Near-threshold structures in the $D_s^+D_s^-$ mass distribution of the decay $B^+ \to D_s^+D_s^-K^+$

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Two near-threshold peaking structures with spin-parities $J^{PC} = 0^{++}$ were recently discovered by the LHCb Collaboration in the $D_s^+D_s^-$ invariant mass distribution of the decay $B^+ \to D_s^+D_s^-K^+$. The first of them is the resonance $X(3960)$, whereas the second one, $X_0(4140)$, is a structure with the mass around 4140 MeV. To explore their natures and model them, we study the hadronic molecule $\mathcal{M} = D_s^+D_s^-$ and calculate its mass, current coupling, and width. The mass and current coupling of the molecule are extracted from the QCD two-point sum rule analyses by taking into account vacuum condensates up to dimension 10. To evaluate its full width, we consider the processes $\mathcal{M} \to D_s^+D_s^-$, $\mathcal{M} \to \eta_c\eta^{(\prime)}$, and $\mathcal{M} \to J/\psi\phi$. Partial widths of these decays are determined by the strong couplings $g_i, i = 1, 2, 3, 4$ at vertices $\mathcal{M}D_s^+D_s^-$, $\mathcal{M}\eta_c\eta^{(\prime)}$, and $\mathcal{M}J/\psi\phi$. They are computed by means of the three-point sum rule method. Predictions for the mass $m = (4117 \pm 85)$ MeV and width $\Gamma_{\mathcal{M}} = (62 \pm 12)$ MeV of the molecule $\mathcal{M}$ are compared with the corresponding LHCb data, and also with our results for the diquark-antidiquark state $X = [cs][\bar{c}\,\bar{s}]$. We argue that the structure $X_0(4140)$ may be interpreted as the hadronic molecule $D_s^+D_s^-$, whereas the resonance $X(3960)$ can be identified with the tetraquark X.

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I. INTRODUCTION

Different X resonances discovered and studied during past years by LHCb Collaboration became important part of exotic meson spectroscopy. Thus, the resonances $X(4140)$, $X(4274)$, $X(4500)$, and $X(4700)$ were seen in the process $B^+ \to J/\psi\phi K^+$ as peaks in the $J/\psi\phi$ invariant mass distribution [1]. The structures $X(4140)$ and $X(4274)$ bear the quantum numbers $J^{PC} = 1^{++}$, and in the tetraquark model are composed of the quarks $c\bar{c}s\bar{s}$. The resonances $X(4500)$ and $X(4700)$ are scalar particles with spin-parities $J^{PC} = 0^{++}$ and the same quark content. It is worth noting that $X(4140)$ and $X(4274)$ were observed previously in the decays $B^\pm \to J/\psi\phi K^\pm$ by different collaborations [2–4], whereas the scalar resonances $X(4500)$ and $X(4700)$ were discovered for the first time by LHCb. The resonance $X(4630)$ fixed in the $J/\psi\phi$ invariant mass distribution of the decay $B^+ \to J/\psi\phi K^+$ is a vector member of the X tetraquarks' family [5].

Recently, LHCb reported new hidden charm-strange structures in the $D_s^+D_s^-$ invariant mass distribution of the decay $B^+ \to D_s^+D_s^-K^+$ [6]. One of them, $X(3960)$, is presumably a tetraquark $c\bar{c}s\bar{s}$ with quantum numbers $J^{PC} = 0^{++}$, and the following parameters:

$$m_{1\,\mathrm{exp}} = (3956 \pm 5 \pm 10)\ \mathrm{MeV},$$
$$\Gamma_{1\,\mathrm{exp}} = (43 \pm 13 \pm 8)\ \mathrm{MeV}. \quad (1)$$

This structure is approximately 20 MeV above the $D_s^+D_s^-$ threshold. The LHCb also found evidence for a second structure $X_0(4140)$ with the mass around 4140 MeV and higher ∼17 MeV than the $J/\psi\phi$ threshold. The mass and full width of this state are

$$m_{2\,\mathrm{exp}} = (4133 \pm 6 \pm 6)\ \mathrm{MeV},$$
$$\Gamma_{2\,\mathrm{exp}} = (67 \pm 17 \pm 7)\ \mathrm{MeV}. \quad (2)$$

The $X_0(4140)$ may be interpreted as a new resonance with either a $J^{PC} = 0^{++}$ assignment or a $J/\psi\phi \leftrightarrow D_s^+D_s^-$ coupled-channel effect [6]. In the present work, we assume

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that the structure $X_0(4140)$ is a second resonance seen by LHCb in the $D_s^+D_s^-$ mass distribution.

The X resonances are interesting objects for theoretical investigations: Features of exotic mesons $c\bar{c}s\bar{s}$ were studied in numerous articles by employing different models and technical methods [7–14]. Some of these states have undergone rather detailed exploration, which is also provided in our publications. Thus, the axial-vector resonances $X(4140)$ and $X(4274)$ were analyzed in Ref. [12], in which they were modeled as states composed of scalar and axial-vector (anti)diquarks. In the case of $X(4140)$ the constituent diquark (antidiquark) is the antitriplet (triplet) state of the color group $SU_c(3)$, whereas to model $X(4274)$ we used (anti)diquarks from the sextet representation of $SU_c(3)$. Predictions for masses and full widths of these tetraquarks were compared with the LHCb data. It turned out for masses of the resonances $X(4140)$ and $X(4274)$ that these models led to nice agreements with the LHCb data. The model based on color triplet (anti)diquarks also reproduced the full width of $X(4140)$; therefore, it could be considered as a serious candidate to resonance $X(4140)$. The width of the construction with color sextet constituents is wider than that of $X(4274)$, which excludes it from a list of possible pretenders.

The vector resonances $Y(4660)$ and $X(4630)$ were explored in our articles (see Refs. [13,14]) as tetraquarks $[cs][\bar{c}\,\bar{s}]$ with spin-parities $J^{\rm PC}=1^{--}$ and $J^{\rm PC}=1^{-+}$, respectively. Predictions for the masses and full widths of these states allowed us to interpret $Y(4660)$ and $X(4630)$ as exotic mesons with diquark-antidiquark composition.

The discovery of structures $X(3960)$ and $X_0(4140)$ activated investigations of hidden charm-strange resonances to account for features of these new states [15–20]. The $X(3960)$ was considered as a coupled-channel effect [15], or as near the $D_s^+D_s^-$ threshold enhancement by the conventional P-wave charmonium $\chi_{c0}(2P)$ [20]. The hadronic $D_s^+D_s^-$ molecule model was suggested in Ref. [17] to explain observed properties of the resonance $X(3960)$.

In our paper [21], we examined the tetraquark $X=[cs][\bar{c}\,\bar{s}]$ with quantum numbers $J^{\rm PC}=0^{++}$ and calculated its mass and full width. The spectroscopic parameters of X, i.e., its mass and current coupling, were found by means of the QCD two-point sum rule method. The full width of this state was evaluated using decay channels $X\to D_s^+D_s^-$ and $X\to\eta_c\eta^{(\prime)}$. Predictions for the mass $m=(3976\pm85)$ MeV and width $\Gamma_{\rm X}=(42.2\pm8.3)$ MeV obtained in Ref. [21] allowed us to consider the diquark-antidiquark state X as an acceptable model for $X(3960)$.

In the present article, we continue our studies of the resonance $X(3960)$ and include in the analysis the structure $X_0(4140)$. We investigate the hadronic molecule $\mathcal{M}=D_s^+D_s^-$ by computing its mass, current coupling, and full width. The mass and current coupling of this state are calculated in the context of the QCD two-point sum rule approach. The full width of $\mathcal{M}$ is estimated by considering the decay channels $\mathcal{M}\to D_s^+D_s^-$, $\mathcal{M}\to\eta_c\eta$, $\mathcal{M}\to\eta_c\eta^{\prime}$, and $\mathcal{M}\to J/\psi\phi$. Partial widths of these processes, apart from parameters of the initial and final particles, depend also on strong couplings $g_i, i=1,\ 2,\ 3,\ 4$ at vertices $\mathcal{M}D_s^+D_s^-$, $\mathcal{M}\eta_c\eta^{\prime}$, $\mathcal{M}\eta_c\eta$, and $\mathcal{M}J/\psi\phi$, respectively. To extract numerical values of g_i, we use the QCD three-point sum rule method. Predictions for the mass and width of the molecule $\mathcal{M}$ are compared with the LHCb data for the resonances $X(3960)$ and $X_0(4140)$. They are also confronted with parameters of the diquark-antidiquark state $X=[cs][\bar{c}\,\bar{s}]$.

This paper is organized in the following way: In Sec. II, we compute the mass and current coupling of the molecule $\mathcal{M}$ by employing the QCD two-point sum rule method. The dominant process $\mathcal{M}\to D_s^+D_s^-$ is considered in Sec. III, where we determine the coupling g_1 and partial width of this decay. The decays $\mathcal{M}\to\eta_c\eta^{(\prime)}$ and $\mathcal{M}\to J/\psi\phi$ are analyzed in Sec. IV. The full width of $\mathcal{M}$ is evaluated also in this section. Section V is reserved for discussion and summing up.

II. SPECTROSCOPIC PARAMETERS OF THE MOLECULE $\mathcal{M}=D_s^+D_s^-$

The mass and current coupling of the molecule $\mathcal{M}=D_s^+D_s^-$ can be evaluated using the QCD two-point sum rule method [22,23]. This approach works quite well not only for analysis of conventional particles but also leads to reliable predictions in the case of multiquark hadrons.

The principal quantity in this method is an interpolating current for a hadron under analysis. In the case of the molecule $D_s^+D_s^-$, this current $J(x)$ has a rather simple form

$$J(x)=[\bar{s}_a(x)i\gamma_5c_a(x)][\bar{c}_b(x)i\gamma_5s_b(x)], \tag{3}$$

where a and b are color indices. This current belongs to the $[\mathbf{1}_c]_{\bar{s}c}\otimes[\mathbf{1}_c]_{\bar{c}s}$ representation of the color group $SU_c(3)$. It corresponds to a molecule structure with spin-parities $J^{\rm PC}=0^{++}$, but may also couple to different diquark-antidiquark structures and other four-quark hadronic molecules [24,25].

To find sum rules for the mass m and coupling f of the molecule $\mathcal{M}$, one has to start from the calculation of the following correlation function:

$$\Pi(p)=i\int d^4x e^{ipx}\langle 0|\mathcal{T}\{J(x)J^{\dagger}(0)\}|0\rangle. \tag{4}$$

At the first stage, we express $\Pi(p)$ in terms of the spectral parameters of $\mathcal{M}$. To this end, it is necessary to insert a complete set of states with $J^{\rm PC}=0^{++}$ into the correlation function $\Pi(p)$ and carry out integration over x. These operations lead to the simple formula

$$\Pi^{\rm Phys}(p)=\frac{\langle 0|J|\mathcal{M}(p)\rangle\langle\mathcal{M}(p)|J^{\dagger}|0\rangle}{m^2-p^2}+\cdots. \tag{5}$$

The expression derived by this method is a hadronic representation of the correlator $\Pi(p)$, which forms the phenomenological side of sum rules. The term written down explicitly in Eq. (5) is a contribution of the ground-state particle $\mathcal{M}$, whereas contributions coming from higher resonances and continuum states are denoted by dots.

The function $\Pi^{\rm Phys}(p)$ can be rewritten in a more convenient form using the matrix element

$$\langle 0|J|\mathcal{M}(p)\rangle = fm. \tag{6}$$

Then, it is not difficult to find $\Pi^{\rm Phys}(p)$ in terms of the parameters m and f,

$$\Pi^{\rm Phys}(p) = \frac{m^2 f^2}{m^2 - p^2} + \cdots. \tag{7}$$

The Lorentz structure of $\Pi^{\rm Phys}(p)$ has a simple form and consists of a term proportional to I. Then, the invariant amplitude $\Pi^{\rm Phys}(p^2)$ corresponding to this structure is given by the expression in the right-hand side of Eq. (7).

The QCD side of the sum rules $\Pi^{\rm OPE}(p)$ is determined by Eq. (4) calculated using the c and s-quarks propagators. To this end, we insert the explicit form of $J(x)$ into Eq. (4), contract heavy and light quark fields, and write the obtained expression using quark propagators. After these operations, we get

$$\Pi^{\rm OPE}(p) = i\int d^4x e^{ipx}{\rm Tr}[\gamma_5 S_c^{aa'}(x)\gamma_5 S_s^{a'a}(-x)] \times {\rm Tr}[\gamma_5 S_c^{b'b}(-x)\gamma_5 S_s^{bb'}(x)]. \tag{8}$$

Here $S_c(x)$ and $S_s(x)$ are the c and s-quark propagators, explicit expressions of which can be found in Ref. [26].

The correlation function $\Pi^{\rm OPE}(p)$ is calculated by employing the quark propagators with some fixed accuracy of the operator product expansion (OPE). The $\Pi^{\rm OPE}(p)$ has a trivial Lorentz structure ∼I as well. Having denoted the corresponding invariant amplitude by $\Pi^{\rm OPE}(p^2)$ and equated it to $\Pi^{\rm Phys}(p^2)$, we get a sum rule equality, which can undergo further processing. The ground-state term and ones due to higher resonances and continuum states contribute to this sum rule equality on equal footing. There is a necessity to suppress unwanted contributions of higher resonances and subtract them from this expression. For these purposes, we apply the Borel transformation to both its sides. This operation suppresses effects of higher resonances and continuum states, but at the same time generates dependence of the obtained equality on the Borel parameter M^2. Afterwards, using the assumption about quark-hadron duality, we perform continuum subtraction, which leads to additional parameter s_0 in formulas.

The Borel transformation of the main term in $\Pi^{\rm Phys}(p^2)$ has a simple form,

$$\Pi^{\rm Phys}(M^2) = m^2 f^2 e^{-m^2/M^2}. \tag{9}$$

For the Borel transformed and subtracted amplitude $\Pi^{\rm OPE}(p^2)$, we find

$$\Pi(M^2, s_0) = \int_{4(m_c+m_s)^2}^{s_0} ds\rho^{\rm OPE}(s)e^{-s/M^2} + \Pi(M^2). \tag{10}$$

The first term in Eq. (10) contains an essential part of contributions and is expressed using the two-point spectral density $\rho^{\rm OPE}(s)$, derived as an imaginary part of the correlation function. The second term $\Pi(M^2)$ collects nonperturbative contributions extracted directly from $\Pi^{\rm OPE}(p)$.

The sum rules for m and f read

$$m^2 = \frac{\Pi'(M^2, s_0)}{\Pi(M^2, s_0)} \tag{11}$$

and

$$f^2 = \frac{e^{m^2/M^2}}{m^2}\Pi(M^2, s_0), \tag{12}$$

where $\Pi'(M^2, s_0) = d\Pi(M^2, s_0)/d(-1/M^2)$.

Numerical calculations of m and f should be performed in accordance with Eqs. (11) and (12), but only after fixing different vacuum condensates and working windows for the parameters M^2 and s_0. The quark, gluon, and mixed condensates are universal and well-known quantities [22,23,27–29]. Their numerical values, extracted from numerous processes are listed below:

$$\begin{aligned}
\langle\bar{q}q\rangle &= -(0.24 \pm 0.01)^3\ {\rm GeV}^3, \qquad \langle\bar{s}s\rangle = (0.8 \pm 0.1)\langle\bar{q}q\rangle,\\
\langle\bar{s}g_s\sigma Gs\rangle &= m_0^2\langle\bar{s}s\rangle, \qquad m_0^2 = (0.8 \pm 0.1)\ {\rm GeV}^2,\\
\left\langle\frac{\alpha_s G^2}{\pi}\right\rangle &= (0.012 \pm 0.004)\ {\rm GeV}^4,\\
\langle g_s^3 G^3\rangle &= (0.57 \pm 0.29)\ {\rm GeV}^6,\\
m_c &= (1.27 \pm 0.02)\ {\rm GeV}, \qquad m_s = 93^{+11}_{-5}\ {\rm MeV}.
\end{aligned} \tag{13}$$

We have included the masses of the c and s-quarks into Eq. (13) as well. The correlation function $\Pi(M^2, s_0)$ is calculated by taking into account vacuum condensates up to dimension ten. The expression of $\Pi(M^2, s_0)$ is rather lengthy; therefore, we do not provide it here explicitly. In numerical analysis we set $m_s^2 = 0$, but take into account contributions proportional to m_s.

To carry out numerical analysis one also needs to choose working regions for the Borel and continuum subtraction parameters M^2 and s_0. They should satisfy standard constraints of sum rule calculations. Thus, the parameters M^2 and s_0 employed in calculations have to guarantee the dominance of the pole contribution (PC) and convergence of OPE. The former can be defined by the expression

$$\text{PC} = \frac{\Pi(M^2, s_0)}{\Pi(M^2, \infty)}, \tag{14}$$

whereas to make sure the operator product expansion converges, we utilize the ratio

$$R(M^2) = \frac{\Pi^{\text{DimN}}(M^2, s_0)}{\Pi(M^2, s_0)}, \tag{15}$$

where $\Pi^{\text{DimN}}(M^2, s_0)$ is a sum of a few last terms in OPE.

The $R(M^2)$ and PC are used to restrict the lower and upper bounds for the Borel parameter, respectively. In fact, at M^2_{min} the function $R(M^2_{\text{min}})$ should be less than some fixed value, whereas at M^2_{max} the pole contribution PC has to overshoot the minimally acceptable limit for this parameter. In our present analysis, we impose on PC and $R(M^2)$ the following constraints:

$$\text{PC} \geq 0.5, \qquad R(M^2_{\text{min}}) \leq 0.05. \tag{16}$$

The first criterion in Eq. (16) is usual for investigations of conventional hadrons and ensures the dominance of the pole contribution. It may be used in the analysis of multiquark hadrons as well. But in the case of multiquark particles this constraint reduces a window for M^2. The second condition is required to enforce the convergence of OPE.

Dominance of a perturbative contribution to $\Pi(M^2, s_0)$ over a nonperturbative term, as well as maximum stability of extracted physical quantities when varying M^2, is among constraints to determine working regions for the parameters M^2 and s_0. Numerical tests carried out by taking into account these aspects of the sum rule analysis demonstrate that the windows for M^2 and s_0,

$$M^2 \in [3, 4]\ \text{GeV}^2, \qquad s_0 \in [21, 22]\ \text{GeV}^2, \tag{17}$$

comply with aforementioned restrictions. Indeed, at $M^2 = 4\ \text{GeV}^2$ the pole contribution on average in s_0 equals 0.52 and amounts to 0.88 at $M^2 = 3\ \text{GeV}^2$. To visualize dynamics of the pole contribution, in Fig. 1 we depict PC as a function of M^2 at different s_0. One can see that the pole contribution overshoots 0.5 for all values of the parameters M^2 and s_0 from Eq. (17).

To be convinced in convergence of OPE, we calculate $R(M^2_{\text{min}})$ at the minimum point $M^2_{\text{min}} = 3\ \text{GeV}^2$ using in Eq. (15) dimension-eight, -nine, and -ten contributions. At the minimal value of M^2, we find $R(3\ \text{GeV}^2) \approx 0.01$ in accordance with the constraint from Eq. (17). Results of a more detailed analysis are shown in Fig. 2, where one sees contributions to the correlation function $\Pi(M^2, s_0)$ arising from the perturbative and nonperturbative terms up to dimension eight. The perturbative contribution forms the 0.65 part of $\Pi(M^2, s_0)$ at $M^2 = 3\ \text{GeV}^2$ and exceeds the sum of nonperturbative terms in the whole region of M^2. The Dim3 term overshoots effects of other nonperturbative operators, which enter to $\Pi(M^2, s_0)$ with different signs. The Dim9 and Dim10 terms are numerically very small and not demonstrated in the figure.

Our predictions for the mass m and coupling f read

$$\begin{aligned} m &= (4117 \pm 85)\ \text{MeV}, \\ f &= (5.9 \pm 0.7) \times 10^{-3}\ \text{GeV}^4. \end{aligned} \tag{18}$$

The results for m and f are calculated as their values averaged over the working regions Eq. (17). Effectively they correspond to the sum rule predictions at $M^2 = 3.5\ \text{GeV}^2$ and $s_0 = 21.5\ \text{GeV}^2$, which is a middle point of the regions of Eq. (17): the red triangle in Fig. 1 marks exactly this point. The pole contribution there is equal to $\text{PC} \approx 0.68$, which in conjunction with other constraints ensures the ground-state nature of the molecule $D_s^+ D_s^-$ and credibility of obtained predictions.

In Fig. 3, we depict the mass m of the molecule $D_s^+ D_s^-$ as a function of the Borel and continuum subtraction parameters. Physical quantities obtained from the sum rule analysis should be stable against variations of the Borel parameter. But the mass m depends on working windows chosen for their calculations. In fact, although the region for the Borel parameter M^2 leads to an approximately stable prediction for m, there is still residual dependence on it. This effect generates theoretical uncertainties of the sum rule calculations. It is worth noting that the uncertainties of m are smaller than ones for the coupling f. The reason is that the mass m is given by the ratio of the correlation functions which compensates changes of m against M^2 and s_0. The coupling f, at the same time, depends on $\Pi(M^2, s_0)$ and is open for an impact of the parameters M^2 and s_0. As a result, uncertainties of calculations are equal to $\pm 2\%$ in the case of the mass, and to $\pm 12\%$ for the current coupling.

The region for s_0 together with M^2 has to provide the dominance of PC and convergence of the operator product

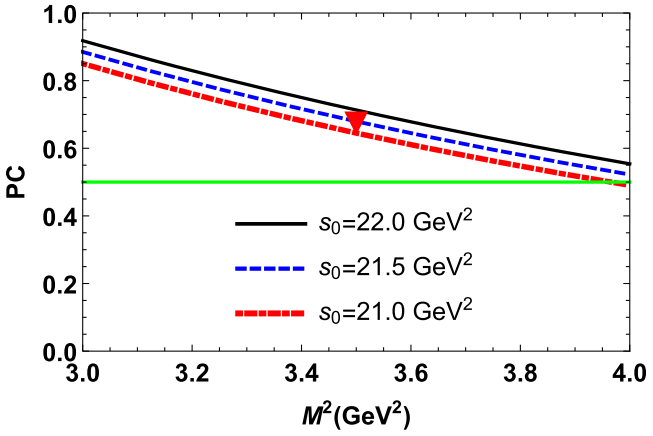

FIG. 1. Dependence of the pole contribution PC on the Borel parameter M^2 at different s_0. The limit PC $= 0.5$ is shown by the horizontal black line. The red triangle shows the point where the mass m of the molecule $\mathcal{M} = D_s^+ D_s^-$ has effectively been calculated.

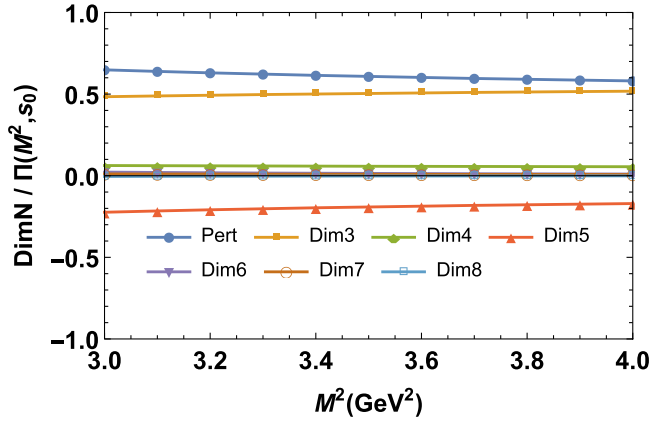

FIG. 2. Normalized contributions to $\Pi(M^2, s_0)$ as functions of the Borel parameter M^2. All curves have been calculated at $s_0 = 21.5$ GeV^2.

expansion. The parameter $\sqrt{s_0}$ also carries useful information on a mass m^* of the first radial excitation of the molecule $D_s^+ D_s^-$. Thus, in the "ground-state +continuum" scheme adopted in the present work, $\sqrt{s_0}$ should be less than the mass m^* of the first excited state. This fact allows us to estimate the low limit for $m^* \geq m + 480$ MeV.

Our result for the mass of the molecule $\mathcal{M}$ overshoots the LHCb datum $m_{1\,\mathrm{exp}}$, but nicely agrees with $m_{2\,\mathrm{exp}}$.

III. PARTIAL WIDTH OF THE PROCESS $\mathcal{M} \to D_s^+ D_s^-$

The mass and current coupling of the molecule $\mathcal{M}$ calculated in the previous section provide information to select its possible decay modes. Besides, one should take into account its quantum numbers $J^{\mathrm{PC}} = 0^{++}$. Because the structures $X(3960)$ and $X_0(4140)$ were discovered in the invariant mass distribution of the $D_s^+ D_s^-$ mesons, we consider $\mathcal{M} \to D_s^+ D_s^-$ as a dominant decay mode of $\mathcal{M}$. The two-meson threshold for this decay is equal approximately to 3937 MeV, which makes $\mathcal{M} \to D_s^+ D_s^-$ the kinematically allowed channel for $\mathcal{M}$.

The partial width of the decay $\mathcal{M} \to D_s^+ D_s^-$ is governed by a coupling g_1 that describes strong interaction at the vertex $\mathcal{M} D_s^+ D_s^-$. This partial width depends also on masses and decay constants of the molecule $\mathcal{M}$ and mesons D_s^+ and D_s^-. The mass and current coupling of $\mathcal{M}$ have been calculated in the present article, whereas physical parameters of the mesons D_s^+ and D_s^- are known from independent sources. Therefore, the only physical quantity to be found is the strong coupling g_1.

To determine g_1, we employ the QCD three-point sum rule method and begin our exploration from the correlation function

$$\Pi(p, p') = i^2 \int d^4x d^4y e^{i(p'y - px)} \langle 0|\mathcal{T}\{J^{D_s^+}(y) \times J^{D_s^-}(0) J^\dagger(x)\}|0\rangle. \quad (19)$$

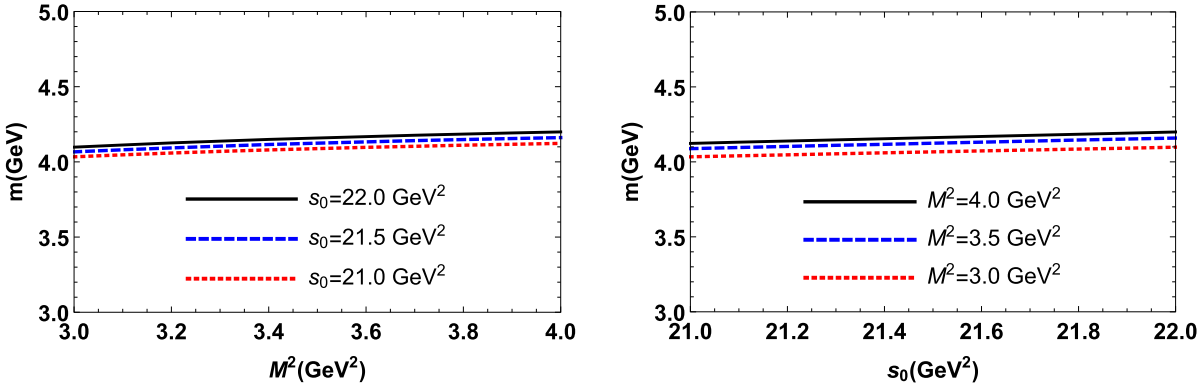

FIG. 3. Mass m of the molecule $\mathcal{M}$ as a function of the Borel M^2 (left panel), and continuum threshold s_0 parameters (right panel).

TABLE I. Masses and decay constants of the mesons $D_s^{\pm}$, η_c, η', η, J/ψ, and ϕ which are employed in numerical calculations.

Quantity	Value [in MeV units]
m_{D_s}	1969.0 ± 1.4
m_{η_c}	2983.9 ± 0.4
$m_{\eta'}$	957.78 ± 0.06
m_{η}	547.862 ± 0.017
$m_{J/\psi}$	3096.900 ± 0.006
m_{ϕ}	1019.461 ± 0.019
f_{D_s}	249.9 ± 0.5
f_{η_c}	320 ± 40
$f_{J/\psi}$	409 ± 15
f_{ϕ}	228.5 ± 3.6

Here, $J(x)$, $J^{D_s^+}(y)$, and $J^{D_s^-}(0)$ are the interpolating currents for $\mathcal{M}$ and pseudoscalar mesons D_s^+ and D_s^-, respectively. The currents $J^{D_s^+}$ and $J^{D_s^-}$ are given by the expressions

$$J^{D_s^+}(x) = \bar{s}_j(x) i\gamma_5 c_j(x),$$
$$J^{D_s^-}(x) = \bar{c}_i(x) i\gamma_5 s_i(x), \tag{20}$$

where i and j are color indices. The four-momenta of $\mathcal{M}$ and D_s^+ are labeled by p and p': Then, the momentum of the meson D_s^- is equal to $q = p - p'$.

To find g_1, we apply standard recipes of the sum rule method and calculate the correlation function $\Pi(p, p')$. For these purposes, we use the physical parameters of the molecule $\mathcal{M}$ and mesons involved in this decay. The correlator $\Pi(p, p')$ obtained by this manner forms the physical side $\Pi^{\rm Phys}(p, p')$ of the sum rule. It is easy to see that

$$\Pi^{\rm Phys}(p, p') = \frac{\langle 0|J^{D_s^+}|D_s^+(p')\rangle \langle 0|J^{D_s^-}|D_s^-(q)\rangle}{(p'^2 - m_{D_s}^2)(q^2 - m_{D_s}^2)} \times \frac{\langle D_s^-(q) D_s^+(p')|\mathcal{M}(p)\rangle \langle \mathcal{M}(p)|J^{\dagger}|0\rangle}{(p^2 - m^2)} + \cdots, \tag{21}$$

with m_{D_s} being the mass of the mesons $D_s^{\pm}$. To get Eq. (21), we isolate contributions of the ground-state and higher resonances and continuum state particles from each other. In Eq. (21) the ground-state term is written down explicitly, whereas other contributions are denoted by ellipses.

The function $\Pi^{\rm Phys}(p, p')$ can be rewritten in terms of the $D_s^{\pm}$ mesons matrix elements

$$\langle 0|J^{D_s^{\pm}}|D_s^{\pm}\rangle = \frac{m_{D_s}^2 f_{D_s}}{m_c + m_s}, \tag{22}$$

where f_{D_s} is their decay constant. We model the vertex $\mathcal{M}D_s^+D_s^-$ by the matrix element

$$\langle D_s^-(q) D_s^+(p')|\mathcal{M}(p)\rangle = g_1(q^2) p \cdot p'. \tag{23}$$

Using these expressions, it is not difficult to calculate the new expression of the correlation function $\Pi^{\rm Phys}(p, p')$,

$$\Pi^{\rm Phys}(p, p') = g_1(q^2) \frac{m_{D_s}^4 f_{D_s}^2 fm}{(m_c + m_s)^2 (p^2 - m^2)} \times \frac{1}{(p'^2 - m_{D_s}^2)(q^2 - m_{D_s}^2)} \times \frac{m^2 + m_{D_s}^2 - q^2}{2} + \cdots. \tag{24}$$

The double Borel transformation of the function $\Pi^{\rm Phys}(p, p')$ over variables p^2 and p'^2 is given by the formula

$$\mathcal{B}\Pi^{\rm Phys}(p, p') = g_1(q^2) \frac{m_{D_s}^4 f_{D_s}^2 fm}{(m_c + m_s)^2 (q^2 - m_{D_s}^2)} e^{-m^2/M_1^2} \times e^{-m_{D_s}^2/M_2^2} \frac{m^2 + m_{D_s}^2 - q^2}{2} + \cdots. \tag{25}$$

The correlator $\Pi^{\rm Phys}(p, p')$ and its Borel transformation has simple Lorentz structure $\sim$I. Then the whole expression in Eq. (24) determines the invariant amplitude $\Pi^{\rm Phys}(p^2, p'^2, q^2)$.

To find the QCD side of the three-point sum rule, we express $\Pi(p, p')$ in terms of quark propagators and get

$$\Pi^{\rm OPE}(p, p') = i^2 \int d^4x d^4y e^{i(p'y - px)} \times {\rm Tr}[\gamma_5 S_c^{ia}(y - x)\gamma_5 S_s^{ai}(x - y)] \times {\rm Tr}[\gamma_5 S_s^{jb}(-x)\gamma_5 S_c^{bj}(x)]. \tag{26}$$

The correlator $\Pi^{\rm OPE}(p, p')$ is computed by taking into account nonperturbative contributions up to dimension 6, and as $\Pi^{\rm Phys}(p, p')$, contains the Lorentz structure proportional to I. Denoting relevant invariant amplitude by $\Pi^{\rm OPE}(p^2, p'^2, q^2)$, equating its double Borel transformation $\mathcal{B}\Pi^{\rm OPE}(p^2, p'^2, q^2)$ to $\mathcal{B}\Pi^{\rm Phys}(p^2, p'^2, q^2)$, and performing continuum subtraction, we get the sum rule for the coupling $g_1(q^2)$.

After these manipulations, the amplitude $\Pi^{\rm OPE}(p^2, p'^2, q^2)$ can be rewritten using the spectral density $\rho(s, s', q^2)$, which is extracted as an imaginary part of $\Pi^{\rm OPE}(p, p')$,

$$\Pi(\mathbf{M}^2, \mathbf{s}_0, q^2) = \int_{4(m_c + m_s)^2}^{s_0} ds \int_{(m_c + m_s)^2}^{s_0'} ds' \rho(s, s', q^2) \times e^{-s/M_1^2} e^{-s'/M_2^2}, \tag{27}$$

where $\mathbf{M}^2 = (M_1^2, M_2^2)$ and $\mathbf{s}_0 = (s_0, s_0')$ are the Borel and continuum threshold parameters. The couples (M_1^2, s_0) and $\mathbf{s}_0 = (M_2^2, s_0')$ correspond to the molecule M and D_s^+ meson channels, respectively.

The sum rule for $g_1(q^2)$ is determined by the formula

$$g_1(q^2) = \frac{2(m_c + m_s)^2}{m_{D_s}^4 f_{D_s}^2 f m} \frac{q^2 - m_{D_s}^2}{m^2 + m_{D_s}^2 - q^2} \times e^{m^2/M_1^2} e^{m_{D_s}^2/M_2^2} \Pi(\mathbf{M}^2, \mathbf{s}_0, q^2). \quad (28)$$

The expression of $g_1(q^2)$ depends on the spectroscopic parameters of the molecule $\mathcal{M}$, as well as the masses and decay constants of the mesons $D_s^{\pm}$: they are input parameters of numerical computations. Values of these parameters as well as masses and decay constants f_{D_s}, f_{η_c}, f_ϕ, and $f_{J/\psi}$ that are necessary to study other decays are collected in Table I. The masses all of mesons are borrowed from Ref. [30]. For the decay constant of the mesons $D_s^{\pm}$, we employ information from the same source, whereas for f_{η_c} we use a prediction made in Ref. [31] on the basis of the sum rule method. As the decay constants f_ϕ and $f_{J/\psi}$ of the vector mesons ϕ and J/ψ, we utilize the experimental values reported in Refs. [32,33], respectively.

The partial width of the decay $\mathcal{M} \to D_s^+ D_s^-$ besides various input parameters also is determined by the strong coupling $g_1(m_{D_s}^2)$ at the mass shell $q^2 = m_{D_s}^2$ of the meson D_s^-. At the same time, the sum rule computations of g_1 can be carried out in the deep-Euclidean region $q^2 < 0$. For simplicity, we introduce a variable $Q^2 = -q^2$ and in what follows label the obtained function by $g_1(Q^2)$. An explored range of Q^2 covers the region $Q^2 = 1$–5 GeV^2.

Numerical calculations also require choosing the working regions for the Borel and continuum subtraction parameters $\mathbf{M}^2$ and $\mathbf{s}_0$. Limits imposed on $\mathbf{M}^2$ and $\mathbf{s}_0$ are standard for sum rule calculations and were considered in the previous section. The regions for M_1^2 and s_0 associated with the $\mathcal{M}$ channel are fixed in accordance with Eq. (17). The parameters (M_2^2, s_0') for the D_s^+ meson channel are varied within limits

$$M_2^2 \in [2.5, 3.5] \text{ GeV}^2, \qquad s_0' \in [5, 6] \text{ GeV}^2. \quad (29)$$

Regions for $\mathbf{M}^2$ and $\mathbf{s}_0$ are chosen in such a way to minimize their effects of the coupling $g_1(Q^2)$.

Results of computations are pictured in Fig. 4. It is seen that results for $g_1(Q^2)$ are extracted at the region $Q^2 > 0$, where the sum rule gives reliable predictions. It has just been explained above that we need $g_1(Q^2)$ at $Q^2 = -m_{D_s}^2$. To this end, one has to introduce some fit function $\mathcal{G}_1(Q^2)$, which at the momenta $Q^2 > 0$ gives the same values as the sum rule computations, but can easily be extrapolated to the region $Q^2 < 0$. There are different choices for such functions. In this article, we use $\mathcal{G}_i(Q^2)$, $i = 1, 2, 3$,

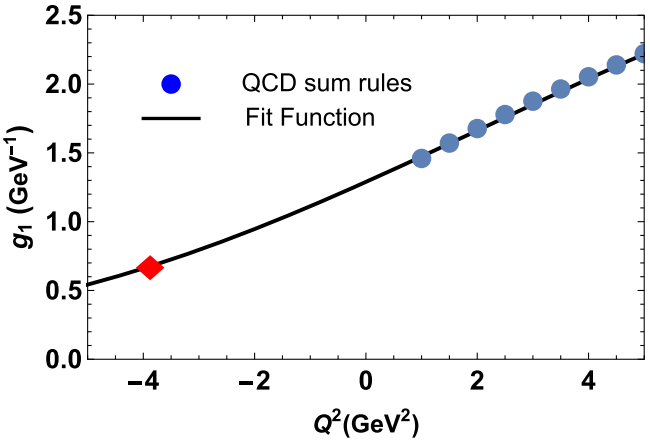

FIG. 4. The sum rule results and fit function for the strong coupling $g_1(Q^2)$. The red diamond denotes the point $Q^2 = -m_{D_s}^2$.

$$\mathcal{G}_i(Q^2) = \mathcal{G}_i^0 \exp\left[c_i^1 \frac{Q^2}{m^2} + c_i^2 \left(\frac{Q^2}{m^2}\right)^2\right], \quad (30)$$

where $\mathcal{G}_i^0$, c_i^1, and c_i^2 are parameters, which should be extracted from fitting procedures. Calculations demonstrate that $\mathcal{G}_1^0 = 1.29$ GeV^{-1}, $c_1^1 = 2.38$, and $c_1^2 = -1.84$ lead to a reasonable agreement with the sum rule's data (see Fig. 4).

At the mass shell $q^2 = m_{D_s}^2$ this function gives

$$g_1 \equiv \mathcal{G}_1(-m_{D_s}^2) = (6.8 \pm 1.6) \times 10^{-1} \text{ GeV}^{-1}. \quad (31)$$

The width of the decay $\mathcal{M} \to D_s^+ D_s^-$ is calculated by means of the formula

$$\Gamma[\mathcal{M} \to D_s^+ D_s^-] = g_1^2 \frac{m_{D_s}^2 \lambda}{8\pi} \left(1 + \frac{\lambda^2}{m_{D_s}^2}\right), \quad (32)$$

where $\lambda = \lambda(m, m_{D_s}, m_{D_s})$ and

$$\lambda(a, b, c) = \frac{1}{2a}[a^4 + b^4 + c^4 - 2(a^2 b^2 + a^2 c^2 + b^2 c^2)]^{1/2}. \quad (33)$$

Using the coupling Eq. (31), it is not difficult to compute the width of the decay $\mathcal{M} \to D_s^+ D_s^-$,

$$\Gamma[\mathcal{M} \to D_s^+ D_s^-] = (46.5 \pm 11.6) \text{ MeV}. \quad (34)$$

IV. DECAYS $\mathcal{M} \to \eta_c \eta'$, $\mathcal{M} \to \eta_c \eta$, AND $\mathcal{M} \to J/\psi\phi$

Other processes that we study here are decays $\mathcal{M} \to \eta_c \eta'$, $\mathcal{M} \to \eta_c \eta$, and $\mathcal{M} \to J/\psi\phi$. The two-meson threshold 3941 MeV for the first two decays is below the

mass m of the molecule $\mathcal{M}$. The threshold for the decay $\mathcal{M} \to J/\psi\phi$ is less than m as well. It is easy to prove that these processes conserve the P and C parities of the initial particle $\mathcal{M}$.

A. $\mathcal{M} \to \eta_c\eta'$ and $\mathcal{M} \to \eta_c\eta$

The decays $\mathcal{M} \to \eta_c\eta'$ and $\mathcal{M} \to \eta_c\eta$ are studied by a method described above. But, here we take into account peculiarities of the $\eta - \eta'$ mesons connected with mixing in this system due to the $U(1)$ anomaly [34]. This effect modifies a choice of interpolating currents and matrix elements for these particles. Although $\eta - \eta'$ mixing can be considered in different approaches, we use the quark-flavor basis $|\eta_q\rangle = (\bar{u}u + \bar{d}d)/\sqrt{2}$ and $|\eta_s\rangle = \bar{s}s$, where the physical particles η and η' have simple decompositions [34–36]

$$\begin{aligned} \eta &= |\eta_q\rangle \cos\varphi - |\eta_s\rangle \sin\varphi, \\ \eta' &= |\eta_q\rangle \sin\varphi + |\eta_s\rangle \cos\varphi, \end{aligned} \tag{35}$$

where φ is the mixing angle in the $\{|\eta_q\rangle, |\eta_s\rangle\}$ basis. Such state mixing implies that the same assumption is applicable to their currents, decay constants, and matrix elements.

Because in the decays $\mathcal{M} \to \eta_c\eta', \eta_c\eta$ participate only $\bar{s}s$ components of the mesons η and η', relevant interpolating currents are given by the expressions

$$\begin{aligned} J^{\eta}(x) &= -\sin\varphi \bar{s}_j(x) i\gamma_5 s_j(x), \\ J^{\eta'}(x) &= \cos\varphi \bar{s}_j(x) i\gamma_5 s_j(x), \end{aligned} \tag{36}$$

where j is the color index.

We start our analysis from the process $\mathcal{M} \to \eta_c\eta'$. Then, we should consider the correlation function

$$\begin{aligned} \tilde{\Pi}(p, p') = i^2 \int d^4x d^4y e^{i(p'y - px)} \langle 0|\mathcal{T}\{J^{\eta_c}(y) \\ \times J^{\eta'}(0) J^{\dagger}(x)\}|0\rangle, \end{aligned} \tag{37}$$

where $J^{\eta_c}(y)$ is the interpolating current of the meson η_c

$$J^{\eta_c}(x) = \bar{c}_i(x) i\gamma_5 c_i(x). \tag{38}$$

The main contribution to the correlation function $\tilde{\Pi}(p, p')$ has the following form:

$$\begin{aligned} \tilde{\Pi}^{\rm Phys}(p, p') = \frac{\langle 0|J^{\eta_c}|\eta_c(p')\rangle \langle 0|J^{\eta'}|\eta'(q)\rangle}{(p'^2 - m_{\eta_c}^2)(q^2 - m_{\eta'}^2)} \\ \times \frac{\langle \eta'(q)\eta_c(p')|\mathcal{M}(p)\rangle \langle \mathcal{M}(p)|J^{\dagger}|0\rangle}{(p^2 - m^2)} + \cdots, \end{aligned} \tag{39}$$

where the ellipses stand for contributions of higher resonances and continuum states. The function $\tilde{\Pi}^{\rm Phys}(p, p')$ can be rewritten using the matrix elements

$$\begin{aligned} \langle 0|J^{\eta_c}|\eta_c\rangle &= \frac{m_{\eta_c}^2 f_{\eta_c}}{2m_c}, \\ 2m_s \langle \eta'|\bar{s}i\gamma_5 s|0\rangle &= h_{\eta'}^s, \end{aligned} \tag{40}$$

where m_{η_c} and f_{η_c} are the mass and decay constant of the η_c meson. The matrix element of local operator $\bar{s}i\gamma_5 s$ sandwiched between the meson η' and vacuum states is denoted by $h_{\eta'}^s$ [35]. The parameter $h_{\eta'}^s$ follows the $\eta - \eta'$ state-mixing pattern, and we get

$$h_{\eta'}^s = h_s \cos\varphi. \tag{41}$$

The parameter h_s can be defined theoretically [35], but for our purposes it is enough to use values of h_s and φ extracted from phenomenological analyses

$$\begin{aligned} h_s &= (0.087 \pm 0.006)\ \text{GeV}^3, \\ \varphi &= 39.3^{\circ} \pm 1.0^{\circ}. \end{aligned} \tag{42}$$

The vertex $\mathcal{M}\eta_c\eta'$ has the following form:

$$\langle \eta'(q)\eta_c(p')|\mathcal{M}(p)\rangle = g_2(q^2) p \cdot p', \tag{43}$$

where g_2 is the strong coupling at the vertex $\mathcal{M}\eta_c\eta'$. By employing these matrix elements, we obtain a new expression for $\tilde{\Pi}^{\rm Phys}(p, p')$:

$$\begin{aligned} \tilde{\Pi}^{\rm Phys}(p, p') = g_2(q^2) \frac{f m m_{\eta_c}^2 f_{\eta_c} h_s \cos^2\varphi}{4m_c m_s (p^2 - m^2)} \\ \times \frac{1}{(p'^2 - m_{\eta_c}^2)(q^2 - m_{\eta'}^2)} \frac{m^2 + m_{\eta_c}^2 - q^2}{2} \\ + \cdots. \end{aligned} \tag{44}$$

The QCD side of the sum rule for $g_2(q^2)$ is given by the formula

$$\begin{aligned} \tilde{\Pi}^{\rm OPE}(p, p') = -\cos\varphi \int d^4x d^4y e^{i(p'y - px)} \\ \times \text{Tr}[\gamma_5 S_c^{ia}(y - x)\gamma_5 S_s^{ai}(x - y)] \\ \times \text{Tr}[\gamma_5 S_s^{jb}(-x)\gamma_5 S_c^{bj}(x - y)]. \end{aligned} \tag{45}$$

The sum rule for the coupling $g_2(q^2)$ is obtained using the Borel transformations of invariant amplitudes $\tilde{\Pi}^{\rm Phys}(p^2, p'^2, q^2)$ and $\tilde{\Pi}^{\rm OPE}(p^2, p'^2, q^2)$ and is equal to

$$g_2(q^2) = -\frac{8m_c m_s}{f m m_{\eta_c}^2 f_{\eta_c} h_s \cos\varphi} \frac{q^2 - m_{\eta_c}^2}{m^2 + m_{\eta_c}^2 - q^2} \times e^{m^2/M_1^2} e^{m_{\eta_c}^2/M_2^2} \tilde{\Pi}(\mathbf{M}^2, \mathbf{s}_0, q^2). \quad (46)$$

Here, $\tilde{\Pi}(\mathbf{M}^2, \mathbf{s}_0, q^2)$ is the Borel transformed and subtracted amplitude $\tilde{\Pi}^{\rm OPE}(p^2, p'^2, q^2)$.

The coupling $g_1(q^2)$ is calculated using the following Borel and continuum threshold parameters in the η_c channel

$$M_2^2 \in [3, 4]\ \text{GeV}^2, \qquad s_0' \in [9.5, 10.5]\ \text{GeV}^2, \quad (47)$$

whereas for the $\mathcal{M}$ channel, we employ M_1^2 and s_0 from Eq. (17). The strong coupling g_2 is defined at the mass shell of the η' meson. The fit function $\mathcal{G}_2(Q^2)$ given by Eq. (30) has the parameters $\mathcal{G}_2^0 = 0.21\ \text{GeV}^{-1}$, $c_2^1 = 5.08$, and $c_2^2 = -4.04$. Computations yield

$$g_2 \equiv \mathcal{G}_2(-m_{\eta'}^2) = (1.6 \pm 0.3) \times 10^{-1}\ \text{GeV}^{-1}. \quad (48)$$

The partial width of this decay can be evaluated using the formula Eq. (32), in which one should make substitutions $g_1 \to g_2$, $m_{D_s}^2 \to m_{\eta_c}^2$, and $\lambda(m, m_{D_s}, m_{D_s}) \to \tilde{\lambda}(m, m_{\eta_c}, m_{\eta'})$. Then, for the width of the decay $\mathcal{M} \to \eta_c \eta'$, we find

$$\Gamma[\mathcal{M} \to \eta_c \eta'] = (4.9 \pm 1.1)\ \text{MeV}. \quad (49)$$

Analysis of the decay $\mathcal{M} \to \eta_c \eta$ can be performed in a similar manner. Avoiding further details, let us write down the predictions obtained for key quantities. Thus, the strong coupling g_3 at the vertex $\mathcal{M}\eta_c\eta$ is given by the formula

$$g_3 \equiv |\mathcal{G}_3(-m_\eta^2)| = (1.5 \pm 0.3) \times 10^{-1}\ \text{GeV}^{-1}, \quad (50)$$

where parameters of the fit function are $\mathcal{G}_3^0 = -0.17\ \text{GeV}^{-1}$, $c_3^1 = 7.29$, and $c_3^2 = -7.33$. The width of the decay $\mathcal{M} \to \eta_c \eta$ is

$$\Gamma[\mathcal{M} \to \eta_c \eta] = (7.7 \pm 1.8)\ \text{MeV}. \quad (51)$$

B. $\mathcal{M} \to J/\psi\phi$

The process $\mathcal{M} \to J/\psi\phi$ is the kinematically allowed decay channel of the molecule $\mathcal{M}$. The hadronic molecule $\mathcal{M}$ can decay also to $J/\psi\omega$ mesons, because through a mixing phenomenon ω acquires a strange-quark component. As in the case of η and η' mesons, the $\omega - \phi$ mixing can be defined in the following form:

$$\omega = |\bar{q}q\rangle \cos\psi_{\rm V} - |\bar{s}s\rangle \sin\psi_{\rm V},$$
$$\phi = |\bar{q}q\rangle \sin\psi_{\rm V} + |\bar{s}s\rangle \cos\psi_{\rm V}, \quad (52)$$

where $|\bar{q}q\rangle$ and $|\bar{s}s\rangle$ are vector counterparts of the basic states $|\eta_q\rangle$ and $|\eta_s\rangle$, and $\psi_{\rm V}$ is the $\omega - \phi$ mixing angle. But in contrast to φ, the mixing angle $\psi_{\rm V}$ is numerically very small [37],

$$\psi_{\rm V} = (3.32 \pm 0.09)^\circ. \quad (53)$$

As a result, a $|\bar{s}s\rangle$ component of the meson ω is small as well. In other words, the ϕ and ω mesons are almost purely strange and nonstrange vector particles, respectively. Therefore, we consider only the decay $\mathcal{M} \to J/\psi\phi$ and neglect the contribution to the full width of $\mathcal{M}$ coming from the process $\mathcal{M} \to J/\psi\omega$.

The correlation function to be examined in the case of decay $\mathcal{M} \to J/\psi\phi$ is

$$\hat{\Pi}_{\mu\nu}(p, p') = i^2 \int d^4x d^4y e^{i(p'y - px)} \langle 0|\mathcal{T}\{J_\mu^{J/\psi}(y) \times J_\nu^\phi(0) J^\dagger(x)\}|0\rangle, \quad (54)$$

where $J_\mu^{J/\psi}$ and J_ν^ϕ are interpolating currents for the vector mesons J/ψ and ϕ, respectively. As is seen, the momenta of the molecule $\mathcal{M}$ and meson J/ψ are equal to p and p', respectively. Consequently, the momentum of the ϕ meson is $q = p - p'$.

The interpolating currents $J_\mu^{J/\psi}$ and J_ν^ϕ are defined by the following expressions:

$$J_\mu^{J/\psi}(x) = \bar{c}_i(x)\gamma_\mu c_i(x),$$
$$J_\nu^\phi(x) = \bar{s}_j(x)\gamma_\nu s_j(x). \quad (55)$$

To find the physical side of the sum rule, we express $\hat{\Pi}_{\mu\nu}(p, p')$ using parameters of the involved particles through their matrix elements

$$\langle 0|J_\mu^{J/\psi}|J/\psi(p')\rangle = f_{J/\psi} m_{J/\psi} \varepsilon_\mu(p'),$$
$$\langle 0|J_\nu^\phi|\phi(q)\rangle = f_\phi m_\phi \varepsilon_\nu(q). \quad (56)$$

For the vertex $\langle \phi'(q) J/\psi(p')|\mathcal{M}(p)\rangle$, we employ the matrix element

$$\langle \phi'(q) J/\psi(p')|\mathcal{M}(p)\rangle = g_4(q^2)[p' \cdot \varepsilon^*(q) \times q \cdot \varepsilon^*(p') - p' \cdot q \varepsilon^*(q) \cdot \varepsilon^*(p')]. \quad (57)$$

In the formulas above, $m_{J/\psi}$, m_ϕ, $f_{J/\psi}$, f_ϕ, and $\varepsilon_\mu(p')$, $\varepsilon_\nu(q)$ are the masses, decay constants, and polarization vectors of the mesons $J_\mu^{J/\psi}$ and J_ν^ϕ, respectively. The strong coupling g_4 in Eq. (57) corresponds to the vertex $\mathcal{M}J/\psi\phi$.

Then the physical side $\hat{\Pi}_{\mu\nu}^{\rm Phys}(p, p')$ of the sum rule takes the form

$$\hat{\Pi}_{\mu\nu}^{\rm Phys}(p,p') = -g_4(q^2)\frac{mm_{J/\psi}m_\phi f f_{J/\psi} f_\phi}{(p^2-m^2)(p'^2-m_{J/\psi}^2)} \times \frac{1}{(q^2-m_\phi^2)}\left[\frac{m^2-m_{J/\psi}^2-q^2}{2}g_{\mu\nu} - p'_\nu q_\mu\right] + \cdots. \quad (58)$$

The same correlation function in terms of quark propagators is given by the formula

$$\hat{\Pi}_{\mu\nu}^{\rm OPE}(p,p') = i^2\int d^4x d^4y e^{i(p'y-px)} \times {\rm Tr}[\gamma_\mu S_c^{ia}(y-x)\gamma_5 S_s^{aj}(x)\gamma_\nu \times S_s^{jb}(-x)\gamma_5 S_c^{bi}(x-y)]. \quad (59)$$

Remaining operations with functions $\hat{\Pi}_{\mu\nu}^{\rm Phys}(p,p')$ and $\hat{\Pi}_{\mu\nu}^{\rm OPE}(p,p')$ are standard manipulations of the sum rule analysis. Let us note only that the sum rule for g_4 is derived using invariant amplitudes which correspond to structures $\sim g_{\mu\nu}$ in these correlators. In numerical analysis the second pair of the parameters (M_2^2, s_0') related to the J/ψ channel is chosen as

$$M_2^2 \in [3,4]\ {\rm GeV}^2, \qquad s_0' \in [11,12]\ {\rm GeV}^2. \quad (60)$$

The strong coupling g_4 is determined at the mass shell of the ϕ meson, i.e., at $q^2 = m_\phi^2$.

Our computations yield

$$g_4 \equiv \mathcal{G}_4(-m_\phi^2) = (6.7\pm 1.2)\times 10^{-1}\ {\rm GeV}^{-1}. \quad (61)$$

The parameters of the fit function are $\mathcal{G}_4^0 = 0.66\ {\rm GeV}^{-1}$, $c_4^1 = -0.09$, and $c_4^2 = -0.02$.

The width of the decay $\mathcal{M} \to J/\psi\phi$ is equal to

$$\Gamma[\mathcal{M} \to J/\psi\phi] = (2.6\pm 0.6)\ {\rm MeV}. \quad (62)$$

Then, it is not difficult to find the full width of $\mathcal{M}$,

$$\Gamma_{\mathcal{M}} = (62\pm 12)\ {\rm MeV}. \quad (63)$$

The width of the hadronic molecule $\mathcal{M}$, within errors of calculations and measurements, agrees with the LHCb datum from Eq. (2).

V. DISCUSSION AND SUMMING UP

In this work, we have calculated the mass and width of the scalar molecule $\mathcal{M} = D_s^+ D_s^-$ in the framework of the QCD sum rule methods. The mass of $\mathcal{M}$ has been evaluated using the two-point sum rule approach. The full width of $\mathcal{M}$ has been computed by taking into account decay modes $\mathcal{M} \to D_s^+ D_s^-$, $\mathcal{M} \to \eta_c \eta^{(\prime)}$, and $\mathcal{M} \to J/\psi\phi$. Strong couplings g_i that determine the width of these decays have been found in the framework of the three-point sum rule method.

Our result for the mass $m = (4117\pm 85)$ MeV of the molecule $\mathcal{M}$ exceeds considerably the corresponding LHCb datum $m_{1\,\rm exp}$, but is consistent with $m_{2\,\rm exp}$. It is evident that $\mathcal{M}$ is significantly heavier than the resonance $X(3960)$, which makes problematic its interpretation as $X(3960)$. The full width $\Gamma_{\mathcal{M}} = (62\pm 12)$ MeV of $\mathcal{M}$ is consistent with the LHCb measurement $\Gamma_{2\,\rm exp}$ as well.

The resonance $X(3960)$ was examined in our article [21] as the tetraquark $X = [cs][\bar{c}\,\bar{s}]$ with quantum numbers $J^{\rm PC} = 0^{++}$. We obtained the following predictions for the parameters of X: the mass $m = (3976\pm 85)$ MeV and the width $\Gamma_{\rm X} = (42.2\pm 8.3)$ MeV. The parameters of the diquark-antidiquark state X are in nice agreement with the LHCb data given by Eq. (1); therefore, in Ref. [21], it was identified with the resonance $X(3960)$.

In both the molecule and diquark-antidiquark pictures dominant decay modes of the scalar four-quark meson $\bar{c}c\bar{s}s$ are channels $\mathcal{M} \to D_s^+ D_s^-$ and $X \to D_s^+ D_s^-$, respectively. Decays of molecular-type resonances to constituent mesons by falling apart are, naturally, preferable channels for such states. The interpolating current for $\mathcal{M}$ given by Eq. (3) has an explicitly $D_s^+ D_s^-$ type structure. Therefore, it couples mainly to the physical mesons D_s^+ and D_s^-. But the current $J(x)$ couples also to other two-meson states. Formally, this can be demonstrated using Fierz transformations of $J(x)$. To this end, it is convenient to rewrite it in the following form:

$$J = \delta_{am}\delta_{bn}[\bar{s}_a i\gamma_5 c_m][\bar{c}_b i\gamma_5 s_n]. \quad (64)$$

After Fierz transformation it contains the following components:

$$J(x) = \frac{\delta_{am}\delta_{bn}}{4}\{[\bar{s}_a\gamma_\mu s_n][\bar{c}_b\gamma^\mu c_m] - [\bar{s}_a s_n][\bar{c}_b c_m] + [\bar{s}_a i\gamma_5 s_n][\bar{c}_b i\gamma_5 c_m] + \cdots\}, \quad (65)$$

where the ellipses stand for axial-vector and tensor structures. Then, rearranging the color indices by means of the equality

$$\delta_{am}\delta_{bn} = \delta_{an}\delta_{bm} + \epsilon_{abk}\epsilon_{mnk}, \quad (66)$$

with ϵ_{ijk} being the Levi-Civita epsilon, we find

$$J(x) = \frac{1}{4}\{[\bar{s}_a\gamma_\mu s_a][\bar{c}_b\gamma^\mu c_b] - [\bar{s}_a s_a][\bar{c}_b c_b] + [\bar{s}_a i\gamma_5 s_a][\bar{c}_b i\gamma_5 c_b] + \cdots\}. \quad (67)$$

The terms above are S-S, V-V, and PS-PS type interpolating currents that couple to relevant meson pairs. For example, V-V and PS-PS currents couple to $J/\psi\phi$, $J/\psi\omega$, $\psi'\phi$, and

$\eta_c\eta^{(\prime)}$ meson pairs (this list can be extended), respectively. Relative significance of the $J(x)$ current's components can be seen by comparing strong couplings at vertices $\mathcal{M}D_s^+D_s^-$, $\mathcal{M}\eta_c\eta^{(\prime)}$, $\mathcal{M}J/\psi\phi$, etc., because only relevant components of $J(x)$ contribute to three-point correlators. From a chain of inequalities $g_1 > g_4 > g_2 > g_3$, it is clear that $J(x)$ couples dominantly to $D_s^+D_s^-$ and $J/\psi\phi$ mesons. Smallness of the partial width $\Gamma[\mathcal{M} \to J/\psi\phi]$ is connected with parameters (masses, decay constants) and quantum numbers $J^{\mathrm{PC}} = 1^{--}$ of the final-state mesons.

In its turn, a diquark-antidiquark current can be expressed in terms of molecule currents [24,25]. Now, comparing strong couplings of the diquark-antidiquark state $X = [cs][\bar{c}\,\bar{s}]$ with mesons $D_s^+D_s^-$ and $\eta_c\eta^{(\prime)}$, we see that $G > g_1 > g_2$ [21]. In general, one might explore the vertex $XJ/\psi\omega$ and evaluate corresponding coupling, but the contribution of the decay $X \to J/\psi\omega$ [$X \to J/\psi\phi$ is forbidden kinematically] to Γ_{X} would be negligible. Summing up, we can state that decays to $D_s^+D_s^-$ are dominant channels for both the diquark-antidiquark structure X and the hadronic molecule $\mathcal{M}$: The resonances $X(3960)$ and $X_0(4140)$ were discovered in the $D_s^+D_s^-$ mass distribution.

In the context of the sum rule approach the molecule $D_s^+D_s^-$ was also studied in Ref. [17]. In accordance with this paper, the mass of such a hadronic molecule is equal to (3980 ± 100) MeV and agrees with the LHCb data. It is worth noting that the authors did not analyze quantitatively the width of this state. Our results for parameters of $\mathcal{M}$, even within existing errors of calculations, does not support molecule assignment for $X(3960)$.

By taking into account predictions for the mass m and full width $\Gamma_{\mathcal{M}}$ of the molecule $\mathcal{M} = D_s^+D_s^-$ obtained in the present work, we argue that the molecule $\mathcal{M}$ may be a candidate to the structure $X_0(4140)$.

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