessary conditions, which were denoted C_1 to C_4. The first two conditions, C_1 and C_2, are that the scheme transformation must map a real positive g to a real positive g' and should not map a moderate value of g, for which perturbation theory may be reliable, to a value of g' that is so large that perturbation theory is unreliable. The third condition, C_3, is that the Jacobian should not vanish in the region of g and g' of interest or else the transformation would be singular. A fourth condition given in [41,42] is that, since the existence of a UV zero of the beta function is a physical property, a scheme transformation should satisfy the property that β_g has a UV zero if and only if $\beta_{g'}$ has a UV zero. These conditions can easily be satisfied by scheme transformations applied in the vicinity of a zero of the beta function at sufficiently small coupling, but they are not automatically satisfied, and are a significant restriction, on a scheme transformation applied in the vicinity of a generic zero of the beta function away from the origin. These results on scheme transformations have been applied to the study of an IR zero in the beta function of an asymptotically free theory, such as a non-Abelian gauge theory with a certain content of massless Dirac fermions in $d = 4$ dimensions [42–49] and the Gross-Neveu model [50] in $d = 2$ [51]. They have also been applied to assess scheme dependence in probing for a possible UV zero in the beta function of an IR-free theory such as $O(N)$ $|\vec{\phi}|^4$ theory in $d = 4$ [52,53] (reviews include [54,55]).

Since $\beta_{g,6\ell}$ does not have a UV zero if $N < N_c$, whereas $\beta_{g,4\ell}$ has, at least formally, a UV zero for all physical N, a natural method to use to study the effects of scheme dependence is to construct a scheme transformation that eliminates the $O(g^4)$ term in $\beta_{g',6\ell}$ and thus yields a beta function consisting of just the first two (scheme-independent) terms. Since the beta function in the transformed scheme always has a UV zero, this scheme transformation would not satisfy condition C_4. However, by applying it, one can at least gain some information about the degree of scheme dependence in the evidence for or against the property that, at a given $N \lesssim N_c$, the six-loop beta function in a particular scheme has a UV zero.

To carry out this procedure, we will use the results of Refs. [42–44], which presented scheme transformations that can be used to set $b'_\ell = 0$ for $\ell \geq 3$, thereby reducing the beta function to its two scheme-independent terms (the 't Hooft scheme). The simplest way to do this is to set $k_1 = 0$ in Eq. (4.1) and then solve the equation $b'_3 = 0$ for k_2, obtaining

$$k_2 = \frac{b_3}{b_1}. \tag{4.5}$$

Although we will only need to apply the procedure here to set $b'_3 = 0$, since this is the highest-order coefficient that has been calculated for this theory, we briefly review how the procedure works if one has a beta function calculated to higher order. One next substitutes the value of k_2 from Eq. (4.5) into the equation for b'_4, Eq. (4.4), and solves the equation $b'_4 = 0$ for k_3. This procedure is applied iteratively to solve for k_j with $j \geq 4$ so as to render $b'_{j+1} = 0$. At least formally, a solution is guaranteed, since the condition that $b'_{j+1} = 0$ is a linear equation for k_j for all $j \geq 2$. However, while this procedure can be carried out for sufficiently weak coupling, e.g., in the deep UV limit of a UV-free theory such as QCD, as originally noted by 't Hooft [56], Refs. [42–44] showed that it can be more difficult to do this with a physically acceptable scheme transformation when studying a zero of the beta function away from the origin in coupling constant space (an IR zero in a UV-free theory or a UV zero in an IR-free theory).

Given that we take $k_1 = 0$ in $f(g')$, the procedure for constructing and applying this scheme transformation here requires only the determination of a single parameter, k_2, since we only have to eliminate b'_3 to reduce the six-loop beta function to its minimal scheme-independent first two terms in the transformed scheme. Thus, we construct a scheme transformation with $j_{\max} = 2$, $k_1 = 0$, and $k_2 = b_3/b_1$, as in Eq. (4.5), so as to render $b'_3 = 0$. This is the transformation with $f(g') = 1 + (b_3/b_1)g'^2$, namely

$$g = g'\left[1 + \left(\frac{b_3}{b_1}\right)g'^2\right]. \tag{4.6}$$

Then, since $b'_3 = 0$, the beta function in the transformed scheme is

$$\beta_{g',6\ell} = b_1 g'^2 + b_2 g'^3. \tag{4.7}$$

Note that $b_3/b_1 \sim 1/N$ for large N, so as $N \to \infty$, the scheme transformation (4.6) approaches the identity mapping.

To check whether, for a given N, this scheme transformation satisfies at least the first three conditions for acceptability, one then calculates how close g and the corresponding g' are to each other. For a given N and resultant UV zero of the four-loop beta function, $g_{\mathrm{UV},4\ell}$, one thus solves the cubic equation (4.6) for g' with $g = g_{\mathrm{UV},4\ell}$, using the minimal positive real root as g'. In Table II we list illustrative values of N and $g_{\mathrm{UV},4\ell}$, together with the solution for g' from Eq. (4.6) with $g = g_{\mathrm{UV},4\ell}$ and the fractional difference $\Delta(g_{\mathrm{UV},4\ell}, g')$. Since the scheme transformation (4.6) approaches the identity as $N \to \infty$, it follows that $\lim_{N\to\infty} g' = \lim_{N\to\infty} g$ for all g. In particular, as is evident from Table II, in the region $N \gtrsim 10^3$, if one sets $g = g_{\mathrm{UV},4\ell}$, then the corresponding value of g' is close to this value. For example, for $N = 2 \times 10^3$ and $N = 10^4$, the respective fractional differences between $g_{\mathrm{UV},4\ell}$ and the corresponding g' are approximately 8% and 2% in magnitude. Furthermore, since b_1 and b_3 are both positive, the condition that the Jacobian J should not vanish is satisfied. However, we find that in the region of $N \lesssim N_c$, although

TABLE II. From left to right, the columns of this table list (i) N; (ii) the UV zero, $g_{\mathrm{UV},4\ell}$, of the four-loop beta function, $\beta_{g,4\ell}$; (iii) g', the value of g in the transformed scheme with $b'_3 = 0$ obtained via the solution of Eq. (4.6) with g set equal to $g_{\mathrm{UV},4\ell}$; and (iv) the fractional difference between these values, denoted for short as $\Delta_{\mathrm{tran}} \equiv \Delta(g_{\mathrm{UV},4\ell}, g')$. The last row lists the limiting values as $N \to \infty$. We use the standard notation 1.0e3 for 1.0×10^3, etc.

N	$g_{\mathrm{UV},4\ell}$	g'	Δ_{tran}
100	108.09	69.901	−0.353
300	154.71	116.09	−0.250
900	178.05	152.90	−0.141
1.0e3	179.37	155.67	−0.132
2.0e3	185.505	170.50	−0.0809
4.0e3	188.71	180.04	−0.0459
1.0e4	190.67	186.84	−0.0201
∞	192	192	0

one can formally apply this scheme transformation, thereby switching to a scheme in which the six-loop beta function has a UV zero, the value of the coupling at this UV zero, g', is substantially different from $g_{\mathrm{UV},4\ell}$. Hence, in the region $N \lesssim N_c$, this theory does not satisfy a necessary requirement for a reliably calculable UV zero of the beta function, namely that the values calculated in different schemes should be close to each other. In this region of N, the scheme transformation obviously also fails to satisfy the fourth condition, C_4, for acceptability discussed above. These results are consistent with the conclusion that if $N \lesssim N_c$, then the beta function, calculated to $O(g^4)$, does not exhibit evidence for an ultraviolet zero.

V. CONCLUSIONS

In this paper we have investigated the ultraviolet behavior of the $|\vec{\phi}|_3^6$ theory, focusing on the question of whether, for a given N, this theory exhibits robust evidence of an ultraviolet zero in the beta function, as calculated to the six-loop [i.e., $O(g^4)$] order. We make use of the result for the six-loop beta function calculated in the minimal subtraction scheme in [4]. Early work [6,7,9] established that this theory has a UV zero $g_{\mathrm{UV},4\ell}$ in the four-loop beta function, which is reliably calculable for large N. We find that the six-loop beta function from [4] has a UV zero if $N > N_c$, where $N_c \simeq 796$. From studying the fractional difference between $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},6\ell}$ as a function of N, we conclude that this zero in the six-loop beta function is robust for N well above N_c. To study the properties of the theory for finite N further, we have analyzed the Padé approximant to the (reduced) six-loop beta function, $[1,1]_{6\ell}$. Although this approximant does have a UV zero for a range of N below N_c, the value of the coupling at this UV zero, $g_{\mathrm{UV},[1,1]_{6\ell}}$, is not close to the value $g_{\mathrm{UV},4\ell}$ obtained from the four-loop beta function, so this does not constitute evidence that the theory actually has a reliably calculable UV zero in this range of N. Our application of a scheme transformation to the minimal two-term beta function ('t Hooft scheme) yields the same conclusion. Quantitatively, if one imposes the criterion that the fractional difference between $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},6\ell}$ should be smaller than, say, 15% for the calculation of the UV zero in the beta function to be reasonably reliable, then our results show that this criterion is satisfied for $N \gtrsim 2 \times 10^3$. Clearly, there is some arbitrariness in this benchmark value of 15% for the relative agreement of these couplings; imposing a more (less) stringent requirement on the relative agreement of $g_{\mathrm{UV},4\ell}$ and $g_{\mathrm{UV},6\ell}$ would shift the estimated minimal value of N for a reliable calculation to a higher (lower) value than 2×10^3. A property of our result is that it is derived from a perturbative expansion of the beta function up to $O(g^4)$ (six-loop) order; further valuable information could be obtained by calculating the $O(g^5)$ (eight-loop) term in the beta function. We again note that nonperturbative effects may be important for this theory. However, we believe that it is useful at least to investigate the basic perturbative question of the range in N for which the beta function, calculated to the highest order to which it is known, yields robust evidence for an ultraviolet zero. We have addressed this question in the present paper.

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APPENDIX: CONVERSIONS BETWEEN DIFFERENT NORMALIZATION CONVENTIONS

In the literature, several different normalization conventions have been used for the interaction coupling in $|\vec{\phi}|_3^6$ theories. We list some conversion relations here and remark on the consequences of these normalizations for the respective beta functions. All of the works included here used a real N-component scalar field ϕ except for [8,9], which used a complex N-component scalar field, equivalent to a $2N$-component real field. Aside from numerical prefactors, there have been two general classes of normalization conventions. The first class of normalizations involves division of the coupling by N^2 in the interaction term $\mathcal{L}_{\mathrm{int}}$., while the second does not. We list the interaction terms below, in the notation used in the original papers, with superscripts added for clarity. Reference [6] by Townsend (T) used the interaction term

$$\mathcal{L}_{\mathrm{int}}^{(\mathrm{T})} = \frac{1}{6!N^2}\eta|\vec{\phi}|^6, \tag{A1}$$

Ref. [9] by Appelquist and Heinz (AH) used the interaction term (with a complex field $\vec{\phi}$)

$$\mathcal{L}_{\mathrm{int}}^{(\mathrm{AH})} = \frac{1}{6N^2}g(\vec{\phi}^{\dagger}\cdot\vec{\phi})^3, \tag{A2}$$

and Ref. [10] by Bardeen, Moshe, and Bander (BMB) used

$$\mathcal{L}_{\mathrm{int}}^{(\mathrm{BMB})} = \frac{1}{6N^2}\eta|\vec{\phi}|^6. \tag{A3}$$

Among the second class of normalizations, Ref. [7] by Pisarski (P) used

$$\mathcal{L}_{\mathrm{int}}^{(\mathrm{P})} = \frac{\pi^2}{3}\lambda|\vec{\phi}|^6, \tag{A4}$$

while Ref. [4] by Hager (H) used

$$\mathcal{L}_{\mathrm{int}}^{(\mathrm{H})} = \frac{1}{6!}w|\vec{\phi}|^6, \tag{A5}$$

and also the rescaling

$$\bar{w} \equiv \frac{w}{32\pi^2}. \tag{A6}$$

[The reader should not confuse the sextic coupling $\lambda^{(P)}$ used in [7] with the quartic coupling λ that we have used in Eq. (1.1).] We have employed the BMB normalization convention in our Eq. (1.1) but with the symbol g rather than η. These couplings are related to each other as follows, where we use the notation in the original papers:

$$\eta^{(\mathrm{BMB})} = \frac{1}{2} g^{(\mathrm{AH})} = \frac{1}{5!}\eta^{(\mathrm{T})} = 2\pi^2 N^2 \lambda^{(\mathrm{P})} = \frac{4\pi^2 N^2}{15}\bar{w}^{(\mathrm{H})}. \tag{A7}$$

These different normalizations affect the definition of the respective beta functions. In general, consider two $|\vec{\phi}|^6$ interaction couplings c and c' that are related to each other according to

$$c' = rc, \tag{A8}$$

where r is a multiplicative factor. The corresponding beta functions are $\beta_c = dc/d\ln\mu$ and $\beta_{c'} = dc'/d\ln\mu$, with respective series expansions

$$\beta_c = c\sum_{j=1}^{\infty} b_{c,j} c^j \tag{A9}$$

and

$$\beta_{c'} = c'\sum_{j=1}^{\infty} b_{c',j} c'^j. \tag{A10}$$

Then, since $b_{c,j}c^j = b_{c',j}c'^j = b_{c',j}(rc)^j$, it follows that these expansion coefficients are related according to

$$b_{c',j} = r^{-j} b_{c,j}. \tag{A11}$$

Consequently, as is evident in Eqs. (2.2), (2.3), and (2.6), with the T, AH, or BMB normalizations of the coupling, the corresponding beta function vanishes in the limit $N \to \infty$. Hence, if m^2 and λ are tuned to zero, in this limit the theory is scale-invariant, and it is this scale invariance that was found in [10] to be spontaneously broken if the BMB coupling is larger than $(4\pi)^2$. In contrast, with the normalization used in [7,4], the respective beta functions $\beta_\lambda = d\lambda/d\ln\mu$ and $\beta_{\bar{w}} = d\bar{w}/d\ln\mu$ do not vanish for large N. Note that a ratio such as $b_1 b_3/b_2^2$ is invariant under these changes in normalizations.

For reference, the conversion relations for the UV zero of the four-loop beta function in the large-N limit are

$$\eta^{(\mathrm{BMB})}_{\mathrm{UV},4\ell} = 192 \Leftrightarrow g^{(\mathrm{AH})}_{\mathrm{UV},4\ell} = 384 \Leftrightarrow \lambda^{(\mathrm{P})}_{\mathrm{UV},4\ell} = \frac{96}{\pi^2 N^2} \Leftrightarrow \bar{w}_{\mathrm{UV},4\ell} = \frac{720}{\pi^2 N^2}. \tag{A12}$$

[1] M. J. Stephen and J. L. McCauley, Jr., Feynman graph expansion for tricritical exponents, Phys. Lett. **44A**, 89 (1973).

[2] A. L. Lewis and F. W. Adams, Tricritical behavior in two dimensions. II. Universal quantities from the ϵ expansion, Phys. Rev. B **18**, 5099 (1978).

[3] I. D. Lawrie and S. Sarbach, Theory of tricritical points, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1984), Vol. 9, pp. 1–161.

[4] J. S. Hager, Six-loop renormalization group functions of $O(n)$-symmetric ϕ^6-theory and ϵ -expansions of tricritical exponents up to ϵ^3, J. Phys. A **35**, 2703 (2002).

[5] Kenneth G. Wilson, Renormalization of a scalar field theory in strong coupling, Phys. Rev. D **6**, 419 (1972).

[6] P. K. Townsend, Consistency of the $1/N$ expansion for three-dimensional ϕ^6 theory, Nucl. Phys. **B118**, 199 (1977).

[7] R. D. Pisarski, Fixed Point Structure of ϕ^6 in Three-Dimensions at Large N, Phys. Rev. Lett. **48**, 574 (1982).

[8] Thomas Appelquist and Ulrich W. Heinz, Three-dimensional O(N) theories at large distances, Phys. Rev. D **24**, 2169 (1981).

[9] Thomas Appelquist and Ulrich W. Heinz, Vacuum stability in three-dimensional O(N) theories, Phys. Rev. D **25**, 2620 (1982).

[10] William A. Bardeen, Moshe Moshe, and Myron Bander, Spontaneous Breaking of Scale Invariance and the Ultraviolet Fixed Point in O(N) Symmetric ($\bar{\phi}^6_3$ in Three-Dimensions) Theory, Phys. Rev. Lett. **52**, 1188 (1984).

[11] Curtis G. Callan, Jr., Broken scale invariance in scalar field theory, Phys. Rev. D **2**, 1541 (1970).

[12] K. Symanzik, Small distance behavior in field theory and power counting, Commun. Math. Phys. **18**, 227 (1970).

[13] Kenneth G. Wilson, The renormalization group and strong interactions, Phys. Rev. D **3**, 1818 (1971).

[14] E. Brézin and Jean. Zinn-Justin, Renormalization of the Nonlinear Sigma Model in $2+\epsilon$ Dimensions. Application to the Heisenberg Ferromagnets, Phys. Rev. Lett. **36**, 691 (1976).

[15] E. Brézin and J. Zinn-Justin, Spontaneous breakdown of continuous symmetries near two-dimensions, Phys. Rev. B **14**, 3110 (1976).

[16] William A. Bardeen, Benjamin W. Lee, and Robert E. Shrock, Phase transition in the nonlinear σ model in $2+\epsilon$ dimensional continuum, Phys. Rev. D **14**, 985 (1976).

[17] Alexander M. Polyakov, Interaction of Goldstone particles in two-dimensions. Applications to ferromagnets and massive Yang-Mills fields, Phys. Lett. **59B**, 79 (1975).

[18] Francois David, David A. Kessler, and Herbert Neuberger, The Bardeen-Moshe-Bander Fixed Point and the Ultraviolet Triviality of ϕ^6 in Three-Dimensions, Phys. Rev. Lett. **53**, 2071 (1984).

[19] Francois David, David A. Kessler, and Herbert Neuberger, A study of $(\phi^2)^3$ in three-dimensions at $N =$ infinity, Nucl. Phys. **B257**, 695 (1985).

[20] Daniel J. Amit and Eliezer Rabinovici, Breaking of scale invariance in ϕ^6 theory: Tricriticality and critical end points, Nucl. Phys. **B257**, 371 (1985).

[21] David A. Kessler and Herbert Neuberger, Infinite $N(\phi^2)^3$ in three-dimensions on the lattice, Phys. Lett. B **157B**, 416 (1985).

[22] Frithjof Karsch and Hildegard Meyer-Ortmanns, Phase structure of $O(N)$ symmetric ϕ^6 in three-dimensions: Models at small and intermediate N, Phys. Lett. B **193**, 489 (1987).

[23] Ragnheidur Gudmundsdottir, Gunnar Rydnell, and Per Salomonson, More on $O(N)$-Symmetric ϕ_3^6 Theory, Phys. Rev. Lett. **53**, 2529 (1984).

[24] Ragnheidur Gudmundsdottir, Gunnar Rydnell, and Per Salomonson, On $1/N$ expansion in $(\phi^2)^3$ in three-dimensions field theory, Ann. Phys. (N.Y.) **162**, 72 (1985).

[25] Hamid Omid, Gordon W. Semenoff, and L. C. R. Wijewardhana, Light dilaton in the large N tricritical $O(N)$ model, Phys. Rev. D **94**, 125017 (2016).

[26] Daniel F. Litim, Edouard Marchais, and Péter Mati, Fixed points and the spontaneous breaking of scale invariance, Phys. Rev. D **95**, 125006 (2017).

[27] Daniel F. Litim and Matthew J. Trott, Asymptotic safety of scalar field theories, Phys. Rev. D **98**, 125006 (2018).

[28] Shunsuke Yabunaka and Bertrand Delamotte, Surprises in $O(N)$ Models: Nonperturbative Fixed Points, Large N Limits, and Multicriticality, Phys. Rev. Lett. **119**, 191602 (2017).

[29] Shunsuke Yabunaka and Bertrand Delamotte, Why Might the Standard Large N Analysis Fail in the O(N) Model: The Role of Cusps in the Fixed Point Potentials, Phys. Rev. Lett. **121**, 231601 (2018).

[30] C. Fleming, B. Delamotte, and S. Yabunaka, Finite N origin of the Bardeen-Moshe-Bander phenomenon and its extension at $N = \infty$ by singular fixed points, Phys. Rev. D **102**, 065008 (2020).

[31] Stanley J. Brodsky, Matin Mojaza, and Xing-Gang Wu, Systematic scale-setting to all orders: The principle of maximum conformality and commensurate scale relations, Phys. Rev. D **89**, 014027 (2014).

[32] Simeon Hellerman, Domenico Orlando, Susanne Reffert, and Masataka Watanabe, On the CFT operator spectrum at large global charge, J. High Energy Phys. 12 (2015) 071.

[33] Luis Alvarez-Gaumé, Orestis Loukas, Domenico Orlando, and Susanne Reffert, Compensating strong coupling with large charge, J. High Energy Phys. 04 (2017) 059.

[34] Gil Badel, Gabriel Cuomo, Alexander Monin, and Riccardo Rattazzi, Feynman diagrams and the large charge expansion in $3 - \varepsilon$ dimensions, Phys. Lett. B **802**, 135202 (2020).

[35] Oleg Antipin, Jahmall Bersini, Francesco Sannino, Zhi-Wei Wang, and Chen Zhang, Charging the $O(N)$ model, Phys. Rev. D **102**, 045011 (2020).

[36] I. Jack and D. R. T. Jones, Anomalous dimensions for ϕ^n in scale invariant $d = 3$ theory, Phys. Rev. D **102**, 085012 (2020).

[37] L. N. Lipatov, Divergence of the perturbation theory series and the quasiclassical theory, Sov. Phys. JETP **45**, 216 (1977).

[38] E. Brézin, J. C. Le Guillou, and Jean Zinn-Justin, Perturbation theory at large order. 1. The ϕ^{2N} interaction, Phys. Rev. D **15**, 1544 (1977).

[39] Gerard 't Hooft and M. J. G. Veltman, Regularization and renormalization of gauge fields, Nucl. Phys. **B44**, 189 (1972).

[40] William A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Deep inelastic scattering beyond the leading order in asymptotically free gauge theories, Phys. Rev. D **18**, 3998 (1978).

[41] Thomas A. Ryttov and Robert Shrock, Scheme transformations in the vicinity of an infrared fixed point, Phys. Rev. D **86**, 065032 (2012).

[42] Thomas A. Ryttov and Robert Shrock, An analysis of scheme transformations in the vicinity of an infrared fixed point, Phys. Rev. D **86**, 085005 (2012).

[43] Robert Shrock, Study of scheme transformations to remove higher-loop terms in the β function of a gauge theory, Phys. Rev. D **88**, 036003 (2013).

[44] Robert Shrock, Generalized scheme transformations for the elimination of higher-loop terms in the beta function of a gauge theory, Phys. Rev. D **90**, 045011 (2014).

[45] Gongjun Choi and Robert Shrock, New scheme transformations and application to study scheme dependence of an infrared zero of the beta function in gauge theories, Phys. Rev. D **90**, 125029 (2014).

[46] Gongjun Choi and Robert Shrock, Integral formalism for the construction of scheme transformations in quantum field theory, Phys. Rev. D **94**, 065038 (2016).

[47] Thomas A. Ryttov, Higher loop corrections to the infrared evolution of fermionic gauge theories in the RI' scheme, Phys. Rev. D **89**, 016013 (2014).

[48] Thomas A. Ryttov, Conformal behavior at four loops and scheme (in)dependence, Phys. Rev. D **90**, 056007 (2014).

[49] J. A. Gracey and R. M. Simms, Banks-Zaks fixed point analysis in momentum subtraction schemes, Phys. Rev. D **91**, 085037 (2015).

[50] David J. Gross and André Neveu, Dynamical symmetry breaking in asymptotically free field theories, Phys. Rev. D **10**, 3235 (1974).

[51] Gongjun Choi, Thomas A. Ryttov, and Robert Shrock, Question of a possible infrared zero in the beta function of the finite-N Gross-Neveu model, Phys. Rev. D **95**, 025012 (2017).

[52] Robert Shrock, Question of an ultraviolet zero of the beta function of the $\lambda(\vec{\phi}^2)^2_4$ theory, Phys. Rev. D **90**, 065023 (2014).
[53] Robert Shrock, Study of the question of an ultraviolet zero in the six-loop beta function of the O(N) $\lambda|\vec{\phi}|^4$ theory, Phys. Rev. D **96**, 056010 (2017).
[54] Jean Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, UK, 1989).
[55] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of* ϕ^4 *Theories* (World Scientific, Singapore, 2001).
[56] Gerard 't Hooft, Can we make sense out of quantum chromodynamics?, Subnuclear series **15**, 943 (1979).