# Renormalization group for nonminimal $\phi^2 R$ couplings and gravitational contact interactions

Dumitru Ghilencea[1,*] and Christopher T. Hill[2,†]

[1]*Department of Theoretical Physics, National Institute of Physics and Nuclear Engineering (IFIN-HH), Bucharest 077125, Romania*

[2]*Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510, USA*

(Received 1 November 2022; accepted 8 March 2023; published 19 April 2023)

Theories of scalars and gravity, with an Einstein-Hilbert term and nonminimal interactions, $M^2R/2 - \alpha\phi^2R/12$, have graviton exchange-induced contact interactions. These modify the renormalization group, leading to a discrepancy between the conventional calculations in the Jordan frame that ignore this effect (and are found to be incorrect), and the Einstein frame in which $\alpha$ does not exist. Thus, the calculation of quantum effects in the Jordan and Einstein frames does not generally commute with the transition from the Jordan to the Einstein frame. In the Einstein frame, though $\alpha$ is absent, for small steps in scale $\delta\mu/\mu$ infinitesimal contact terms $\sim\delta\alpha$ are induced, that are then absorbed back into other couplings by the contact terms. This modifies the $\beta$-functions in the Einstein frame. We show how correct results can be obtained in a simple model by including this effect.

DOI: 10.1103/PhysRevD.107.085013

## I. INTRODUCTION

Over the years there has been considerable interest in Brans-Dicke, scalar-tensor, and scale- or Weyl-invariant theories. These have in common fundamental scalar fields, $\phi_i$, that couple to gravity through nonminimal interactions, $F(\phi_i)R$. Unless forbidden by a symmetry (scale/Weyl), the Einstein term $M^2R$ and the Planck mass, $M$, can coexist with these nonminimal couplings, otherwise $M$ is generated dynamically by the vacuum expectation values (VEVs) of some of these scalars [1]. When the theory is prescribed with nonminimal interactions we say it is given in a "Jordan frame".

A key tool in the analysis of these models is the Weyl transformation, [2]. This involves a redefinition of the metric, $g' = \Omega(\phi_i)g$, in which $g$ comingles with the scalars. $\Omega$ can be chosen to lead to a new effective theory, typically one that has a pure Einstein-Hilbert action, $\sim M^2R$, in which the nonminimal interactions have been removed. This is called the "Einstein frame". Alternatively, one might use a Weyl transformation to partially remove a subset of scalars from the nonminimal interactions $\sim M^2R + F'(\phi_i)R$, where $F'$ is optimized for some particular application.

It is *a priori* unclear, however, how or whether the original Jordan frame theory can be physically equivalent to the Einstein frame form and how the Weyl transformation is compatible with a full quantum theory [3,4]. Nonetheless, many authors consider this to be a valid transformation and a symmetry of Weyl invariant theories, and many loop calculations permeate the literature which attempt to exploit apparent simplifications offered by the Jordan frame.

It has been shown that any theory with nonminimal couplings and a Planck mass, $M$, contains contact terms [5]. These are generated by the graviton exchange amplitudes in tree approximation and they are therefore $\mathcal{O}(\hbar^0)$ and therefore classical. The contact terms occur because emission vertices from the nonminimal interaction are proportional to $q^2$ of the graviton, while the Feynman propagator is proportional to $1/q^2$. The cancellation of $q^2 \times 1/q^2$ therefore leads to pointlike interactions that must be included into the effective action of the theory at any given order of perturbation theory. The result is that the nonminimal interactions disappear from the theory and Planck-suppressed higher-dimension operators appear with modified couplings.

The structural form of the theory when the contact term interactions are included corresponds formally to a Weyl transformation of the metric that takes the theory to the Einstein frame. In the pure Einstein-Hilbert action there are no classical contact terms, but at loop level they will be generated, and must be removed as part of the renormalization group. However, by virtue of the contact terms, nowhere is a metric redefinition performed (hence the issue of a Jacobian in the measure of the gravitational path integral in going to the Einstein frame becomes moot).

This means that, provided we are interested in the theory on mass scales below $M$, the Jordan frame is an illusion and does not really exist physically. In the Jordan frame the contact terms are hidden, but they are always present. Ergo,

*dumitru.ghilencea@cern.ch
†hill@fnal.gov

 

even though the action superficially appears to have nonminimal couplings, it does not, and remains always in an Einstein frame.

Efforts to compute quantities, such as effective potentials (or equivalently, $\beta$-functions), in the Jordan frame, while ignoring the contact terms, will yield incorrect results. Nonetheless, though the nonminimal interactions are not present in the classical Einstein frame, they are regenerated by loops and the potentials and Renormalization Group (RG) equations are modified by this effect.

In a simple theory in the Jordan frame, where the nonminimal coupling is $-\alpha\phi^2 R/12$, this raises the question of how to understand the fate of $\alpha$? With a single scalar with quartic and other interactions, $\lambda_i$, the usual naive calculation of a $\beta$-function in the Jordan frame ("naive" means ignoring contact terms) yields the form,

$$\frac{\partial\alpha(\mu)}{\partial\ln(\mu)} = \beta_\alpha(\lambda_i) \equiv (1-\alpha)\gamma_\alpha(\lambda_i), \tag{1}$$

where the factor $(1-\alpha)$ reflects the fact that when $M^2 = 0$ and $\alpha = 1$ (conformal limit) the kinetic term of $\phi$ disappears and $\phi$ becomes static parameter.

However, the contact terms (or a Weyl transformation) remove $\alpha$ and leave an Einstein frame with only the Einstein-Hilbert term, $M^2 R$ and the $\phi$ couplings $\lambda'_i$ (in what follows primed couplings refer to the Einstein frame and unprimed to Jordan frame). This means that $N$ couplings, $(\alpha, \lambda_i)$, in the Jordan frame have become $N-1$ couplings, $(\lambda'_i)$, in the Einstein frame. Therefore any physical meaning ascribed to $\alpha$ or the $\beta_\alpha$ function is apparently lost.

Three Feynman diagrams [shown below as D1, D2, D3, in Figs. 2–4] contribute to $\beta_\alpha$ in the Jordan frame. One of them multiplies $\alpha$ in the Jordan frame (D3) and yields the $(1-\alpha)$ factor in Eq. (1), but even with $\alpha = 0$ in the Einstein frame, two diagrams (D1 and D2) exist and reintroduce a perturbative $\delta\alpha$ for a small step in scale $\delta\mu/\mu$. This is then removed by the contact terms, but leads to correction terms in the renormalization of the $\lambda'_i$. We are therefore sensitive to the same scale breaking information in the Einstein frame that one has in the Jordan frame, which is encoded into $\gamma_\alpha$. This does not, however, imply that the resulting calculations in the Jordan and Einstein frames are then consistent. We explicitly demonstrate the inconsistency through calculation of effective potentials (the RG equations of the couplings can always be read off from the effective potentials).

If we stayed in the Jordan frame, with nonzero $\alpha$, and naively computed the same effective potential ("naively" means ignoring contact terms), into an Einstein frame, we would obtain a different result. The difference is a term proportional to $\alpha$ in the Jordan frame. Equivalently, going initially to the Einstein frame and running with the RG, does not commute with running initially in the Jordan frame and subsequently going to the Einstein frame.

We turn presently to a brief discussion of contact terms in general and review a simple toy model from [5] that is structurally similar to the gravitational case. We then summarize gravitational contact terms (and refer the reader to [5,6] for details and applications). We then exemplify the Einstein frame renormalization group compared to the naive Jordan frame result, which ignores contact terms, and illustrate the discrepancy.

## II. CONTACT INTERACTIONS

Generally speaking "contact interactions" are pointlike operators that are generated in the effective action of the theory in perturbation theory. They may arise in the UV from ultra-heavy fields that are integrated out, such as the Fermi weak interaction that arises from integrating out the heavy $W$-boson. They may also arise in the IR when a vertex in the theory is proportional to $q^2$ and cancels against a $1/q^2$ propagator. The gravitational contact term we discuss presently is of the IR form.

### A. Contact terms in nongravitational physics

Contact terms arise in a number of phenomena. Diagrammatically they can arise in the IR when a vertex for the emission of, e.g., a massless quantum, of momentum $q_\mu$, is proportional to $q^2$. This vertex then cancels the $1/q^2$ from a massless propagator when the quantum is exchanged. This $q^2/q^2$ cancellation leads to an effective pointlike operator from an otherwise single-particle reducible diagram.

For example, in electroweak physics a vertex correction by a $W$-boson to a massless gluon emission induces a quark flavor changing operator, e.g., describing $s \to d + \text{gluon}$, where $s(d)$ is a strange (down) quark. This has the form of a local operator [7,8],

$$g\kappa\bar{s}\gamma_\mu T^A d_L D_\nu G^{A\mu\nu}, \tag{2}$$

where $G^{A\mu\nu}$ is the color-octet gluon-field strength and $\kappa \propto G_{\text{Fermi}}$.

This implies a vertex for an emitted gluon of 4-momentum $q$ and polarization and color, $\epsilon^{A\mu}$, of the form $g\kappa\bar{s}\gamma_\mu T^A d_L \epsilon^{A\mu} \times q^2 + \ldots$. However, the gluon propagates, $\sim 1/q^2$, and couples to a quark current $\sim g\epsilon^{A\mu}\bar{q}\gamma_\mu T^A q$. This results in a contact term

$$g^2\kappa\left(\frac{q^2}{q^2}\right)\bar{s}\gamma^\mu T^A d_L \bar{q}\gamma_\mu T^A q = g^2\kappa\bar{s}\gamma^\mu T^A d_L \bar{q}\gamma_\mu T^A q. \tag{3}$$

The result is a 4-body local operator which mediates electroweak transitions between, e.g., kaons and pions [7], also known as "penguin diagrams" [8]. Note the we can rigorously obtain the contact term result by use of the gluon field equation within the operator of Eq. (2),

$$D_\nu G^{A\mu\nu} = g\bar{q}\gamma^\mu T^A q. \qquad (4)$$

This is justified as operators that vanish by equations of motion, known as "null operators", will generally have gauge noninvariant anomalous dimensions and are unphysical [9].

Another example occurs in the case of a cosmic axion, described by an oscillating classical field, $\theta(t) = \theta_0 \cos(m_a t)$, interacting with a magnetic moment, $\vec{\mu}(x)\cdot\vec{B}$, through the electromagnetic anomaly $\kappa\theta(t)\vec{E}\cdot\vec{B}$. A static magnetic moment emits a virtual spacelike photon of momentum $(0,\vec{q})$. The anomaly absorbs the virtual photon and emits an on shell photon of polarization $\vec{\epsilon}$, inheriting energy $\sim m_a$ from the cosmic axion. The Feynman diagram, with the exchanged virtual photon, yields an amplitude, $\propto (\theta_0\mu^i\epsilon_{ijk}q^j)(1/\vec{q}^2)(\kappa\epsilon^{k\ell h}q_\ell m_a\epsilon_h) \sim (\kappa\theta_0 m_a\vec{q}^2/\vec{q}^2)\vec{\mu}\cdot\vec{\epsilon}$. The $\vec{q}^2$ factor then cancels the $1/\vec{q}^2$ in the photon propagator, resulting in a contact term which is an induced, parity violating, oscillating electric dipole interaction: $\sim\kappa\theta(t)\vec{\mu}\cdot\vec{E}$. This results in cosmic axion-induced electric dipole radiation from any magnet, including an electron [10].

### B. Illustrative toy model of contact terms

To illustrate the general IR contact term phenomenon, consider a single massless real scalar field $\phi$ and operators $A$ and $B$, which can be functions of other fields, with the action given by

$$S = \int \frac{1}{2}\partial\phi\partial\phi - A\partial^2\phi - B\phi, \qquad (5)$$

where $A$ and $B$ are functions of other fields.

Here $\phi$ has a propagator $i/q^2$, but the vertex of a diagram involving $A$ has a factor of $\partial^2 \sim -q^2$. This yields a pointlike interaction, $\sim q^2 \times (i/q^2)$, in a single particle exchange of $\phi$, and therefore implies contact terms.

At lowest order in perturbation theory consider the diagram with $\phi$ exchange in Fig. 1. This involves two time-ordered products of interaction operators,

$$T\, i\int A\partial^2\phi \times i\int B\phi \to \frac{iq^2}{q^2}AB = i\int AB,$$
$$\frac{1}{2}T\, i\int A\partial^2\phi \times i\int A\partial^2\phi \to \frac{i(q^2)^2}{2q^2}A^2 = \frac{i}{2}\int A\partial^2 A, \qquad (6)$$

where $d^4x$ is understood in the integrals and the $\frac{1}{2}$ factor in the $A\partial^2 A$ term comes from the second order in the expansion of the path integral $\exp(i\int A\partial^2\phi)$. Note that we also produce a nonlocal interaction $-iB^2/2q^2$.

We thus see that we have diagrammatically obtained a local effective action,

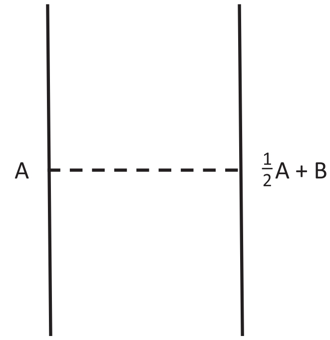


FIG. 1. Contact terms in the toy model are generated by diagrams with exchange of $\phi$ (dashed). In gravity, with nonminimal term $\sim\int\sqrt{-g}F(\phi)R$ and matter field Lagrangian $\sim\int\sqrt{-g}L(\phi)$ then $A$ is replaced by $F(\phi)$ and $B$ is replaced by $L(\phi)$, and the dashed line is a graviton propagator.

$$S = \int \frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}A\partial^2 A + AB + \text{long distance}. \qquad (7)$$

Of course, we can see this straightforwardly by "solving the theory" by defining a shifted field,

$$\phi = \phi' - \frac{1}{\partial^2}(\partial^2 A + B). \qquad (8)$$

Substituting this into the action $S$ and integrating by parts yields,

$$S = \int \frac{1}{2}\partial\phi'\partial\phi' + \frac{1}{2}A\partial^2 A + AB + \frac{1}{2}B\frac{1}{\partial^2}B. \qquad (9)$$

An equivalent effective local action that describes both short and large distance is then,

$$S = \int \frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}A\partial^2 A + AB - B\phi. \qquad (10)$$

The contact terms have become pointlike components of the effective action, while the remaining long distance effects are produced by the usual massless $\phi$ exchange. Note that the derivatively coupled operator $A$ has no long distance interactions due to $\phi$ exchange. Moreover, in the effective action of Eq. (10) we have implicitly integrated out the $A\partial^2\phi$, which is no longer part of the action and is replaced by new operators $\frac{1}{2}A\partial^2 A + AB$. We will see that this is exactly what happens with gravity, where the $A\partial^2\phi$ term is schematically the nonminimal $F(\phi)R(g) \sim F(\phi)\partial^2 h$ term in a weak-field expansion of gravity $g = \eta + h$.

One can also adapt the use of equations of motion to obtain Eq. (10) from the action Eq. (5) but this requires care. For example, the insertion of the $\phi$ equation of motion, into $A\partial^2\phi$ correctly gives the $AB$ term but misses the factor of $1/2$ in the $A\partial^2 A$ term. We can therefore do a trick of defining a modified equation of motion where we supply a factor of $1/2$ on the term $A\partial^2 A$, e.g., substitute

$$\partial^2\phi = -\partial^2 A - B \rightarrow -\frac{1}{2}\partial^2 A - B \tag{11}$$

in place of the $\partial^2\phi$ in the second term of Eq. (5) to obtain Eq. (10).

## III. GRAVITATIONAL CONTACT TERMS

Consider a general theory involving scalar fields $\phi_i$, an Einstein-Hilbert term and a nonminimal interaction,

$$S = \int \sqrt{-g}\left(\frac{1}{2}M^2 R(g_{\mu\nu}) + \frac{1}{2}F(\phi_i)R(g_{\mu\nu}) + L(\phi_i)\right) \\ = S_1 + S_2 + S_3, \tag{12}$$

where we use the metric signature and curvature tensor conventions of [11]. $S_1$ is the kinetic term of gravitons

$$S_1 = \frac{1}{2}M^2\int\sqrt{-g}R \tag{13}$$

and becomes the Fierz-Pauli action in a weak-field expansion.

$S_2$ is the nonminimal interaction, and takes the form

$$S_2 = \frac{1}{2}\int\sqrt{-g}F(\phi_i)R(g_{\mu\nu}), \tag{14}$$

where $F$ is polynomial in fields.

$S_3$ is the matter action with couplings to the gravitational weak field,

$$S_3 = \int\sqrt{-g}L(\phi_i). \tag{15}$$

The Lagrangian takes the form

$$L(\phi_i) = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - W(\phi_i) \tag{16}$$

with potential $W(\phi_i)$. The matter Lagrangian has stress tensor and stress-tensor trace,

$$T_{\mu\nu} = \partial_\mu\phi_i\partial_\nu\phi_i - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi_i\partial_\sigma\phi_i - W(\phi_i)\right) \\ T = -\partial^\sigma\phi_i\partial_\sigma\phi_i + 4W(\phi_i). \tag{17}$$

There are then three ways to obtain the contact term:

(1) **Graviton exchange contact term**

In Ref. [5] the corresponding Feynman diagrams of Fig. 1 are evaluated, arising from a single graviton exchange between the interaction terms.

We treat the theory perturbatively, expanding around flat space. Hence we linearize gravity with a weak field $h_{\mu\nu}$,

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M}. \tag{18}$$

The scalar curvature is then

$$R = R_1 + R_2, \\ MR_1 = (\partial^2 h - \partial^\mu\partial^\nu h_{\mu\nu}), \\ M^2R_2 = -\frac{3}{4}\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} - \frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu} + \cdots, \tag{19}$$

(see [5] for the complete expression for $R_2$). $S_1$ then becomes

$$S_1 = \frac{1}{2}M^2\int\sqrt{-g}R = \frac{1}{2}M^2\int\left(R_1 + R_2 + \frac{1}{2}\frac{h}{M}R_1\right) \\ = \frac{1}{2}\int h^{\mu\nu}\left(\frac{1}{4}\partial^2\eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{4}\partial^2\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{2}\partial_\rho\partial_\sigma\eta_{\mu\nu} \\ + \frac{1}{2}\partial_\mu\partial_\rho\eta_{\nu\sigma}\right)h^{\rho\sigma}. \tag{20}$$

Note that the leading term, $\int R_1$, is a total divergence and is therefore zero in the Einstein-Hilbert action, and what remains of Eq. (20) is the Fierz-Pauli action. This is key to the origin of the contact terms. The nonminimal interaction, $S_2$, then takes the leading form,

$$S_2 = \frac{1}{2}\int\sqrt{-g}F(\phi)R(g) \rightarrow \frac{1}{2}\int F(\phi)R_1(g) \\ = \int\frac{1}{2M}F(\phi)\Pi^{\mu\nu}h_{\mu\nu}, \tag{21}$$

where it is useful to introduce the transverse derivative,

$$\Pi^{\mu\nu} = \partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu. \tag{22}$$

$S_2$ involves derivatives, since $R_1$ is now active, and is the analog of the $A\partial^2\phi$ term in Eq. (5). It will therefore generate contact terms in the gravitational potential due to single graviton exchange. $S_3$ is the analog of the $B\phi$ term in Eq. (5), and this situation will closely parallel the toy model.

In [5] we developed the graviton propagator, following the nice lecture notes of Donoghue *et al.*,[12]. We remark that we found a particularly useful gauge choice,

$$\partial_\mu h^{\mu\nu} = w\partial^\nu h \tag{23}$$

where $w$ defines a single parameter family of gauges. The familiar De Donder gauge corresponds to $w = \frac{1}{2}$, while the choice $w = \frac{1}{4}$ is particularly natural in this application, and the gauge invariance of the result is verified by the $w$

independence (we verify the Newtonian potential from graviton exchange between static masses in $w$ gauge; see [5]).

We can then compute single graviton exchange between the interaction terms of the theory. A diagram with a single $S_2$ vertex and single $S_3$ vertex is the analog of $AB$ in the toy model and yields,

$$-i\langle T S_2 S_3\rangle = \int d^4x \frac{F(\phi_i)}{2M^2} T(\phi_i). \tag{24}$$

Also we have the pair $\langle S_2 S_2\rangle$ which corresponds to $\frac{1}{2}A\partial^2 A$ in the toy model and yields

$$-i\langle T\, S_2 S_2\rangle = -\int d^4x \frac{3}{4M^2} F(\phi_i)\partial^2 F(\phi_i). \tag{25}$$

The action becomes

$$S = S_1 + S_3 + S_{CT}, \tag{26}$$

where

$$S_{CT} = \int d^4x \left( -\frac{3}{4M^2} F\partial^2 F + \frac{1}{2M^2} FT \right). \tag{27}$$

Note the sign of the $F\partial^2 F$ is opposite (repulsive) to that of the toy model $A\partial^2 A$.

### (2) Weyl transformation

Define

$$\Omega^2 = \left( 1 + \frac{F(\phi_i)}{M^2} \right) \tag{28}$$

and perform a Weyl transformation on the metric,

$$\begin{aligned} &g_{\mu\nu}(x) \to \Omega^{-2} g_{\mu\nu}(x), \qquad g^{\mu\nu}(x) \to \Omega^2 g^{\mu\nu}(x), \\ &\sqrt{-g} \to \sqrt{-g}\,\Omega^{-4}, \\ &R(g) \to \Omega^2 R(g) + 6\Omega^3 \Box \Omega^{-1}, \end{aligned} \tag{29}$$

and the action of Eq. (12) becomes

$$\begin{aligned} S \to \int \sqrt{-g} \Bigg( &\frac{1}{2} M^2 R(g) \\ &- 3M^2 \partial_\mu \left( 1 + \frac{F}{M^2} \right)^{1/2} \partial^\mu \left( 1 + \frac{F}{M^2} \right)^{-1/2} \\ &+ \frac{1}{2} \left( 1 + \frac{F}{M^2} \right)^{-1} \partial_\mu \phi_i \partial^\nu \phi^i - \left( 1 + \frac{F}{M^2} \right)^{-2} W(\phi_i) \Bigg). \end{aligned} \tag{30}$$

Keeping terms to $O(1/M^2)$ and integrating by parts we have

$$S = S_1 + S_3 + \int d^4x \left( -\frac{3F(\phi_i)\partial^2 F(\phi_i)}{4M^2} + \frac{F(\phi_i)T(\phi_i)}{2M^2} \right). \tag{31}$$

The Weyl transformed action is identically consistent with the contact terms of Eq. (27) above, to first order in $1/M^2$.

Hence, contact terms arise in gravity with nonminimal couplings to scalar fields due to graviton exchange. Their form is equivalent to a Weyl redefinition of the theory to the Einstein frame action and reinforces their role as induced components of the effective action. Hence any theory with a nonminimal interaction $\sim F(\phi)R$ will lead to contact terms at order $1/M^2$.

The Weyl transformation is non-perturbative. It is technically simpler than the gravitational potential calculation, and it confirms the tricky normalization factors and phases in the graviton exchange calculation. As the Weyl transformation makes no reference to a gauge choice, a calculation of the contact terms in other gauges should yield the equivalent results. To check the invariance we turn now to a calculation in an alternative gauge which sheds further light on the origin of their structure.

### (3) Use of modified $R$ equation of motion

A trick can be used to simplify the calculations below. The Einstein equation with the nonminimal term is

$$\begin{aligned} &M^2 G_{\alpha\beta} = -T_{\alpha\beta} - \nabla_\mu (\nabla_\nu F(\phi_i)) + g_{\mu\nu} \nabla^2 F(\phi_i), \\ &\quad M^2 R = T - 3\nabla^2 F(\phi_i). \end{aligned} \tag{32}$$

To use a modified "equation of motion" we first supply a factor of $1/2$ in the last term which is the analog of the $\partial^2 A$ term as in Eq. (11),

$$R' = \frac{1}{M^2} \left( T - \frac{3}{2} \nabla^2 F(\phi_i) \right). \tag{33}$$

Then substitute $R'$ for $R$ in the nonminimal term $FR$ of the action of Eq. (12)

$$\begin{aligned} S &\to \int \sqrt{-g} \left( \frac{1}{2} M^2 R(g_{\mu\nu}) + \frac{1}{2} F(\phi_i) R'(g_{\mu\nu}) + L(\phi_i) \right) \\ &= S_1 + S_3 + \int d^4x \left( -\frac{3F(\phi_i)\partial^2 F(\phi_i)}{4M^2} + \frac{F(\phi_i)T(\phi_i)}{2M^2} \right). \end{aligned} \tag{34}$$

In the RG calculation we will only need the exact equation of motion for $R$ in the Einstein frame (without the pseudo $-3\nabla^2 F/2$ term), so this ambiguity does not arise.

## IV. CONTACT TERM IN A SIMPLE MODEL

Consider the following action for a single real scalar field $\phi$:

$$S_{\rm Jordan} = \int \sqrt{-g}\bigg(\frac{1}{2}\partial\phi\partial\phi - \frac{\lambda_1}{4}\phi^4 - \frac{\lambda_2}{12M^2}\phi^6 - \frac{\lambda_3}{12M^2}\phi^2\partial\phi\partial\phi - \frac{\alpha}{12}\phi^2 R + \frac{1}{2}M^2 R\bigg), \quad (35)$$

$\partial^2\phi\partial^2\phi$ terms can be dealt with by adding a $k^4$ term in the kinetic term of $\hat{\phi}$. It is then possible to see this does not modify the present operator basis in the loops, and it would be equivalent to using the equations of motion $\partial^2\phi = \lambda_1\phi^3$ which can be absorbed into a redefinition of $\lambda_2$.

This is the most general action for $\phi$ with a $Z_2$ symmetry $\phi \to -\phi$ valid to $\mathrm{O}(M^{-2})$ with Einstein gravity and assuming $\phi$ is massless, $m^2 = 0$. We will study this model to leading order $1/M^2$ and one loop, $\mathcal{O}\hbar$. Hence, we do not include a term $\phi^4 R/M^2$ since, after use of equations of motion, $R \sim M^{-2}$ such a term would enter the physics at $\mathrm{O}(M^{-4})$. Also note that, by integration by parts, $\int \phi^2\partial^2\phi^2 = (-4)\int \phi^2(\partial\phi)^2$ and $\int \phi^3\partial^2\phi = -3\int \phi^2(\partial\phi)^2$.

The matter Lagrangian has stress tensor and stress tensor trace

$$T_{\mu\nu} = \bigg(1 - \frac{\lambda_3\phi^2}{6M^2}\bigg)\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\bigg(\frac{1}{2}(\partial\phi)^2 - W(\phi)\bigg).$$
$$T = -(\partial\phi)^2 + 4W(\phi)$$
$$W = \frac{\lambda_1}{4}\phi^4 + \frac{\lambda_2}{12M^2}\phi^6 + \frac{\lambda_3}{12M^2}\phi^2(\partial\phi)^2 \quad (36)$$

and we have

$$F = -\frac{\alpha}{6}\phi^2, \quad (37)$$

which leads to the contact terms

$$-\frac{3F\partial^2 F}{4M^2} = \frac{\alpha^2(\phi\partial\phi)^2}{12M^2},$$
$$\frac{FT}{2M^2} = \frac{\alpha\phi^2}{12M^2}((\partial\phi)^2 - \lambda_1\phi^4). \quad (38)$$

Therefore, the effect of single graviton exchange to Eq. (35) yields the Einstein frame action to order $M^{-2}$,

$$S_{\rm Einstein} = \int \sqrt{-g}\bigg(\frac{1}{2}\partial\phi\partial\phi - \frac{\lambda_1'}{4}\phi^4 - \frac{\lambda_2'}{12M^2}\phi^6 - \frac{\lambda_3'}{12M^2}\phi^2\partial\phi\partial\phi + \frac{1}{2}M^2 R\bigg), \quad (39)$$

where

$$\lambda_3' = \lambda_3 - \alpha - \alpha^2,$$
$$\lambda_1' = \lambda_1,$$
$$\lambda_2' = \lambda_2 + \alpha\lambda_1. \quad (40)$$

Thus, the Planck-suppressed terms in Jordan frame (of couplings $\lambda_{2,3}$) that are usually ignored to a leading approximation, are actually of same order $(1/M^2)$ to the nonminimal term ($\phi^2 R$) when written in the Einstein frame. We see that to first order in $M^{-2}$ in $S_{\rm Einstein}$ we have three interaction terms, though the original action $S_{\rm Jordan}$ displayed four interaction terms. In the latter action we see that $\alpha$ has disappeared having been absorbed into redefining the primed coupling constants. This indicates that the nonminimal term in $S_{\rm Jordan}$ with coupling $\alpha$ is unphysical.

### A. Effective action

To compute the effective action for a classical background field $\phi_0$ we expand the action with a shifted field,

$$\phi \to \phi_0 + \sqrt{\hbar}\hat{\phi} \quad (41)$$

to $\mathrm{O}(\hat{\phi}^2)$. We will integrate out the quantum fluctuations and can therefore drop terms odd in $\hat{\phi}$, so we have[1]

$$S_{\rm Einstein} = \int \sqrt{-g}\bigg(\frac{1}{2}\partial\phi_0\partial\phi_0 + \frac{1}{2}\partial\hat{\phi}\partial\hat{\phi} - \frac{1}{2}B_0\hat{\phi}^2 - V_0 + \frac{1}{2}M^2 R - \frac{\lambda_3'}{12M^2}(\hat{\phi}+\phi_0)^2(\partial(\hat{\phi}+\phi_0))^2\bigg), \quad (42)$$

where

$$B_0 = \bigg(3\lambda_1'\phi_0^2 + \frac{5\lambda_2'}{2M^2}\phi_0^4\bigg),$$
$$V_0 = \frac{\lambda_1'}{4}\phi_0^4 + \frac{\lambda_2'}{12M^2}\phi_0^6. \quad (43)$$

To treat the $\lambda_3'$ term in the shifted fields we use integration by parts and the identity $2\phi\partial\phi = \partial(\phi^2)$. We obtain the terms to $\mathrm{O}\hat{\phi}^2$,

$$\int (\hat{\phi}+\phi_0)^2(\partial(\hat{\phi}+\phi_0))^2 \to \int \phi_0^2(\partial\phi_0)^2 - \phi_0^2\hat{\phi}\partial^2\hat{\phi} + \hat{\phi}^2\bigg(\partial\phi_0\partial\phi_0 - \frac{1}{2}\partial^2(\phi_0^2)\bigg). \quad (44)$$

For the $\mathrm{O}(1/M^2)$ terms we can then use the equation of motion,

$$\phi_0^2\hat{\phi}\partial^2\hat{\phi} \approx -B_0\phi_0^2\hat{\phi}^2 \approx -3\lambda_1'\phi_0^4\hat{\phi}^2. \quad (45)$$

[1]This corresponds to using a source term, $J\phi$, to compute $S(J)$, then performing a Legendre transformation to obtain the effective action as a function of $\phi_0 = \delta S(J)/\delta J$ as in [13].

This is a bit tricky, but can be seen by consideration of Feynman diagrams that contribute $\ln(\Lambda^2)$.[2] Then we have

$$S_{\rm Einstein} = S_c + \hbar S_q,$$
$$S_c = \int \sqrt{-g}\left(\frac{1}{2}\partial\phi_0\partial\phi_0 - V' + \frac{1}{2}M^2 R\right),$$
$$S_q = \int \sqrt{-g}\left(\frac{1}{2}\partial\hat{\phi}\partial\hat{\phi} - \frac{1}{2}B'\hat{\phi}^2\right), \tag{46}$$

where

$$B' = 3\lambda_1'\phi_0^2 + \frac{5\lambda_2'}{2M^2}\phi_0^4 + \frac{\lambda_3'}{6M^2}\left(3\lambda_1'\phi_0^4 + (\partial\phi_0)^2 - \frac{1}{2}\partial^2(\phi_0^2)\right),$$
$$V' = \frac{\lambda_1'}{4}\phi_0^4 + \frac{\lambda_2'}{12M^2}\phi_0^6 + \frac{\lambda_3'}{12M^2}\phi_0^2(\partial\phi_0)^2. \tag{47}$$

First consider the noncurvature terms, with the flat Minkowski space metric $g_{\mu\nu} = \eta_{\mu\nu}$. We obtain the effective potential from the log of the path integral discussed in the Appendix [Eq. (A6)],

$$\Gamma_0 = -\frac{1}{2}B'^2 L + O\left(\frac{B^3}{\Lambda^2}\right) \tag{48}$$

and we define the log as (see the Appendix)

$$L = \frac{1}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}, \tag{49}$$

with a generic infrared cutoff mass scale, $\mu$. Hence, squaring $B'$, the resulting potential is to $\mathcal{O}(1/M^2)$,

$$\begin{aligned}\Gamma_0 &= -\left(\frac{9\lambda_1'^2}{2}\phi_0^4 + \frac{15\lambda_2'\lambda_1'}{2M^2}\phi_0^6 + \frac{\lambda_3'\lambda_1'}{2M^2}\phi_0^2\left(3\lambda_1'\phi_0^4 + (\partial\phi_0)^2 - \frac{1}{2}\partial^2(\phi_0^2)\right)\right)L \\ &= -\left(\frac{9\lambda_1'^2}{2}\phi_0^4 + \frac{15\lambda_2'\lambda_1' + 3\lambda_3'\lambda_1'^2}{2M^2}\phi_0^6 + \frac{3\lambda_3'\lambda_1'}{2M^2}\phi_0^2(\partial\phi_0)^2\right)L,\end{aligned} \tag{50}$$

where we integrated by parts the $\phi_0^2(\partial^2\phi_0^2)$ term.

### B. Inclusion of a gravitationally induced contact term in the Einstein frame

Consider a weak-field approximation to gravity where the metric becomes

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M}, \qquad g^{\mu\nu} \approx \eta^{\mu\nu} - \frac{h^{\mu\nu}}{M},$$
$$\sqrt{-g} \approx 1 + \frac{1}{2}\frac{h}{M}, \quad \text{where } h = \eta^{\mu\nu}h_{\mu\nu},$$
$$R = g^{\mu\beta}R_{\mu\beta} = \frac{1}{M}(\partial^2 h - \partial^\nu\partial_\rho h_\nu^\rho) + O(h^2). \tag{51}$$

We choose $w$ gauge (this gauge is developed in [5]; $w = 1/2$ corresponds to the familiar "de Donder gauge", see [12]),

$$\partial_\alpha h^{\alpha\beta} = w\partial^\beta h, \qquad R = (1-w)\partial^2 h/M, \tag{52}$$

and up to linear terms in $h_{\mu\nu}\hat{\phi}^2/M$, Eq. (46) becomes

$$S_q \to \int\left(\frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} + \frac{1}{4}\frac{h}{M}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{h^{\mu\nu}}{2M}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \frac{1}{2}\left(1 + \frac{1}{2}\frac{h}{M}\right)B'\hat{\phi}^2\right). \tag{53}$$

If we now include effects of gravity we see that the action in Eq. (53) generates two diagrams of Figs. 2 and 3 that are linear in the curvature $R$ (other diagrams contribute $\sqrt{-g}$ factors for the resulting potential). We use the Wick-rotated, Euclidean loop momentum with cutoff $\Lambda$ and we thus have for D1

$$\begin{aligned}&T(h^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi})(\hat{\phi}^2) \\ &= (2h^{\mu\nu})\int\frac{d^4\ell}{(2\pi)^4}\frac{i}{(\ell+q)^2}\frac{i}{(\ell)^2}((q+\ell)_\mu(\ell)_\nu) \\ &= i\frac{(1+2w)}{3}(\partial^2 h)L,\end{aligned} \tag{54}$$

where the quadratic divergence is projected away by a factor $P_2$ as discussed in the Appendix. For D2 we have

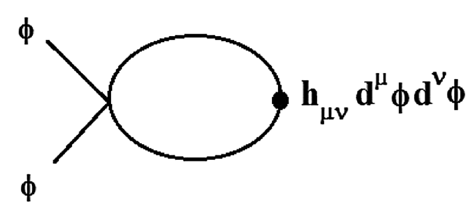


FIG. 2. Diagram D1.

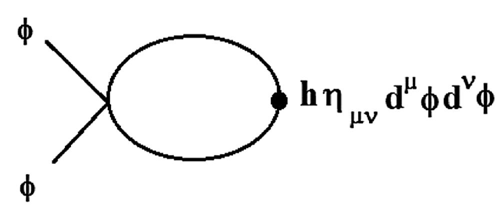


FIG. 3. Diagram D2.

[2] If we restrict ourselves to constant $\phi_0$, as in Coleman-Weinberg [13] we can considerably simplify the analysis and just drop any terms with $\partial\phi_0$ and Eq. (44) becomes $\phi_0^2(\partial\hat{\phi})^2$ and can be absorbed into a wave function renormalization of $\hat{\phi}$; this yields the result we obtain below when $\partial\phi_0 = 0$.

$$T(h\eta^{\mu\upsilon}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi})(\hat{\phi}^2) = 6q^2h\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^4}x(1-x) = i(2\partial^2 h)L. \tag{55}$$

Hence, the contributions to the potential are

$$\begin{aligned}\Gamma_{D1} &= i\frac{B'}{4M}\langle T(h^{\mu\upsilon}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi})(\hat{\phi}^2)\rangle \\ &= -\frac{(1+2w)}{12M}B'\partial^2 hL, \\ \Gamma_{D2} &= -i\frac{B'}{8M}\langle T(h\eta^{\mu\upsilon}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi})(\hat{\phi}^2)\rangle \\ &= \frac{B'}{4M}\partial^2 hL.\end{aligned} \tag{56}$$

The diagrams $\langle T\ hB'\hat{\phi}^2\ B'\hat{\phi}^2\rangle$ generate the covariant $\sqrt{-g}\Gamma$ terms and do not lead to curvature. Note that

$$\frac{B'}{4M}\partial^2 h - \frac{(1+2w)B'}{12M}\partial^2 h = \frac{(1-w)B'}{6M}\partial^2 h = \frac{B'}{6}R. \tag{57}$$

Hence, we have the potential from $D1 + D2$,

$$\Gamma_\alpha \equiv \Gamma_{D1} + \Gamma_{D2} = \frac{1}{6}B'RL \approx \frac{\lambda_1'\phi_0^2}{2}RL, \tag{58}$$

We thus see there is a nonminimal term $(\delta\alpha/12)\phi_0^2R$ in the potential generated by the loops, of the form $\delta\alpha = 6\lambda_1'\delta L$, from the $3\lambda_1'$ term in $B'$, where we only keep leading terms in $1/M^2$ since $R\sim 1/M^2$.

We remove this term by using the contact term. To implement the contact term we use, in Eq. (58), the leading order $R$ equation of motion in the Einstein frame from $S_c$,

$$R = \frac{1}{M^2}T = \frac{1}{M^2}(-(\partial\phi_0)^2 + \lambda_1'\phi_0^4). \tag{59}$$

Here we omit the $\lambda_2$ term in $B'$ which is suppressed by $M^{-2}$, and the $\mathcal{O}\hbar$, $\lambda_1^2L$ term which would lead to a $L^2\sim\hbar^2$ contribution. We then have

$$\Gamma_\alpha = \frac{1}{2}\lambda_1'\phi_0^2RL \to -\frac{\lambda_1'}{2M^2}\phi_0^2(\partial\phi_0)^2L + \frac{\lambda_1'^2}{2M^2}\phi_0^6L. \tag{60}$$

Therefore, combining all effects the effective action becomes our final result,

$$\begin{aligned}S &= \int\frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}M^2R - \Gamma(\phi_0), \\ \Gamma_E(\phi_0) &\equiv \frac{\lambda_1'}{4}\phi_0^4 + \frac{\lambda_2'}{12}\frac{\phi_0^6}{M^2} + \frac{\lambda_3'}{12M^2}\phi_0^2(\partial\phi_0)^2 \\ &\quad - \left(\frac{9\lambda_1'^2}{2}\phi_0^4 + \frac{15\lambda_1'\lambda_2' + 3\lambda_1'^2\lambda_3' - [\lambda_1'^2]}{2M^2}\phi_0^6\right. \\ &\quad \left. + \frac{3\lambda_3'\lambda_1'}{2M^2}\phi_0^2(\partial\phi_0)^2 + \frac{\lambda_1'}{2M^2}\phi_0^2(\partial\phi_0)^2\right)L,\end{aligned} \tag{61}$$

where the term in [...] comes from the gravitational effects of D1 and D2.

From Eq. (61) we can read off the RG equations,

$$\begin{aligned}D\lambda_1' &= 18\lambda_1'^2, \\ D\lambda_2' &= 90\lambda_1'\lambda_2' + 18\lambda_3'\lambda_1'^2 - 6[\lambda_1'^2], \\ D\lambda_3' &= 18\lambda_3'\lambda_1' + 6[\lambda_1'], \qquad D = 16\pi^2\frac{d}{d\ln\mu},\end{aligned} \tag{62}$$

where terms in [...] come from the $\mathcal{O}(\hbar)$ gravitationally induced contact term diagrams, D1 and D2.

The RG equations represent the differential inclusion of a loop induced nonminimal coupling of Eq. (58), $\delta\alpha = (6\lambda_1)\delta L$ occurring in the Einstein frame, back into the $\beta_{\lambda_i}$ functions of the other couplings. Thus, it maintains the reduction from $N$ to $N-1$ couplings (without $\alpha$) in the Einstein frame. The only relevant parameters are the $\lambda_i$ in the Einstein frame and they have a closed set of RG equations. It is not surprising that such effects occur exclusively in the Planck-suppressed operators. One might think that these are not large effects, but they could be relevant when $\lambda_2 \gg \lambda_1$. The main point here is that the gravitational effects are present and must be included in Planck-suppressed terms, but the calculation should be done in the Einstein frame with implementation of the contact terms.

### C. Comparison to a conventional calculation of $\beta_\alpha$ in Jordan frame neglecting contact terms

We now compute the effective potential in the Jordan frame where we naively neglect the contact term, which is often seen in the literature. After evolving the theory in the Jordan frame we can then perform the Weyl transformation to compare with the previous Einstein frame result. As expected, these are found to be inconsistent.

In the Jordan action we shift $\phi\to\phi_0 + \sqrt{\hbar}\hat{\phi}$ and expand to $\mathrm{O}\hat{\phi}^2$ The result is analogous to the Einstein case, but includes the $-\alpha\phi^2R/12$ term and becomes to order $\hbar$,

$$
\begin{aligned}
S_{\text{Jordan}} &= S_{Jc} + \hbar S_{Jq},\\
S_{Jc} &= \int \sqrt{-g}\left(\frac{1}{2}\partial\phi_0\partial\phi_0 - V_J - \frac{\alpha}{12}\phi_0^2 R + \frac{1}{2}M^2 R\right),\\
S_{Jq} &= \int \sqrt{-g}\left(\frac{1}{2}\partial\hat{\phi}\partial\hat{\phi} - \frac{1}{2}B_J\hat{\phi}^2 - \frac{\alpha}{12}(\hat{\phi}^2)R\right), \qquad (63)
\end{aligned}
$$

where

$$
\begin{aligned}
B_J &= 3\lambda_1\phi_0^2 + \frac{5\lambda_2}{2M^2}\phi_0^4 + \frac{\lambda_3}{6M^2}\left(3\lambda_1\phi_0^4 + (\partial\phi_0)^2 - \frac{1}{2}\partial^2(\phi_0^2)\right),\\
V_J &= \frac{\lambda_1}{4}\phi_0^4 + \frac{\lambda_2}{12M^2}\phi_0^6 + \frac{\lambda_3}{12M^2}\phi_0^2(\partial\phi_0)^2. \qquad (64)
\end{aligned}
$$

Expanding to linear terms in $h_{\mu\nu}/M$ in weak-field gravity the action Eq. (63) becomes

$$
S_{Jq} \to \int\left(\frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} + \frac{h}{4M}\eta^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \frac{h^{\mu\nu}}{2M}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \frac{\alpha}{12}R\hat{\phi}^2 - \frac{1}{2}B_J\hat{\phi}^2 - B_J\frac{h}{4M}\hat{\phi}^2\right). \qquad (65)
$$

Neglecting the contact terms we see, in addition to the non-curvature potential obtained previously in Eq. (50), the action Eq. (65) now (naively) generates three diagrams linear in the curvature, D1, D2, and D3 of Figs. 2–4. The additional D3 diagram (which is absent in the Einstein frame) is

$$
\begin{aligned}
\Gamma_{D3} &= i\left\langle T\left(-i\frac{1}{2}A\hat{\phi}^2 R\right)\left(-i\frac{1}{2}B_J\hat{\phi}^2\right)\right\rangle\\
&= -AB_J RL = -\frac{\alpha}{6}B_J RL + \mathcal{O}\frac{R}{M^2}, \qquad (66)
\end{aligned}
$$

where $A = \alpha/6$. Hence we have from Eqs. (58) and (66) for (D1+D2+D3),

$$
\begin{aligned}
\Gamma_{D3} + \Gamma_\alpha &= -\frac{1}{6}(\alpha-1)B_J RL\\
&= -\frac{1}{6}(\alpha-1)\left(3\lambda_1\phi_0^2 + \mathcal{O}\frac{1}{M^2}\right)RL. \qquad (67)
\end{aligned}
$$

Combining all effects at this point we have the potential,

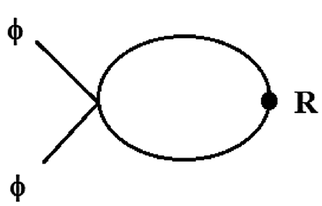


FIG. 4. Diagram D3.

$$
\begin{aligned}
\Gamma_{\text{Jordan}} \equiv{}& \frac{\lambda_1}{4}\phi_0^4 + \frac{\lambda_2}{12M^2}\phi_0^6 + \frac{\lambda_3}{12M^2}\phi_0^2(\partial\phi_0)^2\\
&-\left(\frac{9\lambda_1^2}{2}\phi_0^4 + \frac{15\lambda_1\lambda_2 + 3\lambda_1^2\lambda_3}{2M^2}\phi_0^6 + \frac{3\lambda_3\lambda_1}{2M^2}\phi_0^2(\partial\phi_0)^2\right)L\\
&-\frac{1}{6}(\alpha-1)(3\lambda_1\phi_0^2)RL + \frac{\alpha}{12}\phi_0^2 R. \qquad (68)
\end{aligned}
$$

This then contains the radiative correction and renormalization group running of $\alpha$

$$
\begin{aligned}
&\alpha \to \alpha + 6\lambda_1(1-\alpha)L,\\
&D\alpha = -6\lambda_1(1-\alpha), \qquad D = 16\pi^2\frac{d}{d\ln\mu}. \qquad (69)
\end{aligned}
$$

To compare to the Einstein frame we now implement the contact terms directly from the action,

$$
CT = \int d^4x\left(-\frac{3F\partial^2 F}{4M^2} + \frac{FT(\phi_i)}{2M^2}\right), \qquad (70)
$$

where

$$
\begin{aligned}
T &= -(\partial\phi_0)^2 + (\lambda_1 - 18\lambda_1^2 L)\phi_0^4,\\
F &= -\frac{\alpha}{6}\phi_0^2 + (\alpha-1)\lambda_1\phi_0^2 L, \qquad (71)
\end{aligned}
$$

which leads to the contact term correction in the potential

$$
\begin{aligned}
-CT = -\frac{1}{12M^2}(&(\alpha+\alpha^2)\phi_0^2(\partial\phi_0)^2 - \alpha\lambda_1\phi_0^6\\
&+ (6\alpha + 6 - 12\alpha^2)\lambda_1\phi_0^2(\partial\phi_0)^2 L\\
&+ 6(4\alpha-1)\lambda_1^2\phi_0^6 L) \qquad (72)
\end{aligned}
$$

and combining all effects the potential becomes

$$
\begin{aligned}
\Gamma ={}& \frac{\lambda_1}{4}\phi_0^4 + \frac{\lambda_2}{12M^2}\phi_0^6 + \frac{\lambda_3}{12M^2}\phi_0^2(\partial\phi_0)^2\\
&-\left(\frac{9\lambda_1^2}{2}\phi_0^4 + \frac{15\lambda_1\lambda_2 + 3\lambda_1^2\lambda_3}{2M^2}\phi_0^6 + \frac{3\lambda_3\lambda_1}{2M^2}\phi_0^2(\partial\phi_0)^2\right)L\\
&-\frac{1}{12M^2}((\alpha+\alpha^2)\phi_0^2(\partial\phi_0)^2 - \alpha\lambda_1\phi_0^6\\
&+ (6\alpha + 6 - 12\alpha^2)\lambda_1\phi_0^2(\partial\phi_0)^2 L + (24\alpha - 6)\lambda_1^2\phi_0^6 L). \qquad (73)
\end{aligned}
$$

Converting to the Einstein frame variables,

$$
\begin{aligned}
\lambda_3' &= \lambda_3 - \alpha - \alpha^2,\\
\lambda_1' &= \lambda_1,\\
\lambda_2' &= \lambda_2 + \alpha\lambda_1, \qquad (74)
\end{aligned}
$$

we obtain after a somewhat tedious calculation,

$$\Gamma = \Gamma_E + \left(\frac{(8-3\alpha)\alpha}{2M^2}\lambda_1'^2\phi_0^6 - \frac{(4+\alpha)\alpha}{2M^2}\lambda_1'\phi_0^2(\partial\phi_0)^2\right)L,$$

where $\Gamma_E(\phi_0)$ is given in Eq. (61).

Comparing to Eq. (61) we therefore see an inconsistency between the actions $S_{\rm Einstein}$ and $S_{\rm Jordan}$ at $O(\hbar)$ (opposite sign for potentials). Hence we obtain the "frame anomaly",

$$S_{\rm Jordan} - S_{\rm Einstein} = -\int \frac{(8-3\alpha)\alpha}{2M^2}\lambda_1'^2\phi_0^6 L + \int \frac{(4+\alpha)\alpha}{2M^2}\lambda_1'\phi_0^2(\partial\phi_0)^2 L. \quad (75)$$

We emphasize that the rhs of Eq. (75) represents the mistake of not including the contact term in the initial action of Eq. (63). For nonvanishing $\alpha$, the quantum actions obtained in the two approaches agree if the quartic scalar interaction is absent ($\lambda_1 = 0$).

## V. CONCLUSIONS

The Weyl transformation acting on the Jordan frame, to remove nonminimal interactions, leading to the minimal Einstein frame, is identical to implementing the contact terms [5]. If one didn't know about the Weyl transformation one might discover it in the induced contact terms in the single graviton exchange potential involving nonminimal couplings. The Weyl transformation is powerful as it is fully nonperturbative. Technically, it can provide a useful check on the normalization and implementation of the graviton propagators in various gauges, but the contact term stipulates that the mapping to the Einstein frame is dynamical and inevitable and does not involve field redefinitions. Hence, there is no Jacobian in the path integral associated with going to the Einstein frame action and it uniquely describes the theory.

In a model with nonminimal coupling $-\alpha\phi^2 R/12$ this implies that the parameter $\alpha$ does not exist physically, unless demanded by a symmetry such as Weyl invariance. Computing $\beta$-functions in a Jordan frame without implementing the contact term will yield incorrect results. Implementing the contact term yields the Einstein frame and results computed there will have no contact term ambiguities.

Nonetheless, Einstein frame will have a loop-induced infinitesimal $\alpha$ which can then be absorbed back into the potential terms of the Einstein frame by the contact terms (equivalently, a mini-Weyl transformation, or use of the modified $R$ equation of motion). The use of the modified $R$ equation of motion on the nonminimal term is analogous to the use of the gluon field equation for the electroweak penguin. It is likely that the Deans and Dixon [9] constraints on null operators apply to gravity as well.

We emphasize that our analysis applies strictly to a theory with a Planck mass. A Weyl invariant theory, where $M = 0$, is nonperturbative and our analysis is then inapplicable, and the Jordan frame is then physically relevant. Indeed, there is no conventional gravity in this limit since the usual $M^2 R$ (Fierz-Pauli) graviton kinetic term does not then exist. Hence in this limit one would have to appeal to a UV completion, e.g., string theory, $R^2$ gravity, Weyl conformal geometry, etc.

Contact-term effects will disappear if we can go "on shell." This is demonstrated by the approach of Ruf and Steinwachs [4] (see also [14]), which employs an on shell calculational procedure. However, the calculation of $\beta$-functions and effective actions is intrinsically an off shell problem, since an action is generally a functional of fields that are unconstrained by equations of motion. The contact terms must be implemented for consistency.

In the case of an $R^2$ UV completion theory we view the formation of the Planck mass by, e.g., inertial symmetry breaking, i.e., as a dynamical phase transition, similar to a disorder-order phase transition in a material medium [15]. It is interesting that in a $R_{\mu\nu}R^{\mu\nu}$ UV completion of gravity such as in Ref. [16], the propagator becomes $1/q^4$, nonetheless the contact term effective interactions exists above the Planck scale for fields that couple nonminimally. These then become $q^2 \times (1/q^4) \sim 1/q^2$. Given the sign of $F(\phi_i)$ in Eq. (12) there may exist an inverse square law, pseudogravitational force that can be repulsive. This is one of many issues to develop further in this context.

## ACKNOWLEDGMENTS

We dedicate this paper to our friend and colleague, Graham G. Ross. This work grew out of previous collaborations with him, and we have benefitted greatly from past discussions with him on this and other topics. Part of this work was done at Fermilab, operated by Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the United States Department of Energy. The work of D. G. was supported by a grant of the Romanian Ministry of Education and Research, CNCS-UEFISCDI, Project No. PN-III-P4-ID-PCE-2020-2255.

## APPENDIX: PROJECTION-REGULATED FEYNMAN LOOPS

The loop-induced effective potential for $\phi_0$ provides a useful way to extract all of the $\beta$-functions of the various coupling constants. The potential $\Gamma(\phi_0)$ is the log of the path integral; $\Gamma = i \ln P$. In the case of a real scalar field with mass term we consider the free action,

$$\frac{1}{2}\int d^4x(\partial\phi\partial\phi - m^2\phi^2) \quad (A1)$$

and we have for the path integral,

$$P = \prod_k (k^2 - m^2)^{-1/2} = \det(k^2 - m^2)^{-1/2}, \quad (A2)$$

where $k = (k_0, \vec{k})$ is the 4-momentum. Hence, we have

$$\Gamma = i \ln P = -\frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \ln(k^2 - m^2 + i\epsilon). \quad \text{(A3)}$$

This can be evaluated with a Wick rotation to a Euclidean momentum, $k \to k_E = (ik_0, \vec{k})$, and a Euclidean momentum space cutoff $\Lambda$,

$$\Gamma = \frac{1}{2} \int_0^\Lambda \frac{d^4 k_E}{(2\pi)^4} \ln\left(\frac{k_E^2 + m^2}{\Lambda^2}\right) = \frac{1}{64\pi^2} \left(\Lambda^4 \ln \frac{\Lambda^2 + m^2}{\Lambda^2} - m^4 \ln \frac{\Lambda^2 + m^2}{m^2} - \frac{1}{2}\Lambda^4 + \Lambda^2 m^2\right) + (\text{irrelevant constants}), \quad \text{(A4)}$$

where we inserted $\Lambda^{-2}$ in the arguments of the logs to preserve zero-scale dimension. The cutoff can be viewed as a spurious parameter, introduced to make the integral finite and arguments of logs dimensionless, but it is not part of the defining action. The only physically meaningful dependence upon $\Lambda$ is contained in the logarithm, where it reflects scale symmetry breaking by the quantum trace anomaly. Powers of $\Lambda$, e.g., $\Lambda^4, \Lambda^2 m^2$, spuriously break classical scale symmetry and are not part of the classical action [17].

It is therefore conceptually useful to have a definition of the loops in which the spurious powers of $\Lambda$ do not arise. This can be done by defining the loops applying projection operators on the integrals. The projection operator

$$P_n = \left(1 - \frac{\Lambda}{n} \frac{\partial}{\partial \Lambda}\right) \quad \text{(A5)}$$

removes any terms proportional to $\Lambda^n$. Since the defining classical Lagrangian has mass dimension 4 and involves no terms with $\Lambda^2 m^2$ or $\Lambda^4$, we define the regularized loop integrals as

$$\Gamma \to \frac{1}{2} P_2 P_4 \int_0^\Lambda \frac{d^4 k_E}{(2\pi)^4} \ln\left(\frac{k_E^2 + m^2}{\Lambda^2}\right) = -\frac{1}{64\pi^2} m^4 \left(\ln \frac{\Lambda^2}{m^2}\right) + O\left(\frac{m^6}{\Lambda^2}\right), \quad \text{(A6)}$$

where we then take the limit $\Lambda \gg m$ to suppress $O(m^6/\Lambda^2)$ terms and we are interested only in the log term (not additive constants) This means that the additive, nonlogarithmic terms, e.g., $c' m^2$, are undetermined, and the only physically meaningful result is the $\ln(\Lambda^2/m^2)$ term. $\Lambda$ can be swapped for a running renormalization scale $\mu$. Interestingly, if we define the integral as $\int dX \to P_1 P_2 P_3 \ldots P_\infty \int dX$, the action on the logs will lead to the Euler constant that arises in dimensional regularization, hinting at a mapping to the dimensionally-regularized result.


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