Integrable sigma models on Riemann surfaces

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We consider quantum aspects of a class of generalized Gross-Neveu models, which in special cases reduce to sigma models. We show that, in the case of gauged models, an admissible gauge is $A_\mu = 0$, which is a direct analog of the conformal gauge in string models. Chiral anomalies are a gauge counterpart of the Weyl anomaly, and are required to vanish. Topological effects on the worldsheet lead to an integration over moduli spaces of connections on a Riemann surface.

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I. INTRODUCTION

Recently, a new approach to sigma models with complex homogeneous target spaces such as $\mathbb{CP}^{n-1}$, Grassmannians, flags etc. was proposed [1–4]. The approach, based on an exact equivalence with gauged chiral Gross-Neveu models involving both bosonic and fermionic fields, offers many calculational benefits as compared to the standard formulation, provides a new take on supersymmetry (SUSY) models and should prove useful in the analysis of quantum integrability. A somewhat analogous first-order formulation is also helpful in 4D Yang-Mills theory [5].

Among other matters, in [1] it was observed that the one-loop beta functions of such theories are independent of the gauge fields, so that the same beta function may be shared by several inequivalent models. Motivated by this curious property, in the present paper we systematically study the role of gauge fields in these models. The key result is that they carry only topological degrees of freedom, parametrizing moduli spaces of holomorphic vector bundles over the worldsheet Σ (which is assumed to be a Riemann surface). In particular, for $\Sigma = \mathbb{R}^2$ one can completely eliminate the gauge field by choosing a gauge $A_\mu = 0$. The setup is reminiscent of string sigma models, where the worldsheet metric may be eliminated by going to conformal gauge, and in general one has to integrate over conformal classes of metrics. Our models provide a gauge field version of that gravitational setup [also studied in [6] in the case of gauged Wess-Zumino-Novikov-Witten (WZNW) models]. The gauge counterpart of the Weyl anomaly, which is carefully canceled in string models [7,8], is the chiral anomaly that is also required to vanish. The cancellation may be achieved by including fermions in various ways.

There are several facts hinting at the integrability of the proposed models in flat space. One piece of evidence is the integrability of the fermionic Gross-Neveu model [9] (both classical [10] and quantum [11,12]). Another one, perhaps more familiar from sigma model theory [13,14], has to do with the existence of a family of flat connections (for a review cf. [15,16]). Finally, as is typical for integrable models (see [17–20] for the background and [21] for the relevant modern developments), there are canonical trigonometric/elliptic deformations, and the deformed geometry is stable under RG-flow, at least at one loop [1,3] (for earlier developments cf. [22–27]). It was also conjectured in [1] that the anomalies known to obstruct integrability of the bosonic $\mathbb{CP}^{n-1}$ model [28–30] are canceled in models with vanishing chiral anomalies.

Throughout most of the paper we treat the $\mathbb{CP}^{n-1}$ model in detail, touching upon the non-Abelian case in the last section. Whereas to date little has been known about the quantum theory of integrable sigma models on curved worldsheets, our results suggest that the theories of interest may be consistently placed on Riemann surfaces without spoiling some of the crucial features.

II. THE $\mathbb{CP}^{n-1}$ SIGMA MODEL AS A GROSS-NEVEU MODEL

Classical formulations of the $\mathbb{CP}^{n-1}$ sigma model [31,32] are based on the Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$, with fiber $U(1)$. At the level of the Lagrangian this is implemented

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as follows:

$$\mathcal{L} = \sum_{A=1}^{n} |D_\mu U^A|^2, \qquad \sum_{A=1}^{n} |U^A|^2 = 1. \tag{1}$$

The second constraint defines the sphere S^{2n-1}, whereas the covariant derivative $D_\mu U^A = \partial_\mu U^A - iA_\mu U^A$ ensures the $U(1)$ quotient. On the other hand, the canonical definition of projective space,

$$\mathbb{CP}^{n-1} = (\mathbb{C}^n - \{0\})/\mathbb{C}^*, \tag{2}$$

suggests an alternative approach to the sigma model, based on the gauge group $\mathbb{C}^*$. This leads to a reformulation of the $\mathbb{CP}^{n-1}$ model as a generalized Gross-Neveu model [1–3], which we now recall.

Consider a column vector $U \in \mathbb{C}^n$ and a row vector $V \in \mathbb{C}^n$, together with a complex gauge field $\mathcal{A}$, and write down the Lagrangian

$$\mathcal{L} = V\bar{D}U + \bar{U}D\bar{V} + \varkappa(\bar{U}U)(V\bar{V}), \tag{3}$$

where $\bar{D}U = \bar{\partial}U - i\bar{\mathcal{A}}U$. Here $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ is the derivative with respect to a holomorphic coordinate z on Σ and $\mathcal{A}$ is the (0,1) component of a Hermitian (unitary) connection $A = \mathcal{A}dz + \bar{\mathcal{A}}d\bar{z}$. Grouping the U and $\bar{V}$ fields in a single Dirac spinor, one recognizes in (3) the Lagrangian of a chiral Gross-Neveu model [33,34] in bosonic incarnation. Classically (3) has a $\mathbb{C}^*$ gauge symmetry ($\chi \in \mathbb{C} \bmod 2\pi i$)

$$U \to e^{\chi}U, \qquad V \to e^{-\chi}V, \qquad \bar{\mathcal{A}} \to \bar{\mathcal{A}} - i\bar{\partial}\chi. \tag{4}$$

Here $\mathbb{C}^* = U(1) \times \mathbb{R}^*$ is the chiral symmetry on a Riemannian worldsheet [1,35], where $U(1)$ stands for vectorial, and $\mathbb{R}^*$ for axial, transformations.

One can eliminate the $V, \bar{V}$ variables from (3) using the equations of motion, arriving at the geometric form of the Lagrangian

$$\mathcal{L} \simeq \frac{1}{\varkappa}\frac{|\bar{D}U|^2}{|U|^2}. \tag{5}$$

Using the $\mathbb{R}^*$ part of the gauge symmetry to set the gauge $|U|^2 = 1$, one obtains the standard gauged linear sigma model (GLSM) form of the $\mathbb{CP}^{n-1}$ sigma model, up to a topological term.

III. THE GAUGE $A = 0$ AS ANALOG OF CONFORMAL GAUGE

Let us first study the gauge transformations (4) for the case of the simplest possible worldsheet $\Sigma = \mathbb{R}^2$, assuming decay conditions for $\mathcal{A}$ at infinity. If anomalies are absent, somewhat surprisingly an admissible gauge is

$$A = \mathcal{A} = \bar{\mathcal{A}} = 0, \tag{6}$$

i.e., one can eliminate the gauge field altogether. Indeed, one can explicitly solve the equation $\bar{\mathcal{A}} - i\bar{\partial}\chi = 0$ that leads to (6) by the Cauchy-Green formula,

$$\chi(z,\bar{z}) = \frac{i}{\pi}\int d^2w \frac{1}{z-w}\bar{\mathcal{A}}(w,\bar{w}). \tag{7}$$

The gauge (6) is a direct analog of conformal gauge in string models, where instead the metric may be completely eliminated. Recall that imposing the conformal gauge involves using both Diff- and Weyl-invariance. In the gauge system at hand the analog of Diff-symmetry is the usual G-gauge invariance, where G is a compact reductive group [such as U(1)], and the analog of Weyl symmetry corresponds to axial gauge transformations related to the noncompact part $\mathsf{G}_{\mathbb{C}}/\mathsf{G}$ (such as $\mathbb{R}^*$).

IV. CHIRAL ANOMALIES

Chiral gauge transformations, crucial for imposing the gauge (6), are typically anomalous quantum mechanically, and care should be taken to ensure the anomalies cancel. Recall that vanishing of the Weyl anomaly leads to the central charge being zero. Cancellation of the chiral anomaly means that the level of the corresponding Kac-Moody algebra should vanish, $\mathsf{k} = 0$.

In the present paper we only discuss the critical case, when the anomalies cancel. Just as in string theory, the noncritical case is more complicated, cf. [36]. Here the axial $\mathbb{R}^*$ part of $\mathbb{C}^*$ symmetry may no longer be gauged and leads to a generation of an extra (noncompact) direction in target space, analogous to the Liouville mode. Working out the precise mechanism of this phenomenon is left for the future.

A. BRST quantization

One way of arriving at the condition $\mathsf{k} = 0$ is by performing Becchi-Rouet-Stora-Tyutin (BRST) quantization in the gauge (6). Since the gauge transformation is $\delta\mathcal{A} = i\partial\bar{\chi}$, $\delta\bar{\mathcal{A}} = -i\bar{\partial}\chi$, the part of the action related to gauge fixing has the form

$$\mathcal{S}_{\mathrm{gf}} = i\int d^2z(\bar{\lambda}\mathcal{A} + \lambda\bar{\mathcal{A}} + b\bar{\partial}c - \bar{b}\partial\bar{c}), \tag{8}$$

where $\lambda, \bar{\lambda}$ are Lagrange multipliers and b, c are the ghost fields. This has the obvious off shell invariance $\delta\bar{\mathcal{A}} = -i\varepsilon\bar{\partial}c$, $\delta b = i\varepsilon\lambda$. To ensure invariance of the matter part (3), we additionally postulate the transformation laws $\delta U = \varepsilon cU$, $\delta V = -\varepsilon cV$.

One can eliminate the Lagrange multipliers and pass over to on shell BRST transformations. To this end, we take the variation of the full action with respect to the gauge fields, which leads to $\lambda - VU = 0$, so that on shell transformations take the form

$$\delta b = i\varepsilon V U, \qquad \delta U = \varepsilon c U, \qquad \delta V = -\varepsilon c V. \quad (9)$$

The BRST current is $j_{\mathrm{BRST}} = cJ$, where $J = VU$ is the U(1) chiral algebra current with operator product expansion (OPE) $J(z)J(w) = \frac{\mathsf{k}}{(z-w)^2} + \cdots$ Here $\mathsf{k} = -n$ is the level of the chiral algebra. Using nilpotency of $c(z)$, one finds

$$j_{\mathrm{BRST}}(z) j_{\mathrm{BRST}}(w) = \mathsf{k}\frac{c\partial c(z)}{z-w} + \ldots. \quad (10)$$

On a punctured plane one can define a BRST charge $\mathcal{Q} := \frac{1}{2\pi i}\oint_{S^1} dz j_{\mathrm{BRST}}(z)$. It follows from (10) that

$$\mathcal{Q}^2 = \frac{\mathsf{k}}{2}\sum_{j\in\mathbb{Z}} j c_j c_{-j}, \quad (11)$$

where c_j are the Laurent series coefficients of $c(z)$. Nilpotency of the BRST charge requires $\mathsf{k} = 0$, i.e., that the central extension in the chiral algebra of the gauge group should vanish [37]. This is equivalent to the vanishing of chiral gauge anomalies. The purely bosonic model (3) is anomalous, since here $\mathsf{k} = -n$. Anomaly cancellation may be achieved by adding fermions, which make a positive contribution to the level. More generally, assuming $\mathfrak{h}$ is the Lie (super)algebra of the complex gauge (super)group, the cancellation condition is

$$\mathrm{Str}_{\mathsf{W}}(\tau^a \tau^b) = 0 \quad \text{for } \tau^a, \tau^b \in \mathfrak{h}, \quad (12)$$

where W is the representation of the matter fields (bosonic and fermionic, including ghosts). This reduces to $\mathsf{k} = 0$ if $\mathrm{Str}_{\mathsf{W}}(\tau^a\tau^b) = \mathsf{k}\delta^{ab}$. Condition (12) is a direct counterpart of the anomaly cancellation condition [38] for WZNW models on Riemannian worldsheets (these are encountered in [39–41]), extended to gauge supergroups.

B. Anomaly in the "light cone" gauge

To complete the parallel with gravitational anomalies in string sigma models, recall that the latter manifest themselves in the light cone gauge via an anomaly in the (target space) Lorentz symmetry algebra. The same phenomenon occurs in the model (3). Here the light cone gauge is replaced by the inhomogeneous gauge,

$$U^n = 1. \quad (13)$$

Generically, this can be achieved by the $\mathbb{C}^*$ gauge symmetry. The remaining U^A, $A = 1, \ldots n-1$ coordinates are the inhomogeneous coordinates on $\mathbb{CP}^{n-1}$.

Anomalies are related to the kinetic term in (3), which is invariant not only under $\mathfrak{u}_n$, but also under the extended Kac-Moody symmetry $\widehat{\mathfrak{gl}_n}$. The chosen gauge explicitly breaks $\mathfrak{u}_n$ down to $\mathfrak{u}_{n-1}$, and accordingly $\widehat{\mathfrak{gl}_n}$ down to $\widehat{\mathfrak{gl}_{n-1}}$, and the question is whether the original symmetry is realized nonlinearly.

Every Noether current has two components $\ell^z := \ell(z)$, $\ell^{\bar z} := \overline{\ell(z)}$ but for brevity we will concentrate on the ℓ^z parts. To write them out in the gauge (13), observe that a variation of (3) with respect to the gauge field produces the constraint $\sum_{A=1}^n U^A V^A = 0$. Using (13), one can solve it as $V^n = -\sum_{A=1}^{n-1} U^A V^A$. As a result, the currents may be split as follows ($A, B = 1, \ldots n-1$):

$$\ell^{AB}(z) = -V^A U^B, \qquad \ell^{nn}(z) = \sum_{A=1}^{n-1} U^A V^A,$$

$$\ell^{An}(z) = -V^A, \qquad \ell^{nA}(z) = U^A \sum_{B=1}^{n-1} U^B V^B.$$

The anomaly arises from the cubic generators. Ultimately we are interested in the zero modes of the above currents,

$$\mathsf{L}^{nA} \equiv \frac{1}{2\pi i}\oint_{S^1} dz\, \ell^{nA}. \quad (14)$$

The OPE $\ell^{nA}(z)\ell^{nB}(w)$ would be nonsingular in the $\widehat{\mathfrak{gl}_n}$ symmetry algebra. For the zero modes this would imply $[\mathsf{L}^{nA}, \mathsf{L}^{nB}] = 0$.

Here instead, using the basic OPE $U^A(z)V^B(w) = \frac{\delta^{AB}}{z-w} + \cdots$, we find

$$\begin{aligned}\ell^{nA}(z)\ell^{nB}(w) = &-\frac{n+2}{(z-w)^2} U^A(z) U^B(z) \\ &+ \frac{1}{z-w}((1+n)U^A \partial U^B(z) \\ &+ U^B \partial U^A(z)) + \ldots. \end{aligned} \quad (15)$$

On a punctured plane we may decompose $U^A(z) = \sum_{k\in\mathbb{Z}} U_k^A z^k$. The OPE (15) then implies the following commutator of the zero modes:

$$[\mathsf{L}^{nA}, \mathsf{L}^{nB}] = \frac{n}{2}\sum_{k\in\mathbb{Z}} k\left(U_k^A U_{-k}^B - U_k^B U_{-k}^A\right), \quad (16)$$

which is nonzero unless $n = 2$. This means that, for $n > 2$, the model (3) genuinely loses $SU(n)$ symmetry in the gauge $U^n = 1$ (the above is a proof in the limit $\varkappa = 0$).

The anomaly (15) has been observed in [42,43] in the context of $\beta\gamma$-systems. It can be canceled in the special case $n = 2$. At the level of the zero modes this is clear from (16), since here the indices $A = B = 1$ take only one value, and the commutator vanishes. For the full current algebra one additionally has to modify the generators as $\ell^{nA} \mapsto \ell^{nA} - 2\partial U^A$ (this does not affect the zero modes), which leads to the $\widehat{\mathfrak{sl}_2}$ chiral algebra at the critical level $\mathsf{k}_{\mathfrak{sl}_2} = -2$ [43,44]. As it turns out, this is still problematic, since here the Sugawara energy-momentum tensor is identically zero, making a sigma model interpretation obscure.

Both anomalies (10) and (16) are reminiscent of the expressions occurring in string sigma models [45,46]. The special case $n = 2$ has a counterpart in the string case, where for target space(time) of dimension $D = 3$ the anomaly in the Lorentz algebra cancels as well. This case has its own pathology, namely the spins of the particles arising in the string spectrum are irrational [47–49].

Finally, let us mention that similar anomalies in the $\mathbb{CP}^{n-1}$-model in 4D spacetime have been claimed in [50]. It is an interesting question whether all of the mentioned facts are related in some way.

C. The linear axion

One way of canceling the Weyl anomaly in string models is by introducing a linear dilaton. An Abelian gauge anomaly may be canceled by a similar mechanism. Indeed, Schwinger's effective action for the gauge field in (3) [51] has the form

$$\mathcal{S}_{\text{eff}} = -\frac{n}{4\pi}\int d^2z F \frac{1}{\triangle} F, \tag{17}$$

where $F = i(\bar{\partial}\mathcal{A} - \partial\bar{\mathcal{A}})$. This can be canceled by an additional scalar field ϕ with the linear axion action,

$$\mathcal{S}_{\phi} = \frac{1}{2\pi}\int d^2z\left(\frac{1}{2}(\partial_\alpha\phi)^2 + n^{\frac{1}{2}}\phi\cdot F\right) \tag{18}$$

Elimination of ϕ via its equations of motion provides a contribution equal to (17) but with an opposite sign. Alternatively, one can fermionize $\mathcal{S}_\phi$ [52] arriving at the action of a single Dirac fermion with charge $Q = n^{\frac{1}{2}}$. Therefore, this is another application of the anomaly cancellation mechanism by fermions (we have $\text{Str}(\tau^2) = 1 \times n - Q^2 \times 1 = 0$, in line with (12). This type of coupling features in the model arising in a limit of the $\text{AdS}_4 \times \mathbb{CP}^3$ superstring [52,53].

V. CURVED WORLDSHEET

As a next step, we wish to couple the Gross-Neveu model to a (fixed) worldsheet metric on a Riemann surface Σ. The classical action $S = \int d^2z\mathcal{L}$ with Lagrangian (3) then defines the theory in conformal coordinates. For it to make sense in this extended setup, $VUdz$ should be a section of the canonical bundle K_Σ. Therefore, we may take U and V as sections of $N_{\text{grav}} \otimes N_{\text{gauge}}$ and $K_\Sigma \otimes N_{\text{grav}}^{-1} \otimes N_{\text{gauge}}^{-1}$, respectively, where N_{grav} is a line bundle over Σ characterizing the spin of the matter field, and N_{gauge} is the line bundle corresponding to the gauge field A.

A. The mixed anomaly

Upon coupling the theory to a worldsheet metric, one should make sure that no mixed gauge-gravitational anomaly arises. Denote by $\mathcal{J}$ the matrix of integer-normalized gravitational charges (spins) characterizing the bundle N_{grav} for various fields. $\mathcal{J} = 0$ corresponds to spin-$1/2$ fields, and spins of the dual fields are related by $\mathcal{J} \to -\mathcal{J}$. We assume $\mathcal{J}$ commutes with the gauge generators τ_a and with any global symmetry that one wants to keep. The condition for the vanishing of the mixed anomaly reads (cf. [54]),

$$\text{Str}_{\mathsf{W}}(\mathcal{J}\tau^a) = 0 \quad \text{for } \tau^a \in \mathfrak{h}. \tag{19}$$

In the language of the conformal field theory (CFT) system discussed earlier, this is tantamount to requiring that the current $J(z)$ be a primary operator.

The two conditions (12) and (19) form a set of anomaly cancellation conditions; (12) is a condition on the theory in flat space, whereas (19) restricts the ways how it could couple to the worldsheet metric. It was shown in [2] that in several important cases (the SUSY case and the case of minimally coupled fermions) $\text{Str}_{\mathsf{W}}(\tau_a) = 0$ holds, so that $\mathcal{J} = q\mathbb{1}$ ($q \in \mathbb{Z}$) would satisfy the constraint (19). This means that bosons and fermions in W have the same gravity couplings, which corresponds to the A-type topological twist [55–57].

B. Global aspects

When the topology of Σ is nontrivial, the gauge (6) cannot be imposed. First, observe that the degree $p \coloneqq \frac{1}{2\pi}\int_\Sigma dA$ is gauge-invariant, since $\delta p = \frac{1}{2\pi}\int_\Sigma d * d\text{Re}(\chi) = 0$ as integral of a total derivative (for bounded χ). We may decompose $A = A^{(0)} + \hat{A}$, where $A^{(0)}$ is a fixed background gauge field satisfying $\frac{1}{2\pi}\int_\Sigma dA^{(0)} = p$, and $\frac{1}{2\pi}\int_\Sigma d\hat{A} = 0$. Using chiral gauge transformations, we may then set $d\hat{A} = 0$ [58]. The flat connection leads to additional gauge invariants—the holonomies $\exp\left(i\oint_\gamma \hat{A}\right)$ for $\gamma \in \pi_1(\Sigma)$.

In the Abelian case these invariants parametrize the moduli space of line bundles N over Σ, which is

$$\text{Pic}(\Sigma) \simeq \mathbb{Z} \times \text{Jac}(\Sigma), \tag{20}$$

where $\mathbb{Z}$ corresponds to the degree of N and (assuming the surface is of genus g) $\text{Jac}(\Sigma) \simeq \mathsf{T}^{2g}$ is the Jacobian of Σ (cf. [51]). Holonomies may be thought of as coordinates on $\text{Jac}(\Sigma)$. An explicit description of elements of (20) involves the theory of theta functions [59].

Additional insight comes from the analysis of ghost zero modes. The zero modes of c correspond to residual gauge transformations preserving the gauge $A = 0$. These are holomorphic functions on Σ (constant for compact Σ). Since $b(z)dz$ is a one form on Σ, the zero modes of b are holomorphic one forms. On a surface of genus g there are exactly g of those, and they parametrize the tangent space to $\text{Jac}(\Sigma)$.

C. $\Sigma = S^2$: Bundles of nonzero degree

Let us illustrate the meaning of the discrete part $\mathbb{Z}$ in (20), taking the simplest case $\Sigma = S^2$. Here all holonomies are zero and a line bundle is characterized by its degree $p \in \mathrm{Pic}(S^2) = \mathbb{Z}$. The connection is not flat for $p \neq 0$, but we can still find a suitable gauge, such as

$$A = i\frac{p}{2}\frac{zd\bar{z} - \bar{z}dz}{1 + |z|^2}. \tag{21}$$

Recall that, for a sphere, $K_\Sigma = \mathcal{O}(-2)$. Assuming the background bundle is $N_{\mathrm{grav}} = \mathcal{O}(q)$, we find the following gauge transformations for the fields under the change of variables $z \to z^{-1}$ (here $\varphi = \arg(z)$):

$$U \to z^{-q}e^{-ip\varphi}U, \qquad V \to z^{2+q}e^{ip\varphi}V. \tag{22}$$

Although the gauge field (21) is not flat for $p \neq 0$, one finds $i\bar{\mathcal{A}} = -\frac{p}{2}\bar{\partial}\log(1 + |z|^2)$, so that each component can be gauged away by $U \to (1 + |z|^2)^{-\frac{p}{2}}U$, $V \to (1 + |z|^2)^{\frac{p}{2}}V$. In the new variables the gauge transformations are purely holomorphic,

$$U \to z^{-p-q}U, \qquad V \to z^{2+p+q}V. \tag{23}$$

As a result, U and V are sections of $\mathcal{O}(p + q)$ and $\mathcal{O}(-2 - p - q)$, respectively. It follows that, once all gauge line bundles N_{gauge} are included, there is no invariant meaning of N_{grav}; summing over gauge bundles is the same as summing over all spins.

Quantum mechanically one should take into account the anomaly cancellation conditions. To give an example where they play a role, recall that the central charge of the free CFT system is (cf. [45]) $\mathsf{c} = \mathrm{Str}_{\mathsf{W}}(3\mathcal{J}^2 - \mathbb{1})$. Now, consider the change of the gravitational couplings by elements from the center of $\mathfrak{h}$ (s_a are constants), $\mathcal{J} \to \mathcal{J} + \sum_{\tau_a \in \mathcal{Z}(\mathfrak{h})} s_a\tau_a$. This is a generalization of the shift from $N_{\mathrm{grav}} = \mathcal{O}(q)$ to $N_{\mathrm{grav}} = \mathcal{O}(p + q)$ in (22) and (23). One sees that the conditions (12) and (19) ensure invariance of c under this shift.

Finally, bundles of nonzero degree p exist on arbitrary Riemann surfaces. In that case one can take $A = \frac{ip}{2}\partial\Omega dz - \frac{ip}{2}\bar{\partial}\Omega d\bar{z}$ for the connection (Ω being the potential). One can again get rid of the gauge field by the transformation $U \to e^{p\Omega}U$, $V \to e^{-p\Omega}U$ and assume that U and V are sections of the relevant line bundles. Similar questions have been discussed in the context of the quantum Hall effect on Riemann surfaces, cf. [60–62].

In the following sections we will analyze the role of the Jacobian in (20). As a first example we take up the case when the worldsheet is a cylinder.

D. $\Sigma = \mathbb{R} \times S^1$: The Hilbert space interpretation

On a cylinder $\Sigma = \mathbb{R} \times S^1$ we regard $\mathbb{R}$ as the "space" direction and S^1 as the compactified Euclidean time direction (of circumference β). We assume decay conditions for the curvature F at infinity, so that the integrals $\oint_{S^1} A|_{\pm\infty}$ are independent of the contours. The gauge invariants are the degree $p = \frac{1}{2\pi}(\oint_{S^1} A|_{+\infty} - \oint_{S^1} A|_{-\infty}) \in \mathbb{R}$ and the holonomy $h \coloneqq \exp(i\oint_{S^1} A|_{+\infty}) \in \mathrm{U}(1)$. A particularly simple interpretation exists for the sector $p = 0$. In this case we may eliminate the gauge field at the expense of imposing h-twisted boundary conditions along the circle S^1 (cf. [4]). This leads to the following expression for the partition function in an external gauge field A: $\mathcal{Z}_{p=0}(A) = \mathrm{Tr}(he^{-\beta H})$, where H is the Hamiltonian of the theory on $\mathbb{R}$. Integrating over the gauge field is tantamount to averaging over the twist,

$$\mathcal{Z}_{p=0} = \int dh\mathrm{Tr}(he^{-\beta H}) = \mathrm{Tr}_{\mathrm{inv}}(e^{-\beta H}), \tag{24}$$

i.e., the Hilbert space is projected to the subspace of states invariant with respect to the gauge group.

VI. NON-ABELIAN GAUGE GROUPS

The simple case of the $\mathbb{CP}^{n-1}$ model has a generalization [3] related to quiver varieties [63,64]. The gauge groups would then generally be non-Abelian—an example is provided by the Grassmannian $\mathrm{Gr}_{k,n}$ target space with (complexified) gauge group $\mathrm{GL}(k, \mathbb{C})$. The gauge transformation law (4) is then replaced by

$$\bar{\mathcal{A}} \to g\bar{\mathcal{A}}g^{-1} - i\bar{\partial}gg^{-1}, \qquad g \in \mathrm{GL}(k, \mathbb{C}). \tag{25}$$

Equivalence classes of connections with respect to these transformations define equivalence classes of rank-k holomorphic vector bundles over Σ. As is typical for complex quotients, there is an alternative description as a symplectic quotient, if one considers the natural symplectic form $\omega = \int_\Sigma \mathrm{Tr}(\delta A \wedge \delta A)$ (see [65,66] for a review). The corresponding moment map equation is

$$dA - iA \wedge A = \frac{p}{k}\mathrm{vol}_\Sigma \mathbb{1}_k, \tag{26}$$

where vol_Σ is the 2π-normalized volume form on Σ. One may think of (26) as a partial gauge for the $\mathrm{GL}(k, \mathbb{C})$ gauge symmetry (25), whose residual gauge transformations are the (standard) unitary ones acting on A.

The right-hand side of (26) involves a central Fayet-Iliopoulos term, and the coefficient $\frac{p}{k}$ is chosen so that the integral of the trace of the curvature, $\frac{1}{2\pi}\int \mathrm{Tr}(F) = p$ is equal to the degree of the bundle. For fixed p the space of solutions to (26) may be described via representations of the central extension of $\pi_1(\Sigma)$ [65,67] (again assuming g is the genus),

$$\mathcal{M}^{(p)} = \left\{ \prod_{i=1}^{g} \mathsf{A}_i \mathsf{B}_i \mathsf{A}_i^{-1} \mathsf{B}_i^{-1} = e^{\frac{2\pi i p}{k}} \right\} / \mathrm{U}(k), \qquad (27)$$

Here $\mathsf{A}_i, \mathsf{B}_i$ are $\mathrm{U}(k)$-valued matrices representing holonomies $\mathrm{P}\exp(i\oint A)$ along the A- and B-cycles of Σ. We may as well get rid of the non-Abelian part of the gauge connection at the expense of imposing twisted boundary conditions; for example, one has $U \to \mathsf{A}_i U$ as one moves around the A-cycle on Σ. The remaining Abelian part may be included in the spin of the fields, as in the $\Sigma = S^2$ case studied above.

Calculating the partition function of the theory should therefore proceed in the following steps:

(i) For instanton number $p \in \mathbb{Z}$, calculating the partition function with fixed twists along each cycle.
(ii) Integrating over the moduli space $\mathcal{M}^{(p)}$ of such twists (the torus $\mathrm{Jac}(\Sigma) = \mathsf{T}^{2g}$ in the Abelian case).
(iii) Summing over all instanton numbers p.

This is the generalization to an arbitrary Riemann surface of the procedure encountered in (24) in the case of a cylinder. Calculation of the partition function with generic twists should rely on integrability-related methods, tailored to the case of Riemann surface worldsheets. Curiously, this again leads to the study of flat connections, namely the ones encoding the spectrum (i.e., Lax connections).

VII. OUTLOOK

We have shown that, classically, the gauge sector in the Gross-Neveu reformulation of familiar sigma models (such as $\mathbb{CP}^{n-1}$) is topological. Although this is also true for pure Yang-Mills theory in 2D [68,69], from the point of view of general 2D gauge-matter models this is rather exceptional, cf. [70]. Besides, the mechanism of how this happens is rather different in our case; here we have the complex quotient (25), whereas in 2D Yang-Mills theory one naturally arrives at flat connections. These are of course related by the theorem of [67], but they represent two opposite endpoints thereof.

Upon quantization, the topological nature of the gauge fields may be violated by gauge anomalies. Cancellation of these anomalies imposes conditions on the target space of the sigma model. In this regard our GLSM-based analysis is as well applicable to $\beta\gamma$-systems that arise in the limit $\varkappa \to 0$. It would thus be interesting to relate the anomalies found in [42,43] to the ones of the present paper. Applications to the pure spinor formulation of the superstring [71,72], where such $\beta\gamma$-systems arise, are worth studying as well.

In the gauge $\mathcal{A} = \bar{\mathcal{A}} = 0$ the model (3) (or rather its anomaly-free completion) admits a well-defined perturbative expansion in $\varkappa$, valid for any n. In particular, no $\frac{1}{n}$-expansion is necessary, as compared to the standard approach. This direct perturbative expansion has already been applied to the calculation of β-function in [73] and is expected to yield more results in the future.

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