


Properties of multiple D0-brane systems: 11D origin, equations of motion, and their solutions

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We study the properties of 10D multiple D0-brane (mD0) system described by a recently proposed complete supersymmetric and κ -symmetric nonlinear action which includes an arbitrary positive definite function $\mathcal{M}(\mathcal{H})$ of the relative motion Hamiltonian \mathcal{H} . First we show how the action with a particular nonlinear $\mathcal{M}(\mathcal{H})$ can be obtained from the action for 11D multiple M-wave (multiple M0-brane or mM0) system. Then we obtain the complete set of equations of motion of mD0 system with arbitrary positive definite $\mathcal{M}(\mathcal{H})$, perform a convenient gauge fixing, solve the center of energy equations and establish an interesting correspondence between the relative motion mD0 equations and the equations of maximally supersymmetric $SU(N)$ Yang-Mills theory (SYM). We show that this correspondence does not imply a gauge equivalence but establishes a relation between solutions of the system. In particular, it implies that all the supersymmetric solutions of mD0 equations in its relative motion part coincide with supersymmetric solutions of the SYM equations.

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I. INTRODUCTION

Supersymmetric extended objects, super- p -branes,¹ play very important role in string/M-theory [1–4] and in AdS/CFT duality which has been developed to a much more general gauge-gravity correspondence. Particularly interesting are ten-dimensional ($D = 10$) Dirichlet p -branes or Dp -branes, the world volumes of which are the surfaces where the fundamental string (sometimes called F1-brane) can have its ends.

The (theoretical) discovery of these objects is dated by late 80th [5–7] but the common interest to them was awakened by the famous paper by Polchinski [8] in which he identified them as mysterious supersymmetric solutions of $\mathcal{N} = 2$ (type-IIA and type-IIB) supergravity and showed that they carry the charges of the Ramond-Ramond q -form gauge fields of these supergravity theories. The complete nonlinear world volume actions for the Dp -brane

possessing a worldline supersymmetry and the local fermionic κ -symmetry was found soon by several groups [9–14] but even before the supersymmetric equations of motion had been found by Howe and Sezgin [15] in the frame of a superembedding approach [16,17].

The spacetime-filling D9-brane in type-IIB supergravity superspace corresponds to the case of open superstring, the ends of which can move freely in the (super)spacetime (see [18] and Refs. therein); such an end of the string can be coupled to a vector gauge field. The vector gauge field is also present in the quantum-state spectrum of the open string, so that in this case it is natural to assume that we are dealing with coupling of the string to one of its excitations considered as a background. The equation of motion for such a background gauge field can be found from the requirement of preservation of the conformal symmetry of string theory in such a background (vanishing of the beta functions). Those were found [19] to be the same as followed from the Born-Infeld action [20] which thus can be considered as (the bosonic limit of) an effective action for the open (super)string with free ends and, in modern language, the effective action for the spacetime filling D9-brane.² Indeed, it was shown that Dp -branes with $p \leq 9$ carry out world volume vector gauge fields and

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¹Here p refers to the number of spacial dimensions of the world volume of the object so that $p = 1$ corresponds to strings, $p = 2$ corresponds to membrane, and 0-branes are supersymmetric particles.

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²Also in the case of the bosonic string considered in [19] one speaks about spacetime-filling D25-brane, we allow ourselves to streamline the discussion a bit by referring directly to the generalizations for the case of the superstring.

thus that the effective action for this is the $d = p + 1$ dimensional Dirac-Born-Infeld action (which is to say the generalization of Born-Infeld action to curved space with metric of this induced by the embedding of the world volume in the target spacetime) [21].

Interestingly enough, from the early years of string theory it was known that the ends of open string can be supplied with additional degrees of freedom describing “isospin” [22]. This fact was associated to the possibility of attaching “quarks” to the ends of the string. In the frame of Lagrangian mechanics of the string model those are described by auxiliary fermionic fields (boundary fermions) “living” at the string endpoints [23,24]. In the case of string amplitudes, the “isospin” degrees of freedom manifest themselves as additional factor constructed from the trace of matrices representing the generators of a non-Abelian internal symmetry group which in the simplest case is considered to be $U(N)$ [22].

From this point of view one can find natural the brilliant idea of Witten [25]; that in the limit of N (nearly) coincident Dp -brane, when the strings connecting different branes can be described by (nearly) massless $U(1)$ gauge fields, as the string with both ends on the same brane can, the manifest $(U(1))^{N^2}$ symmetry of the system is enhanced till $U(N)$ and thus in low-energy limit (and after an appropriate gauge fixed with breaking Lorentz invariance) shall be described by non-Abelian supersymmetric Yang-Mills (SYM) model.

The search for a (more) complete nonlinear multiple Dp -brane (mDp) action describing the system of N nearly coincident Dp -branes and strings ending on the same branes and on different branes has more than 25 years of history. Despite many interesting and deep results obtained on this way [26–54] one cannot say that the problem is definitely solved even in the simplest case of mD0 system. The expected properties of this action, which should be a kind of non-Abelian generalization of the single Dp -brane actions from [9–14], are Lorentz invariance, spacetime (target superspace) supersymmetry, the counterpart of local fermionic κ -symmetry of single Dp -brane (which makes the ground state of the system supersymmetric) and appearance of the $U(N)$ SYM action as low-energy description in the suitable gauge.

The so-called “dielectric brane” action by Myers [30], which was widely accepted as a bosonic limit of such an action and was studied and generalized in [35,37–39], still resists its supersymmetric generalization³ (neither Lorentz invariant formulation of this is known). An interesting approach on “–1 quantization” level was developed in

³The bilinear fermionic contributions to the hypothetical supersymmetric generalization of the action [30] were searched for in [46] using the suggestion from the studies of single Dp -brane action in [55]. Notice that the approach with non-Abelian version of κ -symmetry [31] successful in the linearized approximation cannot be extended to cubic level as it was shown in [32].

[40–42]. It uses the superembedding approach version of boundary fermion formalism and it looks like to obtain the non-Abelian counterpart of the complete nonlinear action for single Dp -brane on this basis the problem of quantization of the system including supergravity and “–1 quantized” Dp -brane should be addressed (more discussion on this approach can be found in [51]).

The superembedding approach to mD0 and related mM0 (multiple M-wave or multiple M0-brane) system in type-IIA 10D and in 11D superspaces was developed in [43–45]. It was based on the standard superembedding equations, the same as was used to describe single D0-brane, and seemed to provide some approximation to the mD0 and mM0 equations. The suggestion on how to possibly modify the superembedding equations might come from the action principle for the 11D mM0 system which was constructed in [47] and studied in [48].

In [54] we have constructed the nonlinear doubly-supersymmetric action which possesses the properties expected for 10D mD0 brane system (its $D = 3$ counterpart had been found in [53]). An unexpected property was the presence in the action of an arbitrary positively definite function $\mathcal{M}(\mathcal{H})$ of the relative motion Hamiltonian \mathcal{H} [constructed from the matrix fields of the $d = 1$ $\mathcal{N} = 16$ $SU(N)$ SYM supermultiplet].⁴ The model corresponding to constant $\mathcal{M}(\mathcal{H}) = m$ has been constructed before in [51] by adding to the single D0-brane action (describing then the center-of-mass motion of mD0 system) of the action for 1d $\mathcal{N} = 16$ $SU(N)$ SYM supermultiplet coupled to supergravity induced by the embedding of the center-of-mass worldline into the target type-IIA superspace. The problem of the relation of this candidate mD0 action to the previously known action for 11D multiple M0-brane (multiple M-wave or mM0) system from [47] was posed but not solved in [51].

In this paper we will show how the dimensional reduction of the mM0 action, performed after a suitable redefinition of the matrix fields, produces a specific representative of the family of candidate mD0 action from [54] with a particular form of the function $\mathcal{M}(\mathcal{H})$,

$$\mathcal{M} = \frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{\mathcal{H}}{\mu^6}}, \quad (1)$$

where m and μ are the constants of dimension of mass. The first of these is associated with the center-of-mass motion while the second is the coupling constant of the relative-motion and center-of-energy variables.

Then we study the equations of motion which follows from the action of [54]. In particular we show that equations of motion imply the preservation of the relative motion

⁴Similar property is possessed by the 0-brane model proposed in [36]. See [51] for the discussion on this model in the perspective of our approach.

Hamiltonian, $d\mathcal{H} = 0$ and, hence, of the on shell value of the mass of the mD0 system

$$\mathfrak{M} = m + \frac{2}{\mu^6} \frac{\mathcal{H}}{\mathcal{M}(\mathcal{H})}, \quad (2)$$

$d\mathfrak{M} = 0$. We also show that the Bogomol'nyi-Prasad-Sommerfield (BPS) condition appearing as consistency condition for the existence of supersymmetric purely bosonic solution of the system of mD0 equations is expressed by the vanishing of the relative motion Hamiltonian, $\mathcal{H}_{\text{susy}} = 0$, so that the mass of supersymmetric mD0 configuration is defined by the above mentioned constant m from the center-of-mass part of the action,

$$\mathfrak{M}_{\text{susy}} = m. \quad (3)$$

We also find a convenient gauge fixing, which simplifies essentially the relative-motion equations, and also an interesting correspondence of these with the equations of motion of the maximally supersymmetric $d = 1$ $SU(N)$ SYM model. In particular we show that all the supersymmetric solutions of the mD0 equations preserve just a half of the spacetime supersymmetry (this is to say are 1/2 BPS states) and, in its relative-motion sector, are in one-to-one correspondence with BPS states of the SYM theory.

The rest of the paper is organized as follows. In Sec. II we describe the action from [54] which possesses the properties expected from the complete nonlinear supersymmetric action of multiple D0-brane system so that we call it candidate mD0 action or, simplifying the terminology, mD0-action. It contains an arbitrary positive-definite function $\mathcal{M}(\mathcal{H})$ of the relative motion Hamiltonian \mathcal{H} and in Appendix A we present the explicit form of the κ -symmetry transformation for any choice of $\mathcal{M}(\mathcal{H})$. In Sec. III B we show that a particular representative of this family with $\mathcal{M}(\mathcal{H})$ given in (1) can be obtained by dimensional reduction of the action for multiple M0-brane (mM0) [47] which is described in Sec. III A. The dimensional reduction essentially uses the solution of the constraints for the 11D spinor moving-frame variables (Lorentz harmonics) in terms of their 10D counterparts given in Eq. (63). In Sec. IV we vary our mD0 action to obtain the complete set of equations of motion and study their properties. In Sec. IV E we describe convenient gauge fixing in which the equations simplify essentially and in IV G establish an interesting relation of the gauge fixed equations with the equations of maximally-supersymmetric 1d $SU(N)$ SYM theory. Supersymmetric bosonic solutions of our mD0 equations are discussed in Sec. V where it is shown that they obey the BPS equation $\mathcal{H} = 0$ and, in their relative motion part, are in one-to-one correspondence with BPS states of the SYM theory. We conclude and discuss the obtained results in Sec. VI. Some technical details can be

found in Appendixes B and C. In Appendix D we consider a particular nonsupersymmetric solution of mD0 equations.

II. 10D mD0 ACTION AND ITS SYMMETRIES

A. mD0 fields

The field content of the mD p system is suggested by its very low-energy gauge fixed description given, according to [25], by the $U(N)$ SYM action (where N is the number of nearly coincident D p -branes in the system), and by the known fact that in the case of a single D p -brane such a description is given by the Abelian $U(1)$ SYM action. Hence, the field content of the Lorentz-covariant formulation of mD p -brane system should be given by the fields known from the description of single D p -brane, which are essentially coordinate functions and an Abelian gauge field, and by the fields of maximally-supersymmetric $d = p + 1$ $SU(N)$ SYM multiplet.

In the case of the mD0-brane, the worldline gauge fields (all depending on the proper time τ which parametrizes the center-of-mass worldline \mathcal{W}^1) are coordinate functions

$$\begin{aligned} Z^M(\tau) &= (x^\mu(\tau), \theta^{\alpha 1}(\tau), \theta_\alpha^2(\tau)), \quad \mu = 0, \dots, 9, \\ &\alpha = 1, \dots, 16, \end{aligned} \quad (4)$$

describing the embedding of the worldline \mathcal{W}^1 in target type-IIA superspace $\Sigma^{(10|32)}$ with bosonic and fermionic coordinates $Z^M = (x^\mu, \theta^{\alpha 1}, \theta_\alpha^2)^5$

$$\mathcal{W}^1 \subset \Sigma^{(10|32)}: \quad Z^M = Z^M(\tau), \quad (5)$$

the matrix fields of the $d = 1$ $\mathcal{N} = 16$ $SU(N)$ SYM multiplet, and some auxiliary fields. The set of these latter includes the spinor moving frame fields, which are described below, and the matrix momentum field which allows us to write the SYM action in first-order form.

The set of matrix fields of mD0 system includes nine bosonic traceless Hermitian $N \times N$ matrices \mathbb{X}^i enumerated by the $SO(9)$ vector index $i = 1, \dots, 9$, sixteen Hermitian fermionic $N \times N$ matrices Ψ_q enumerated by spinor $SO(9)$ [i.e., $Spin(9)$] index $q = 1, \dots, 16$ and the anti-Hermitian bosonic traceless $N \times N$ matrix 1-form $\mathbb{A} = d\tau \mathbb{A}_\tau$. In addition we introduce a 9-vector Hermitian $N \times N$ field \mathbb{P}^i which will play a role of momentum conjugate to the matrix field \mathbb{X}^i .

⁵Greek indices from the middle of alphabet, in particular, $\mu = 0, \dots, 9$ are spacetime 10-vector indices. In the case of curved target superspace these should be distinguished from the target superspace indices which we denote by symbols from the beginning of Latin alphabet, $a = 0, \dots, 9$. We find convenient also use both these types of indices in the case of flat-target superspace, although in this case they can be identified. The letters from the beginning of Greek alphabet denote 10D Majorana-Weyl spinor indices, e.g. $\alpha = 1, \dots, 16$.

B. Moving-frame and spinor moving-frame fields (Lorentz harmonics)

Finally, to write the candidate mD0 action from [54] we need to introduce the moving frame and spinor moving-frame fields. These are given by SO(1,9) valued and Spin(1,9) valued matrices

$$(u_\mu^0, u_\mu^i) \in \text{SO}(1,9) \quad \text{and} \quad v_\alpha^q \in \text{Spin}(1,9) \quad (6)$$

which are related by the conditions of the Lorentz invariance of sigma matrices

$$u_\mu^{(b)} \sigma_{\alpha\beta}^\mu = v_\alpha^q \sigma_{qp}^{(b)} v_\beta^p, \quad u_\mu^{(b)} \tilde{\sigma}_{(b)}^{qp} = v_\alpha^q \tilde{\sigma}_\mu^{\alpha\beta} v_\beta^p. \quad (7)$$

Here $q, p = 1, \dots, 16$ should be identified as the spinor indices of SO(9) which is the group of symmetry preserving the splitting of the moving-frame matrix in (6),

$$u^{\mu 0}(\tau) \mapsto u^{\mu 0}(\tau), \quad u^{\mu i}(\tau) \mapsto u^{\mu j}(\tau) \mathcal{O}^{ji}(\tau), \quad \mathcal{O}^{ji} \mathcal{O}^{jk} = \delta^{ik}. \quad (8)$$

This will be one of the gauge symmetries of our model.

Notice that the first of the relations in (6) implies

$$u_\mu^0 u^{\mu 0} = 1, \quad u_\mu^0 u^{\mu i} = 0, \quad u_\mu^i u^{\mu j} = -\delta^{ij}. \quad (9)$$

Choosing the SO(9) invariant representation

$$\sigma_{qp}^{(a)} = (\delta_{qp}, \gamma_{qp}^i) = \tilde{\sigma}_{(a)}^{qp}, \quad \gamma_{qp}^i = \gamma_{pq}^i, \quad \gamma^{(i}\gamma^{j)} = \delta^{ij}, \quad (10)$$

in which $\gamma_{qp}^i = \gamma_{pq}^i$ are $d = 9$ gamma matrices, we find that (7) implies

$$\sigma_{\alpha\beta}^0 := u_\mu^0 \sigma_{\alpha\beta}^\mu = v_\alpha^q v_\beta^q, \quad \sigma_{\alpha\beta}^i := u_\mu^i \sigma_{\alpha\beta}^\mu = v_\alpha^q \gamma_{qp}^i v_\beta^p, \\ \text{and} \quad v_\alpha^q \tilde{\sigma}_\mu^{\alpha\beta} v_\beta^p = u_\mu^0 \delta_{qp} + u_\mu^i \gamma_{qp}^i. \quad (11)$$

We can also find

$$\tilde{\sigma}^{0\alpha\beta} := u_\mu^0 \tilde{\sigma}^{\mu\alpha\beta} = v_q^\alpha v_q^\beta, \quad \tilde{\sigma}^{i\alpha\beta} := u_\mu^i \tilde{\sigma}^{\mu\alpha\beta} = -v_q^\alpha \gamma_{qp}^i v_p^\beta, \quad (12)$$

where

$$v_\alpha^q \tilde{\sigma}^{0\alpha\beta} = v_q^\beta, \quad \sigma_{\alpha\beta}^0 v_q^\beta = v_\alpha^q, \quad (13)$$

obey

$$v_q^\alpha v_\alpha^p = \delta_q^p \Leftrightarrow v_\alpha^q v_q^\beta = \delta_\alpha^\beta \quad (14)$$

and hence can be identified as the inverse spinor-frame matrix.

Below we will use the Cartan forms

$$\Omega^i = u_\mu^0 du^{\mu i}, \quad (15)$$

$$\Omega^{ij} = u_\mu^i du^{\mu j}, \quad (16)$$

where Eq. (15) transforms covariantly under SO(9) gauge group (8) while Eq. (16) has the property of SO(9) connection. Using (9) we can express the derivatives of the moving-frame variables in terms of the Cartan forms,

$$Du_\mu^0 := du_\mu^0 = u_\mu^i \Omega^i, \quad Du_\mu^i := du_\mu^i + u_\mu^j \Omega^{ji} = u_\mu^0 \Omega^i. \quad (17)$$

Here, we have also introduced the SO(9) covariant derivatives with the composite connection given by the Cartan form Ω^{ji} (16).

The derivatives of the spinor moving-frame matrix is also expressed in terms of the same SO(1,9) Cartan forms by

$$Dv_\alpha^q := dv_\alpha^q + \frac{1}{4} \Omega^{ij} v_\alpha^p \gamma_{pq}^{ij} = \frac{1}{2} \gamma_{qp}^i v_\alpha^p \Omega^i \quad (18)$$

which implies

$$Dv_q^\alpha := dv_q^\alpha - \frac{1}{4} \Omega^{ij} \gamma_{qp}^{ij} v_p^\alpha = -\frac{1}{2} v_p^\alpha \gamma_{pq}^i \Omega^i. \quad (19)$$

The variation of the moving-frame and spinor moving-frame variables can be also expressed in terms of Cartan forms or, more precisely, in terms of the contraction of these with the variational symbol,

$$i_\delta \Omega^i = u^{\mu 0} \delta u_\mu^i, \quad i_\delta \Omega^{ij} = u^{\mu i} \delta u_\mu^j. \quad (20)$$

The latter parametrizes the SO(9) transformations which will be the manifest gauge symmetry of our construction so that the essential variations of the moving-frame vectors and of the spinor moving-frame matrix are given by

$$\delta u_\mu^0 = u_\mu^i i_\delta \Omega^i, \quad \delta u_\mu^i = u_\mu^0 i_\delta \Omega^i, \quad (21)$$

$$\delta v_\alpha^q = \frac{1}{2} \gamma_{qp}^i v_\alpha^p i_\delta \Omega^i, \quad \delta v_q^\alpha = -\frac{1}{2} \gamma_{qp}^i v_p^\alpha i_\delta \Omega^i. \quad (22)$$

The moving-frame formalism allows us to define the Lorentz invariant 1-forms on the worldline by contracting the pullback of the Volkov-Akulov (VA) 1-form of the type-IIA superspace

$$\Pi^\mu = dx^\mu - id\theta^1 \sigma^\mu \theta^1 - id\theta^2 \tilde{\sigma}^\mu \theta^2 \quad (23)$$

with moving frame vectors,

$$E^0 = \Pi^\mu u_\mu^0, \quad E^i = \Pi^\mu u_\mu^i. \quad (24)$$

The spinor moving-frame formalism allows us to define on the worldline the following Lorentz-invariant fermionic 1-forms,

$$E^{1q} = d\theta^{\alpha 1} v_\alpha^q, \quad E_q^2 = d\theta_\alpha^2 v_q^\alpha. \quad (25)$$

C. Candidate mD0 action(s)

The action for the N nearly-coincident D0-brane (multiple D0-brane or mD0) system proposed in [54] involves two constants of dimension mass, m and μ , and has the form,

$$S_{\text{mD0}} = m \int_{\mathcal{W}^1} E^0 - im \int_{\mathcal{W}^1} (d\theta^1 \theta^2 - \theta^1 d\theta^2) + \frac{1}{\mu^6} \int_{\mathcal{W}^1} \left(\text{tr}(\mathbb{P}^i D \mathbb{X}^i + 4i \Psi_q D \Psi_q) + \frac{2}{\mathcal{M}} E^0 \mathcal{H} \right) - \frac{1}{\mu^6} \int_{\mathcal{W}^1} \frac{d\mathcal{M}}{\mathcal{M}} \text{tr}(\mathbb{P}^i \mathbb{X}^i) + \frac{1}{\mu^6} \int_{\mathcal{W}^1} \frac{1}{\sqrt{2\mathcal{M}}} (E^{1q} - E_q^2) \text{tr} \left(-4i (\gamma^i \Psi)_q \mathbb{P}^i + \frac{1}{2} (\gamma^{ij} \Psi)_q [\mathbb{X}^i, \mathbb{X}^j] \right). \quad (26)$$

In it E^0 is the projection (24) of the VA 1-form (23) to the vector $u_\mu^0(\tau)$ of the moving frame (6), E^{1q} and E_q^2 are the contractions (25) of the differentials of the first and the second fermionic coordinate functions with a spinor moving-frame matrix and with its inverse, respectively. The bosonic \mathbb{P}^i and \mathbb{X}^i and 16 fermionic Ψ_q are Hermitian traceless $N \times N$ matrix fields and

$$\mathcal{H} = \frac{1}{2} \text{tr}(\mathbb{P}^i \mathbb{P}^i) - \frac{1}{64} \text{tr}[\mathbb{X}^i, \mathbb{X}^j]^2 - 2 \text{tr}(\mathbb{X}^i \Psi_q \gamma^i \Psi) \quad (27)$$

has the meaning of the Hamiltonian of the relative motion of the mD0 constituents.

$$\mathcal{M} = \mathcal{M}(\mathcal{H}/\mu^6)$$

is an arbitrary nonvanishing function of this Hamiltonian. Actually the consistency requires the function $\mathcal{M} = \mathcal{M}(\mathcal{H}/\mu^6)$ to be positive definite and we will assume this below,

$$\mathcal{M}(\mathcal{H}/\mu^6) > 0. \quad (28)$$

The covariant derivatives of the matrix fields

$$D \mathbb{X}^i = d\mathbb{X}^i - \Omega^{ij} \mathbb{X}^j + [\mathbb{A}, \mathbb{X}^i], \quad (29)$$

$$D \Psi_q = d\Psi_q - \frac{1}{4} \Omega^{ij} \gamma_{qp}^{ij} \Psi_p + [\mathbb{A}, \Psi_q], \quad (30)$$

include the composite SO(9) connection given by the Cartan form Ω^{ij} (15) and the SU(N) connection which is the anti-Hermitian traceless $N \times N$ matrix 1-form $\mathbb{A} = d\tau \mathbb{A}_\tau$.

The formal Ricci identities for such a covariant derivatives⁶ read

⁶These can be calculated on the extension of the worldline to some space of two or more dimensions. Such an extension can be realized by considering the forms depending on both differentials and variations of the variables, i.e., on $dx^\mu(\tau)$ and $\delta x^\mu(\tau)$ etc.

$$DD \mathbb{X}^i = \Omega^i \wedge \Omega^j \mathbb{X}^j + [\mathbb{F}, \mathbb{X}^i],$$

$$DD \Psi_q = \frac{1}{4} \Omega^i \wedge \Omega^j (\gamma^{ij} \Psi)_q + [\mathbb{F}, \Psi_q], \quad (31)$$

where $\mathbb{F} = d\mathbb{A} - \mathbb{A} \wedge \mathbb{A}$ is the formal 2-form field strength of the 1-form gauge field \mathbb{A} . When deriving (31) we have used the Maurer-Cartan equations

$$D\Omega^i = d\Omega^i - \Omega^j \wedge \Omega^{ji} = 0,$$

$$d\Omega^{ij} + \Omega^{ik} \wedge \Omega^{kj} = -\Omega^i \wedge \Omega^j, \quad (32)$$

which can be found by taking formal exterior derivatives of (17). These formal expressions are useful to find the variation of the Lagrangian forms by the method described in Appendix C of [54].

The action (26) is manifestly invariant under 10D type-IIA super-Poincaré transformations, including space-time (actually target superspace) supersymmetry, as well as under SU(N) gauge symmetry and SO(9) symmetry acting on the suitable indices of moving-frame, spinor moving-frame, and matrix matter fields. Moreover, it is invariant under the worldline supersymmetry transformations the explicit form of which can be found in [54] as well as in Appendix A. This invariance generalizes κ -symmetry of single D0-brane action and guarantees that the ground state of the system described by the action (26) preserves a part (1/2) of the spacetime supersymmetry.

III. 11D mM0 ACTION AND ITS DIMENSIONAL REDUCTION TO D = 10

In this section we will show that (as was announced in [54]) a particular representative of the family of the candidate mD0 actions (26), that with $\mathcal{M}(\mathcal{H})$ given in (1), can be obtained by dimensional reduction of the 11D mM0 action of [47].

A. 11D mM0 action and its symmetries

1. mM0 center of energy variables

The mM0 brane system which was assumed to be decompactification limit (M-theory lifting) of mD0, carries the same matrix fields of $d = 1$ $\mathcal{N} = 16$ SYM on its worldline, but the set of fields describing its center of energy motion is different. It includes the coordinate functions

$$Z^{\underline{M}}(\tau) = (X^{\underline{\mu}}(\tau), \Theta^{\underline{\alpha}}(\tau)), \quad \underline{\mu} = 0, 1, \dots, 9, 10, \\ \underline{\alpha} = 1, \dots, 32, \quad (33)$$

which describe the embedding of the worldline in the 11D superspace $\Sigma^{(11|32)}$ with coordinates $Z^{\underline{M}} = (X^{\underline{\mu}}, \Theta^{\underline{\alpha}})$, the spinor moving frame fields, which we describe below, and the Lagrange multiplier $\rho^{\#}(\tau)$, the role of which will be clarified below.

The 11D Volkov-Akulov (VA) 1-form

$$\Pi^{\underline{\mu}} = dX^{\underline{\mu}} - id\Theta\Gamma^{\underline{\mu}}\Theta = (\Pi^{\underline{\mu}}, \Pi^*), \quad (34)$$

involves real symmetric 32×32 matrices $\Gamma_{\underline{\alpha}\underline{\beta}}^{\underline{\mu}} = \Gamma_{\underline{\beta}\underline{\alpha}}^{\underline{\mu}} = \Gamma_{\underline{\alpha}\underline{\beta}}^{\underline{\mu}} C_{\underline{\gamma}\underline{\beta}}$ constructed from 11D Dirac matrices $\Gamma_{\underline{\alpha}\underline{\gamma}}^{\underline{\mu}} = -(\Gamma_{\underline{\alpha}\underline{\gamma}}^{\underline{\mu}})^*$ obeying the Clifford algebra

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\eta^{ab}\mathbb{1}_{32 \times 32} \quad (35)$$

and the conjugation matrix C which is antisymmetric and imaginary in our mostly minus notation, $C_{\underline{\gamma}\underline{\beta}} = -C_{\underline{\beta}\underline{\gamma}} = -(C_{\underline{\gamma}\underline{\beta}})^*$. We will also need the symmetric matrices with upper indices $\tilde{\Gamma}^{\underline{\mu}\underline{\alpha}\underline{\beta}} = \tilde{\Gamma}^{\underline{\mu}\underline{\beta}\underline{\alpha}} = C^{\underline{\alpha}\underline{\gamma}}\Gamma_{\underline{\gamma}\underline{\delta}}^{\underline{\mu}}\tilde{\Gamma}^{\underline{\delta}\underline{\beta}}$ which can be used to write the Clifford algebra (35) in the form $\Gamma_{\underline{\alpha}\underline{\delta}}^{(\underline{\mu}}\tilde{\Gamma}^{\underline{\nu})\underline{\delta}\underline{\beta}} = \eta^{\underline{\mu}\underline{\nu}}\delta_{\underline{\alpha}}^{\underline{\beta}}$.

We use the following $\text{SO}(1,9)$ invariant decomposition of 11D fermionic Majorana spinor coordinates on two 10D Majorana-Weyl spinor coordinates of opposite chiralities

$$\Theta^{\underline{\alpha}} = \begin{pmatrix} \theta^{1\alpha} \\ \theta^2_{\alpha} \end{pmatrix}. \quad (36)$$

Then, with the appropriate $\text{SO}(1,9)$ invariant representation of the 11D gamma matrices which is presented in Appendix C, the 11D VA 1-form splits, as indicated already in (34), in 10D VA 1-forms (23)

$$\Pi^{\underline{\mu}} = dX^{\underline{\mu}} - id\theta^1\sigma^{\underline{\mu}}\theta^1 - id\theta^2\tilde{\sigma}^{\underline{\mu}}\theta^2 \quad (37)$$

and the scalar 1-form

$$\Pi^* = dX^* + id\theta^1\theta^2 + id\theta^2\theta^1. \quad (38)$$

The 11D $\text{SO}(1,9)/[\text{SO}(1,1) \times \text{SO}(9)]$ spinor moving frame variables (or 11D Lorentz harmonics [56]; see [47,48] and refs therein) are defined as rectangular blocks of $\text{Spin}(1,10)$ -valued matrix

$$V_{\underline{\alpha}}^{(\underline{\beta})} = (v_{\underline{\alpha}q}^+, v_{\underline{\alpha}q}^-) \in \text{Spin}(1,10) \quad (39)$$

which provides a kind of square root of the $\text{SO}(1,10)$ valued moving-frame matrix

$$U_{\underline{\mu}}^{(\underline{a})} = \left(\frac{1}{2}(U_{\underline{\mu}}^{\#} + U_{\underline{\mu}}^{-}), U_{\underline{\mu}}^i, \frac{1}{2}(U_{\underline{\mu}}^{\#} - U_{\underline{\mu}}^{-}) \right) \in \text{SO}(1,10) \quad (40)$$

constructed from two lightlike vectors normalized in their contraction and nine orthonormal spacelike vectors orthogonal to that two,

$$U_{\underline{\mu}}^{-}U^{\underline{\mu}\#} = 0, \quad U_{\underline{\mu}}^{\#}U^{\underline{\mu}\#} = 0, \quad U_{\underline{\mu}}^{-}U^{\underline{\mu}\#} = 2, \quad (41)$$

$$U_{\underline{\mu}}^iU^{\underline{\mu}\#} = 0, \quad U_{\underline{\mu}}^iU^{\underline{\mu}\#} = 0, \quad U_{\underline{\mu}}^iU^{\underline{\mu}j} = -\delta^{ij}. \quad (42)$$

The moving frame vectors also obey $U_{\underline{\mu}}^{(\underline{c})}\eta_{(\underline{c})(\underline{d})}U_{\underline{\nu}}^{(\underline{d})} = \eta_{\underline{\mu}\underline{\nu}}$, which can be written in the form of

$$\delta_{\underline{\mu}\underline{\nu}} = \frac{1}{2}U_{\underline{\mu}}^{-}U^{\underline{\nu}\#} + \frac{1}{2}U_{\underline{\mu}}^{\#}U^{\underline{\nu}\#} - U_{\underline{\mu}}^iU^{\underline{\nu}i}. \quad (43)$$

The above mentioned square root relations have the form of the Lorentz invariance statement for the 11D Dirac matrices

$$V\Gamma_{\underline{\mu}}V^T = U_{\underline{\mu}}^{(\underline{a})}\Gamma_{(\underline{a})}, \quad V^T\tilde{\Gamma}^{(\underline{a})}V = \tilde{\Gamma}^{\underline{\mu}}U_{\underline{\mu}}^{(\underline{a})}. \quad (44)$$

The spinor moving frame variables also obey the constraint

$$VCV^T = C \quad (45)$$

manifesting the Lorentz invariance of the charge conjugation matrix. This implies that

$$v_q^{+\underline{\alpha}} = iC^{\underline{\alpha}\underline{\beta}}v_{\underline{\beta}q}^+, \quad v_q^{-\underline{\alpha}} = -iC^{\underline{\alpha}\underline{\beta}}v_{\underline{\beta}q}^- \quad (46)$$

define the inverse spinor moving-frame matrix, i.e., that

$$v_q^{-\underline{\alpha}}v_{\underline{a}p}^+ = \delta_{qp}, \quad v_q^{-\underline{\alpha}}v_{\underline{a}q}^- = 0, \\ v_q^{+\underline{\alpha}}v_{\underline{a}p}^+ = 0, \quad v_q^{+\underline{\alpha}}v_{\underline{a}p}^- = \delta_{qp}. \quad (47)$$

With an appropriate $[\text{SO}(1,1) \times \text{SO}(9)]$ invariant representation of 11D gamma matrices (see Appendix C) Eq. (44) split into

$$v_q^- \tilde{\Gamma}_\mu^- v_p^- = U_\mu^- \delta_{qp}, \quad U_\mu^- \Gamma_{\underline{\alpha}\underline{\beta}}^\mu = 2v_{\underline{\alpha}q}^- v_{\underline{\beta}q}^-, \quad (48)$$

$$v_q^+ \tilde{\Gamma}_\mu^+ v_p^+ = U_\mu^+ \delta_{qp}, \quad U_\mu^+ \Gamma_{\underline{\alpha}\underline{\beta}}^\mu = 2v_{\underline{\alpha}q}^+ v_{\underline{\beta}q}^+, \quad (49)$$

$$v_q^- \tilde{\Gamma}_\mu^- v_p^+ = U_\mu^i \gamma_{qp}^i, \quad U_\mu^i \Gamma_{\underline{\alpha}\underline{\beta}}^\mu = 2v_{(\underline{\alpha}|q}^- \gamma_{qp}^i v_{|\underline{\beta})p}^+, \quad (50)$$

where $\gamma_{qp}^i = \gamma_{pq}^i$ are SO(9) Dirac matrices,

$$\gamma_{qr}^i \gamma_{rp}^j + \gamma_{qr}^j \gamma_{rp}^i = 2\delta^{ij} \delta_{qp}. \quad (51)$$

The derivative of the moving frame and spinor moving frame variables can be expressed in terms of Cartan forms

$$\underline{\Omega}^{ij} := U^{\underline{\mu}i} dU_\mu^j, \quad (52)$$

$$\underline{\Omega}^{(0)} := U^{\underline{\mu}=-} dU_\mu^{\#}, \quad (53)$$

$$\underline{\Omega}^{\#j} := U^{\underline{\mu}\#} dU_\mu^j, \quad \underline{\Omega}^{\#j} := U^{\underline{\mu}=j} dU_\mu^j, \quad (54)$$

(see e.g., [57,58] and references therein for more details). The form (52) transforms as connection under the SO(9) symmetry acting on the 9-vector indices i, j and 16 component spinor indices q, p of the moving-frame variables and matrix fields. This is the counterpart of the Cartan form (16) of the 10D SO(1,9)/SO(9) spinor moving-frame (Lorentz harmonics) formalism.

The form (53) transforms as the connection under the SO(1,1) group acting on the moving frame, spinor moving frame variables and Lagrangian multiplier $\rho^\#$ according to the weights indicated by their sign indices. In particular $\# = ++$ so that under SO(1,1)

$$\begin{aligned} U_\mu^\# &\mapsto e^{2\beta} U_\mu^\#, & U_\mu^- &\mapsto e^{-2\beta} U_\mu^-, & U_\mu^i &\mapsto U_\mu^i, \\ \underline{\Omega}^{(0)} &\mapsto \underline{\Omega}^{(0)} + d\beta, & \underline{\Omega}^{ij} &\mapsto \underline{\Omega}^{ij}, \\ \rho^\# &\mapsto e^{2\beta} \rho^\#. \end{aligned} \quad (55)$$

The forms (54) are covariant with respect to both symmetries which will serve as gauge symmetry of the mM0 action.

2. Matrix fields describing the relative motion of mM0 constituents

Matrix fields describing the relative motion of the mM0 constituents are exactly the same as those of the mD0-system; the bosonic traceless $N \times N$ matrices \mathbb{X}^i and \mathbb{P}^i , carrying the SO(9) vector indices, fermionic traceless $N \times N$ matrices Ψ_q carrying SO(9) spinor index $q = 1, \dots, 16$, and bosonic traceless $N \times N$ matrix 1-form $\mathbb{A} = d\tau \mathbb{A}_\tau$.

3. mM0 action

The action for the 11D mM0 system proposed in [47] can be written in the form,

$$\begin{aligned} S_{\text{mM0}} &= \int_{\mathcal{W}^1} \rho^\# \underline{E}^- + \frac{1}{\mu^6} \int_{\mathcal{W}^1} \left(\text{tr}(\mathbb{P}^i \mathbb{D} \mathbb{X}^i + 4i \Psi_q \mathbb{D} \Psi_q) + \underline{E}^\# \frac{1}{\rho^\#} \mathcal{H} \right) \\ &\quad + \frac{1}{\mu^6} \int_{\mathcal{W}^1} \underline{E}^{+q} \frac{1}{\sqrt{\rho^\#}} \text{tr} \left(-4i (\gamma^i \Psi)_q \mathbb{P}^i + \frac{1}{2} (\gamma^{ij} \Psi)_q [\mathbb{X}^i, \mathbb{X}^j] \right) - \frac{1}{\mu^6} \int_{\mathcal{W}^1} \frac{\mathbb{D} \rho^\#}{\rho^\#} \text{tr}(\mathbb{P}^i \mathbb{X}^i), \end{aligned} \quad (56)$$

in which the matrix fields are redefined (with respect to the ones used in [47]) to be inert under the SO(1,1) symmetry acting on spinor moving frame variables and Lagrange multiplier $\rho^\#$ (55).

The covariant derivatives of the matrix fields in (56)

$$\mathbb{D} \mathbb{X}^i := d\mathbb{X}^i - \underline{\Omega}^{ij} \mathbb{X}^j + [\mathbb{A}, \mathbb{X}^i], \quad (57)$$

$$\mathbb{D} \Psi_q := d\Psi_q - \frac{1}{4} \underline{\Omega}^{ij} \gamma_{qp}^{ij} \Psi_p + [\mathbb{A}, \Psi_q], \quad (58)$$

include, besides the SU(N) connection 1-form \mathbb{A} , also the composite SO(9) connection $\underline{\Omega}^{ij}$ (52), while

$$\mathbb{D} \rho^\# = d\rho^\# - 2\underline{\Omega}^{(0)} \rho^\# \quad (59)$$

includes the SO(1,1) connection (53). Let us recall that these composite connections are Cartan forms representing the nonvanishing components of the derivatives of the moving frame and spinor moving-frame variables.

The moving frame and spinor moving-frame variables enter the action also explicitly through the projections

$$\underline{E}^\# = \Pi^\mu U_\mu^\#, \quad \underline{E}^- = \Pi^\mu U_\mu^-, \quad (60)$$

of the VA 1-form (34), which are the bosonic supervielbein forms of flat 11D superspace, and

$$\underline{E}^{+q} = d\Theta^\alpha v_{\underline{\alpha}q}^+ \quad (61)$$

of the pullback of the fermionic supervielbein form of this superspace.

The relative motion Hamiltonian in (56)

$$\mathcal{H} = \frac{1}{2} \text{tr}(\mathbb{P}^i \mathbb{P}^i) - \frac{1}{64} \text{tr}[\mathbb{X}^i, \mathbb{X}^j]^2 - 2\text{tr}(\mathbb{X}^i \Psi \gamma^i \Psi) \quad (62)$$

coincides with that of the mD0 system, Eq. (27).

The action (56) has the manifest 11D target superspace supersymmetry and, as it was shown in [47], is invariant under 16 parametric worldline supersymmetry generalizing the κ -symmetry of single M0-brane action in its spinor moving frame formulation of [59,60].

The properties of this mM0 system described by the action (56) were further studied in [48]. The problem of its dimensional reduction to $D = 10$ was addressed in [51] but was not solved there. The reason was that a suitable convenient choice of the basic matrix fields was not found in [51]; we have found it more recently, first for the simplified case of $D = 4$ counterpart of mM0 system [53], and we have already written the mM0 action of [47] in terms of these fields in Eq. (56). The dimensional reduction of this mM0 brane action is the subject of the next (sub) section.

B. Dimensional reduction of mM0 action to $D=10$

As we have already noticed, with the suitable representation for 11D Gamma matrices and charge conjugation matrix which can be found in Appendix C [see Eqs. (C1), (C2), and (C3)] the 11D Volkov-Akulov (VA) 1-form is split as in (34) into 10D VA 1-form (37) and $\text{SO}(1,9)$ invariant 1-form (38).

1. $\text{SO}(1, 9)$ invariant expressions for 11D spinor moving frame and basic 1-forms

The next stage is to solve the constraints defining the above described 11D spinor moving frame variables in terms of $\text{Spin}(1,9)/\text{Spin}(9)$ spinor moving frame variables (6). The convenient form of the solution is

$$\begin{aligned} \frac{1}{\sqrt{\rho^\#}} v_{\underline{a}q}^+ &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\mathcal{M}}} \begin{pmatrix} v_a^q \\ -v_q^a \end{pmatrix}, \\ \sqrt{\rho^\#} v_{\underline{a}q}^- &= \frac{1}{\sqrt{2}} \sqrt{\mathcal{M}} \begin{pmatrix} v_a^q \\ v_q^a \end{pmatrix}, \end{aligned} \quad (63)$$

which implies the complementary relations

$$\begin{aligned} \frac{1}{\sqrt{\rho^\#}} v_q^{+\alpha} &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\mathcal{M}}} \begin{pmatrix} v_q^\alpha \\ v_a^q \end{pmatrix}, \\ \sqrt{\rho^\#} v_q^{-\alpha} &= \frac{1}{\sqrt{2}} \sqrt{\mathcal{M}} \begin{pmatrix} v_q^\alpha \\ -v_a^q \end{pmatrix}. \end{aligned} \quad (64)$$

Equations (63) involve the Lagrange multiplier $\rho^\#$ of the mM0 action and also an arbitrary function $\mathcal{M} = \mathcal{M}(\tau)$ of

the proper time parametrizing the center-of-energy worldline of the mM0 system.

Let us stress that this is not an ansatz but rather a general solution of the constraints (48)–(50) and (45). This can be easily checked by counting the number of degrees of freedom, modulo natural gauge symmetries, in the left and right-hand sides of the relation (63) which gives $1 + 9 = 1 + 9$. Indeed, both sides include scalar functions, $\rho^\#$ and \mathcal{M} respectively, and spinor frame variables parametrizing cosets isomorphic to \mathbb{S}^9 sphere: $\frac{\text{SO}(1,10)}{(\text{SO}(1,1) \times \text{SO}(9)) \otimes K_9} \simeq \mathbb{S}^9$ and $\frac{\text{SO}(1,9)}{\text{SO}(9)} \simeq \mathbb{S}^9$, respectively (see e.g., [57,58] and references therein for more details). Notice also that the lhs of (63) preserves the characteristic $\text{SO}(1,1)$ gauge symmetry (55) of the 11D spinor moving-frame formalism (which was taken into account in the above counting of the degrees of freedom).

As far as the 11D Lorentz symmetry $\text{SO}(1,10)$ is concerned, the expressions (63) break this down to its $\text{SO}(1,9)$ subgroup which becomes 10D Lorentz symmetry of the reduced theory.

Using (63) we obtain the following expression for (the pullbacks of) 11D fermionic supervielbein forms which enter Eq. (56)

$$\underline{E}^{+q} = \sqrt{\rho^\#} \frac{1}{\sqrt{2}\mathcal{M}} (E^{1q} - E_q^2), \quad (65)$$

where

$$E^{1q} = d\theta^{1\alpha} v_a^q, \quad E_q^2 = d\theta_{\alpha}^2 v_q^a, \quad (66)$$

can be naturally identified with the pullbacks of the fermionic $D = 10$ type-IIA supervielbein forms (25).

Now, substituting (63) into Eqs. (48) and (49) and using the suitable representation for 11D gamma matrices (see Appendix C) as well as (11) we find that the 11D moving-frame vectors are related to its 10D counterparts by

$$U_\mu^- = \frac{\mathcal{M}}{\rho^\#} u_\mu^0, \quad U_*^- = -\frac{\mathcal{M}}{\rho^\#}, \quad (67)$$

$$U_\mu^\# = \frac{\rho^\#}{\mathcal{M}} u_\mu^0, \quad U_*^\# = \frac{\rho^\#}{\mathcal{M}}, \quad (68)$$

so that

$$\rho^\# \underline{E}^- = \mathcal{M}(E^0 - \Pi^*), \quad \frac{\underline{E}^\#}{\rho^\#} = \frac{1}{\mathcal{M}}(E^0 + \Pi^*). \quad (69)$$

In the same way we find from (50)

$$U_\mu^i = \delta_\mu^\nu u_\nu^i, \quad (70)$$

which can be used to check that the composite $\text{SO}(9)$ connection of the mM0 system coincides with its 10D counterpart (16),

$$\underline{\Omega}^{ij} := U^{\mu i} dU_{\mu}^j = u^{\mu i} du_{\mu}^j =: \Omega^{ij}. \quad (71)$$

Thus the covariant derivative of the matrix fields in the mM0 action (56), Eqs. (57) and (58), coincide with (29) and (30) used in the mD0 action (26).

As far as SO(1, 1) connection is concerned, we find from (67) and (68)

$$\underline{\Omega}^{(0)} = \frac{1}{4} U^{\mu} dU_{\mu}^{\#} = \frac{1}{2} \left(\frac{d\rho^{\#}}{\rho^{\#}} - \frac{d\mathcal{M}}{\mathcal{M}} \right). \quad (72)$$

In particular, this implies that

$$D\rho^{\#} = d\rho^{\#} - 2\rho^{\#}\underline{\Omega}^{(0)} = \rho^{\#} \frac{d\mathcal{M}}{\mathcal{M}}. \quad (73)$$

At this stage one can observe that, after using in (56) the relations (63) and their consequences, $\rho^{\#}$ disappears from the 11D action being replaced by \mathcal{M} .

2. Dimensional reduction of mM0 action

Thus with the solution (63) and the splitting (34) the mM0 action (56) acquires the form

$$\begin{aligned} S_{\text{mM0}}|_{(63)} = & \int_{\mathcal{W}^1} \mathcal{M}(E^0 - \Pi^*) + \frac{1}{\mu^6} \int_{\mathcal{W}^1} \frac{1}{\mathcal{M}} (E^0 + \Pi^*) \mathcal{H} + \frac{1}{\mu^6} \int_{\mathcal{W}^1} \text{tr}(\mathbb{P}^i D \mathbb{X}^i + 4i \Psi_q D \Psi_q) - \frac{1}{\mu^6} \int_{\mathcal{W}^1} \frac{d\mathcal{M}}{\mathcal{M}} \text{tr}(\mathbb{P}^i \mathbb{X}^i) \\ & + \frac{1}{\mu^6} \int_{\mathcal{W}^1} \frac{1}{2\sqrt{\mathcal{M}}} (E^{1q} - E_q^2) \text{tr} \left(-4i (\gamma^i \Psi)_q \mathbb{P}^i + \frac{1}{2} (\gamma^{ij} \Psi)_q [\mathbb{X}^i, \mathbb{X}^j] \right), \end{aligned} \quad (74)$$

where the covariant derivatives are defined in (29), (30), the relative motion Hamiltonian \mathcal{H} has the form of (27),

$$E^0 = \Pi^{\mu} u_{\mu}^0 \quad (75)$$

with Π^{μ} from (37) [see (24)], Π^* is given in (38) and, according to (66),

$$E^{1q} - E_q^2 = d\theta^{1\alpha} v_{\alpha}^q - d\theta_{\alpha}^2 v_q^{\alpha}. \quad (76)$$

The dimensional reduction is then completed by deriving from the action (74) the equation of motion for eleventh bosonic coordinate field X^* ,

$$d \left(\mathcal{M} - \frac{1}{\mathcal{M} \mu^6} \mathcal{H} \right) = 0, \quad (77)$$

and substituting its solution back into the functional (74).

Equation (77) can be equivalently written in the form

$$\mathcal{M} - \frac{1}{\mathcal{M} \mu^6} \mathcal{H} = m \quad (78)$$

with some constant m . Its solution which has a non-vanishing limit when $\mathcal{H} \mapsto 0$ is

$$\mathcal{M} = \frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{\mathcal{H}}{\mu^6}}. \quad (79)$$

Substituting the result back into (74) we find the action (26) with a particular function $\mathcal{M} = \mathcal{M}(\mathcal{H}/\mu^6)$ given by (79).

Thus just this particular case of (26) model has the apparent 11D origin.

IV. EQUATIONS OF MOTION OF 10D mD0

In this section we write the complete set of equations of motion for the multiple D0-brane system which follow from our action (26).

A. Equations for the center-of-energy variables

Beginning from the center of energy variables, we regroup the variation with respect to the coordinate functions, $\delta Z^M(\tau) = (\delta x^{\mu}(\tau), \delta \theta^{\alpha 1}(\tau), \delta \theta_{\alpha}^2(\tau))$, into

$$\begin{aligned} i_{\delta} E^i &= \delta Z^M(\tau) E_M^a(Z^M(\tau)) u_a^i(\tau), \\ i_{\delta} E^0 &= \delta Z^M(\tau) E_M^a(Z^M(\tau)) u_a^0(\tau), \end{aligned} \quad (80)$$

and

$$i_{\delta}(E^{q1} - E_q^2), \quad i_{\delta}(E^{q1} + E_q^2), \quad (81)$$

where

$$i_{\delta} E^{q1} = \delta \theta^{\alpha 1} v_{\alpha}^q, \quad i_{\delta} E_q^2 = \delta \theta_{\alpha}^2 v_q^{\alpha}. \quad (82)$$

This choice simplifies the form of the equations of motion for the coordinate functions [originally defined as $\frac{\delta S_{\text{mD0}}}{\delta Z^M(\tau)} = 0$] and splits their set in a Lorentz-covariant manner into

$$\Omega^i \left(m + \frac{2}{\mu^6} \frac{\mathcal{H}}{\mathcal{M}} \right) = 0, \quad (83)$$

$$\left(m + \frac{1}{\mu^6} \frac{\mathcal{H}}{\mathcal{M}} \right) (E^{1q} + E_q^2) = \frac{-i}{4\sqrt{2}\mathcal{M}\mu^6} \gamma_{qp}^i i\nu_p \Omega^i, \quad (84)$$

and

$$\frac{2}{\mathcal{M}} \left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H} \right) d\mathcal{H} = 0, \quad (85)$$

$$\frac{1}{\sqrt{2\mathcal{M}}} iD\nu_q - \frac{1}{\mu^6} \frac{1}{2\sqrt{2\mathcal{M}}} \frac{\mathcal{M}'}{\mathcal{M}} i\nu_q d\mathcal{H} - \frac{2i}{\mathcal{M}} (E^{1q} - E_q^2) \mathcal{H} = 0, \quad (86)$$

where

$$i\nu_q := \text{tr} \left(-4i(\gamma^i \Psi)_q \mathbb{P}^i + \frac{1}{2} (\gamma^{ij} \Psi)_q [\mathbb{X}^i, \mathbb{X}^j] \right). \quad (87)$$

As we will show below, Eqs. (85) and (86) are dependent, this is to say they are satisfied identically when other equations are taken into account. This statement is the content of Noether identities for the worldline reparametrization gauge symmetry and for the worldline

supersymmetry (κ -symmetry) the ‘‘local parameters’’ of which can be identified with $i_\delta E^0$ and $i_\delta (E^{q1} - E_q^2)$, respectively. Keeping this in mind, we will first discuss (83), (84), and the equations for spinor frame variables and then turn to the matrix equations [which will then allow us to make the above statements about dependence of (85) and (86)].

As far as the function $\mathcal{M}(\mathcal{H}/\mu^6)$ in our action is positive definite, (28), Eq. (83) implies

$$\Omega^i = 0. \quad (88)$$

Taking this into account, we find that the fermionic equations (84) simplifies to

$$E^{1q} + E_q^2 = 0. \quad (89)$$

Finally, the essential variations of the moving-frame and spinor moving-frame variables,

$$\delta u_\mu^0 = u_\mu^i i_\delta \Omega^i, \quad \delta v_\alpha^q = \frac{1}{2} i_\delta \Omega^i \gamma_{qp}^i v_\alpha^p, \quad \delta v_q^\alpha = -\frac{1}{2} i_\delta \Omega^i v_p^\alpha \gamma_{pq}^i, \quad (90)$$

result in

$$E^i \left(m + \frac{2}{\mu^6} \frac{\mathcal{H}}{\mathcal{M}} \right) - \frac{1}{2\mu^6 \sqrt{2\mathcal{M}}} (E^{1q} + E_q^2) \gamma_{qp}^i i\nu_p - \frac{2}{\mu^6} \text{tr}(\mathbb{P}^i \mathbb{X}^j + i\Psi \gamma^{ij} \Psi) \Omega^j = 0, \quad (91)$$

where $i\nu_q$ is defined in (87). Taking into account (88) and the fermionic equations (89), we see that (91) finally simplifies to

$$E^i = 0. \quad (92)$$

Thus the set of equations for the center of energy variables is given by Eqs. (88), (89), and (92). This coincides with the set of equations of motion for single D0-brane in its spinor moving frame formulation [61].

B. Equations of the relative motion of the constituents of mD0 system

The equations of motion for the matrix fields describing the relative motion of the mD0 constituents read

$$D\mathbb{X}^i = -\frac{2}{\mathcal{M}} \left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H} \right) E^0 \mathbb{P}^i + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} (\mathbb{X}^i d\mathcal{H} - \mathbb{P}^i d\mathcal{K}) + \frac{1}{\sqrt{2\mathcal{M}}} (E^{1q} - E_q^2) \left(4i(\gamma^i \Psi)_q - \frac{1}{2\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} i\nu_q \mathbb{P}^i \right), \quad (93)$$

$$D\mathbb{P}^i = \frac{2}{\mathcal{M}} \left[\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H} \right) E^0 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{4\sqrt{2\mathcal{M}}} (E^{1q} - E_q^2) i\nu_q \right] \left(\frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i \{ \Psi_p, \Psi_r \} \right) - \frac{1}{\sqrt{2\mathcal{M}}} (E^{1q} - E_q^2) [(\gamma^{ij} \Psi)_q, \mathbb{X}^j] + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} d\mathcal{K} \left(\frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i \{ \Psi_p, \Psi_r \} \right) - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathbb{P}^i d\mathcal{H}, \quad (94)$$

$$D\Psi_q = -\frac{1}{2\sqrt{2\mathcal{M}}} (E^{1p} - E_p^2) \left(\gamma_{pq}^i \mathbb{P}^i + \frac{i}{8} \gamma_{pq}^{ij} [\mathbb{X}^i, \mathbb{X}^j] \right) - \frac{i}{2\mathcal{M}} \left(\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H} \right) E^0 + \frac{\mathcal{M}'}{2\mu^6} d\mathcal{K} + \frac{1}{4\mu^6} \frac{\mathcal{M}'}{\sqrt{2\mathcal{M}}} (E^{1p} - E_p^2) i\nu_p \right) [(\gamma^i \Psi)_q, \mathbb{X}^i], \quad (95)$$

and

$$[\mathbb{X}^i, \mathbb{P}^i] = 4i\{\Psi_q, \Psi_q\}. \quad (96) \quad d\mathcal{H} = 0 \quad (98)$$

In Eqs. (93)–(95) \mathcal{H} is given in (27),

and

$$\mathcal{K} = \text{tr}(\mathbb{X}^i \mathbb{P}^i) \quad (97) \quad iD\nu_q = \frac{2\sqrt{2}}{\sqrt{\mathcal{M}}}(E^{1q} - E_q^2)\mathcal{H}, \quad (99)$$

and $i\nu_q$ is defined in (87). Equation (96) appears as a result of variation with respect to the worldline gauge field $\mathbb{A} = d\tau\mathbb{A}_\tau$ and is the (non-Abelian version of the) Gauss law of 10D SYM reduced to 1d.

Using Eqs. (93)–(95) and (96) one can check (by straightforward although a bit involving calculations) that

which implies that Eqs. (85) and (86) are satisfied identically (as we have announced and discussed just after their derivation).

Taking into account (98) we find that the equations for bosonic matrix matter fields (93) and (94) simplify,

$$D\mathbb{X}^i = -\frac{2}{\mathcal{M}}\left(1 - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathcal{H}\right)E^0\mathbb{P}^i - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathbb{P}^i d\mathcal{K} + \frac{1}{\sqrt{2}\mathcal{M}}(E^{1q} - E_q^2)\left(4i(\gamma^i\Psi)_q - \frac{1}{2\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}i\nu_q\mathbb{P}^i\right), \quad (100)$$

$$D\mathbb{P}^i = \frac{2}{\mathcal{M}}\left[\left(1 - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathcal{H}\right)E^0 + \frac{1}{\mu^6}\frac{\mathcal{M}'}{4\sqrt{2}\mathcal{M}}(E^{1q} - E_q^2)i\nu_q\right]\left(\frac{1}{16}[[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i\{\Psi_p, \Psi_r\}\right) - \frac{1}{\sqrt{2}\mathcal{M}}(E^{1q} - E_q^2)[(\gamma^{ij}\Psi)_q, \mathbb{X}^j] + \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}d\mathcal{K}\left(\frac{1}{16}[[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i\{\Psi_p, \Psi_r\}\right). \quad (101)$$

This form of equations is not yet final since it involves $d\mathcal{K} = \text{tr}(D\mathbb{X}^i\mathbb{P}^i + \mathbb{X}^i D\mathbb{P}^i)$. Calculating the rhs of this expression with the use of (100) and (101), we find

$$d\mathcal{K} = -\frac{2}{\mathcal{M}}\frac{\mathfrak{S}}{\left(1 + \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathfrak{S}\right)}\left(1 - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathcal{H}\right)E^0 + \frac{(E^{1q} - E_q^2)\text{tr}(4i(\gamma^i\Psi)_q\mathbb{P}^i + (\gamma^{ij}\Psi)_q[\mathbb{X}^i, \mathbb{X}^j]) - \frac{1}{2\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathfrak{S}i\nu_q}{\left(1 + \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathfrak{S}\right)}, \quad (102)$$

with

$$\mathfrak{S} := \text{tr}(\mathbb{P}^i\mathbb{P}^i) + \frac{1}{16}\text{tr}[\mathbb{X}^i, \mathbb{X}^j]^2 + 2\text{tr}(\mathbb{X}^i\Psi\gamma^i\Psi). \quad (103)$$

Thus the final form of the equations for the matrix matter fields is given by (100), (101) and (95) with $d\mathcal{K}$ substituted by the rhs of (102). This looks frighteningly complicated but, as we will show, the equations can be simplified essentially by fixing in a convenient manner the gauge symmetries of our dynamical system. However, before turning to the gauge fixing, we have to discuss the mass of the mD0 system.

C. Mass of mD0 system and its center-of-mass motion

It is instructive to calculate the canonical momentum for the center-of-energy coordinate function of mD0 system

$$p_\mu = \left(m + \frac{2}{\mu^6}\frac{\mathcal{H}}{\mathcal{M}}\right)u_\mu^0 =: \mathfrak{M}u_\mu^0. \quad (104)$$

The mass \mathfrak{M} of the mD0 is defined by the square of this 10-momentum, $p_\mu p^\mu = \mathfrak{M}^2$, and thus is given by

$$\mathfrak{M} = m + \frac{2}{\mu^6}\frac{\mathcal{H}}{\mathcal{M}}. \quad (105)$$

Note that it depends essentially on the choice of the positively definite function $\mathcal{M}(\mathcal{H}/\mu^6)$ in the action (56). However, as a consequence of (98), this mass is a constant of motion,

$$d\mathfrak{M} = 0. \quad (106)$$

It is important that the mass depends on the relative motion of the mD0 constituents so that in our supersymmetric model the relative motion does influence the center of energy motion like it does in the purely bosonic dynamical system from [30].

The constant parameter m in the action (26) is thus the mass of the center-of-energy motion of mD0 the relative motion sector of which is in the ground state, i.e., with $\mathcal{H} = 0$. As we will see below, the supersymmetric states of mD0 systems have this property. Thus, exciting the state of relative motion of the mD0 constituents we inevitably increase the mass of the center of energy of mD0 system [remember (28)] and, as we will see below, inevitably break supersymmetry.

Now let us also observe that, by definition of Cartan forms, $du_\mu^0 = u_\mu^i \Omega^i$ and hence Eq. (88) implies that on the mass shell

$$du_\mu^0 = 0. \quad (107)$$

This together with (106) implies that the momentum (104) is a constant of motion

$$dp_\mu = 0. \quad (108)$$

D. Gauge fixing of the local SO(9) and SU(N) symmetries

For the future use, let us also notice that on the surface of Eq. (88) the derivatives of the orthogonal moving frame vectors u_μ^i are decomposed on the set of these vectors only,

$$du_\mu^i = \Omega^{ij} u_\mu^j. \quad (109)$$

Furthermore, as Ω^{ij} transforms as the connection under local gauge SO(9) symmetry of the mD0 action (26), and as any one-dimensional connection can be gauged away, we can always fix the gauge

$$\Omega^{ij} = 0 \quad (110)$$

in which also nine spacelike vectors of the moving frame become constant

$$du_\mu^i = 0. \quad (111)$$

Similarly, we can use the SU(N) gauge symmetry to fix the gauge in which the $\mathfrak{su}(N)$ valued gauge field vanishes,

$$\mathbb{A} = 0, \quad (112)$$

so that the covariant derivative $D = d\tau D_\tau$ reduces to time derivative,

$$\begin{aligned} D\mathbb{X}^i &= d\tau \frac{d}{d\tau} \mathbb{X}^i = d\tau \dot{\mathbb{X}}^i, & D\mathbb{P}^i &= d\tau \frac{d}{d\tau} \mathbb{P}^i = d\tau \dot{\mathbb{P}}^i, \\ D\Psi_q &= d\tau \frac{d}{d\tau} \Psi_q = d\tau \dot{\Psi}_q. \end{aligned} \quad (113)$$

Notice that, as far as moving frame and spinor moving-frame variables are now (proper-)time independent,

$$\begin{aligned} E^0 &= d\tau E_\tau^0 = dx^0 - id\theta^{1q} \theta^{1q} - id\theta_q^2 \theta_q^2, \\ E^{q1} &= d\theta^{1q}, & E_q^2 &= d\theta_q^2, \end{aligned} \quad (114)$$

and

$$E^i = d\tau E_\tau^i = dx^i - id\theta^{1q} \gamma_{qp}^i \theta^{1p} + id\theta_q^2 \gamma_{qp}^i \theta_p^2, \quad (115)$$

where

$$x^0 = x^\mu u_\mu^0, \quad x^i = x^\mu u_\mu^i, \quad \theta^{1q} = \theta^{\alpha 1} v_\alpha^q, \quad \theta_q^2 = \theta_\alpha^2 v_q^\alpha, \quad (116)$$

describe supersymmetric generalization of the comoving coordinate system of the center-of-mass of mD0-brane.

E. Gauge fixing of κ -symmetry and reparametrization symmetry

The last two of consequences (114) of Eqs. (88) and the gauge-fixing conditions (110) allow us to reduce (89) to

$$d\theta^{1q} = -d\theta_q^2. \quad (117)$$

This equation is clearly invariant under spacetime supersymmetry and κ -symmetry which in coordinate basis (116) implies

$$\begin{aligned} \delta x^0 &= id\theta^{1q} (\epsilon^{1q} - \kappa^q / \sqrt{2}) + i\theta_q^2 (\epsilon_q^2 + \kappa^q / \sqrt{2}), \\ \delta \theta^{1q} &= \epsilon^{1q} + \kappa^q / \sqrt{2}, & \delta \theta_q^2 &= \epsilon_q^2 - \kappa^q / \sqrt{2}, \end{aligned} \quad (118)$$

and

$$\delta E^0 = -2i(d\theta^{1q} - d\theta_q^2) \kappa^q / \sqrt{2}. \quad (119)$$

Clearly, the worldline supersymmetry can be used to set one of two (but not both) fermionic coordinate functions equal to zero (as the κ -symmetry of single D0-brane can be). Let us choose

$$\theta_q^2 = 0. \quad (120)$$

This gauge is preserved by the combination of supersymmetry and worldline supersymmetry which obey

$$\epsilon_q^2 = \kappa^q / \sqrt{2}. \quad (121)$$

This relation also implies that the parametric function of the worldline supersymmetry becomes a fermionic constant spinor at this stage.

The gauge choice (120) simplifies Eq. (117) to

$$d\theta^{1q} = 0, \quad (122)$$

which in its turn reduces E^0 form (114) to dx^0 ,

$$E^0 = dx^0, \quad (123)$$

and E^i form (115) to dx^i ,

$$E^i = dx^i. \quad (124)$$

This is supersymmetric because now

$$\delta x^0 = i(\epsilon^{1q} + \epsilon_q^2)\theta^{1q}, \quad \delta x^i = i(\epsilon^{1q} + \epsilon_q^2)\gamma_{qp}^i \theta^{1p}, \quad (125)$$

and the right-hand sides of these expressions are constants [due to (122), $dv_\alpha^q = 0$ and $d\epsilon^{\alpha 1,2} = 0$].

In this gauge with respect to the κ -symmetry we can fix the gauge with respect to the reparametrization symmetry by setting

$$d\tau = dx^0 = E^0 \quad \Rightarrow \quad E_\tau^0 = \dot{x}^0 = 1 \quad (126)$$

still preserving the supersymmetry. We however find convenient to do not do this and preserve explicit τ -reparametrization symmetry at the next stages.

F. Gauge fixed form of the field equations

Thus the above gauge fixing and the field equations for the center of energy variables imply that

$$du_\mu^0 = 0, \quad du_\mu^i = 0, \quad dv_\alpha^q = 0, \quad (127)$$

$$E^i = dx^i = 0, \quad (128)$$

$$E^{1q} = d\theta^{1q} = 0, \quad E_q^2 = 0, \quad (129)$$

and $E^0 = d\tau \dot{x}^0$. With this in mind the equations for the matrix fields reduce to

$$\dot{\mathbb{X}}^i = -\frac{2}{\mathcal{M}} \left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right) \dot{x}^0 \mathbb{P}^i - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathbb{P}^i \dot{\mathcal{K}}, \quad (130)$$

$$\begin{aligned} \dot{\mathbb{P}}^i &= \left[\frac{2}{\mathcal{M}} \left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right) \dot{x}^0 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \dot{\mathcal{K}} \right] \\ &\times \left(\frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i \{\Psi_p, \Psi_r\} \right), \end{aligned} \quad (131)$$

$$\dot{\Psi}_q = -\frac{i}{2\mathcal{M}} \left(\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right) \dot{x}^0 + \frac{\mathcal{M}'}{2\mu^6} \dot{\mathcal{K}} \right) [(\gamma^i \Psi)_q, \mathbb{X}^i], \quad (132)$$

where

$$\dot{\mathcal{K}} = -\frac{2}{\mathcal{M}} \frac{\mathfrak{H}}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right) \dot{x}^0 \quad (133)$$

with \mathfrak{H} defined in (103).

Substituting (133) we find that the final form of the gauge fixed Eqs. (130)–(132) is

$$\dot{\mathbb{X}}^i = -\frac{2}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \dot{x}^0 \mathbb{P}^i, \quad (134)$$

$$\dot{\mathbb{P}}^i = \frac{2}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \dot{x}^0 \left(\frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i \{\Psi_p, \Psi_r\} \right), \quad (135)$$

$$\dot{\Psi}_q = -\frac{i}{2\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \dot{x}^0 [(\gamma^i \Psi)_q, \mathbb{X}^i]. \quad (136)$$

Notice that, despite the possibility to fix the gauge $\dot{x}^0 = 1$ (126), we prefer to keep it unfixed to stress the invariance of our equations of motion under the τ reparametrizations, the fact which will be important for our discussion below.

G. Relation of the relative motion equations of mD0 with SYM equations

Formally, we can define the new time variable by

$$dt = dx^0 \frac{2}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \Leftrightarrow \frac{dt(\tau)}{d\tau} = \dot{x}^0 \frac{2}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)}, \quad (137)$$

and write the above equations in terms of derivative $\frac{d}{dt}$ arriving at

$$\frac{d}{dt} \mathbb{X}^i = -\mathbb{P}^i, \quad (138)$$

$$\frac{d}{dt} \mathbb{P}^i = \frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i \{\Psi_p, \Psi_r\}, \quad (139)$$

$$\frac{d}{dt} \Psi_q = -\frac{i}{4} [(\gamma^i \Psi)_q, \mathbb{X}^i], \quad (140)$$

which has the form of the equations of motion of $d = 1$ reduction of 10D YM theory in the gauge $\mathbb{A}_0 = 0$.

Notice, however, that this procedure cannot be considered as reparametrization (1d general coordinate) transformations of the proper time τ . The simplest way to see this is to appreciate that the defining equation (137) for the “new time” is actually invariant under the τ reparametrizations and hence cannot be a gauge fixing condition for this symmetry.⁷

The other, more physical observation is that a change of the form of the positively definite function $\mathcal{M}(\mathcal{H})$ implies the changes of the physical characteristic of the system. Namely such a passage will change the mass \mathfrak{M} of the mD0 system (105) which is different for different forms of $\mathcal{M}(\mathcal{H})$. Hence, such a change cannot be achieved by transformation of a gauge symmetry of the system.

Thus what we have found is not a kind of gauge equivalence, but an interesting correspondence of the gauge fixed form of the relative motion equations of our mD0 system with the equation of maximally supersymmetric $\mathcal{N} = 16$ $d = 1$ $SU(N)$ SYM theory. Notice that the formal definition (137) of new time variable involves the matrix fields \mathbb{X}^i , \mathbb{P}^i , Ψ_q and thus generically it is different for different solutions of the field equations. Indeed, although \mathcal{H} , and hence $\mathcal{M}(\mathcal{H})$, are constants on the mass shell, this is not the case for $\mathfrak{S} = \mathfrak{S}(\mathbb{X}^i, \mathbb{P}^i, \Psi_q)$. However, in the particular case of solutions with $\mathcal{H} = 0$, also $\mathfrak{S} = 0$, t is related with x^0 by a constant rescaling and the correspondence between such solutions of relative motion mD0 and $SU(N)$ SYM equations is one-to-one.

As we will see in a moment, this is the case for bosonic supersymmetric solutions of mD0 equations.

V. SUPERSYMMETRIC BOSONIC SOLUTIONS

As we have discussed in Sec. IV E, on the mass shell one of two center-of-energy fermions can be gauged away by the worldline supersymmetry (κ -symmetry), (120), and then the remaining fermionic coordinate function becomes constant (122). Setting this also to zero, thus arriving at purely bosonic center-of-mass configuration with

$$\theta^{\alpha 1} = 0, \quad \theta_\alpha^2 = 0, \quad (141)$$

we find that such a choice is still preserved by a half of the target (super)space supersymmetry with two spinor parameters expressed in terms of one $SO(9)$ spinor parameter of the worldline supersymmetry and spinor frame variables [which are constant in the gauge under consideration, (127)] by

$$\epsilon^{\alpha 1} = -\kappa^q v_q^\alpha / \sqrt{2}, \quad \epsilon_\alpha^2 = \kappa^q v_\alpha^q / \sqrt{2}. \quad (142)$$

Then, after fixing the reparametrization invariance by setting $x^0(\tau) = \tau$, the solution of the bosonic center-of-energy equations can be written in the form

$$x^\mu(\tau) = x_0^\mu + \frac{p^\mu}{\mathfrak{M}} \tau, \quad (143)$$

where p^μ and \mathfrak{M} are given in (104) and (105). These are constants due to (98) which is the Noether identity for the reparametrization symmetry. Let us stress that any of these bosonic solutions of the equations of center of energy motion preserves 1/2 of the type-IIA spacetime supersymmetry, which can be then broken or preserved by the relative motion of the mD0 constituents.

Originally the matrix fields which describe the relative motion of the mD0 constituents are inert under spacetime supersymmetry. However, after gauge fixing of the worldline supersymmetry (κ -symmetry) by (120), this is no longer the case since the parameter of the worldline supersymmetry, which does act on the matrix fields, becomes identified with the parameter of the second spacetime supersymmetry, Eq. (121). Furthermore, setting the remaining fermionic coordinate function to zero, (141), also identifies with that the parameter of the first spacetime supersymmetry (142), so that the action of spacetime supersymmetry on the matrix fields becomes nontrivial but coincides with the action of the worldline supersymmetry which is now parametrized by a constant fermionic κ^q .

Setting the fermionic matrix field to zero

$$\Psi_q = 0 \quad (144)$$

we can still preserve the worldline supersymmetry (completely or partially), and hence also a part of the spacetime supersymmetry if the following Killing spinor equation [see (A4) in Appendix A]

$$\delta_\kappa \Psi_q = -\frac{1}{2\sqrt{\mathcal{M}}} (\kappa \gamma^i)_q \mathbb{P}^i - \frac{i}{16\sqrt{\mathcal{M}}} (\kappa \gamma^{ij})_q [\mathbb{X}^i, \mathbb{X}^j] = 0 \quad (145)$$

has a nontrivial solution. Notice that the contribution of the nonvanishing function $\mathcal{M}(\mathcal{H})$ can be factored out thus reducing the equation to

⁷Actually the equations (130)–(132) are manifestly reparametrization invariant themselves and thus cannot be transformed into the other form (138)–(140) by τ reparametrizations.

$$\kappa^p \mathbb{L}_{pq} := \kappa^p \left(\gamma_{pq}^i \mathbb{P}^i - \frac{i}{8} (\gamma^{ij})_{pq} [\mathbb{X}^i, \mathbb{X}^j] \right) = 0, \quad (146)$$

which is formally the same as defining the potentially-preserved supersymmetry in the case of (the first-order formulation of) the maximally-supersymmetric $SU(N)$ SYM theory.⁸ It also formally coincides with the equation defining the supersymmetry which can be preserved by purely bosonic solutions of 11D mM0 equations [47] so that our discussion below will be close to the one in [47].

The preservation of the 1/2 of the target superspace supersymmetry implies, in the light of (142), that Eq. (146) does not impose any restriction on the parameter κ^q which implies that the matrix \mathbb{L}_{pq} vanishes, $\gamma_{pq}^i \mathbb{P}^i - \frac{i}{8} (\gamma^{ij})_{pq} [\mathbb{X}^i, \mathbb{X}^j] = 0$. This in its turn implies that

$$\mathbb{P}^i = 0, \quad [\mathbb{X}^i, \mathbb{X}^j] = 0, \quad (147)$$

which describes the vacuum of the relative motion of the mD0 constituents. In it $\mathcal{H} = 0$ and, actually, (147) can be obtained from this equation.

Now let us check whether other parts of spacetime supersymmetry can be preserved. To this end we have to study Eq. (146) which is actually a set of 16 traceless $N \times N$ matrix equations. It is convenient to extract from this just 16 equations by multiplying it by the matrix $\tilde{\mathbb{L}}_{qr} := (\gamma_{qr}^i \mathbb{P}^i + \frac{i}{8} (\gamma^{ij})_{qr} [\mathbb{X}^i, \mathbb{X}^j])$, which differs by relative sign from the matrix \mathbb{L}_{qr} in (146), and tracing the result with respect to the $SU(N)$ indices. In this way we arrive at

$$\begin{aligned} \kappa^p \text{tr}(\mathbb{L}\tilde{\mathbb{L}})_{pq} &:= \kappa^q \left(\text{tr}(\mathbb{P}^i \mathbb{P}^i) - \frac{1}{32} \text{tr}([\mathbb{X}^i, \mathbb{X}^j]^2) \right) \\ &+ \frac{i}{4} \kappa^p (\gamma^j)_{pq} \text{tr}(\mathbb{P}^i [\mathbb{X}^i, \mathbb{X}^j]) = 0, \end{aligned} \quad (148)$$

where we have used $\gamma^{ijkl} \text{tr}([\mathbb{X}^i, \mathbb{X}^j][\mathbb{X}^k, \mathbb{X}^l]) = \gamma^{ijkl} \text{tr}(\mathbb{X}^i [\mathbb{X}^j, [\mathbb{X}^k, \mathbb{X}^l]]) = 0$ (which follows from the Jacobi identity $[\mathbb{X}^j, [\mathbb{X}^k, \mathbb{X}^l]] \equiv 0$). Now one can recognize that the multiplier of the first term the bosonic limit of the relative-motion Hamiltonian \mathcal{H} (27) is multiplied by 2 and also appreciate that the coefficient for the second term vanishes as a consequence of the (bosonic limit of the) Gauss constraint (96), $\text{tr}(\mathbb{P}^i [\mathbb{X}^i, \mathbb{X}^j]) = \text{tr}([\mathbb{P}^i, \mathbb{X}^i] \mathbb{X}^j) = 0$. Thus, if we use this constraint, the final form of the consistency condition (148) for the supersymmetry preservation by mD0 configuration simplifies to

⁸The word formally refers to the fact that the relation between \mathbb{P}^i and \mathbb{X}^i in the case of the generic mD0 action includes the contribution of $\mathcal{M}(\mathcal{H})$ and, if $\mathcal{M}^i \neq 0$, also of \mathbb{P}^i and \mathbb{X}^i inside of \mathfrak{S} function (103), see (135).

$$\kappa^q \mathcal{H} = 0. \quad (149)$$

Hence, the BPS equations determining the supersymmetric pure bosonic solution of the mD0 equations is just the vanishing of the (bosonic limit of the) relative motion Hamiltonian,

$$\mathcal{H} = \frac{1}{2} \text{tr}(\mathbb{P}^i \mathbb{P}^i) - \frac{1}{64} \text{tr}[\mathbb{X}^i, \mathbb{X}^j]^2 = 0. \quad (150)$$

As we have already noticed, the general solution of this equation is given by the vacuum (147) so that only 1/2 of the spacetime supersymmetry can be conserved by the mD0 system.

Furthermore, this allows us to conclude that any supersymmetric solution of the maximally-supersymmetric $SU(N)$ SYM theory gives rise to the set of supersymmetric solutions of the mD0 equations with arbitrary nonvanishing function $\mathcal{M}(\mathcal{H})$ since the relative motion mD0 equations then differ from the SYM equations by a rescaling of the time variable by the constant factor $\mathcal{M}(0)/2$. Here we are speaking about a family of solutions since these can have different characteristics of the center-of-mass motion x_0^μ and $p^\mu = \mathfrak{M}u^{0\mu}$ in (143). As far as the supersymmetric solutions of relative motion equations of mD0 constituents is concerned, these are in one-to-one correspondence with such solutions of the $SU(N)$ SYM equations.

The correspondence of the relative motion equations of the mD0 brane constituents and the maximally-supersymmetric $d = 1$ $\mathcal{N} = 16$ $SU(N)$ SYM equations can be also used to search for nonsupersymmetric solutions of the former. See Appendix D for an example.

VI. CONCLUSION

In this paper we have studied the properties of the dynamical system described by recently constructed in [54] doubly-supersymmetric nonlinear action (26) which possesses the properties expected from the action for multiple super-D0-brane (mD0) system (hence the name candidate mD0 action or simply mD0 action which we use above).

Doubly supersymmetric means that the action possesses both spacetime (or more precisely, target superspace) supersymmetry and worldline supersymmetry, generalizing the famous κ -symmetry of the action for single D0-brane.⁹ This property guarantees that the ground state of the dynamical system preserves a part (one half) of the target space supersymmetry, and thus is a stable 1/2 BPS state, the property expected for mD0 brane system.

⁹This was found for the first time in [62] for $\mathcal{N} = 2$ $D = 4$ massive superparticle, lower-dimensional counterpart of 10D D0-brane [13].

The action of [54], Eq. (26), which is under study in this paper, includes an arbitrary positive definite function $\mathcal{M}(\mathcal{H}/\mu^6)$ of the relative motion Hamiltonian. The simpler candidate mD0 action previously proposed in [51] can be obtained by choosing \mathcal{M} to be equal to the constant mass parameter m which appears in the center-of-energy part of the action, $\mathcal{M}(\mathcal{H}/\mu^6) = m$. This center-of-energy part of the action coincides formally with the spinor moving-frame action [61] of a single D0-brane with mass m .

As we have shown in this paper, the action (26) with other particular choice $\mathcal{M} = \frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{\mathcal{H}}{\mu^6}}$ (where μ is a parameter of dimension of mass entering the relative motion sector of the action and characterizing its interaction with the center of energy sector), can be obtained by dimensional reduction from the 11D mM0 action of [47], Eq. (56) (this fact was announced in [54] but not proved there).

We have also obtained the complete set of equations of motion which follow from Eq. (26) with an arbitrary positive $\mathcal{M}(\mathcal{H}/\mu^6)$ and show that their part describing the center-of-energy motion formally coincides with the set of equations for a single D0-brane [but with effective mass $\mathfrak{M} = m + \frac{2}{\mu^6} \frac{\mathcal{H}}{\mathcal{M}}$ (105)] expressed in terms of the above mentioned positive definite function $\mathcal{M}(\mathcal{H}/\mu^6)$. The complete gauge-invariant form of the relative motion equations is quite complicated but we have shown that they imply that the relative-motion Hamiltonian is a constant of motion, $d\mathcal{H} = 0$. This relation, which is the Noether identity manifesting the reparametrization symmetry (1d general coordinate invariance) of our mD0 action (26), is particularly important as it guarantees that the effective mass of mD0 system is constant.

The relative motion equations simplify essentially in a certain gauge fixed on the center-of-energy variables and 1d $SU(N)$ gauge field. In this gauge we have established an interesting relation of the gauge fixed form of our equations of relative motion with the equations of $\mathcal{N} = 16$ d = 1 $SU(N)$ SYM model. This relation, which is valid for arbitrary positive $\mathcal{M}(\mathcal{H}/\mu^6)$, does not imply a gauge equivalence. This fact can be seen from that it can be used to relate the (equations of relative motion of the) models with different \mathcal{M} and hence with different values of the mass \mathfrak{M} of our mD0 system, Eq. (105).

The established relation allows us to show that all supersymmetric bosonic solutions of our mD0 equations (in its relative motion sector) are given essentially by supersymmetric solutions of the SYM model. The BPS equations which are obeyed by that supersymmetric solutions are shown to reduce to single equation as the relative motion Hamiltonian $\mathcal{H} = 0$ vanishes. Hence, according to (105), the effective mass \mathfrak{M} of the configurations described by these supersymmetric solutions are given by center-of-energy parameter m .

Thus, in other words, for any choice of positive function $\mathcal{M}(\mathcal{H}/\mu^6)$ the spectrum of the BPS sector of our mD0 model essentially coincides (as far as the relative motion sector is considered) with the set of vacua of maximally-supersymmetric $\mathcal{N} = 16$ d = 1 $SU(N)$ SYM theory. The established correspondence also allows us to study the nonsupersymmetric solutions of our mD0 equations using the knowledge of the SYM solutions (see Appendix D for a simple example).

The enigma of surprising multiplicity of massive $p = 0$ supersymmetric objects still cannot be claimed as resolved.¹⁰ However, our present study of the equations following from the candidate mD0 actions (26) [54] has allowed us to establish an interesting relation of the gauge fixed equations for any $\mathcal{M}(\mathcal{H}/\mu^6)$ and maximally-supersymmetric SYM equations which, in its turn, suggests a possible reason why such a multiplicity becomes possible. In particular, it allows us to show that the BPS spectra of the model (26) with different $\mathcal{M}(\mathcal{H}/\mu^6)$ coincides and in its relative-motion sector is given by the BPS spectrum of the SYM model.

One of the most interesting directions of the development of our approach is to attack the problem of its generalization for the case of higher p multiple Dp -brane system, beginning from $p = 1$ case of multiple Dirichlet strings (mD1). This simplest case is particularly interesting as an attempt to indicate what animal in the 10D type-IIA 0-brane zoo describes the true mD0 system as this has to be related to mD1 by T-duality transformations. Such a study could also deepen our comprehension of T-duality.

In this respect it is also interesting to note the appearance of an indication of a new possible inhabitant of the type II 1-brane zoo. It comes from [52] where a DBI-like non-Abelian 2d model was recently obtained by the so-called $T\bar{T}$ -deformation of a 2d non-Abelian SYM model. It was noticed in [52] that its properties are quite different from the non-Abelian generalization of the DBI action which would describe a mD1 system, but its origin in $T\bar{T}$ -deformation suggests, using the arguments from [63–66], an existence of possible maximally-supersymmetric generalizations. An interesting direction for future study is to construct explicitly such a supersymmetric generalization, to try to extend it to an action for a new 10D multiple 1-brane system and to find what representative of supersymmetric multiple 0-brane family is related to it by T-duality.

A search for the generalization of our approach to mD1 case, as well as the above-mentioned search for new exotic non-Abelian 1-branes, will require the use of the appropriate spinor moving-frame (Lorentz harmonic) formalism

¹⁰Notice that the set of these also includes, besides our mD0 models with different positive $\mathcal{M}(\mathcal{H}/\mu^6)$, a beautiful 0-brane model of [36] including arbitrary function of relative motion variables $\mathcal{M}(\times^i, \Psi_q)$.

which was developed in [67–69]. This is suggested by a special role which is played by $SO(1,9)/SO(9)$ spinor moving-frame formalism in the construction of our candidate mD0 actions and in studying their properties.¹¹

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APPENDIX A: WORLDLINE SUPERSYMMETRY TRANSFORMATION

The action of the worldline gauge supersymmetry on the center-of-energy variables imitates the action of the

κ -symmetry on the variables of single D0-brane

$$\begin{aligned}\delta_\kappa \theta^{1\alpha} &= \kappa^q v_\alpha^q / \sqrt{2}, & \delta_\kappa \theta_\alpha^2 &= -\kappa^q v_\alpha^q / \sqrt{2}, \\ \delta_\kappa x^\mu &= i\delta_\kappa \theta^1 \sigma^\mu \theta^1 + i\delta_\kappa \theta^2 \tilde{\sigma}^\mu \theta^2, \\ \delta_\kappa v_\alpha^q &= 0 \quad \Rightarrow \quad \delta_\kappa u_\mu^0 &= 0 = \delta_\kappa u_\mu^i.\end{aligned}\tag{A1}$$

The worldline supersymmetry transformations of the matrix matter fields are [54]

$$\delta_\kappa \mathbb{X}^i = \frac{4i}{\sqrt{\mathcal{M}}} \kappa \gamma^i \Psi + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \delta_\kappa \mathcal{H} \mathbb{X}^i - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \Delta_\kappa \mathcal{K} \mathbb{P}^i,\tag{A2}$$

$$\delta_\kappa \mathbb{P}^i = -\frac{1}{\sqrt{\mathcal{M}}} [\kappa \gamma^{ij} \Psi, \mathbb{X}^j] - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \delta_\kappa \mathcal{H} \mathbb{P}^i + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \Delta_\kappa \mathcal{K} \left(\frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pq}^i \{ \Psi_p, \Psi_q \} \right),\tag{A3}$$

$$\delta_\kappa \Psi_q = -\frac{1}{2\sqrt{\mathcal{M}}} (\kappa \gamma^i)_q \mathbb{P}^i - \frac{i}{16\sqrt{\mathcal{M}}} (\kappa \gamma^{ij})_q [\mathbb{X}^i, \mathbb{X}^j] - \frac{i}{4\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \Delta_\kappa \mathcal{K} [(\gamma^i \Psi)_q, \mathbb{X}^i]\tag{A4}$$

where

$$\delta_\kappa \mathcal{H} = \frac{1}{\sqrt{\mathcal{M}}} \frac{\text{tr}(\kappa^q \Psi_q ([\mathbb{X}^i, \mathbb{P}^i] - 4i \{ \Psi_q, \Psi_q \}))}{1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}}\tag{A5}$$

with

$$\mathfrak{H} := \text{tr}(\mathbb{P}^i \mathbb{P}^i) + \frac{1}{16} \text{tr}[\mathbb{X}^i, \mathbb{X}^j]^2 + 2\text{tr}(\mathbb{X}^i \Psi \gamma^i \Psi)\tag{A6}$$

is the worldline-supersymmetry variation of the relative-motion Hamiltonian (27) and

$$\Delta_\kappa \mathcal{K} = \frac{1}{2\sqrt{\mathcal{M}}} \frac{\text{tr}(4i(\kappa \gamma^i \Psi) \mathbb{P}^i + \frac{5}{2} (\kappa \gamma^{ij} \Psi) [\mathbb{X}^i, \mathbb{X}^j])}{1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}}.\tag{A7}$$

¹¹To stress the importance of the spinor moving-frame variables, let us discuss the way of possible derivation of our model (26) starting from the maximally-supersymmetric $d = 1$ $SU(N)$ SYM model which possesses $SO(16)$ symmetry and rigid $\mathcal{N} = 16$ supersymmetry, or from its nonlinear deformations, preserving all the supersymmetry and at least $SO(9)$ parts of $SO(16)$. First we observe that spinor moving-frame variables allow us to convert the $SO(9)$ spinor index of the fermionic parameter of rigid SYM supersymmetry into the $SO(1,9)$ Majorana-Weyl spinor $[\text{Spin}(1,9)]$ index which is carried by the coordinates of target superspace and also by a parameter of κ -symmetry of a single D0-brane. We can also find that they can be used to provide us with a composite supergravity induced by the embedding of the D0-brane worldline into the target superspace. Then we add the single super-D0-brane action to the SYM action coupled to supergravity, thus possessing reparametrization symmetry and local supersymmetry due to this coupling, to arrive at our candidate mD0 action. For the simplest case of $\mathcal{M} = m$ this program was realized in [51].

Finally, the transformations of the worldline gauge field are

$$\begin{aligned} \delta_\kappa \mathbb{A} = & -\frac{2}{\mathcal{M}\sqrt{\mathcal{M}}} E^0 (\kappa^q \Psi_q) \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}\right)} + \frac{1}{\sqrt{2}\mathcal{M}} (E^{1q} - E_q^2) (\gamma^i \kappa)_q \mathbb{X}^i \\ & - (E^{1q} - E_q^2) \frac{1}{\mu^6} \frac{\mathcal{M}'}{\sqrt{2}\mathcal{M}^2} \frac{1}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}\right)} \kappa^p \Psi_{(q} \text{tr} \left(4i(\gamma^i \Psi)_p \mathbb{P}^i + \frac{5}{2} (\gamma^{ij} \Psi)_p [\mathbb{X}^i, \mathbb{X}^j] \right). \end{aligned} \quad (\text{A8})$$

APPENDIX B: COMPLETE SET OF GAUGE INVARIANT mD0 EQUATIONS

The center-of-mass equations are the same as in the case of single D0-brane,

$$E^i = 0, \quad (\text{B1})$$

$$E^{1q} + E_q^2 = 0, \quad (\text{B2})$$

$$\Omega^i = 0. \quad (\text{B3})$$

The relative-motion equations are the Gauss constraint

$$[\mathbb{X}^i, \mathbb{P}^i] = 4i\{\Psi_q, \Psi_q\}, \quad (\text{B4})$$

and

$$\text{D}\mathbb{X}^i = -\frac{2}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}\right)} E^0 \mathbb{P}^i + \frac{(E^{1q} - E_q^2)}{\sqrt{2}\mathcal{M}} \left[4i(\gamma^i \Psi)_q - \frac{\frac{1}{2\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \text{tr}(4i(\gamma^j \Psi)_q \mathbb{P}^j + \frac{5}{2} (\gamma^{jk} \Psi)_q [\mathbb{X}^j, \mathbb{X}^k])}{1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}} \mathbb{P}^i \right] \quad (\text{B5})$$

$$\begin{aligned} \text{D}\mathbb{P}^i = & \frac{2}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}\right)} E^0 \left(\frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i \{\Psi_p, \Psi_r\} \right) - \frac{(E^{1q} - E_q^2)}{\sqrt{2}\mathcal{M}} [(\gamma^{ij} \Psi)_q, \mathbb{X}^j] \\ & + \frac{(E^{1q} - E_q^2)}{\sqrt{2}\mathcal{M}} \frac{1}{2\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \frac{\text{tr}(4i(\gamma^k \Psi)_q \mathbb{P}^k + \frac{5}{2} (\gamma^{kl} \Psi)_q [\mathbb{X}^k, \mathbb{X}^l])}{1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}} \left(\frac{1}{16} [[\mathbb{X}^i, \mathbb{X}^j], \mathbb{X}^j] - \gamma_{pr}^i \{\Psi_p, \Psi_r\} \right), \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} \text{D}\Psi_q = & -\frac{i}{2\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}\right)} E^0 [(\gamma^i \Psi)_q, \mathbb{X}^i] - \frac{(E^{1q} - E_q^2)}{2\sqrt{2}\mathcal{M}} \left(\gamma_{pq}^i \mathbb{P}^i + \frac{i}{8} \gamma_{pq}^{ij} [\mathbb{X}^i, \mathbb{X}^j] \right) \\ & - \frac{(E^{1p} - E_p^2)}{\sqrt{2}\mathcal{M}} \frac{i}{8\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \frac{\text{tr}(4i(\gamma^j \Psi)_p \mathbb{P}^j + \frac{5}{2} (\gamma^{jk} \Psi)_p [\mathbb{X}^j, \mathbb{X}^k])}{1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{S}} [(\gamma^i \Psi)_q, \mathbb{X}^i]. \end{aligned} \quad (\text{B7})$$

Writing the relative-motion equations we have already taken into account the fact that they and the center-of-mass equations imply the preservation of the relative-motion Hamiltonian (27),

$$d\mathcal{H} = 0. \quad (\text{B8})$$

This is the Noether identity for the reparametrization symmetry. The Noether identity for the worldline supersymmetry,

$$iD\nu_q = \frac{2\sqrt{2}}{\sqrt{\mathcal{M}}}(E^{1q} - E_q^2)\mathcal{H} \quad (\text{B9})$$

where $i\nu_q$ is defined in (87), also holds.

APPENDIX C: CONVENIENT REPRESENTATIONS FOR 11D GAMMA MATRICES

1. SO(1,9) covariant representation

$$\Gamma_{\underline{\alpha}\underline{\beta}}^\mu = \begin{pmatrix} \sigma_{\alpha\beta}^\mu & 0 \\ 0 & \tilde{\sigma}^{\mu\alpha\beta} \end{pmatrix}, \quad \Gamma_{\underline{\alpha}\underline{\beta}}^{*\mu} = \begin{pmatrix} 0 & -\delta_{\alpha}^{\beta} \\ -\delta_{\beta}^{\alpha} & 0 \end{pmatrix}, \quad (\text{C1})$$

$$\tilde{\Gamma}^{\mu\alpha\beta} = \begin{pmatrix} \tilde{\sigma}^{\mu\alpha\beta} & 0 \\ 0 & \sigma_{\alpha\beta}^\mu \end{pmatrix}, \quad \tilde{\Gamma}^{*\alpha\beta} = \begin{pmatrix} 0 & \delta_{\beta}^{\alpha} \\ \delta_{\alpha}^{\beta} & 0 \end{pmatrix}, \quad (\text{C2})$$

$$C_{\underline{\alpha}\underline{\beta}} = i \begin{pmatrix} 0 & \delta_{\alpha}^{\beta} \\ -\delta_{\beta}^{\alpha} & 0 \end{pmatrix}, \quad C^{\alpha\beta} = i \begin{pmatrix} 0 & \delta_{\beta}^{\alpha} \\ -\delta_{\alpha}^{\beta} & 0 \end{pmatrix}, \quad (\text{C3})$$

$$\tilde{\Gamma}^\mu = C\Gamma^\mu C. \quad (\text{C4})$$

2. SO(1,1) × SO(9) covariant representation

$$(\Gamma^\#)_{\underline{\alpha}\underline{\beta}} = \begin{pmatrix} 2\delta_{pq} & 0 \\ 0 & 0 \end{pmatrix} = \tilde{\Gamma}^{-\alpha\beta},$$

$$(\Gamma^\ominus)_{\underline{\alpha}\underline{\beta}} = \begin{pmatrix} 0 & 0 \\ 0 & 2\delta_{pq} \end{pmatrix} = \tilde{\Gamma}^{\#\alpha\beta}, \quad (\text{C5})$$

$$(\Gamma^i)_{\underline{\alpha}\underline{\beta}} = \begin{pmatrix} 0 & \gamma^i_{pq} \\ \gamma^i_{pq} & 0 \end{pmatrix} = -\tilde{\Gamma}^{i\alpha\beta}, \quad (\text{C6})$$

$$C_{\underline{\alpha}\underline{\beta}} = -C_{\underline{\beta}\underline{\alpha}} = \begin{pmatrix} 0 & i\delta_{pq} \\ -i\delta_{pq} & 0 \end{pmatrix} = (C^{-1})^{\alpha\beta} =: C^{\alpha\beta}. \quad (\text{C7})$$

APPENDIX D: A NONSUPERSYMMETRIC SOLUTION OF mD0 EQUATIONS

As an example of using the correspondence with 1d SYM of the relative motion equations of mD0 system to find nonsupersymmetric solutions of these, let us discuss the SYM solution used in [70] to study the possibility to describe cosmology in the frame of the BFSS matrix model [71]. It uses the ansatz

$$\mathbb{X}^i(\tau) = a(\tau)\mathbb{Y}^i, \quad (\text{D1})$$

where \mathbb{Y}^i are nine constant traceless $N \times N$ matrices which obey

$$[[\mathbb{Y}^i, \mathbb{Y}^j]\mathbb{Y}^j] = 16\lambda\mathbb{Y}^i \quad (\text{D2})$$

with some constant λ .

Let us search for a solution of our equations of motion with this ansatz. Equation (134), after fixing the gauge $\dot{x}^0 = 1$, implies that

$$\mathbb{P}^i(\tau) = b(\tau)\mathbb{Y}^i, \quad (\text{D3})$$

where $b(\tau)$ should be found from solving Eq. (134) with (D1) and (D3). Then, with the above ansatz

$$\mathcal{H} = c \left(b^2 + \frac{\lambda}{2} a^4 \right),$$

$$\mathfrak{H} = 2\mathcal{H} + \frac{3}{32} \text{tr}([\mathbb{X}^i, \mathbb{X}^i]^2) = 2\mathcal{H} - 3c\lambda a^4$$

$$= 2c(b^2 - \lambda a^4), \quad (\text{D4})$$

where

$$c = \frac{1}{2} \text{tr}(\mathbb{Y}^i \mathbb{Y}^i), \quad (\text{D5})$$

so that the straightforward approach to (134) does not look too promising. However, at this stage we can use Eq. (98) stating that on the mass shell \mathcal{H} is constant and conclude from (D4) that

$$b^2 = \frac{\mathcal{H}}{c} - \frac{\lambda}{2} a^4, \quad \mathfrak{H} = 2\mathcal{H} - \frac{3\lambda}{2} a^4. \quad (\text{D6})$$

Furthermore, the approach in [70] allows for the field-dependent redefinition of the time variable similar to (137) so that we use this with the chosen ansatz to define

$$dt = d\tau \frac{2}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} (2\mathcal{H} - \frac{3\lambda}{2} a^4(\tau))\right)} \quad (\text{D7})$$

and to obtain in such a way the equations

$$\frac{d}{dt} a = -b, \quad \frac{d^2 a}{dt^2} = -\frac{\lambda^3}{a}. \quad (\text{D8})$$

Following [70], we can now multiply the last equation by $\frac{da}{dt}$ and integrate it over dt arriving at

$$\frac{1}{2} \left(\frac{da}{dt} \right)^2 = -\frac{\lambda}{4} a^4 + k \quad (\text{D9})$$

with constant k . Next we can redefine the time variable once more

$$d\tilde{t} = dt a(t) \quad (\text{D10})$$

thus arriving at the Friedmann equation

$$\frac{1}{a^2} \left(\frac{da}{d\tilde{t}} \right)^2 = -\frac{\lambda}{2} + \frac{2k}{a^4}. \quad (\text{D11})$$

This was the basis of the discussion of “BFSS cosmology” in [70].

Although the change of variables (D10) is similar in spirit with our (D7), for our model with nonconstant \mathcal{M} the

complete equation for the change-of-time variable is much more complicated

$$d\tilde{t} = d\tau \frac{2a(\tau)}{\mathcal{M}} \frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 + \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} (2\mathcal{H} - \frac{3\lambda}{2} a^4(\tau))\right)} \quad (\text{D12})$$

so that to find the evolution in proper time is not an easy problem.

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- [1] J. Polchinski, *String Theory. Vol. 1: An Introduction to the Bosonic String* (Cambridge University Press, Cambridge, England, 2007), ISBN 978-0-511-25227-3, 978-0-521-67227-6, 978-0-521-63303-1.
- [2] J. Polchinski, *String Theory. Vol. 2: Superstring Theory and Beyond* (Cambridge University Press, Cambridge, England, 2007), ISBN 978-0-511-25228-0, 978-0-521-63304-8, 978-0-521-67228-3.
- [3] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press, Cambridge, England, 2006), ISBN 978-0-511-25486-4, 978-0-521-86069-7, 978-0-511-81608-6.
- [4] P. West, *Introduction to Strings and Branes* (Cambridge University Press, Cambridge, England, 2012), ISBN 978-0-521-81747-9, 978-1-139-41529-3, 978-0-521-81747-9.
- [5] A. Sagnotti, Open Strings and their Symmetry Groups, Contribution to: NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory (Cargese Summer Institute), 1987, pp. 0521–528; Report No. ROM2F-87/25; arXiv:hep-th/0208020.
- [6] J. Dai, R. G. Leigh, and J. Polchinski, New connections between string theories, *Mod. Phys. Lett. A* **04**, 2073 (1989).
- [7] P. Horava, Background duality of open string models, *Phys. Lett. B* **231**, 251 (1989).
- [8] J. Polchinski, Dirichlet Branes and Ramond-Ramond charges, *Phys. Rev. Lett.* **75**, 4724 (1995).
- [9] M. Cederwall, A. von Gussich, B. E. W. Nilsson, and A. Westerberg, The Dirichlet super three-brane in ten-dimensional type IIB supergravity, *Nucl. Phys.* **B490**, 163 (1997).
- [10] M. Aganagic, C. Popescu, and J. H. Schwarz, D-brane actions with local kappa symmetry, *Phys. Lett. B* **393**, 311 (1997).
- [11] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell, and A. Westerberg, The Dirichlet super p-branes in ten-dimensional type IIA and IIB supergravity, *Nucl. Phys.* **B490**, 179 (1997).
- [12] M. Aganagic, C. Popescu, and J. H. Schwarz, Gauge invariant and gauge fixed D-brane actions, *Nucl. Phys.* **B495**, 99 (1997).
- [13] E. Bergshoeff and P. K. Townsend, Super D-branes, *Nucl. Phys.* **B490**, 145 (1987).
- [14] I. A. Bandos, D. P. Sorokin, and M. Tonin, Generalized action principle and superfield equations of motion for $D = 10$ Dp-branes, *Nucl. Phys.* **B497**, 275 (1997).
- [15] P. S. Howe and E. Sezgin, Superbranes, *Phys. Lett. B* **390**, 133 (1997).
- [16] I. A. Bandos, D. P. Sorokin, M. Tonin, P. Pasti, and D. V. Volkov, Superstrings and supermembranes in the doubly supersymmetric geometrical approach, *Nucl. Phys.* **B446**, 79 (1995).
- [17] D. P. Sorokin, Superbranes and superembeddings, *Phys. Rep.* **329**, 1 (2000).
- [18] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory VI* (Cambridge University Press, Cambridge, England, 1987).
- [19] E. S. Fradkin and A. A. Tseytlin, Nonlinear electrodynamics from quantized strings, *Phys. Lett.* **163B**, 123 (1985).
- [20] M. Born and L. Infeld, Foundations of the new field theory, *Proc. R. Soc. A* **144**, 425 (1934).
- [21] R. G. Leigh, Dirac-born-infeld action from Dirichlet sigma model, *Mod. Phys. Lett. A* **04**, 2767 (1989).
- [22] J. E. Paton and H. M. Chan, Generalized Veneziano model with isospin, *Nucl. Phys.* **B10**, 516 (1969).
- [23] I. Bars and A. J. Hanson, Quarks at the ends of the string, *Phys. Rev. D* **13**, 1744 (1976).
- [24] N. Marcus and A. Sagnotti, Group theory from quarks at the ends of strings, *Phys. Lett. B* **188**, 58 (1987).
- [25] E. Witten, Bound states of strings and p-branes, *Nucl. Phys.* **B460**, 335 (1996).
- [26] A. A. Tseytlin, On nonAbelian generalization of Born-Infeld action in string theory, *Nucl. Phys.* **B501**, 41 (1997).
- [27] R. Emparan, Born-Infeld strings tunneling to D-branes, *Phys. Lett. B* **423**, 71 (1998).
- [28] W. Taylor and M. Van Raamsdonk, Multiple D0-branes in weakly curved backgrounds, *Nucl. Phys.* **B558**, 63 (1999).
- [29] W. Taylor and M. Van Raamsdonk, Multiple Dp-branes in weak background fields, *Nucl. Phys.* **B573**, 703 (2000).
- [30] R. C. Myers, Dielectric branes, *J. High Energy Phys.* **12** (1999) 022.

- [31] E. A. Bergshoeff, M. de Roo, and A. Sevrin, NonAbelian Born-Infeld and kappa symmetry, *J. Math. Phys. (N.Y.)* **42**, 2872 (2001).
- [32] E. A. Bergshoeff, A. Bilal, M. de Roo, and A. Sevrin, Supersymmetric nonAbelian Born-Infeld revisited, *J. High Energy Phys.* **07** (2001) 029.
- [33] D. P. Sorokin, Coincident (super)Dp-branes of codimension one, *J. High Energy Phys.* **08** (2001) 022.
- [34] J. M. Drummond, P. S. Howe, and U. Lindstrom, Kappa symmetric nonAbelian Born-Infeld actions in three-dimensions, *Classical Quantum Gravity* **19**, 6477 (2002).
- [35] B. Janssen and Y. Lozano, On the dielectric effect for gravitational waves, *Nucl. Phys.* **B643**, 399 (2002).
- [36] S. Panda and D. Sorokin, Supersymmetric and kappa invariant coincident D0-branes, *J. High Energy Phys.* **02** (2003) 055.
- [37] B. Janssen and Y. Lozano, A microscopical description of giant gravitons, *Nucl. Phys.* **B658**, 281 (2003).
- [38] B. Janssen, Y. Lozano, and D. Rodriguez-Gomez, A microscopical description of giant gravitons. 2. The AdS(5) x S**5 background, *Nucl. Phys.* **B669**, 363 (2003).
- [39] Y. Lozano and D. Rodriguez-Gomez, Fuzzy 5-spheres and pp-wave matrix actions, *J. High Energy Phys.* **08** (2005) 044.
- [40] P. S. Howe, U. Lindstrom, and L. Wulff, Superstrings with boundary fermions, *J. High Energy Phys.* **08** (2005) 041.
- [41] P. S. Howe, U. Lindstrom, and L. Wulff, On the covariance of the Dirac-Born-Infeld-Myers action, *J. High Energy Phys.* **02** (2007) 070.
- [42] P. S. Howe, U. Lindstrom, and L. Wulff, Kappa-symmetry for coincident D-branes, *J. High Energy Phys.* **09** (2007) 010.
- [43] I. A. Bandos, On superembedding approach to multiple D-brane system. D0 story, *Phys. Lett. B* **680**, 267 (2009).
- [44] I. A. Bandos, Superembedding approach to M0-brane and multiple M0-brane system, *Phys. Lett. B* **687**, 258 (2010).
- [45] I. A. Bandos, Multiple M0-brane system in an arbitrary eleven dimensional supergravity background, *Phys. Rev. D* **82**, 105030 (2010).
- [46] P. McGuirk, G. Shiu, and F. Ye, Soft branes in supersymmetry-breaking backgrounds, *J. High Energy Phys.* **07** (2012) 188.
- [47] I. A. Bandos, Action for the eleven dimensional multiple M-wave system, *J. High Energy Phys.* **01** (2013) 074.
- [48] I. A. Bandos and C. Meliveo, Covariant action and equations of motion for the eleven dimensional multiple M0-brane system, *Phys. Rev. D* **87**, 126011 (2013).
- [49] J. Choi, J. J. Fernandez-Melgarejo, and S. Sugimoto, Supersymmetric gauge theory with space-time-dependent couplings, *Prog. Theor. Exp. Phys.* **2018**, 013B01 (2018).
- [50] J. Choi, J. J. Fernández-Melgarejo, and S. Sugimoto, Deformation of $\mathcal{N} = 4$ SYM with varying couplings via fluxes and intersecting branes, *J. High Energy Phys.* **03** (2018) 128.
- [51] I. Bandos, Supersymmetric action for multiple D0-brane system, *J. High Energy Phys.* **11** (2018) 189.
- [52] T. D. Brennan, C. Ferko, and S. Sethi, A non-abelian analogue of DBI from $T\bar{T}$, *SciPost Phys.* **8**, 052 (2020).
- [53] I. Bandos and U. D. M. Sarraga, 3D supersymmetric non-linear multiple D0-brane action and 4D counterpart of multiple M-wave system, *J. High Energy Phys.* **03** (2022) 122.
- [54] I. Bandos and U. D. M. Sarraga, Complete nonlinear action for a supersymmetric multiple D0-brane system, *Phys. Rev. D* **106**, 066004 (2022).
- [55] L. Martucci, J. Rosseel, D. Van den Bleeken, and A. Van Proeyen, Dirac actions for D-branes on backgrounds with fluxes, *Classical Quantum Gravity* **22**, 2745 (2005).
- [56] A. S. Galperin, P. S. Howe, and P. K. Townsend, Twistor transform for superfields, *Nucl. Phys.* **B402**, 531 (1993).
- [57] I. Bandos, An analytic superfield formalism for tree superamplitudes in $D = 10$ and $D = 11$, *J. High Energy Phys.* **05** (2018) 103.
- [58] I. Bandos, Spinor frame formalism for amplitudes and constrained superamplitudes of 10D SYM and 11D supergravity, *J. High Energy Phys.* **11** (2018) 017.
- [59] I. A. Bandos, Spinor moving frame, M0-brane covariant BRST quantization and intrinsic complexity of the pure spinor approach, *Phys. Lett. B* **659**, 388 (2008).
- [60] I. A. Bandos, $D = 11$ massless superparticle covariant quantization, pure spinor BRST charge and hidden symmetries, *Nucl. Phys.* **B796**, 360 (2008).
- [61] I. A. Bandos, Super D0-branes at the endpoints of fundamental superstring: An Example of interacting brane system, in *Proceedings, International Workshop on Supersymmetries and Quantum Symmetries (SQS'99): Moscow, Russia, 1999* (JINR Publishing, Dubna, 2000), arXiv:hep-th/0001150.
- [62] J. A. de Azcarraga and J. Lukierski, Supersymmetric particles with internal symmetries and central charges, *Phys. Lett.* **113B**, 170 (1982).
- [63] M. Baggio, A. Sfondrini, G. Tartaglino-Mazzucchelli, and H. Walsh, On $T\bar{T}$ deformations and supersymmetry, *J. High Energy Phys.* **06** (2019) 063.
- [64] C. K. Chang, C. Ferko, and S. Sethi, Supersymmetry and $T\bar{T}$ deformations, *J. High Energy Phys.* **04** (2019) 131.
- [65] C. K. Chang, C. Ferko, S. Sethi, A. Sfondrini, and G. Tartaglino-Mazzucchelli, $T\bar{T}$ flows and (2,2) supersymmetry, *Phys. Rev. D* **101**, 026008 (2020).
- [66] E. A. Coleman, J. Aguilera-Damia, D. Z. Freedman, and R. M. Soni, $T\bar{T}$ -deformed actions and (1,1) supersymmetry, *J. High Energy Phys.* **10** (2019) 080.
- [67] I. A. Bandos and A. A. Zheltukhin, Spinor Cartan moving n hedron, Lorentz harmonic formulations of superstrings, and kappa symmetry, *JETP Lett.* **54**, 421 (1991).
- [68] I. A. Bandos and A. A. Zheltukhin, Green-Schwarz superstrings in spinor moving frame formalism, *Phys. Lett. B* **288**, 77 (1992).
- [69] I. A. Bandos and A. A. Zheltukhin, Twistor-like approach in the Green-Schwarz $D = 10$ superstring theory, *Phys. Part. Nucl.* **25**, 453 (1994).
- [70] S. Brahma, R. Brandenberger, and S. Laliberte, Emergent metric space-time from matrix theory, *J. High Energy Phys.* **09** (2022) 031.
- [71] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, M theory as a matrix model: A Conjecture, *Phys. Rev. D* **55**, 5112 (1997).