Fractional analytic QCD beyond leading order in the timelike region

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In this paper we show that, as in the spacelike case, the inverse logarithmic expansion is applicable for all values of the argument of the analytic coupling constant. We present two different approaches, one of which is based primarily on trigonometric functions, and the latter is based on dispersion integrals. The results obtained up to the 5th order of perturbation theory have a compact form and their acquiring is much easier than the methods that have been used before. As an example, we apply our results to study the Higgs boson decay into a $b\bar{b}$ pair.

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I. INTRODUCTION

The perturbative expansion in QCD works well only for estimation of the quantities in the region of large squared momentum Q^2 (here and further $Q^2 = -q^2$, where q^2 is the transferred momentum in the Euclidean domain for spacelike processes). However, for transferred momenta less than 1 GeV^2, the situation changes dramatically. The reason for this is the presence of the singularity of the coupling constant (couplant) $\alpha_s(Q^2)$ at the point $Q^2 = \Lambda^2$ which is widely known as Landau (ghost) pole. This singularity is especially important when we expand various physical observables in terms of the couplant, which makes the behavior of the observables nonanalytic in the Q^2-plane. For the correct description of QCD observables in the region of small Q^2 values, it is necessary to construct a new everywhere continuous couplant.

The renormalization group (RG) method allows us to sufficiently improve the expressions obtained in the frame of perturbation theory (PT). To show that the RG method cannot solve the above-mentioned problem, we first write the differential equation

$$\frac{d}{dL}\bar{a}_s(Q^2) = \beta(\bar{a}_s), \qquad \bar{a}_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}, \qquad L = \frac{Q^2}{\Lambda^2}, \tag{1}$$

with the QCD β-function

$$\beta(\bar{a}_s) = -\sum_{i=0}\beta_i \bar{a}_s^{i+2} = -\beta_0 \bar{a}_s^2\left(1 + \sum_{i=1} b_i \beta_0^i \bar{a}_s^i\right), \qquad b_i = \frac{\beta_i}{\beta_0^{i+1}}, \tag{2}$$

where the first fifth coefficients, i.e. β_i with $i \leq 4$, are exactly known [1–3]. Here we use the following definition of strong couplant:

$$a_s(Q^2) = \frac{\beta_0 \alpha_s(Q^2)}{4\pi} = \beta_0 \bar{a}_s(Q^2), \tag{3}$$

where we absorb the first coefficient of the QCD β-function into the a_s definition, as is usually the case of analytic couplants (see, e.g., Refs. [4–12]).

Solving Eq. (1) for $\bar{a}_s(Q^2)$ with the only leading order (LO) term on the right side, one can obtain the one-loop expression

$$a_s^{(1)}(Q^2) = \frac{1}{L}, \tag{4}$$

i.e. $a_s^{(1)}(Q^2)$ contains the pole at $Q^2 = \Lambda^2$ that indicates the inability of the RG approach to remove the Landau pole.

In the timelike region ($q^2 > 0$) (i.e., in the Minkowski space), the determination of the running coupling turns out to be quite difficult. The reason for the problem is that, strictly speaking, the expansion of perturbation theory in QCD cannot be determined directly in this area. Indeed, since the early days of QCD, much effort has been made to determine the appropriate coupling parameter in the Minkowski space to describe important timelike processes such as, for example, the

e^+e^--annihilation into hadrons, quarkonium and τ-lepton decays into hadrons. Most of the attempts (see, for example, [13]) were based on the analytical continuation of the strong couplant from the deep Euclidean region, where QCD perturbative calculations can be performed, to the Minkowski space, where physical measurements are performed. Over the time, it became clear that in the infrared (IR) regime, the strong couplant can reach a stable fixed point and stop increasing. This behavior would imply that the color forces can saturate at low momenta. So, for example, Cornwall [14] already in 1982 obtained the appearance of the gluon effective mass, which behaves as IR regulator in the region of small momenta. Similar results were obtained by others in subsequent years (see, for example, [15]) using different methods.

In other developments, analytical expressions for the LO couplant directly in the Minkowski space were obtained [16] using an integral transformation from the spacelike to the timelike region for the Adler D-function (more information can be found in Ref. [17]).

The systematic approach, called the analytical perturbation theory (APT), arose in the Shirkov and Solovtsov studies [4]. In this paper authors proposed to use new everywhere continuous analytic couplant $A_{\rm MA}(Q^2)$ in the form of spectral integral

$$A^{(i)}_{\rm MA}(Q^2) = \frac{1}{\pi}\int_0^{+\infty} \frac{d\sigma}{(\sigma+Q^2)}\, r^{(1)}_{\rm pt}(\sigma), \tag{5}$$

which is directly related with the appropriate PT order via the spectral function $r_{\rm pt}(s)$

$$r^{(i)}_{\rm pt}(\sigma) = {\rm Im}\, a^{(i)}_s(-\sigma - i\epsilon). \tag{6}$$

Similarly, the analytical images of a running coupling in Minkowski space are defined using another linear operation

$$U^{(i)}_{\rm MA}(s) = \frac{1}{\pi}\int_s^{+\infty} \frac{d\sigma}{\sigma}\, r^{(i)}_{\rm pt}(\sigma), \tag{7}$$

This method, called the *minimal approach* (MA) (see, e.g., [12]), contains a spectral function of a pure perturbative nature.[1]

The analytic couplants $A_{\rm MA}(Q^2)$ and $U_{\rm MA}(s)$ take almost the same values as $a_s(Q^2)$ when $Q^2(s) \gg \Lambda^2$ and completely different finite values at $Q^2 \leq \Lambda^2$. Moreover, the MA couplants $A_{\rm MA}(Q^2)$ and $U_{\rm MA}(Q^2)$ are related each other as [8]

$$A^{(i)}_{\rm MA}(Q^2) = \int_0^{+\infty} \frac{d\sigma Q^2}{(\sigma+Q^2)^2}\, U^{(i)}_{\rm MA}(\sigma),$$
$$U^{(i)}_{\rm MA}(s) = \frac{1}{2\pi i}\int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{d\sigma}{\sigma}\, A^{(i)}_{\rm MA}(\sigma). \tag{8}$$

The APT were extended for the case of noninteger power of the couplant, which appears in the QFT framework for quantities with nonzero anomalous dimensions (see the famous papers [7–9], some previous study [10] and reviews in Ref. [11]). For these purposes the fractional APT (FAPT) was developed. Due to the complexity of FAPT, the main results here until recently were obtained mostly in LO, however, it was also used in higher orders by reexpanding the corresponding coupling constants in the terms of LO ones, as well as using some approximations.

Following our recent paper [20] devoted to the couplant in the Euclidean domain, in this article we extend the FAPT in the Minkowski space to higher PT orders using the so-called $1/L$-expansion of the usual couplant. For an ordinary couplant, this expansion is valid only for the large values of L, i.e. for $Q^2 \gg \Lambda^2$; however, as it was shown in [20], if we consider an analytic couplant, this expansion is applicable throughout whole axis of squared transfered momentum. This becomes possible due to the smallness of the corrections for MA couplant which disappear when $Q^2 \to \infty$ and also $Q^2 \to 0$.[2] Thus, only in the region $Q^2 \sim \Lambda^2$ corrections turn out to be important enough (see also detailed discussions in Sec. III below).

Below we represent two different forms for the MA couplant in the Minkowski space calculated up to the 5th order of PT both of which contain the coefficients of the QCD β-function as parameters (some short version with the results based on the first three orders can be found in Ref. [21]).

The paper is organized as follows. In Sec. II we shortly review the basic properties of the usual strong couplant, its fractional derivatives (i.e., the ν-derivatives) and the $1/L$-expansions, which can be represented as some operators acting on the ν-derivatives of the LO strong couplant. This was the key idea of the paper [20], which makes it possible here to construct $1/L$-expansions of the ν-derivatives of MA couplant in the Minkowski space for high-order perturbation theory, see Sec. III. In Sec. IV we applied our new derivative operators to an integral representations of the MA couplant in the Minkowski space and in this manner continued it at the high PT orders. Section V contains an application of this approach to the Higgs boson decay into a $b\bar{b}$ pair. In conclusion, some final discussions are given. In addition, we have several

[1] An overview of other similar approaches can be found in [11] including approaches [18,19] close to APT.

[2] The absence of high-order corrections for $Q^2 \to 0$ was also discussed in Refs. [4–6].

Appendices. Appendix A presents some alternative results for the ν-derivatives of the MA couplant $U^{(1)}_{\rm MA}(s)$, which may be useful for some applications. Some details related to the derivation of the coefficients of the running quark mass are gathered in Appendix B. Appendix C contains formulas for restoring noninteger ν-powers of the usual strong couplant as a series of its $(\nu+m)$-derivatives.

II. STRONG COUPLING CONSTANT AND ITS FRACTIONAL DERIVATIVES

The strong couplant $a_s(Q^2)$ can be represented as $1/L$-series when $Q^2 \gg \Lambda^2$. Here we give the first five terms of the expansion in an agreement with the number of known coefficients β_i in the following short form

$$a^{(1)}_s(Q^2) = \frac{1}{L}, \qquad a^{(i+1)}_s(Q^2) = a^{(1)}_s(Q^2) + \sum_{m=2}^{i+1} \delta^{(m)}_s(Q^2), \quad (i = 0, 1, 2, \ldots) \tag{9}$$

where L is defined in Eq. (1).

The corresponding corrections $\delta^{(m)}_s(Q^2)$ are represented in [20]. At any PT order, the couplant $a_s(Q^2)$ contains its own parameter Λ of dimensional transmutation, which is fitted from experimental data for every single case.

The coefficients β_i depend on the number f of flavors, which increases or decreases at thresholds $Q^2_f \sim m^2_f$, where some new quark appears at $Q^2 > Q^2_f$. Here m_f is the $\overline{\rm MS}$ mass of f quark, for example, $m_b = 4.18 + 0.003 - 0.002$ GeV from PDG20 [22].[3] Thus, the couplant a_s is f-dependent and its f-dependence can be incorporated into Λ, as Λ^f, where f indicates the number of active flavors. In the $\overline{\rm MS}$ scheme, the relations between Λ^f_i and Λ^{f-1}_i are known up to the four-loop order [23–25] and they are usually used at $Q^2_f = m^2_f$, where the relations are simplified (for a recent review, see e.g. [26,27]).

Below we mainly deal with the region of low Q^2, where the only three first lightest quarks appear. Since in this case we will use the set of $\Lambda^{f=3}_i$ $(i = 0, 1, 2, 3)$ taken from the recent Ref. [28]. Further, since we will consider the $H \to b\bar{b}$ decay as an application, we will use also the results for $\Lambda^{f=5}_i$ taken also from [28]

$$\begin{aligned} &\Lambda^{f=3}_0 = 142\ \text{MeV}, \quad \Lambda^{f=3}_1 = 367\ \text{MeV}, \quad \Lambda^{f=3}_2 = 324\ \text{MeV}, \quad \Lambda^{f=3}_3 = 328\ \text{MeV}, \\ &\Lambda^{f=5}_0 = 87\ \text{MeV}, \quad \Lambda^{f=5}_1 = 224\ \text{MeV}, \quad \Lambda^{f=5}_2 = 207\ \text{MeV}, \quad \Lambda^{f=5}_3 = 207\ \text{MeV}. \end{aligned} \tag{10}$$

We use also $\Lambda_4 = \Lambda_3$, since in the highest orders Λ_i values become very similar.

A. Fractional derivatives

As it was done in [29,30], we first introduce the derivatives of couplant (in the $(i+1)$-order of PT)

$$\tilde{a}^{(i+1)}_{n+1}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n a^{(i+1)}_s(Q^2)}{(dL)^n}, \tag{11}$$

which is a key element in construction of FAPT (see, e.g., Ref. [31] and discussions therein).

The derivatives $\tilde{a}_n(Q^2)$ can be successfully used instead of a_s-powers in the decomposition of QCD observables. Although every derivative decreases the power of a_s, it produces the additional β-function $\sim a^2_s$, appeared from the term da_s/dL. At LO, the series of derivatives exactly coincide with the series of powers. Beyond LO, the relation between $\tilde{a}_n(Q^2)$ and $a^n_s(Q^2)$ was established [30,32] (the corresponding expansion $a^{n+1}(Q^2)$ in the terms $\tilde{a}^{(i+1)}_{n+m+1}(Q^2)$ can be found in Appendix C) and extended to the fractional case, where n is replaced for a noninteger ν, in Ref. [33]. The results for evaluation of $\tilde{a}^{(i+1)}_{n+1}(Q^2)$ are shown in (11) was considered in details in Appendix B of [20].

Here we write only the final results of calculations, which are represented in the form similar to given in Eq. (9)[4]:

$$\begin{aligned} \tilde{a}^{(1)}_\nu(Q^2) &= (a^{(1)}_s(Q^2))^\nu = \frac{1}{L^\nu}, \\ \tilde{a}^{(i+1)}_\nu(Q^2) &= \tilde{a}^{(1)}_\nu(Q^2) + \sum_{m=1}^{i} C^{\nu+m}_m \tilde{\delta}^{(m+1)}_\nu(Q^2), \\ \tilde{\delta}^{(m+1)}_\nu(Q^2) &= \hat{R}_m \frac{1}{L^{\nu+m}}, \qquad C^{\nu+m}_m = \frac{\Gamma(\nu+m)}{m!\Gamma(\nu)}, \end{aligned} \tag{12}$$

where

[3] Strictly speaking, the quark masses are Q^2-dependent in $\overline{\rm MS}$-scheme and $m_f = m_f(Q^2 = m^2_f)$. However, the Q^2-dependence is quite slow and it is not shown in the present study.

[4] The expansion (12) is very similar to those used in Refs. [7,8] for the expansion of $(a^{(i+1)}_s(Q^2))^\nu$ in terms of powers of $a^{(1)}_s(Q^2)$.

$$\hat{R}_1 = b_1\left[\hat{Z}_1(\nu) + \frac{d}{d\nu}\right], \qquad \hat{R}_2 = b_2 + b_1^2\left[\frac{d^2}{(d\nu)^2} + 2\hat{Z}_1(\nu+1)\frac{d}{d\nu} + \hat{Z}_2(\nu+1)\right],$$

$$\hat{R}_3 = \frac{b_3}{2} + 3b_2 b_1\left[Z_1(\nu+2) - \frac{11}{6} + \frac{d}{d\nu}\right] + b_1^3\left[\frac{d^3}{(d\nu)^3} + 3\hat{Z}_1(\nu+2)\frac{d^2}{(d\nu)^2} + 3\hat{Z}_2(\nu+2)\frac{d}{d\nu} + \hat{Z}_3(\nu+2)\right],$$

$$\hat{R}_4 = \frac{1}{3}(b_4 + 5b_2^2) + 2b_3 b_1\left[Z_1(\nu+3) - \frac{13}{6} + \frac{d}{d\nu}\right] + 6b_1^2 b_2\left[\frac{d^2}{(d\nu)^2} + 2\left(Z_1(\nu+3) - \frac{11}{6}\right)\frac{d}{d\nu} + Z_2(\nu+3) - \frac{11}{3}Z_1(\nu+3) + \frac{38}{9}\right] + b_1^4\left[\frac{d^4}{(d\nu)^4} + 4\hat{Z}_1(\nu+3)\frac{d^3}{(d\nu)^3} + 6\hat{Z}_2(\nu+3)\frac{d^2}{(d\nu)^2} + 4\hat{Z}_3(\nu+3)\frac{d}{d\nu} + \hat{Z}_4(\nu+3)\right]. \tag{13}$$

The representation (12) of the $\tilde{\delta}_\nu^{(m+1)}(Q^2)$ corrections as $\hat{R}_m$-operators plays a very important role in this paper.[5] Hereinafter, acting these operators on the analytic couplant in the Minkowski space, we will obtain the results for high-order corrections.

III. MINIMAL ANALYTIC COUPLING IN MINKOWSKI SPACE

There are several ways to obtain analytical versions of the strong couplant a_s (see, e.g. [11]). Here we will follow the MA approach [4–6] as discussed in the Introduction. To the fractional case, the MA approach was generalized by Bakulev, Mikhailov and Stefanis (hereinafter referred to as the BMS approach), that is presented in three famous papers [7–9] (see also the previous paper [10], the reviews [11,12] and *Mathematica* package in [35]).

We first show the leading order BMS results, and later we will go beyond LO, following our results for the usual strong couplant obtained in the previous section [see Eq. (12)].

A. LO

The LO MA coupling $U^{(1)}_{\mathrm{MA},\nu}(s)$ in the Minkowski space has the following form [8]

$$U_\nu^{(1)}(\mathrm{s}) = \tilde{U}_\nu^{(1)}(\mathrm{s}) = \frac{\sin[(\nu-1)g(s)]}{\pi(\nu-1)(\pi^2+L_s^2)^{(\nu-1)/2}}, \quad (\nu>0), \tag{14}$$

where

$$L_s = \ln\frac{s}{\Lambda^2}, \qquad g(s) = \arccos\left(\frac{L_s}{\sqrt{\pi^2+L_s^2}}\right). \tag{15}$$

The fact that Eq. (14) is applicable only for $\nu > 0$ will be discussed later.

For the cases $\nu = 0.5, 1, 1.5$, $U^{(1)}_{\mathrm{MA},\nu}(Q^2)$ is shown in Fig. 3. Strictly speaking, the value of the parameter Λ is obtained by fitting experimental data. To obtain its values (one of the two MA couplants $A_{\mathrm{MA}}(Q^2)$ and $U_{\mathrm{MA}}(s)$ can be fitted as they are very close to each other, as will be shown on Figs. 7 and 8 below) within the framework of analytical QCD, it is necessary to fit experimental data for various processes[6] by using, for instance, formulas obtained in this paper that simplify the form of higher-order terms. This, however, requires additional special research. In this article we use the values $\Lambda_{f=3}$ and $\Lambda_{f=5}$ [see Eq. (10)] obtained in the

[5]The results for $\hat{R}_m$-operators contain the transcendental principle [34]: the corresponding functions $\hat{Z}_k(\nu)$ ($k \leq m$) contain the Polygamma-functions $\Psi_k(\nu)$ and their products, such as $\Psi_{k-l}(\nu)\Psi_l(\nu)$, and also with a larger number of factors) with the same total index k. However, the importance of this property is not clear yet.

[6]One of the most important applications is fitting experimental data for the DIS structure functions (SFs) $F_2(x,Q^2)$ and $F_3(x,Q^2)$ (see, e.g., Refs. [36–39] and [40,41], respectively). One can use the ν-derivatives of the MA couplant $\tilde{A}^{(i)}_{\mathrm{MA},\nu}(Q^2)$, which is indeed possible, because when fitting we study the SF Mellin moments (following Ref. [42]) and only at the end reconstruct the SF themselves. This differs from the more popular approaches [43] based on numerical solutions of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [44]. In the case of using the [42] approach, the Q^2-dependence of the SF moments is known exactly in analytical form (see, e.g., [45]): it can be expressed in terms of the ν-derivatives $\tilde{A}^{(i)}_{\mathrm{MA},\nu}(Q^2)$, where the corresponding ν-variable becomes to be N-dependent (here N is the Mellin moment number), and the using of the ν-derivatives should be crucial. Beyond LO, in order to obtain complete analytic results for Mellin moments, we will use their analytic continuation [46].

framework of a conventional perturbative QCD since PT and FAPT couplants must coincide in the limit of large Q^2 and this requirement is fulfilled. It is clearly seen that at low Q^2 $U^{(1)}_{\mathrm{MA},\nu}(Q^2)$ agrees with its asymptotic values:

$$U^{(1)}_{\mathrm{MA},\nu}(Q^2=0) = \begin{cases} 0 & \text{when} \quad \nu > 1, \\ 1 & \text{when} \quad \nu = 1, \\ \infty & \text{when} \quad \nu < 1, \end{cases} \tag{16}$$

obtained in Ref. [47]. The corresponding results in the Euclidean space for $A^{(1)}_{\mathrm{MA},\nu}(Q^2)$ $\nu = 0.5, 1, 1.5$ were numerically obtained and shown on Fig. 1 in [20]. They are very close to those shown above in Fig. 1. Moreover, the asymptotic values of $A^{(1)}_{\mathrm{MA},\nu}(Q^2=0)$ and $U^{(1)}_{\mathrm{MA},\nu}(s=0)$ are completely identical to each other.

B. Beyond LO

Hereafter we repeat for $U_{\mathrm{MA},\nu}(s)$ the procedure that was applied to $\tilde{a}^{(i)}_\nu(Q^2)$. For this purpose, following to the representation (14) for the LO MA couplant in the Minkowski space, we consider its derivatives

$$\tilde{U}_{\mathrm{MA},n+1}(s) = \frac{(-1)^n}{n!}\frac{d^n U_{\mathrm{MA}}(s)}{(dL_s)^n}. \tag{17}$$

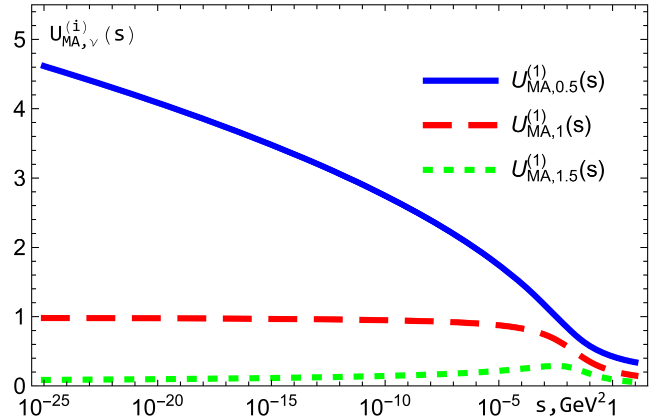

FIG. 1. The results for $U^{(1)}_{\mathrm{MA},\nu}(Q^2)$ with $\nu = 0.5, 1, 1.5$ in logarithmic scale.

Using the results (12) for the usual couplant we have

$$\tilde{U}^{\mathrm{MA},(i+1)}_\nu(\mathrm{s}) = \tilde{U}^{(1)}_{\mathrm{MA},\nu}(\mathrm{s}) + \sum_{m=1}^{i} C^{\nu+m}_m \tilde{\delta}^{(m+1)}_\nu(\mathrm{s}),$$
$$\tilde{\delta}^{(m+1)}_\nu(\mathrm{s}) = \hat{R}_m \tilde{U}^{(1)}_{\mathrm{MA},\nu+m}(\mathrm{s}), \tag{18}$$

where $\tilde{U}^{(1)}_{\mathrm{MA},\nu}(\mathrm{s})$ is given in Eq. (14).

This approach allows to express the high order corrections in explicit form

$$\tilde{\delta}^{(m+2)}_\nu(\mathrm{s}) = \frac{1}{(\nu+m)\pi(\pi^2+L_s^2)^{(\nu+m)/2}}\left\{\bar{\delta}^{(m+2)}_{\nu+m-1}(\mathrm{s})\sin((\nu+m)g) + \hat{\delta}^{(m+2)}_{\nu+m-1}(\mathrm{s})g\cos((\nu+m)g)\right\}, \tag{19}$$

where $\bar{\delta}^{(m+2)}_\nu(\mathrm{s})$ and $\hat{\delta}^{(m+2)}_\nu(\mathrm{s})$ are

$$\begin{aligned}
\bar{\delta}^{(2)}_\nu(\mathrm{s}) &= b_1[\hat{Z}_1(\nu) - G], \qquad \hat{\delta}^{(2)}_\nu(\mathrm{s}) = b_1, \\
\bar{\delta}^{(3)}_\nu(\mathrm{s}) &= b_2 + b_1^2[\hat{Z}_2(\nu) - 2G\hat{Z}_1(\nu) + G^2 - g^2], \qquad \hat{\delta}^{(3)}_\nu(\mathrm{s}) = 2b_1^2[\hat{Z}_1(\nu) - G], \\
\bar{\delta}^{(4)}_\nu(\mathrm{s}) &= \frac{b_3}{2} + 3b_1b_2\left[Z_1(\nu) - \frac{11}{6} - G\right] + b_1^3[\hat{Z}_3(\nu) - 3G\hat{Z}_2(\nu) + 3(G^2-g^2)\hat{Z}_1(\nu) - G(G^2-3g^2)], \\
\hat{\delta}^{(4)}_\nu(\mathrm{s}) &= 3b_1b_2 + b_1^3[3\hat{Z}_2(\nu) - 6G\hat{Z}_1(\nu) + (3G^2-g^2)], \\
\bar{\delta}^{(5)}_\nu(\mathrm{s}) &= \frac{1}{3}(b_4 + 5b_2^2) + 2b_1b_3\left[Z_1(\nu) - \frac{13}{6} - G\right] + 6b_1^2b_2\left[Z_2(\nu) - \frac{11}{3}Z_1(\nu) + \frac{38}{9} - 2G\left(Z_1(\nu) - \frac{11}{6}\right) + G^2 - g^2\right] \\
&\quad + b_1^4[\hat{Z}_4(\nu) - 4G\hat{Z}_3(\nu) + 6(G^2-g^2)\hat{Z}_2(\nu) - 4G(G^2-3g^2)\hat{Z}_1(\nu) + G^4 - 6G^2g^2 + g^4], \\
\hat{\delta}^{(5)}_\nu(\mathrm{s}) &= 2b_1b_3 + 12b_1^2b_2\left[Z_1(\nu) - \frac{11}{6} - G\right] + 4b_1^4[\hat{Z}_3(\nu) - 3G\hat{Z}_2(\nu) + (3G^2-g^2)\hat{Z}_1(\nu) - G(G^2-g^2)]
\end{aligned} \tag{20}$$

and

$$G(\mathrm{s}) = \frac{1}{2}\ln(\pi^2 + L_s^2). \tag{21}$$

C. The case $\nu = 1$

For the case $\nu = 1$ we get

$$\tilde{U}^{(i+1)}_{\mathrm{MA},\nu=1}(\mathrm{s}) = \tilde{U}^{(1)}_{\mathrm{MA},\nu=1}(\mathrm{s}) + \sum_{m=1}^{i} \tilde{\delta}^{(m+1)}_{\nu=1}(\mathrm{s}), \tag{22}$$

where LO gives the famous Shirkov-Solovtsov result [4,6]

$$U^{(1)}_{\mathrm{MA}}(\mathrm{s}) = \tilde{U}^{(1)}_{\mathrm{MA},\nu=1}(\mathrm{s}) = \frac{g(s)}{\pi} = \frac{1}{\pi}\arccos\left(\frac{L_s}{\sqrt{L_s^2+\pi^2}}\right) = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan\left(\frac{L_s}{\pi}\right)\right) \tag{23}$$

and the high order corrections sufficiently simplify

$$\tilde{\delta}^{(m+1)}_{\nu=1}(\mathrm{s}) = \frac{1}{m\pi(\pi^2+L_s^2)^{m/2}}\{\bar{\delta}^{(m+1)}_{m-1}(\mathrm{s})\sin(mg) + \hat{\delta}^{(m+1)}_{m-1}(\mathrm{s})g\cos(mg)\}, \tag{24}$$

where $\bar{\delta}^{(m+1)}_{m-1}(\mathrm{s})$ and $\hat{\delta}^{(m+1)}_{m-1}(\mathrm{s})$ can be obtained from the corresponding values in Eq. (20) with $\nu = 1$. Using Eqs. (A1) and (A2) we get

$$\begin{aligned}
&\bar{\delta}^{(2)}_0(\mathrm{s}) = -b_1[1+G], \qquad \hat{\delta}^{(2)}_0(\mathrm{s}) = b_1,\\
&\bar{\delta}^{(3)}_1(\mathrm{s}) = b_2 + b_1^2[G^2 - g^2 - 1], \qquad \hat{\delta}^{(3)}_1(\mathrm{s}) = -2Gb_1^2,\\
&\bar{\delta}^{(4)}_2(\mathrm{s}) = \frac{b_3}{2} - b_1b_2[1+3G] + \frac{b_1^3}{2}[1+6G+3(G^2-g^2) - 2G(G^2-3g^2)],\\
&\hat{\delta}^{(4)}_2(\mathrm{s}) = 3b_1b_2 + b_1^3[3G^2 - g^2 - 3G - 3],\\
&\bar{\delta}^{(5)}_3(\mathrm{s}) = \frac{1}{3}(b_4 + 5b_2^2) - \frac{2}{3}b_1b_3[1+3G] + 3b_1^2b_2[2G^2 - 2g^2 - 1]\\
&\qquad + b_1^4\left[\frac{5}{3} + 2G - 4(G^2-g^2) - \frac{10}{3}G(G^2-3g^2) + G^4 - 6G^2g^2 + g^4\right],\\
&\hat{\delta}^{(5)}_3(\mathrm{s}) = 2b_1b_3 - 12Gb_1^2b_2 + 2b_1^4\left[4G - 1 + \frac{5}{3}(3G^2-g^2) - 2G(G^2-g^2)\right].
\end{aligned} \tag{25}$$

Another form of $\tilde{\delta}^{(i+1)}_{\nu=1}(\mathrm{s})$ is given in Appendix A [see Eq. (A3)].

At the point $s = \Lambda^2$ the above results are simplified. They are

$$\tilde{U}^{(i+1)}_{\mathrm{MA},\nu=1}(\mathrm{s}=\Lambda^2) = \tilde{U}^{(1)}_{\mathrm{MA},\nu=1}(\mathrm{s}=\Lambda^2) + \sum_{m=1}^{i} \tilde{\delta}^{(m+1)}_{\nu=1}(\mathrm{s}=\Lambda^2), \tag{26}$$

where LO gives

$$U^{(1)}_{\mathrm{MA},1}(\mathrm{s}=\Lambda^2) = \frac{1}{2}, \tag{27}$$

and the high order corrections are

$$\begin{aligned}
\tilde{\delta}^{(2m+1)}_{\nu=1}(\mathrm{s}=\Lambda^2) &= \frac{(-1)^m}{4m(\pi^2+L_s^2)^m}\hat{\delta}^{(2m+1)}_{2m-1}(\mathrm{s}=\Lambda^2),\\
\tilde{\delta}^{(2m+2)}_{\nu=1}(\mathrm{s}) &= \frac{(-1)^m}{(2m+1)\pi(\pi^2+L_s^2)^{m+1/2}}\bar{\delta}^{(2m+2)}_{2m}(\mathrm{s}=\Lambda^2)
\end{aligned} \tag{28}$$

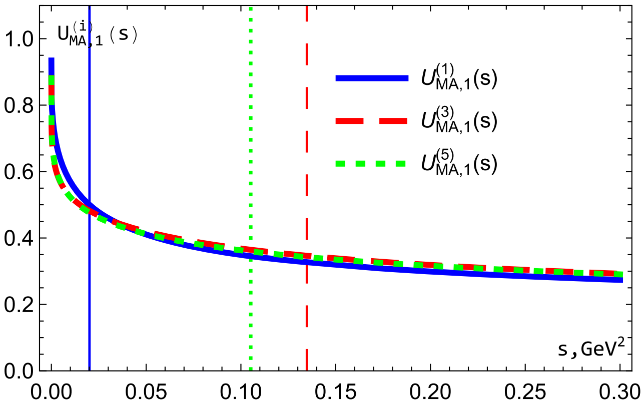

FIG. 2. 1, 3, and 5 orders of $U^{(i)}_{\mathrm{MA},\nu=1}$. The vertical lines indicate $(\Lambda^{f=3}_{i-1})^2$.

where $\bar{\delta}^{(2m+2)}_{2m}(s=\Lambda^2)$ and $\hat{\delta}^{(2m+1)}_{2m-1}(s=\Lambda^2)$ can be taken from Eq. (25) with the following replacement:

$$G(s=\Lambda^2)=\ln(\pi), \qquad g(s=\Lambda^2)=\frac{\pi}{2}. \qquad (29)$$

D. Discussions

This subsection provides graphical results of couplant construction. Figures 2 and 3 show the results for $U^{(i)}_{\mathrm{MA},\nu=1}(s)$ with $i=1,3,5$ in usual and logarithmic scales (the last one was chosen to stress the limit $U^{(i)}_{\mathrm{MA},\nu=1}(s\to 0)\to 1$). From Figs. 4 and 5 we can see the differences between $U^{(i)}_{\mathrm{MA},\nu=1}(Q^2)$ with $i=1,\ldots,5$, which are rather small and have nonzero values around the position $Q^2=\Lambda_i^2$. In Figs. 2, 4, 5, and 8 the values of $(\Lambda^{f=3}_i)^2$ $(i=0,2,4)$ are shown by vertical lines with color matching in each order. Note that Fig. 5 contains only one vertical line since $(\Lambda^{f=3}_4)^2=(\Lambda^{f=3}_5)^2$.

So, Figs. 2–5 point out that the difference between $U^{(i+1)}_{\mathrm{MA},\nu=1}(s)$ and $U^{(i)}_{\mathrm{MA},\nu=1}(s)$ is essentially less then the

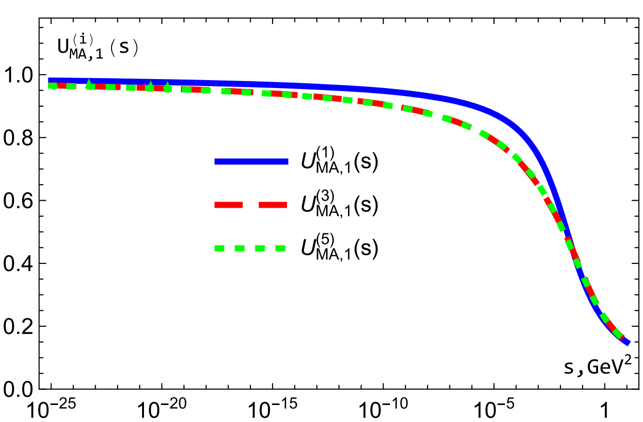

FIG. 3. 1, 3, and 5 orders of $U^{(i)}_{\mathrm{MA},\nu=1}$ with logarithmic scale of s.

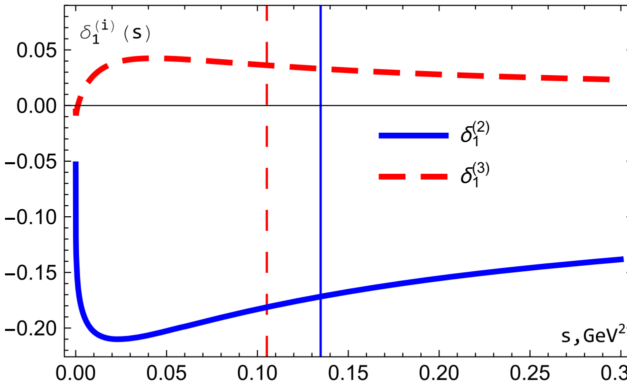

FIG. 4. $\delta^{(i)}_{\mathrm{MA},\nu=1}$ with $i=2,3$. The vertical lines indicate $(\Lambda^{f=3}_{i-1})^2$.

couplants themselves. From Figs. 3, 4, and 5 it is clear that for $s\to 0$ the asymptotic behavior of $U^{(1)}_{\mathrm{MA},\nu=1}(s)$, $U^{(3)}_{\mathrm{MA},\nu=1}(s)$, and $U^{(5)}_{\mathrm{MA},\nu=1}(s)$ coincides [and is equal to behavior considered in (16)], i.e. the differences $\delta^{(i)}_{\mathrm{MA},\nu=1}(s\to 0)$ are negligible. Also Figs. 4 and 5 show the differences $\delta^{(i+1)}_{\mathrm{MA},\nu=1}(s)$ $(i\geq 2)$ essentially less then $\delta^{(2)}_{\mathrm{MA},\nu=1}(s)$. We note that general form of the results is exactly the same as in the case of the MA couplants $A^{(i+1)}_{\mathrm{MA},\nu,\mathrm{i}}(Q^2)$, which have been studied earlier in [20]. Indeed, the similarity is shown in Figs. 6 and 7. In Fig. 6 the results for $U^{(i)}_{\mathrm{MA},\nu=1}(s)$ and $A^{(i)}_{\mathrm{MA},\nu=1}(Q^2)$ $(i=1,3,5)$ are shown in the so-called mirror form, which is in accordance with the similar one presented earlier in [8]. Figure 7 contains $U^{(1)}_{\mathrm{MA},\nu=1}(s)$, $A^{(1)}_{\mathrm{MA},\nu=1}(Q^2)$, $U^{(2)}_{\mathrm{MA},\nu=1}(s)$ and $A^{(2)}_{\mathrm{MA},\nu=1}(Q^2)$ which are very close to each others but have different limit values when $Q^2\to 0$. Moreover, the differences $\delta^{(2)}_{\mathrm{MA},\nu=1}(Q^2)$ in the cases $U^{(2)}_{\mathrm{MA},\nu=1}(s)$ and $A^{(2)}_{\mathrm{MA},\nu=1}(Q^2)$ are almost the same although correction of the spacelike couplant decreases

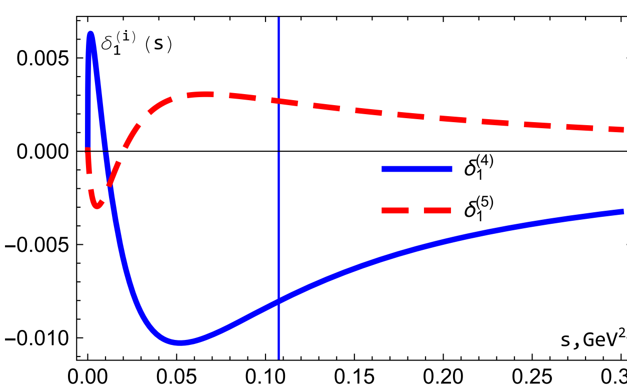

FIG. 5. $\delta^{(i)}_{\mathrm{MA},\nu=1}$ with $i=4$, 5. The vertical line indicates $(\Lambda^{f=3}_3)^2=\Lambda^{f=3}_4)^2$.

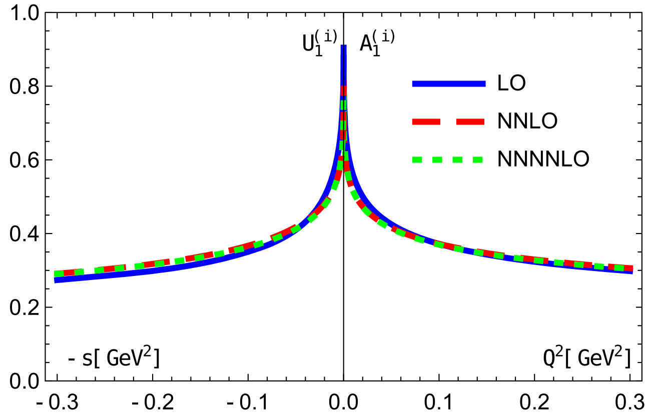

FIG. 6. 1, 3 and 5 orders of $U^{(i)}_{\mathrm{MA},\nu=1}$ and $A^{(i)}_{\mathrm{MA},\nu=1}$.

more rapidly. The direct relation between $A^{(i)}_{\mathrm{MA},\nu=1}(Q^2)$ and $U^{(i)}_{\mathrm{MA},\nu=1}(Q^2)$ gives an interesting picture (see Fig. 8). Obviously we have $\frac{A^{(i)}_{\mathrm{MA},\nu=1}(Q^2=0)}{U^{(i)}_{\mathrm{MA},\nu=1}(Q^2=0)}=1$ for any order and the second similar point

$$\frac{A^{(i)}_{\mathrm{MA},\nu=1}(Q^2=(\Lambda^{f=3}_{i-1})^2)}{U^{(i)}_{\mathrm{MA},\nu=1}(Q^2=(\Lambda^{f=3}_{i-1})^2)}=1 \tag{30}$$

for $i=1$. Higher order corrections break the identity (30), shifting the second point from $(\Lambda^{f=3}_{i})^2$. As we can see in Fig. 8, the shift is quite small. As can be seen from Fig. 8, the ratio (30) asymptotically approaches 1 when $\mathrm{Q}^2\to\infty$.

Thus, we can conclude that contrary to the case of the usual couplant, the $1/L$-expansion of the MA couplant is very good approximation at any $Q^2(s)$ values. Moreover, the differences between $U^{(i+1)}_{\mathrm{MA},\nu=1}(s)$ and $U^{(i)}_{\mathrm{MA},\nu=1}(s)$ become smaller with the increase of order. So, the expansions of $U^{(i+1)}_{\mathrm{MA},\nu=1}(s)$ $i\geq 1$ through the $U^{(1)}_{\mathrm{MA},\nu=1}(s)$ done in Refs. [7–9] are very good approximations.

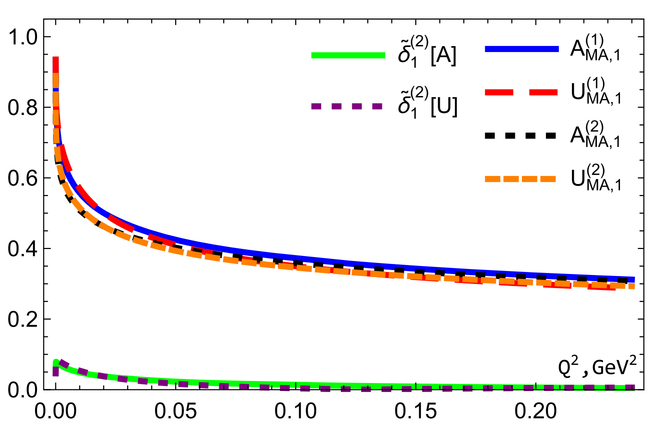

FIG. 7. 1 and 2 orders of $U^{(i)}_{\mathrm{MA},\nu=1}$, $A^{(i)}_{\mathrm{MA},\nu=1}$ and $\delta^{(2)}_{\mathrm{MA},\nu=1}$ in Euclidean and Minkowski spaces.

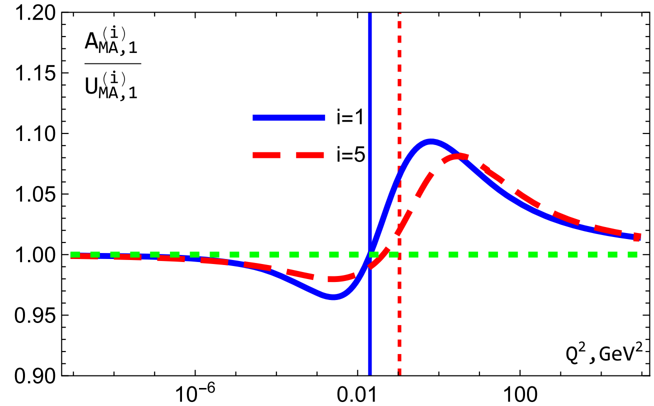

FIG. 8. The relation $A^{(i)}_{\mathrm{MA},\nu=1}/U^{(i)}_{\mathrm{MA},\nu=1}$ for $i=1,5$. The vertical lines indicate $(\Lambda^{f=3}_{i-1})^2$.

IV. INTEGRAL REPRESENTATIONS FOR MINIMAL ANALYTIC COUPLING

As it was mentioned in the Introduction, the MA couplants $A^{(1)}_{\mathrm{MA},\nu}(Q^2)$ and $U^{(1)}_{\mathrm{MA},\nu}(s)$ are constructed as follows: the LO spectral function is taken directly from perturbation theory but the MA couplants $A^{(1)}_{\mathrm{MA},\nu}(Q^2)$ and $U^{(1)}_{\mathrm{MA},\nu}(s)$ themselves were obtained using the correct integration contours. Thus, at LO, the MA couplants $A^{(1)}_{\mathrm{MA},\nu}(Q^2)$ and $U^{(1)}_{\mathrm{MA},\nu}(s)$ obey Eqs. (5) and (7) presented in the Introduction.

To check Eqs. (24) and (20) we compare them with an integral form

$$U^{(i)}_1(\mathrm{s})=\frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}r^{(i)}_{\mathrm{pt}}(\sigma). \tag{31}$$

For LO, we can take the integral form from [8]

$$U^{(1)}_\nu(\mathrm{s})=\frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}r^{(1)}_{\nu}(\sigma), \tag{32}$$

where

$$r^{(1)}_\nu(\mathrm{s})=\frac{\sin[\nu g(s)]}{\pi(\pi^2+L_s^2)^{(\nu-1)/2}}=\nu U^{(1)}_{\nu+1}(\mathrm{s}), \tag{33}$$

In (14) only the case $\nu\geq 0$ is considered, it means that the integral (32) converges to zero at the upper limit. We would like to note, that dispersion integral (31) does not converge for some ν and, in this case, we will introduce constant, which corresponds to the upper limit of the integral. In general, it is better to replace the integral (31) by the one

$$U^{(1)}_\nu(\mathrm{s})=\frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}r^{(1)}_{\nu}(\sigma)-U^{(1)}_\nu(\infty), \tag{34}$$

where

$$U_\nu^{(1)}(\infty) = \begin{cases} 0 & \text{when} \quad \nu > 0, \\ 1 & \text{when} \quad \nu = 0, \\ \infty & \text{when} \quad \nu < 0. \end{cases} \tag{35}$$

We see that the expression (32) diverges for $\nu < 0$ and requires additional constant for $\nu = 0$. Therefore Eq. (14) is applicable only when $\nu > 0$. Further in this paper we will only consider the region $\nu > 0$.

Using our approach to obtain high-order terms from LO (32), we can extend the LO integral (32) to the one

$$\tilde{U}_\nu^{(i)}(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma} r_\nu^{(i)}(\sigma), \tag{36}$$

where obviously

$$r_\nu^{(i)}(s) = \nu \tilde{U}_{\nu+1}^{(i)}(s). \tag{37}$$

The spectral function $r_1^{(i)}(s)$ has the form

$$r_1^{(i)}(s) = r_1^{(1)}(s) + \sum_{m=1}^{i} \delta_1^{(m+1)}(s) \tag{38}$$

where

$$r_1^{(1)}(s) = U_2^{(1)}(s), \quad \delta_1^{(m+1)}(s) = (m+1)\tilde{\delta}_{\nu=2}^{(m+1)}(s). \tag{39}$$

In the explicit form:

$$\begin{aligned} r_1^{(i)}(s) &= \frac{\sin(g)}{\pi(\pi^2+L_s^2)^{1/2}} = \frac{1}{\pi^2+L_s^2}, \\ \tilde{\delta}_1^{(m+1)}(s) &= \frac{1}{\pi(\pi^2+L_s^2)^{(m+1)/2}} \{\bar{\delta}_m^{(m+1)}(s)\sin((m+1)g) \\ &\quad + \hat{\delta}_m^{(m+1)}(s) g\cos((m+1)g)\}, \end{aligned} \tag{40}$$

where $\bar{\delta}_m^{(m+1)}(s)$ and $\hat{\delta}_m^{(m+1)}(s)$ can be obtained from the results in (20) with $\nu = 2$. They are

$$\begin{aligned} \bar{\delta}_1^{(2)}(s) &= -Gb_1, \qquad \hat{\delta}_1^{(2)}(s) = b_1, \\ \bar{\delta}_2^{(3)}(s) &= b_2 + b_1^2[G^2 - g^2 - G - 1], \qquad \hat{\delta}_2^{(3)}(s) = b_1^2[1-2G], \\ \bar{\delta}_3^{(4)}(s) &= \frac{b_3}{2} - 3Gb_1b_2 + \frac{b_1^3}{2}[4G - 1 + 5(G^2 - g^2) - 2G(G^2 - 3g^2)], \\ \hat{\delta}_3^{(4)}(s) &= 3b_1b_2 + b_1^3[3G^2 - g^2 - 5G - 2], \\ \bar{\delta}_4^{(5)}(s) &= \frac{1}{3}(b_4 + 5b_2^2) - 2b_1b_3\left[\frac{1}{12} + G\right] + 3b_1^2b_2[2G^2 - 2g^2 - G - 1] \\ &\quad + b_1^4\left[\frac{7}{6} + 4G - \frac{3}{2}(G^2 - g^2) - \frac{13}{3}G(G^2 - 3g^2) + G^4 - 6G^2g^2 + g^4\right], \\ \hat{\delta}_4^{(5)}(s) &= 2b_1b_3 + 3b_1^2b_2[1-4G] + b_1^4\left[3G - 4 + \frac{13}{3}(3G^2 - g^2) - 4G(G^2 - g^2)\right]. \end{aligned} \tag{41}$$

Using the results (A1) and (A2) for $\cos(ng)$ and $\sin(ng)$ ($n \leq 4$), we see that with the results [48,49] (see also Sec. VI in [20]) give more compact results for $r_1^{(i)}(s)$. We think that Eqs. (40) and (41) give apparently very compact results for $r_1^{(i)}(s)$.

Note that the results (36) for $\tilde{U}_\nu^{(i)}(s)$ are exactly the same as the results in Eq. (18) done in the form of trigonometric factions. However the results (36) should be very handy in case of nonminimal versions of analytic couplants (see Refs. [29,30,32]).

V. $H \to b\bar{b}$ DECAY

In Ref. [20] we used the polarized Bjorken sum rule [50] as an example for the application of the MA couplant $A_{\rm MA}(Q^2)$, which is a popular object of study in the framework of analytic QCD (see [51–54]). Here we consider the decay of the Higgs boson into a bottom-antibottom pair, which is also a popular application of the MA couplant $U_{\rm MA}(Q^2)$ (see, e.g., [8] and reviews in Ref. [11]).

The Higgs-boson decay into a bottom-antibottom pair can be expressed in QCD by means of the correlator

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|T[J_b^S(x)J_b^S(0)]0\rangle \tag{42}$$

of two quark scalar (S) currents in terms of the discontinuity of its imaginary part, i.e., $R_S(s) = \mathbf{Im}\Pi(-s - i\varepsilon)/(2\pi s)$, so that the width reads

$$\Gamma(H \to b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi} M_H m_b^2(M_H^2) R_s(s = M_H^2). \qquad (43)$$

Direct multiloop calculations were performed in the Euclidean (spacelike) domain for the corresponding Adler function D_S (see Refs. [55–58]). Hence, we write ($D_s \to \tilde{D}_s$ and $R_s \to \tilde{R}_s$ because the additional factor m_b^2)

$$\tilde{D}(Q^2) = 3m_b^2(Q^2)\left[1 + \sum_{n\geq 1} d_n a_s^n(Q^2)\right], \qquad (44)$$

where for $f = 5$ the coefficients d_n are

$$d_1 = 2.96, \quad d_2 = 11.44, \quad d_2 = 50.17, \quad d_4 = 260.24, \qquad (45)$$

Taking the imagine part, one has

$$\tilde{R}_s(s) = 3m_b^2(s)\left[1 + \sum_{n\geq 1} r_n a^n(s)\right], \qquad (46)$$

and for $f = 5$ [57,59]

$$r_1 = 2.96, \quad r_2 = 7.93, \quad r_3 = 5.93, \quad r_4 = -61.84, \qquad (47)$$

Here $\bar{m}_b^2(Q^2)$ has the form (see Appendix B):

$$\bar{m}_b^2(Q^2) = \hat{m}_b^2 a_s^d(Q^2)\left[1 + \sum_{k=1}^{k=4} \bar{e}_k a_s^k(Q^2)\right], \qquad (48)$$

where

$$\bar{e}_k = \frac{\tilde{e}_k}{k(\beta_0)^k} \qquad (49)$$

and $\tilde{e}_k$ are done in Eq. (B8). For $f = 5$ we have

$$\bar{e}_1 = 1.23, \quad \bar{e}_2 = 1.20, \quad \bar{e}_3 = 0.55, \quad \bar{e}_4 = 0.54. \qquad (50)$$

The normalization constant $\hat{m}_b$ cab be obtained as (see, e.g., [11])

$$\hat{m}_b = \bar{m}_b(Q^2 = m_b^2) a_s^{-d/2}(m_b^2)\left[1 + \sum_{k=1}^{k=4} \bar{e}_k a_s^k(Q^2)\right]^{-1/2} = 10.814 \text{ GeV}^2, \qquad (51)$$

since $\bar{m}_b(Q^2 = m_b^2) = m_b = 4.18$ GeV.

So, we have

$$\tilde{R}_s(s) = \tilde{R}_s^{(m=5)}(s), \qquad \tilde{R}_s^{(m+1)}(s) = 3\hat{m}_b^2 a_s^d(s) \times \left[1 + \sum_{k=0}^{m} \bar{r}_k a_s^k(s))\right], \qquad (52)$$

where

$$\bar{r}_k = r_k + \bar{e}_k + \sum_{l=1}^{k-1} r_l \bar{e}_{k-l}. \qquad (53)$$

For $f = 5$ we have

$$\bar{r}_1 = 4.18, \quad \bar{r}_2 = 12.76, \quad \bar{r}_3 = 19.76, \quad \bar{r}_4 = -42.25. \qquad (54)$$

We can express all results through derivatives $\tilde{a}_{d+k}$ (see Appendix B):

$$\tilde{R}_s(s) = \tilde{R}_s^{(m=5)}(s), \quad \tilde{R}_s^{(m+1)}(s) = 3\hat{m}_b^2\left[\tilde{a}_d + \sum_{k=0}^{m} \tilde{r}_k \tilde{a}_{d+k}\right], \qquad (55)$$

where

$$\tilde{r}_k = \bar{r}_k + \tilde{k}_k(d) + \sum_{l=1}^{k-1} \bar{r}_l \tilde{k}_{k-l}(d+l), \qquad (56)$$

where $\tilde{k}_i(\nu)$ are given in Appendix C.

For $d = 24/23$ and $f = 5$, we have

$$\tilde{r}_1 = 4.17, \quad \tilde{r}_2 = 9.86, \quad \tilde{r}_3 = 1.29, \quad \tilde{r}_4 = -71.21. \qquad (57)$$

Performing the same analysis for the Adler function we have

$$\tilde{D}_s = \tilde{D}_s^{(m=5)}, \quad \tilde{D}_s^{(m+1)} = 3\hat{m}_b^2 a_s^d(Q^2)\left[1 + \sum_{k=0}^{m} \bar{d}_k a_s^k(Q^2))\right], \qquad (58)$$

where

$$\bar{d}_k = r_k + \bar{e}_k + \sum_{l=1}^{k-1} d_l \bar{e}_{k-l}. \qquad (59)$$

For $f = 5$ we have

$$\bar{d}_1 = 4.18, \quad \bar{d}_2 = 16.27, \quad \bar{d}_3 = 68.30, \quad \bar{r}_4 = 337.66. \qquad (60)$$

We express all results through derivatives $\tilde{a}_{d+k}$:

$$\tilde{D}_s = \tilde{D}_s^{(m=5)}, \quad \tilde{D}_s^{(m+1)} = 3\hat{m}_b^2\left[\tilde{a}_d(Q^2) + \sum_{k=0}^{m} \tilde{d}_k \tilde{a}_{d+k}(Q^2)\right], \qquad (61)$$

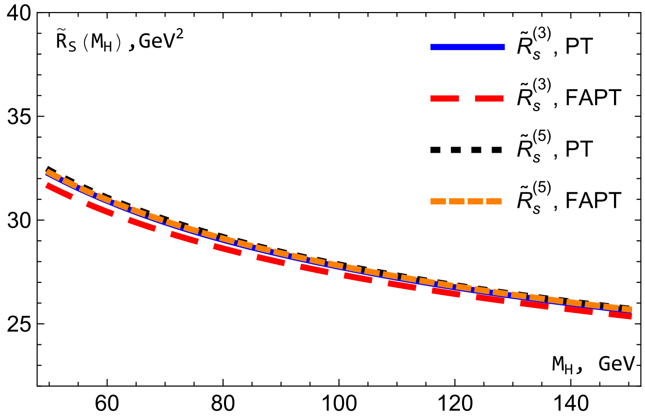

FIG. 9. The results for $\tilde{R}_s^{m+1}$ with $m = 2$ and 4 in the usual PT and FAPT.

where

$$\tilde{d}_k = \bar{d}_k + \tilde{k}_k(d) + \sum_{l=1}^{k-1} \bar{d}_l \tilde{k}_{k-l}(d+l). \tag{62}$$

For $f = 5$ and $d = 24/23$, we have

$$\tilde{d}_1 = 4.17, \quad \tilde{d}_2 = 13.37, \quad \tilde{d}_3 = 43.90, \quad \tilde{d}_4 = 178.18. \tag{63}$$

As it was discussed earlier in [8] in FAPT there are the following representation for $\tilde{R}_s$

$$\tilde{R}_s(s) = \tilde{R}_s^{(m=5)}(s),$$
$$\tilde{R}_s^{(m+1)}(s) = 3\hat{m}_b^2 \left[\tilde{U}_d^{(m+1)}(\mathrm{s}) + \sum_{k=0}^{m} \tilde{d}_k \tilde{U}_{d+k}^{(m+1)}(s) \right], \tag{64}$$

The results for $\tilde{R}_s^{(m+1)}(s)$ are shown in Fig. 9. We see that the FAPT results (64) are lower than those (55) based on the conventional PT. This is in full agreement with arguments given in [11]. But the difference becomes less notable as the PT order increases. Indeed, for N^{3}LO the difference is very small, which proves the assumption about the possibility of using $\tilde{R}_s^{(m+1)}(s)$ expression for $\tilde{D}_s^{(m+1)}(Q^2)$ with $A_{\mathrm{MA}}^{(i)}(Q^2) \to U_{\mathrm{MA}}^{(i)}(s)$, which was done in Ref. [8].

The results for $\Gamma^{(m)}(H \to b\bar{b})$ in the N^{m}LO approximation using $\tilde{R}_s^{(m+1)}(s)$ from Eqs. (52) and (55) are exactly same and have the following form:

$$\Gamma^{(0)} = 1.76\ \mathrm{MeV}, \quad \Gamma^{(1)} = 2.27\ \mathrm{MeV}, \quad \Gamma^{(2)} = 2.37\ \mathrm{MeV},$$
$$\Gamma^{(3)} = 2.38\ \mathrm{MeV}, \quad \Gamma^{(4)} = 2.38\ \mathrm{MeV}. \tag{65}$$

The corresponding results for $\Gamma^{(m)}(H \to b\bar{b})$ with $\tilde{R}_s^{(m+1)}(s)$ form Eq. (64) are very similar to the ones in (65). They are

$$\Gamma^{(0)} = 1.74\ \mathrm{MeV}, \quad \Gamma^{(1)} = 2.23\ \mathrm{MeV}, \quad \Gamma^{(2)} = 2.34\ \mathrm{MeV},$$
$$\Gamma^{(3)} = 2.37\ \mathrm{MeV}, \quad \Gamma^{(4)} = 2.38\ \mathrm{MeV}. \tag{66}$$

So, we see a good agreement between the results obtained in FAPT and in the framework of the usual PT.

It is clearly seen that the results of FAPT are very also close to the results [60] obtained in the framework of the now very popular principle of maximum conformality [61] (for the recent review, see [62]). Indeed, our results are within the band obtained by varying the renormalization scale.

The Standard Model expectation is [63]

$$\Gamma_{H\to b\bar{b}}^{\mathrm{SM}}(M_H = 125.1\ \mathrm{GeV}) = 2.38\ \mathrm{MeV}. \tag{67}$$

The ratios of the measured events yield to the Standard Model expectations are $1.01 \pm 0.12(\mathrm{stat.}) + 0.16 - 0.15(\mathrm{syst.})$ [64] in ATLAS Collaboration and $1.04 \pm 0.14(\mathrm{stat.}) \pm 0.14(\mathrm{syst.})$ [65] in SMC Collaboration (see also [66]).

Thus, our results obtained in both approaches, in the standard perturbation theory and in analytical QCD, are in good agreement both with the Standard Model expectations [63] and with the experimental data [64,65].

VI. CONCLUSIONS

In this paper we have used $1/L$-expansions of the ν-derivatives of the strong couplant a_s expressed [20] as combinations of operators $\hat{R}_m$ (13) applied to the LO couplant $a_s^{(1)}$. Applying the same operators to the ν-derivatives of the LO MA couplant $U_{\mathrm{MA}}^{(1)}$, we obtained two different representations [see Eqs. (24) and (36)] for the ν-derivatives of the MA couplants, i.e. $\tilde{U}_{\mathrm{MA},\nu}^{(i)}$ introduced for timelike processes, in each i-order of perturbation theory: one form contains a combinations of trigonometric functions, and the other is based on dispersion integrals containing the i-order spectral function. All results are presented up to the 5th order of perturbation theory, where the corresponding coefficients of the QCD β-function are well known (see [1,2]).

As in the case of $\tilde{A}_{\mathrm{MA},\nu}^{(i)}$ [20] applied in the Euclidean space, high-order corrections for $\tilde{U}_{\mathrm{MA},\nu}^{(i)}$ are negligible in the $s \to 0$ and $s \to \infty$ limits and are nonzero in the vicinity of the point $s = \Lambda^2$. Thus, in fact, there are actually only small corrections to the LO MA couplant $U_{\mathrm{MA},\nu}^{(1)}(s)$. In particular, this proves the possibility of expansions of high-order couplants $U_{\mathrm{MA},\nu}^{(i)}(s)$ via the LO couplants $U_{\mathrm{MA},\nu}^{(1)}(s)$, which was done in Ref. [9].

As an example, we examined the Higgs boson decay into a $b\bar{b}$ pair and obtained results are in good agreement with the Standard Model expectations [63] and with the experimental data [64,65]. Moreover, our results also in good agreement with studies based on the principle of maximum conformality [61].

As a next step, we plan to include $1/L$-expansions for other MA couplants (see Refs. [8,9,47,67]), as well as for nonminimal analytic couplants (following Refs. [29,30,32,68,69]). In the case of nonminimal analytic couplants, one can use the integral representations (32) and (36) with nonperturbative spectral functions.

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APPENDIX A: ANOTHER FORM FOR $U_1^{(i+1)}(\mathrm{s})$

Using the results in Eq. (15) and transformation rules for $\sin(ng)$ and $\cos(ng)$, we have

$$\sin(ng) = \frac{S(ng)}{(\pi^2 + L_s^2)^{n/2}}, \qquad \cos(ng) = \frac{C(ng)}{(\pi^2 + L_s^2)^{n/2}}, \tag{A1}$$

where

$$\begin{aligned} &S(g) = \pi, \qquad C(g) = L_s, \qquad S(2g) = 2\pi L_s, \qquad C(2g) = L_s^2 - \pi^2, \qquad S(3g) = \pi(3L_s^2 - \pi^2), \\ &C(3g) = L_s(L_s^2 - 3\pi^2), \qquad S(4g) = 4\pi L_s(L_s^2 - \pi^2), \qquad C(4g) = L_s^4 - 6\pi^2 L_s^2 + \pi^4, \\ &S(5g) = \pi(5L_s^4 - 10\pi^2 L_s^2 + \pi^4), \qquad C(5g) = L_s(L_s^4 - 10\pi^2 L_s^2 + 5\pi^4). \end{aligned} \tag{A2}$$

Using Eqs. (A1), (A2), and (25), the results for $\tilde{\delta}_{\nu=1}^{(m+1)}(\mathrm{s})$ in (24) can be rewritten in the following form

$$\begin{aligned} \tilde{\delta}_{\nu=1}^{(2)}(\mathrm{s}) &= \frac{b_1}{(\pi^2 + L_s^2)} \left(\frac{g}{\pi} L_s - (1+G) \right), \\ \tilde{\delta}_{\nu=1}^{(3)}(\mathrm{s}) &= \frac{1}{(\pi^2 + L_s^2)^2} \left[b_2 L_s - b_1^2 \left(\frac{gG}{\pi} (L_s^2 - \pi^2) + L_s(1 + g^2 - G^2) \right) \right], \\ \tilde{\delta}_{\nu=1}^{(4)}(\mathrm{s}) &= \frac{1}{(\pi^2 + L_s^2)^3} \left\{ b_1 b_2 \left(\frac{g L_s}{\pi} (L_s^2 - 3\pi^2) - \left(L_s^2 - \frac{1}{3}\pi^2 \right)(1 + 3G) \right) \right. \\ &\quad + \frac{b_3}{6}(3L_s^2 - \pi^2) + \frac{b_1^3}{6} \left((3L_s^2 - \pi^2)[1 + 6G - 3g^2 + 3G^2 + 6g^2 G - 2G^3] \right. \\ &\quad \left. \left. - \frac{2gL_s}{\pi}(L_s^2 - 3\pi^2)[3 + 3G + g^2 - 3G^2] \right) \right\}, \\ \tilde{\delta}_{\nu=1}^{(5)}(\mathrm{s}) &= \frac{1}{(\pi^2 + L_s^2)^4} \left\{ 3b_1^2 b_2 \left(L_s(\pi^2 - L_s^2)[1 + 2g^2 - 2G^2] - \frac{gG}{\pi}(L_s^4 - 6L_s^2\pi^2 + \pi^4) \right) \right. \\ &\quad + 2b_1 b_3 \left(\frac{L_s}{3}(\pi^2 - L_s^2)[1 + 3G] + \frac{g}{4\pi}(L_s^4 - 6L_s^2\pi^2 + \pi^4) \right) - \frac{5b_2^2 + b_4}{3} L_s(\pi^2 - L_s^2) \\ &\quad + b_1^4 \left(L_s(\pi^2 - L_s^2) \left[-\frac{5}{3} - 2G - 4g^2 + 4G^2 - 10g^2 G - \frac{10}{3} G^3 - g^4 + 6g^2 G^2 - G^4 \right] \right. \\ &\quad \left. \left. + \frac{g}{2\pi}(L_s^4 - 6L_s^2\pi^2 + \pi^4) \left[4G - 1 - \frac{5}{3} g^2 + 5G^2 + 2g^2 G - 2G^3 \right] \right) \right\}, \end{aligned} \tag{A3}$$

which is similar to the results for the spectral function $r_1^{(i)}(\mathrm{s})$ done in Refs. [48,49] (see also Sec. VI in [20]).

APPENDIX B: $\bar{m}_b^2(Q^2)$

Here we present evaluation of $\bar{m}_b^2(Q^2)$, which has the form

$$\bar{m}_b^2(Q^2) = \bar{m}_b^2(Q^2) \exp\left[2 \int_{\bar{a}_s(Q_0^2)}^{\bar{a}_s(Q^2)} \frac{\gamma_m(a)}{\beta(a)} \right], \qquad \bar{a}_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}, \tag{B1}$$

where

$$\gamma_m(a) = -\sum_{k=0}\gamma_k a^{k+1} = -\gamma_0 a\left(1+\sum_{k=1}\delta_k a^k\right), \qquad \delta_k = \frac{\gamma_k}{\gamma_0},$$
$$\beta(a) = -\sum_{k=0}\beta_k a^{k+2} = -\beta_0 a^2\left(1+\sum_{k=1}c_k a^k\right), \qquad c_k = \frac{\beta_k}{\beta_0}. \tag{B2}$$

Evaluating the integral in (B1) we have the following results (see, e.g., also Refs. [8,70])

$$\bar{m}_b^2(Q^2) = \bar{m}_b^2(Q^2)\frac{\bar{a}_s^d(Q^2)}{\bar{a}_s^d(Q_0^2)}\frac{T(\bar{a}_s(Q^2))}{T(\bar{a}_s(Q_0^2))}, \tag{B3}$$

where

$$d = \frac{2\gamma_0}{\beta_0}, \qquad T(\bar{a}_s) = \exp\left[\sum_{k=1}^{k=4}\frac{e_k}{k}\bar{a}_s^k\right] \tag{B4}$$

and

$$e_1 = d\Delta_1, \qquad e_2 = d(\Delta_2 - c_1\Delta_1),$$
$$e_3 = d(\Delta_3 - c_1\Delta_2 - \tilde{c}_2\Delta_1),$$
$$e_4 = d(\Delta_4 - c_1\Delta_3 - \tilde{c}_2\Delta_2 - \tilde{c}_3\Delta_1), \tag{B5}$$

with

$$\Delta_i = \delta_i - c_i, \quad \tilde{c}_2 = c_2 - c_1^2, \quad \tilde{c}_3 = c_3 - 2c_1c_2 + c_1^3 \tag{B6}$$

The result for $T(\bar{a}_s)$ can be rewritten as

$$T(\bar{a}_s) = 1 + \sum_{k=1}^{k=4}\frac{\tilde{e}_k}{k}\bar{a}_s^k, \tag{B7}$$

where

$$\tilde{e}_1 = e_1, \qquad \tilde{e}_2 = e_2 + e_1^2, \qquad \tilde{e}_3 = e_3 + \frac{3}{2}e_1e_2 + \frac{1}{2}e_1^3,$$
$$\tilde{e}_3 = e_4 + e_2^2 + \frac{4}{3}e_1e_3 + \frac{1}{2}e_1^3 + e_1^2e_2 + \frac{1}{6}e_1^4, \tag{B8}$$

APPENDIX C: RELATIONS BETWEEN a_s^ν AND $\tilde{a}_\nu$

Considering Ref. [33] we have

$$a_s^\nu = \tilde{a}_\nu + \sum_{m\geq 1}\tilde{k}_m(\nu)\tilde{a}_{\nu+m}, \tag{C1}$$

where

$$\tilde{k}_1(\nu) = -\nu b_1\tilde{B}_1(\nu),$$
$$\tilde{k}_2(\nu) = \nu(\nu+1)\left(-b_2\tilde{B}_2(\nu) + \frac{b_1^2}{2}\tilde{B}_{1,1}(\nu)\right),$$
$$\tilde{k}_3(\nu) = \frac{\nu(\nu+1)(\nu+2)}{2}\left(-b_3\tilde{B}_3(\nu) + b_1b_2\tilde{B}_{1,2}(\nu) - \frac{b_1^3}{3}\tilde{B}_{1,1,1}(\nu)\right),$$
$$\tilde{k}_4(\nu) = \frac{\nu(\nu+1)(\nu+2)(\nu+3)}{6}\left(-b_4\tilde{B}_4(\nu) + b_2^2\tilde{B}_{2,2}(\nu) + \frac{b_1b_3}{2}\tilde{B}_{1,3}(\nu) - \frac{b_1^2b_2}{2}\tilde{B}_{1,1,2}(\nu) + \frac{b_1^4}{4}\tilde{B}_{1,1,1,1}(\nu)\right), \tag{C2}$$

where

$$\tilde{B}_1(\nu) = \tilde{Z}_1(\nu) - 1, \qquad \tilde{B}_2(\nu) = \frac{\nu-1}{2(\nu+1)}, \qquad \tilde{B}_{1,1}(\nu) = \tilde{Z}_2(\nu) - 2\tilde{Z}_1(\nu+1) + 1,$$
$$\tilde{B}_3(\nu) = \frac{1}{6} - \frac{1}{(\nu+1)(\nu+2)}, \qquad \tilde{B}_{1,2}(\nu) = \frac{\nu-1}{6(\nu+1)}\left(6\tilde{Z}_1(\nu+1) - 1 + \frac{4}{\nu+2}\right),$$
$$\tilde{B}_{1,1,1}(\nu) = \tilde{Z}_3(\nu) - 3\tilde{Z}_2(\nu+1) + 3\tilde{Z}_1(\nu+2) - 1,$$
$$\tilde{B}_4(\nu) = \frac{1}{12} - \frac{2}{(\nu+1)(\nu+2)(\nu+3)}, \qquad \tilde{B}_{2,2}(\nu) = \frac{13}{12} - \frac{1}{\nu+1} - \frac{1}{\nu+2} - \frac{1}{\nu+3},$$

$$\tilde{B}_{1,3}(\nu) = \left(1 - \frac{6}{(\nu+1)(\nu+2)}\right)\tilde{Z}_1(\nu+3) + \frac{1}{6} + \frac{4}{\nu+1} - \frac{5}{\nu+2} - \frac{2}{\nu+3},$$
$$\tilde{B}_{1,1,2}(\nu) = \frac{3(\nu-1)}{2(\nu+1)}\tilde{Z}_2(\nu+2) - \left(1 - \frac{6}{(\nu+1)(\nu+2)}\right)\tilde{Z}_1(\nu+3) + \frac{8}{3} - \frac{2}{\nu+1} + \frac{1}{\nu+2} - \frac{8}{\nu+3},$$
$$\tilde{B}_{1,1,1,1}(\nu) = \tilde{Z}_4(\nu) - 4\tilde{Z}_3(\nu+1) + 6\tilde{Z}_2(\nu+2) - 4\tilde{Z}_1(\nu+3) + 1 \quad \text{(C3)}$$

and

$$\tilde{Z}_1(\nu) = S_1(\nu), \qquad \tilde{Z}_2(\nu) = S_1^2(\nu) + S_2(\nu),$$
$$\tilde{Z}_3(\nu) = S_1^3(\nu) + 3S_2(\nu)S_1(\nu) + 2S_3(\nu),$$
$$\tilde{Z}_4(\nu) = S_1^4(\nu) + 6S_2(\nu)S_1^2(\nu) + 3S_2^2(\nu) + 8S_3(\nu)S_1(\nu) + 6S_4(\nu). \quad \text{(C4)}$$

For arbitrary ν values, $S_i(\nu)$ are expressed through polygamma-functions as

$$S_1(\nu) = \Psi(\nu+1) - \Psi(1), \qquad \Psi(\nu) \equiv \frac{d}{d\nu}\ln\Gamma(\nu), \qquad \Psi^{(i)}(\nu) \equiv \frac{d^i}{(d\nu)^i}\Psi(\nu),$$
$$S_{i+1}(\nu) \equiv \frac{(-1)^i}{i!}(\Psi^{(i)}(\nu+1) - \Psi^{(i)}(1)). \quad \text{(C5)}$$

In the case of integer $\nu = n$,

$$S_i(n) = \sum_{m=1}^{n}\frac{1}{m^i}. \quad \text{(C6)}$$

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