Binary dynamics from worldline QFT for scalar QED

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(Received 29 June 2022; accepted 8 February 2023; published 11 April 2023)

We investigate the worldline quantum field theory (WQFT) formalism for scalar QED and observe that a generating function emerges from WQFT, from which the scattering angle ensues. This generating function bears important similarities to the radial action in that it requires no consideration of exponentiation of lower-order contributions. We demonstrate the computations of this generating function and the resulting scattering angle of a binary system coupled to an electromagnetic field up to the third order in the post-Minkowskian expansion.

DOI: 10.1103/PhysRevD.107.085011

Recent developments in gravitational-wave physics [1–5] call for innovations of theoretical framework that facilitate both numerical [6–8] and analytical [9–36] high-precision computations of the dynamics of binary black hole or neutron star mergers.

It has proven fruitful to extract classical observables from scattering amplitudes in perturbative quantum field theories [37–42], thanks to modern tools based on on-shell techniques [43–52] and effective field theory [38,53]. However, to expose the classical quantity, amplitude-based approaches often requires a delicate analysis which removes quantum and superclassical contributions alike [39,40,54,55]. Alternative methods that capture classical observables more directly are, therefore, in demand, and several explorations in this direction [56–59] have been shown to be beneficial.

It is in this light that the worldline quantum field theory (WQFT) [60], in which worldline degrees of freedom are quantized, is formulated, providing a formal link between black hole observables extracted from scattering amplitudes and time-ordered correlators in WQFT. WQFT Feynman rules circumvent the need for the effective potential in traditional worldline effective field theory (EFT) methods [10,61,62] and streamline loop calculations encountered in amplitude-based approaches to summing over diagrams of tree topologies only, yielding classical observables directly. Recent applications of WQFT involve a series of work on spinning black holes [63–65] and the state-of-the-art derivations of the conservative momentum impulse and the spin kick up to the third order in post-Minkowskian (3PM) expansion and quadratic order in spin have been obtained from WQFT [66].

As established in amplitude-based approaches, conservative and radiative dynamics in classical relativistic scattering can be extracted from the eikonal phase [55,67–70]. Inspired by the eikonal approximation, an amplitude-action relation has been revealed [55], and the *radial action* [71–74] serves as another generating function for the scattering angle. Another closely related generating function is defined in the heavy-particle EFT [59], which agrees with the radial action in their real parts but differs in the imaginary part. One crucial difference between these functions and the standard eikonal exponentiation is that iterations from lower orders can be discarded for the former.

WQFT is expected to have the potential of capturing such generating functions, too. The classical eikonal phase can be obtained from WQFT in various contexts up to 2PM or next-to-leading order [60,65,75,76]. However, the calculation of the eikonal phase at 3PM and beyond in WQFT remains somewhat ambiguous in the $i\epsilon$ prescription of the worldline propagator. On the other hand, it is conceivable that WQFT speaks more directly to a generating function whose classical part is readily isolated than the eikonal. In this paper, we seek to explore the construction of such a generating function from WQFT.

In this paper, we consider the WQFT counterpart of scalar QED as a toy model, which is shown to be a useful playground for higher PM gravitational computations [74,77,78]. We illustrate that a generating function emerges from WQFT in a highly streamlined fashion, which reproduces both the conservative and radiative contributions of the scattering

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angle. This generating function bears similarities to the radial action and the eikonal exponentiation. The WQFT integrands can be made to match with those in the heavy-mass limit of scalar QED in the comparable-masses sector and in those diagrams responsible for the radiation reaction. We expect these observations to carry over straightforwardly to WQFT in a gravitational background.

I. WQFT FORMALISM FOR SCALAR QED

The worldline action describing a charged massive nonspinning point particle in an electromagnetic background reads [79,80]

$$\mathcal{S}_i = -m_i \int \mathsf{d}\sigma \left[\frac{1}{2}(\eta^{-1}\dot{x}_i^2 + \eta) + ie\frac{q_i}{m_i}A_\mu \dot{x}_i^\mu\right], \quad (1)$$

where the worldline coordinate x^μ is parametrized by σ and $\dot{x}^\mu = \mathsf{d}x^\mu/\mathsf{d}\sigma$. q_i and m_i denote the charge and mass, respectively, of the scalar $i = 1, 2$. The worldline is coupled to the electromagnetic (EM) field A_μ, and the bulk theory is simply given by the usual EM action. For convenience, we set the einbein $\eta(\sigma) = 1$. As shown in Ref. [60], specializing the photon to plane waves of fixed momenta and polarizations, the photon-dressed Feynman-Schwinger propagator [81] can be identified with the path integral for the WQFT correlator, with external legs amputated through the Lehmann-Symanzik-Zimmermann reduction.

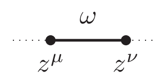
Expanding the worldline around straight-line trajectories $x_i^\mu = b_i^\mu + u_i^\mu \sigma + z_i^\mu(\sigma)$, the WQFT Feynman rules are readily expressed in frequency and momentum space: $z_i^\mu(\sigma) = \int_\omega e^{-i\sigma\omega} z_i^\mu(\sigma)$ and $A_\mu(x) = \int_k e^{-ik\cdot x} A_\mu(-k)$, where we have used the shorthand notations $\int_{\omega/k}$ as introduced in Ref. [60]. The explicit expressions of WQFT Feynman rules for worldline-photon interactions are given in Supplemental Material [82].

Inspired by the eikonal exponentiation [60], we consider the phase identified with the WQFT path integral in the classical limit:

$$e^{i\delta} = \mathcal{Z}_{\mathrm{WQFT}} = \int \mathcal{D}[A] \prod_{j=1}^{2} \mathcal{D}[z_j] e^{i(\mathcal{S}_{\mathrm{EM}} + \sum_{j=1}^{2} \mathcal{S}_j)}, \quad (2)$$

where $\mathcal{S}_{\mathrm{EM}}$ denotes the standard action for the electromagnetic field in the bulk. We note that the identification above is designed to hold in the classical limit. Hence, the phase δ is a purely classical quantity. That is, δ is uniform in $\hbar$ and admits only an expansion in the coupling constant e^2. Taking the logarithm on both sides, we identify δ at each order of e^2 with the sum of connected WQFT diagrams, without iteration corrections from lower orders, which sets it apart from the eikonal approach proposed in Ref. [60]. It may be tempting to identify it with the radial action due to the similar definitions; but preliminary evidence suggests that differences occur in their respective imaginary parts. Similar to the Heavy-mass Effective Field Theory (HEFT) phase [59], we restrict ourselves to the real parts of this generating function and the resulting scattering angle. The imaginary part is beyond the scope of this paper.

The evaluation of WQFT path integrals is normally sensitive to the $i\epsilon$ prescription of the worldline propagator. We observe that only the *principal-value* part of the time-symmetric propagator [60] is relevant for the construction of this generating function. Hence, we propose the principal-value prescription for the worldline propagator, and the propagator simply reads



$$= -\frac{i}{m}\frac{\eta^{\mu\nu}}{\omega^2}. \quad (3)$$

This treatment is reminiscent of Refs. [59,73,75,83].

The Feynman rules are given in terms of kinematic variables b_i^μ and u_i^μ, the interpretation of which depends on the worldline trajectory they describe [60]. The kinematics of the $2 \to 2$ scattering is given by the momenta:

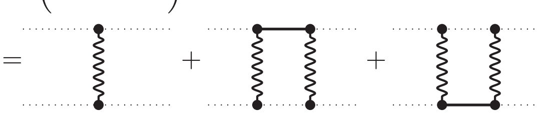
$$p_1 = \bar{p}_1 + q/2, \qquad p_2 = \bar{p}_2 - q/2,$$
$$p_1' = \bar{p}_1 - q/2, \qquad p_2' = \bar{p}_2 + q/2,$$

with $p_i^2 = p_i'^2 = m_i^2$ and $\bar{p}_i^2 = \bar{m}_i^2$. The initial trajectory $(\sigma = -\infty)$ corresponds to $p_i^\mu = m_i u_i^\mu$, and the initial impact parameter is given by $b^\mu = b_1^\mu - b_2^\mu$. The in-between trajectory $(\sigma = 0)$ corresponds to the "barred variable" $\bar{p}_i^\mu = \overline{m_i u_i^\mu}$ and $\bar{b}^\mu$. The differences between the two sets of variables come at $\mathcal{O}(\mathsf{q}^2)$. Similar to the observations in Ref. [59], the phase δ is free from iterations, and, hence, the barred variables can be traded with the unbarred ones at no cost.

II. 1PM AND 2PM

At the leading (1PM) and subleading (2PM) orders, the phase δ is given by

$$i\left(\delta^{(0)} + \delta^{(1)}\right)$$



$$= \int_q e^{ib\cdot q}\, \hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2) \left[-\frac{ie^2 q_1 q_2 \gamma}{q^2} + ie^4 q_1^2 q_2^2 \frac{\left((2D-7)\gamma^2 - 1\right)(m_1+m_2)}{2\left(\gamma^2-1\right)m_1 m_2} G_i^{(1)}\right], \quad (4)$$

where we have adopted the notations in Ref. [60] for the integration measure, $\hat{\delta}(x) \coloneqq 2\pi\delta(x)$, $\gamma = u_1 \cdot u_2$, and D denotes the spacetime dimension. The integral $G_i^{(1)}$ in $D = 4 - 2\epsilon$ reads

$$G_i^{(1)} = \int_{\ell_1} \frac{\delta\!\!\!{}^-(\ell_1 \cdot u_i)}{\ell_1^2 (q-\ell_1)^2} = \frac{(4\pi)^{\epsilon-\frac{3}{2}}\Gamma(\frac{1}{2}-\epsilon)^2\Gamma(\frac{1}{2}+\epsilon)}{(-q^2)^{\frac{1}{2}+\epsilon}\Gamma(1-\epsilon)}. \tag{5}$$

Note that both the impact parameter b^μ and the total momentum transfer q^μ are spacelike and the Fourier transform is performed in $(D-2)$ dimensions due to the two δ functions as follows:

$$\int_q \frac{e^{iq\cdot b}}{(-q^2)^\alpha} = \frac{(-b^2)^{\alpha-1+\epsilon}\Gamma(1-\alpha-\epsilon)}{4^\alpha \pi^{1-\epsilon}\sqrt{\gamma^2-1}\Gamma(\alpha)}. \tag{6}$$

Fourier transforming to the impact parameter space, we obtain

$$\delta^{(0)} = \alpha q_1 q_2 \frac{\gamma}{\sqrt{\gamma^2-1}} \frac{\Gamma(-\epsilon)}{\pi^{-\epsilon}} (\mathsf{b}^2)^\epsilon, \tag{7}$$

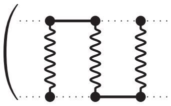
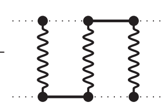
$$\delta^{(1)} = -(\alpha q_1 q_2)^2 \frac{\pi(m_1+m_2)}{2m_1 m_2\sqrt{\gamma^2-1}\mathsf{b}}, \tag{8}$$

where $\mathsf{b} = |b| = \sqrt{-b^2}$ and $\alpha = e^2/(4\pi)$.[1]

Moving on to the next-to-next-to-leading order (3PM), we consider the comparable-masses sector ($m_1 \sim m_2$) and the probe-limit sector ($m_1 \ll m_2$ or $m_1 \gg m_2$). The former contributes to both the conservative and the radiative parts of the scattering angle, whereas the latter contributes only to the conservative part. In addition, the scattering angle begins to receive the so-called *radiation reactions* at 3PM [74,84–89], and we shall consider them separately.

III. 3PM COMPARABLE MASSES

The conservative contribution from this sector is computed by the following diagrams:

$$i\delta^{(2)}\Big|_{m_1 m_2} = \frac{1}{2}\Bigg($$

$$+$$

$$\Bigg)$$

$$= \frac{ie^6 q_1^2 q_2^3}{2m_1 m_2}\int_q e^{ib\cdot q}\prod_{i=1}^{2}\delta(q\cdot u_i)\int_{\ell_1\ell_2}\frac{\delta(\ell_1\cdot u_2)\delta(\ell_2\cdot u_1)}{\ell_1^2\ell_2^2(q-\ell_1-\ell_2)^2}$$
$$\left[-\frac{\gamma(q-\ell_2)^2}{(\ell_1\cdot u_1)^2} - \frac{\gamma(q-\ell_1)^2}{(\ell_2\cdot u_2)^2} + \frac{\gamma^3(q-\ell_1)^2(q-\ell_2)^2}{2(\ell_1\cdot u_1)^2(\ell_2\cdot u_2)^2}\right.$$
$$\left. - \frac{\gamma^2 q^2}{(\ell_1\cdot u_1)(\ell_2\cdot u_2)} - \frac{2(\ell_1\cdot u_1)}{(\ell_2\cdot u_2)} - \frac{2(\ell_2\cdot u_2)}{(\ell_1\cdot u_1)}\right], \tag{9}$$

where we have removed the tadpole terms from the integrand.[2] Here, we note that the two diagrams in the first line agree with the "zigzag" diagrams of the QED counterpart of HEFT.[3] Using LiteRed [90], Eq. (9) is cast by *integration-by-parts* (IBP) relations in the basis of master integrals as follows[4]:

$$i\delta^{(2)}|_{m_1 m_2} = \frac{ie^6 q_1^2 q_2^3}{2m_1 m_2}\int_q e^{ib\cdot q}\prod_{i=2}^{2}\delta\!\!\!{}^-(q\cdot u_i)[a_1 G_{0,0,0,0,1,1,1}$$
$$+ a_2 G_{0,0,0,0,1,2,1} + a_3 G_{0,0,0,0,2,1,1}$$
$$+ a_4 G_{1,1,0,0,1,1,1}]. \tag{10}$$

The integrals are defined as

$$G_{n_1 n_2 \dots n_7} = \int_{\ell_1\ell_2}\frac{\delta\!\!\!{}^-(\ell_1\cdot u_2)\delta\!\!\!{}^-(\ell_2\cdot u_1)}{\rho_1^{n_1}\rho_2^{n_2}\dots\rho_7^{n_7}}, \tag{11}$$

where the propagators are

$$\rho_1 = \ell_1\cdot u_1, \quad \rho_2 = \ell_2\cdot u_2, \quad \rho_3 = \ell_1^2, \quad \rho_4 = \ell_2^2,$$
$$\rho_5 = (q-\ell_1-\ell_2)^2, \quad \rho_6 = (q-\ell_1)^2, \quad \rho_7 = (q-\ell_2)^2. \tag{12}$$

We list the coefficients in $D = 4-2\epsilon$ below:

$$a_1 = -\frac{2\epsilon(\gamma^4(4\epsilon^2-2\epsilon-1)+\gamma^2(2\epsilon+3)-3)}{\gamma(\gamma^2-1)(1-2\epsilon)}, \tag{13}$$

$$a_2 = \frac{(2\epsilon+1)(\gamma^4(2\epsilon(6\epsilon-5)-1)+\gamma^2(6\epsilon+3)-3)q^2}{3\gamma(\gamma^2-1)^2(1-2\epsilon)\epsilon}, \tag{14}$$

$$a_3 = -\frac{(\gamma^4(2\epsilon(6\epsilon-1)-5)+\gamma^2(6\epsilon-3)+3)q^2}{3\gamma(\gamma^2-1)(1-2\epsilon)}, \tag{15}$$

$$a_4 = -\frac{\gamma^2(2\gamma^2\epsilon+3)q^2}{3(\gamma^2-1)}. \tag{16}$$

These integrals are extensively studied in the literature [59,66,67,87,91] using differential equations [92–96]. We display the explicit expressions for the *real* parts of the master integrals present in Eq. (10) in Supplemental Material [82], and Eq. (10) is readily evaluated. Fourier transforming to the impact parameter space and taking $\epsilon \to 0$, we obtain

[1]We multiply a factor of $\frac{i}{(4\pi)^2}(4\pi e^{-\gamma_E})^\epsilon$ per loop in the end to restore the proper normalization [66,68].

[2]Tadpole terms are those that do not have all three massless poles ℓ_1^2, ℓ_2^2, and $(q-\ell_1-\ell_2)^2$ simultaneously, which integrate to zero.

[3]More detailed discussions on the connections between the WQFT and HEFT approaches are given in Supplemental Material [82].

[4]In all three sectors, only the principal-value parts of the time-symmetric propagators contribute to the real part of the phase δ. Take the zigzag diagrams, for example. The master integral $G_{1,1,0,0,1,1,1}$ needs to be dealt with in four sectors $G^{\pm\pm}_{1,1,0,0,1,1,1}$ separately, depending on their respective $i\epsilon$ prescriptions for the two worldline propagators. However, rewriting $1/(x \pm i\epsilon) = \mathrm{pv}(1/x) \mp i\pi\delta(x)$, the imaginary part drops out.

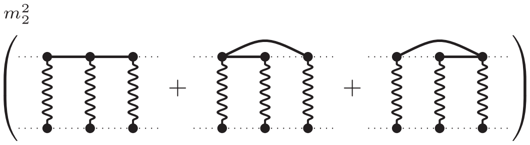
$$\mathrm{Re}\delta^{(2)}|_{m_1 m_2} = -\frac{2(\alpha q_1 q_2)^3(\gamma^4 - 3\gamma^2 + 3)}{3 m_1 m_2 \mathsf{b}^2(\gamma^2-1)^{5/2}} + \frac{2(\alpha q_1 q_2)^3 \gamma^2(\gamma\sqrt{\gamma^2-1} - \mathrm{arccosh}\gamma)}{m_1 m_2 \mathsf{b}^2 (\gamma^2-1)^{5/2}}. \tag{17}$$

The two terms are identified with the conservative and radiative contributions, because they result from boundary values computed in different regions. The first term comes purely from the potential region identified in Ref. [59] while the second from the radiative region. As will be demonstrated shortly, they reproduce the conservative and radiative parts of the scattering angle, respectively.

IV. 3PM PROBE LIMIT

Similarly, in the probe limit we consider the following diagrams:

$$i\delta^{(2)}\Big|_{m_2^2} = \frac{1}{3}\Bigg(\quad + \quad + \quad \Bigg)$$



$$= \frac{ie^6 q_1^3 q_2^3}{12 m_1^2}\int_q e^{ib\cdot q}\prod_{i=1}^{2}\delta(q\cdot u_i)\int_{\ell_1\ell_2}\frac{\delta(\ell_1\cdot u_2)\delta(\ell_2\cdot u_2)}{\ell_1^2\ell_2^2(q-\ell_1-\ell_2)^2}$$
$$\left[\frac{\gamma\big((q-\ell_1)^2-q^2\big)}{(\ell_1\cdot u_1)^2} + \frac{2\gamma\big((q-\ell_2)^2-q^2\big)}{(\ell_2\cdot u_1)^2} + \frac{\gamma^3 q^2 (q-\ell_2)^2}{(\ell_1\cdot u_1)^2(\ell_2\cdot u_1)^2} - \frac{\gamma^3(q-\ell_2)^4}{(\ell_1\cdot u_1)^2(\ell_2\cdot u_1)^2} + \frac{\gamma^3(q-\ell_2)^2\big((q-\ell_2)^2+(q-\ell_1)^2-q^2\big)}{(\ell_1\cdot u_1)^3(\ell_2\cdot u_1)}\right]. \tag{18}$$

Here, we have symmetrized the diagrams by labeling the momenta universally in all three diagrams. This symmetrization helps to reproduce all the pole structures expected explicitly in the classical limit of the corresponding Feynman diagrams in this sector. The two probe-limit sectors are simply related by relabeling $m_1 \leftrightarrow m_2$.

After IBP reduction using LiteRed, the integrand in Eq. (18) is simplified to one single master integral:

$$\frac{ie^6 q_1^3 q_2^3 (6\epsilon - 1)\gamma(\gamma^2(6\epsilon-2)+3)}{6 m_1^2(\gamma^2-1)^2} G_2^{(2)}, \tag{19}$$

where the master integral in $D = 4 - 2\epsilon$ reads

$$G_i^{(2)} = \int_{\ell_1\ell_2}\frac{\delta\!\!\!{}^-(\ell_1\cdot u_i)\delta\!\!\!{}^-(\ell_2\cdot u_i)}{\ell_1^2\ell_2^2(q-\ell_1-\ell_2)^2} = -\frac{(4\pi^2)^{-3+2\epsilon}}{(-q^2)^{2\epsilon}}\frac{\Gamma(\frac{1}{2}-\epsilon)^3\Gamma(2\epsilon)}{\Gamma(\frac{3}{2}-3\epsilon)}. \tag{20}$$

That only one master integral contributes is also observed in the context of gravity [59,73]. We note that the matching with the heavy-mass limit of scalar QED in the probe limit is less manifest. The two integrands can be shown to be equal after IBP reduction. Fourier transforming to impact parameter space, we have

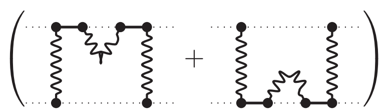
$$\mathrm{Re}\delta^{(2)}|_{m_2^2} = \frac{(\alpha q_1 q_2)^3\gamma(2\gamma^2-3)}{m_1^2 \mathsf{b}^2(\gamma^2-1)^{5/2}}. \tag{21}$$

We will see shortly that this reproduces the conservative part of the scattering angle in the probe limit.

V. 3PM RADIATION REACTION

The radiation reaction is accounted for by the following diagrams:

$$i\delta^{(2)}\Big|_{\mathrm{r.r.}} = \Bigg(\quad + \quad \Bigg)$$



$$= \frac{ie^6 q_1^4 q_2^2}{2 m_1^2}\int_q e^{ib\cdot q}\prod_{i=1}^{2}\delta\!\!\!{}^-(q\cdot u_i)\int_{\ell_1\ell_2}\frac{\delta\!\!\!{}^-(\ell_1\cdot u_2)\delta\!\!\!{}^-(\ell_2\cdot u_1)}{\ell_1^2(q-\ell_1)^2(q-\ell_1-\ell_2)^2}$$
$$\left[1 + \frac{(\ell_2\cdot u_2)^2}{(\ell_1\cdot u_1)^2} - \frac{\gamma^2 q^2}{2(\ell_1\cdot u_1)^2} + \frac{\gamma\ell_2^2(\ell_2\cdot u_2)}{(\ell_1\cdot u_1)^3} + \frac{\gamma^2\ell_2^2(q-\ell_2)^2}{2(\ell_1\cdot u_1)^4}\right] + \big(\{q_1, m_1\}\leftrightarrow\{q_2, m_2\}\big). \tag{22}$$

We again apply IBP reductions to the expression above, which leads to one single master integral $G_{0,0,1,0,1,1,0}$ as defined in Eq. (11). Hence, the radiation reaction contribution reads

$$\mathrm{Re}\delta^{(2)}|_{\mathrm{rr}} = \frac{-2(\alpha q_1 q_2)^3\gamma^2}{3 m_1 m_2 \mathsf{b}^2(\gamma^2-1)}\left[\frac{q_1/m_1}{q_2/m_2} + \frac{q_2/m_2}{q_1/m_1}\right]. \tag{23}$$

VI. SCATTERING ANGLE

It is straightforward to compute the scattering angle in the center of mass frame via

$$\chi = -\frac{\partial\delta}{\partial J}, \tag{24}$$

where J denotes the total angular momentum and we have $J = \mathsf{pb}$ with

$$\mathsf{p} = \frac{m_1 m_2\sqrt{\gamma^2-1}}{E}, \qquad E = \sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \gamma}. \tag{25}$$

Plugging in Eqs. (7) and (8), we obtain the scattering angle at 1PM and 2PM:

$$\chi^{(0)} = \frac{2\alpha q_1 q_2 E\gamma}{m_1 m_2 \mathsf{b}(\gamma^2 - 1)}, \tag{26}$$

$$\chi^{(1)} = -\frac{(\alpha q_1 q_2)^2 \pi E(m_1 + m_2)}{2m_1^2 m_2^2 \mathsf{b}^2(\gamma^2 - 1)}. \tag{27}$$

The conservative part of the 3PM scattering angle follows from the first term of the comparable-mass (17) and the two probe-limit sectors (21):

$$\chi^{(2)}_{\rm con} = -\frac{4(\alpha q_1 q_2)^3 E(\gamma^4 - 3\gamma^2 + 3)}{3m_1^2 m_2^2 \mathsf{b}^3(\gamma^2 - 1)^3} + \frac{2(\alpha q_1 q_2)^3 E(m_1^2 + m_2^2)\gamma(2\gamma^2 - 3)}{3m_1^3 m_2^3 \mathsf{b}^3(\gamma^2 - 1)^3}. \tag{28}$$

Likewise, the radiative part at 3PM follows from the second line in Eq. (17) and the radiation reaction term (23):

$$\chi^{(2)}_{\rm rad} = \frac{4(\alpha q_1 q_2)^3 E\gamma^2}{m_1^2 m_2^2 \mathsf{b}^3}\left(\frac{\gamma}{(\gamma^2 - 1)^{5/2}} - \frac{\mathrm{arccosh}\gamma}{(\gamma^2 - 1)^3}\right) - \frac{4(\alpha q_1 q_2)^3 E\gamma^2}{3m_1^2 m_2^2(\gamma^2 - 1)^{3/2}}\left(\frac{q_1/m_1}{q_2/m_2} + \frac{q_2/m_2}{q_1/m_1}\right). \tag{29}$$

For Eqs. (28) and (29), we find agreement with known results in the literature [74,89].

VII. DISCUSSIONS

We have demonstrated a highly streamlined method for obtaining both the conservative and radiative contributions of the scattering angle in the WQFT formalism for scalar QED. The scattering angle is computed from a generating function that naturally arises from the WQFT path integral. This generating function is constructed to be purely classical by virtue of WQFT and coincides with the recently proposed "HEFT phase," although the precise connection between the two remains to be clarified. Its real part also agrees with the radial action, while the differences between their respective imaginary parts are yet to be investigated. These observations are expected to hold in other WQFTs, especially those in a gravitational background, which we leave to future work. It is also interesting to further clarify the relation between this generating function and the eikonal phase in the context of WQFT. Another immediate followup is to study higher PM orders. In particular, the probe limit involves only one diagram (up to symmetrization) at any order, for which the vertices are known on closed forms. In this limit, it is promising to obtain all-loop results from WQFT.

ACKNOWLEDGMENTS

T. W. is grateful to G. Mogull and J. Plefka for initial collaborations, many discussions, and useful comments on the manuscript. T. W. thanks R. Bonezzi, M. V. S. Saketh, C. Heissenberg, G. Chen, A. Brandhuber, and G. Travaglini for discussions and clarifications on their respective works. The research is supported by the Special Research Assistantship of the Chinese Academy of Sciences and Humboldt æUniversitt zu Berlin. It is also supported in part by the Key Research Program of the Chinese Academy of Sciences, Grant No. XDPB15, and by National Natural Science Foundation of China under Grants No. 11935013, No. 11947301, No. 12047502, and No. 12047503. T. W. is also supported by the Fellowship of China Postdoctoral Science Foundation (No. 2022M713228).

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