

Quantum scale invariance in gauge theories and applications to muon production

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We discuss quantum scale invariance in (scale invariant) gauge theories with both ultraviolet (UV) and infrared (IR) divergences. First, their Becchi-Rouet-Stora-Tyutin invariance is checked in two apparently unrelated approaches using a scale invariant regularization (SIR). These approaches are then shown to be equivalent. Second, for the Abelian case we discuss both UV and IR quantum corrections present in such theories. We present the Feynman rules in a form suitable for off-shell Green's function calculations, together with their one-loop renormalization. This information is then used for the muon production cross section at one-loop in a quantum scale invariant theory. Such a theory contains not only new UV poles but also IR poles. While the UV poles bring new quantum corrections (in the form of counterterms), finite or divergent, that we compute, it is shown that the IR poles do not bring new physics. The IR quantum corrections, both finite and divergent, cancel out similarly to the way the IR poles themselves cancel in the traditional approach to IR divergences (in the cross section, after summing over virtual and real corrections). Hence, the evanescent interactions induced by the scale-invariant analytical continuation of the SIR scheme do not affect IR physics, as illustrated at one-loop for the muon production ($e^+e^- \rightarrow \mu^+\mu^-$) cross section.

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I. INTRODUCTION

In this work we study quantum scale invariance and its implications for scale invariant gauge theories. An example is the Standard Model (SM) with a vanishing Higgs mass [1]. Consider such a theory in four dimensions; its quantum corrections are usually divergent in the ultraviolet (UV) and possibly in the infrared (IR) as well. To make the theory well defined, a regularization is needed and this introduces a dimensionful parameter, i.e., a mass scale, regardless of the regularization [dimensional regularization (DR), cutoff scheme, etc.]. Then the original theory in $d = 4$ that was classically scale invariant has this symmetry broken explicitly by the regularization, e.g., by the analytical continuation to d dimensions in the DR scheme, because the DR subtraction scale μ breaks the classical scale symmetry.

This explicit (anomalous) breaking of scale invariance can be avoided by a manifestly scale invariant regularization (SIR), see Refs. [2–13], leading to a quantum scale invariant theory. One motivation to study such a symmetry is that it can naturally preserve a classical hierarchy of scales generated by field vacuum expectation values (VEVs) [3] (for an example see Ref. [6]). More generally, in gauge theories of scale invariance that include Einstein gravity [14] quantum scale symmetry becomes necessary.

In the SIR scheme an additional scalar field σ is introduced by the analytical continuation¹ to enforce the classical scale symmetry in $d = 4 - 2\epsilon$ dimensions. Here σ is the Goldstone of (global) scale symmetry and will generate the subtraction scale: the “usual” DR subtraction scale is replaced by a dilaton dependent function $\mu \sim z\sigma$ (z is a dimensionless parameter) that enforces manifest scale invariance in d dimensions at the classical and the quantum level. When σ acquires a nonvanishing VEV $\langle\sigma\rangle$, the usual subtraction scale $\mu_0 \sim \langle\sigma\rangle$ is generated via spontaneous breaking of scale symmetry. Because of this there is no anomalous scale symmetry breaking anymore [2,4,6,7].²

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¹An exception is the case where σ is already present in the classical theory in $d = 4$ as a flat direction in the scalar sector.

²Nevertheless, the couplings still run with momentum [15], information captured by dimensionless z .

While the analytical continuation to d dimensions preserves scale invariance, note that the spectrum may then differ from that of the initial $d = 4$ theory by the additional (dynamical) field σ .

The above procedure has strong implications at the quantum level. After quantum corrections are computed in d dimensions, scale invariant counterterms are identified and renormalization is performed. After doing so, the limit $d \rightarrow 4$ is taken. Some quantum corrections depend on the dilaton and these are solely due to quantum scale invariance. Eventually, the theory can now be expanded about the (large) VEV of the dilaton $\langle \sigma \rangle$ which represents the scale of “new physics” (above which scale invariance is restored). For a large dilaton VEV, these corrections are actually expected to vanish. In this way, one has a quantum scale invariant theory that also recovers the usual quantum corrected theory (regularized by “standard” DR) in the decoupling limit of the dilaton i.e., at large $\langle \sigma \rangle$. The two theories have, however, different UV spectrum, UV symmetry and UV behavior.

Since the Lagrangian in $d = 4 - 2\epsilon$ dimensions now depends on $\mu^\epsilon(\sigma)$, where $\mu(\sigma)$ is a mass dimension-one function of σ , by expanding in powers of ϵ one obtains new “evanescent” polynomial interactions proportional to powers of ϵ . Then there are *new*, scale invariant quantum corrections that emerge from evanescent interactions $\propto \epsilon^n$ hitting a pole $1/\epsilon^m$ arising from loop integrals, in the cases when $n \leq m$. If $n = m$, new finite quantum corrections (which hence have the form of finite counterterms) are generated, whereas if $n < m$, new poles are generated leading to new scale invariant counterterms and quantum corrections, suppressed by powers of σ . Such calculations were already performed at the one-loop [3–8], the two-loop [9,10], and even the three-loop level [11,12] (for a review see Ref. [13]).

Quantum scale invariance was studied in the past especially in the scalar sector, whereas gauge theories were less studied except in [4,6,8] and this motivated this work. In particular, one aspect that was overlooked is the Becchi-Rouet-Stora-Tyutin (BRST) invariance of these theories following their analytical continuation to $d = 4 - 2\epsilon$ in a manifestly scale invariant way. Second, the scale invariant analytical continuation for gauge theories can be performed in two different ways, apparently unrelated, which are not known to be equivalent. We show that both approaches lead to the same d dimensional, BRST invariant Lagrangian and are thus equivalent.

The third and strongest motivation of this study is related to the fact that (scale invariant) gauge theories have not only UV but also IR divergences which were not yet discussed in the context of quantum scale invariance. The SIR scheme based on the DR scheme can actually handle the IR divergences as well. A question is whether quantum scale invariance leads to new physical effects related to IR divergences. We show that, while the UV poles bring new quantum corrections due to quantum scale invariance, the IR poles do not bring any new physics. The new infrared

quantum corrections, be they *finite or divergent*, cancel out, similar to the way the IR poles themselves cancel in the traditional approach to IR divergences (at the level of cross sections after summing over virtual and real corrections). Thus, evanescent interactions from the SIR do not impact IR physics. This cancellation is a strong consistency check for quantum scale invariant gauge theories. We illustrate this in detail with a one-loop calculation of the cross section of muon production, $e^-e^+ \rightarrow \mu^-\mu^+$; this is evaluated and analyzed in the framework of quantum electrodynamics extended by the dilaton which enforces quantum scale invariance.

The plan of the paper is as follows: Section II discusses the BRST invariance in SIR schemes for general gauge theories. Since gauge theories also have IR poles, we study these in an Abelian gauge theory, derive Feynman rules (Appendix A) and its renormalization (Appendix B). This information is then used in Sec. III that presents the muon production cross section and the effect of UV and IR corrections, with technical details in Appendixes C and D. Our conclusions are presented in the last section.

II. QUANTUM SCALE INVARIANCE IN GAUGE THEORIES

We check the BRST invariance in general gauge theories in two apparently unrelated approaches in SIR schemes and show the equivalence of their results. Further, gauge theories have not only UV poles, but also IR ones, the study of which is restricted to the Abelian case, in the SIR scheme. We thus consider an Abelian case (quantum electrodynamics extended by the dilaton), for which we give the Feynman diagrams and one-loop renormalization in a manifestly scale invariant form suitable for computer implementation.

A. Non-Abelian theories and BRST

Consider the Lagrangian of a scale invariant $SU(N)$ gauge theory in $d = 4$ given by

$$\mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{GF} + \mathcal{L}_{Ghost} \quad (1)$$

$$\mathcal{L}_{cl} = -\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + i\bar{\psi}_i (\delta_{ij} \not{\partial} - ig \not{A}^a T_{ij}^a) \psi_j \quad (2)$$

$$\mathcal{L}_{GF} = -B^a \not{\partial}^\mu G_\mu^a + \frac{\xi}{2} B^a B^a \quad (3)$$

$$\mathcal{L}_{Ghost} = \partial^\mu \bar{c}^a D_\mu^{ac} c^c, \quad (4)$$

where B^a , c^a and \bar{c}^a are Nakanishi-Lautrup fields, ghosts and antighosts, respectively.

Apparently, there are two different ways to implement a scale invariant regularization:

- (a) *Analytical continuation to $d = 4 - 2\epsilon$ by rescaling the gauge fields.*—First, rescale the gauge fields in $d = 4$ by a factor of the gauge coupling,

$$G_\mu^a \rightarrow \hat{G}_\mu^a = gG_\mu^a. \quad (5)$$

Since the gauge parameter β^a also needs to be rescaled, the ghost c^a is analogously rescaled, as $\beta^a(x) = \theta c^a(x)$, for some Grassmann number θ . Further, it is convenient to rescale the antighost \bar{c}^a as well in order to obtain a ghost term similar to (4). Thus,

$$c^a \rightarrow \hat{c}^a = gc^a, \quad \bar{c}^a \rightarrow \hat{\bar{c}}^a = \frac{1}{g}\bar{c}^a. \quad (6)$$

This leads to

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4g^2} \hat{F}_{\mu\nu}^a \hat{F}^{a,\mu\nu} + i\bar{\psi} \hat{\not{D}} \psi - \frac{1}{g} B^a \partial^\mu \hat{G}_\mu^a \\ & + \frac{\xi}{2} B^a B^a + \partial^\mu \hat{\bar{c}}^a \hat{D}_\mu^{ac} \hat{c}^c, \end{aligned} \quad (7)$$

where the rescaled field strength tensor and covariant derivative do not explicitly depend on the gauge coupling,

$$\hat{F}_{\mu\nu}^a = \partial_\mu \hat{G}_\nu^a - \partial_\nu \hat{G}_\mu^a + f^{abc} \hat{G}_\mu^b \hat{G}_\nu^c \quad (8)$$

$$\hat{D}_\mu \psi_i = (\delta_{ij} \partial_\mu - i \hat{G}_\mu^a T_{ij}^a) \psi_j \quad (9)$$

$$\hat{D}_\mu^{ac} \hat{c}^c = (\delta^{ac} \partial_\mu + f^{abc} \hat{G}_\mu^b) \hat{c}^c. \quad (10)$$

Lagrangian (7) can now be analytically continued to $d = 4 - 2\epsilon$ in a scale invariant way. This is achieved by the addition of a *dynamical* scalar field σ to the spectrum of the initial classical theory in $d = 4$, playing the role of a dilaton. When σ acquires a VEV, the subtraction scale is generated dynamically. Hence, the analytical continuation to d dimensions modifies the spectrum of the initial classical theory in $d = 4$ by an extra degree of freedom, but scale invariance is maintained in d dimensions (and at the quantum level). This is the ‘‘cost’’ of implementing quantum scale invariance. The Lagrangian becomes

$$\begin{aligned} \mathcal{L}^{(d)} = & -\frac{1}{4g^2} \mu^{-2\epsilon}(\sigma) \hat{F}_{\mu\nu}^a \hat{F}^{a,\mu\nu} + i\bar{\psi} \hat{\not{D}} \psi \\ & + \frac{1}{2} (\partial_\mu \sigma) (\partial^\mu \sigma) - \frac{1}{g} \mu^{-\epsilon}(\sigma) B^a \partial^\mu \hat{G}_\mu^a \\ & + \frac{\xi}{2} B^a B^a + \partial^\mu \hat{\bar{c}}^a \hat{D}_\mu^{ac} \hat{c}^c. \end{aligned} \quad (11)$$

The mass dimensions of the fields and couplings are

$$\begin{aligned} [\hat{G}_\mu^a] = 1, \quad [\psi] = \frac{3}{2} - \epsilon, \quad [\sigma] = 1 - \epsilon, \quad [g] = 0, \\ [\hat{c}^a] = 0, \quad [\hat{\bar{c}}^a] = 2 - 2\epsilon, \quad [B^a] = 2 - \epsilon, \quad [\xi] = 0 \end{aligned} \quad (12)$$

and $[\hat{\beta}^a] = 0$, $[\theta] = 0$. Further, integrating out B^a in (11), then

$$\begin{aligned} \mathcal{L}^{(d)} = & -\frac{1}{4g^2} \mu^{-2\epsilon}(\sigma) \hat{F}_{\mu\nu}^a \hat{F}^{a,\mu\nu} + i\bar{\psi} \hat{\not{D}} \psi + \frac{1}{2} (\partial_\mu \sigma) (\partial^\mu \sigma) \\ & - \frac{1}{2\xi g^2} \mu^{-2\epsilon}(\sigma) (\partial^\mu \hat{G}_\mu^a)^2 + \partial^\mu \hat{\bar{c}}^a \hat{D}_\mu^{ac} \hat{c}^c. \end{aligned} \quad (13)$$

In general, the BRST transformations are given by

$$\begin{aligned} \psi_i \mapsto \psi_i + \delta\psi_i, \quad G_\mu^a \mapsto G_\mu^a + \delta G_\mu^a, \quad c^a \mapsto c^a + \delta c^a, \\ \bar{\psi}_i \mapsto \bar{\psi}_i + \delta\bar{\psi}_i, \quad B^a \mapsto B^a + \delta B^a, \quad \bar{c}^a \mapsto \bar{c}^a + \delta\bar{c}^a, \\ \sigma \mapsto \sigma, \end{aligned} \quad (14)$$

where σ transforms trivially under BRST symmetry. One finds the following d dimensional BRST transformations:

$$\begin{aligned} \delta\psi_i &= \theta Q\psi_i = i\theta \hat{c}^a T_{ij}^a \psi_j, \\ \delta\bar{\psi}_i &= \theta Q\bar{\psi}_i = -i\theta \hat{c}^a \bar{\psi}_j T_{ji}^a, \\ \delta\hat{G}_\mu^a &= \theta Q\hat{G}_\mu^a = \theta \hat{D}_\mu^{ac} \hat{c}^c = \theta \partial_\mu \hat{c}^a + \theta f^{abc} \hat{G}_\mu^b \hat{c}^c \\ \delta\hat{c}^a &= \theta Q\hat{c}^a = -\frac{1}{2} \theta f^{abc} \hat{c}^b \hat{c}^c \\ \delta\hat{\bar{c}}^a &= \theta Q\hat{\bar{c}}^a = -\frac{\theta}{g} \mu^{-\epsilon}(\sigma) B^a = -\frac{\theta}{\xi g^2} \mu^{-2\epsilon}(\sigma) \partial^\mu \hat{G}_\mu^a \\ \delta B^a &= \theta Q B^a = 0, \end{aligned} \quad (15)$$

where the first three transformations are given by the infinitesimal gauge transformations, as usual, using $\hat{\beta}^a(x) = \theta \hat{c}^a(x)$.

It is straightforward to show that the BRST operator Q that generates the BRST transformations in (15) is nilpotent as it should be, i.e., $Q^2 = 0$. Furthermore, the d dimensional Lagrangian (11) is indeed BRST invariant, i.e., invariant under transformations (15), which can be shown by direct calculation.

Although a detailed all-orders investigation is beyond the scope of the present work, let us briefly comment on the role of the BRST invariance in the case of renormalization of the considered gauge theories, which are nonrenormalizable due to quantum scale invariance. We expect BRST invariance to be crucial for the consistency of the quantized theory, just as in ordinary gauge theories, which

are strictly renormalizable and thus involve only a finite number of operators in the Lagrangian.

At higher orders, we expect that BRST invariance will control the structure of possible UV divergences, and required counterterms are expected to be BRST invariant as well, even if there will be infinitely many of these. To understand this, note that the situation is similar to the general analysis in the review [16], although our particular case with a quantum scale symmetry is not covered in this reference. However, note that in our theory the dilaton transforms trivially under BRST transformations, i.e., behaves like a BRST singlet; quantum scale invariance imposes additional constraints on the theory and on the structure of its UV-counterterms. In the broken phase of quantum scale symmetry and the decoupling limit of the dilaton i.e., $\langle\sigma\rangle \rightarrow \infty$, the higher dimensional operators vanish (since they are suppressed by the VEV of the dilaton) and one recovers a renormalizable theory where traditional results and BRST constraints of gauge theories apply. These constraints extend however to the symmetric phase of the theory since the quantum scale symmetry is broken only spontaneously. Ultimately, we expect an all-orders Slavnov-Taylor identity to hold, establishing physical properties such as unitarity and gauge-parameter independence of the physical S matrix, cf. [17,18]. The unitarity of the physical S matrix was also verified [19] in more general theories where scale symmetry is *gauged* (e.g., [14]) and quantum scale invariance is thus automatic.

- (b) *A different approach to $d = 4 - 2\epsilon$.*—There is a second approach to a scale invariant regularization

of gauge theories that is somewhat “geometrical.” First, by analytical continuation to d dimensions, the covariant derivative changes:

$$D_\mu \rightarrow \tilde{D}_\mu = \partial_\mu - ig\mu^\epsilon(\sigma)G_\mu^a T^a. \quad (16)$$

The field strength, regarded as the curvature tensor of the internal coordinates, is then

$$\tilde{F}_{\mu\nu} = \tilde{F}_{\mu\nu}^a T^a = \frac{i}{g\mu^\epsilon(\sigma)} [\tilde{D}_\mu, \tilde{D}_\nu]. \quad (17)$$

Evaluating the commutator leads to

$$\begin{aligned} \tilde{F}_{\mu\nu}^a &= \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g\mu^\epsilon(\sigma) f^{abc} G_\mu^b G_\nu^c \\ &+ \epsilon\mu^{-1}(\sigma) \frac{\partial\mu}{\partial\sigma} (\partial_\mu \sigma G_\nu^a - \partial_\nu \sigma G_\mu^a), \end{aligned} \quad (18)$$

so \tilde{F} has received an evanescent correction ($\propto \epsilon$). The Lagrangian $\mathcal{L}_{cl}^{(d)}$ in $d = 4 - 2\epsilon$ is then

$$\begin{aligned} \mathcal{L}_{cl}^{(d)} &= -\frac{1}{4} \tilde{F}_{\mu\nu}^a \tilde{F}^{a,\mu\nu} + i\bar{\psi}_i (\delta_{ij} \not{\partial} - ig\mu^\epsilon(\sigma) \mathcal{G}^a T_{ij}^a) \psi_j \\ &+ \frac{1}{2} (\partial_\mu \sigma) (\partial^\mu \sigma). \end{aligned} \quad (19)$$

Similarly, the combination $\mu^\epsilon(\sigma)G_\mu^a$ is the one which determines the gauge fixing and ghost Lagrangian as well as the BRST transformations.³ The gauge fixing and ghost Lagrangian is

$$\begin{aligned} \mathcal{L}_{GF}^{(d)} + \mathcal{L}_{Ghost}^{(d)} &= \mathcal{Q} \left[-\bar{c}^a \left(\frac{\xi}{2} B^a - \partial^\mu G_\mu^a - \epsilon\mu^{-1}(\sigma) \frac{\partial\mu}{\partial\sigma} \partial^\mu \sigma G_\mu^a \right) \right] \\ &= \frac{\xi}{2} B^a B^a - B^a \partial^\mu G_\mu^a + \partial^\mu \bar{c}^a \tilde{D}_\mu^{ac} c^c - \epsilon\mu^{-1}(\sigma) \frac{\partial\mu}{\partial\sigma} \partial^\mu \sigma [B^a G_\mu^a + \bar{c}^a \tilde{D}_\mu^{ac} c^c - \partial_\mu \bar{c}^a c^a] \\ &\quad - \epsilon^2 \mu^{-2}(\sigma) \left(\frac{\partial\mu}{\partial\sigma} \right)^2 \partial^\mu \sigma \partial_\mu \sigma \bar{c}^a c^a, \end{aligned} \quad (20)$$

where

$$\tilde{D}_\mu^{ac} = \delta^{ac} \partial_\mu + g\mu^\epsilon(\sigma) f^{abc} G_\mu^b. \quad (21)$$

It can be seen that the gauge fixing and the ghost Lagrangian also acquire purely evanescent corrections

³The BRST transformations also involve the combination $\mu^\epsilon(\sigma)c^a$ in an essential way; see later, Eq. (23).

($\propto \epsilon$), analogously to the kinetic term of the gauge field, Eq. (18). Consequently, the d dimensional Lagrangian of the considered $SU(N)$ gauge theory is given by

$$\mathcal{L}^{(d)} = \mathcal{L}_{cl}^{(d)} + \mathcal{L}_{GF}^{(d)} + \mathcal{L}_{Ghost}^{(d)}. \quad (22)$$

The d dimensional BRST transformations in this approach are found to be

$$\begin{aligned}
 \delta\psi_i &= \theta Q\psi_i = i\theta g\mu^\epsilon(\sigma)c^a T_{ij}^a \psi_j \\
 \delta\bar{\psi}_i &= \theta Q\bar{\psi}_i = -i\theta g\mu^\epsilon(\sigma)c^a \bar{\psi}_j T_{ji}^a \\
 \delta G_\mu^a &= \theta Q G_\mu^a = \theta \tilde{D}_\mu^{ac} c^c + \epsilon\theta\mu^{-1}(\sigma) \frac{\partial\mu}{\partial\sigma} \partial_\mu \sigma c^a \\
 &= \theta \partial_\mu c^a + \theta g\mu^\epsilon(\sigma) f^{abc} G_\mu^b c^c + \epsilon\theta\mu^{-1}(\sigma) \frac{\partial\mu}{\partial\sigma} \partial_\mu \sigma c^a \\
 \delta c^a &= \theta Q c^a = -\frac{1}{2} \theta g\mu^\epsilon(\sigma) f^{abc} c^b c^c \\
 \delta \bar{c}^a &= \theta Q \bar{c}^a = -\theta B^a \\
 \delta B^a &= \theta Q B^a = 0,
 \end{aligned} \tag{23}$$

where the first three BRST transformations in (23) are again given by the gauge transformation, as usual, using $\beta^a(x) = \theta c^a(x)$. The new, evanescent correction to the gauge field BRST transformation δG_μ^a has its origin in the dilaton dependent function $\mu(\sigma)$. In particular, this correction originates from $\partial_\mu U$, with $U = \exp(ig\mu^\epsilon(\sigma)\beta^a T^a)$, in the derivation of δG_μ^a . Again, the BRST operator Q , generating the BRST transformations in (23), is nilpotent, $Q^2 = 0$. Moreover, while the gauge fixing and ghost Lagrangian may be written as a Q -exact term, as in (20), the BRST invariance of $\mathcal{L}_{cl}^{(d)}$ in (19) can again be shown by explicit calculation. Thus, $\mathcal{L}^{(d)}$ in (22) is BRST invariant under (23).

The mass dimensions of the fields and parameters in (22) are

$$\begin{aligned}
 [G_\mu^a] &= 1 - \epsilon, & [\psi] &= \frac{3}{2} - \epsilon, & [\sigma] &= 1 - \epsilon, & [g] &= 0, \\
 [c^a] &= -\epsilon, & [\bar{c}^a] &= 2 - \epsilon, & [B^a] &= 2 - \epsilon, & [\xi] &= 0
 \end{aligned} \tag{24}$$

and $[\beta^a] = -\epsilon$, $[\theta] = 0$. Compared to the previous approach, Eq. (12), notice the different dimensions of ghost/antighost, gauge field and $\hat{\beta}^a$.

B. Equivalence of the two approaches

The two approaches of (a) and (b) and their corresponding Lagrangians, Eqs. (11) and (22), must be equivalent. To see this, start with (11), then the gauge coupling g and the subtraction function $\mu^\epsilon(\sigma)$ can be factored out from the fields \hat{G}_μ^a , \hat{c}^a and $\hat{\bar{c}}^a$ leading to Lagrangian (22). Conversely, starting from Eq. (22), g and $\mu^\epsilon(\sigma)$ can be absorbed into G_μ^a , c^a and \bar{c}^a . Then, additional terms coming from commuting derivatives and the subtraction function must be taken into account by subtracting them, leading to Eq. (11). Moreover, the same equivalence holds true for the BRST transformations in (15) and (23).

For practical calculations, Lagrangian (11) takes a more convenient form than (22) as it avoids the evanescent corrections to the gauge kinetic term and to the gauge

fixing and ghost Lagrangians, Eqs. (18) and (20), respectively.

Further, in approach (a), *after* the theory was analytically continued to $d = 4 - 2\epsilon$ a second field redefinition can be applied where the dimensionless gauge coupling g is “extracted” from the gauge field by replacing⁴

$$\hat{G}_\mu^a = g\mu^\epsilon(\sigma) G_\mu^a = g\bar{G}_\mu^a. \tag{25}$$

The mass dimension of the gauge field remains $[\bar{G}_\mu^a] = 1$. However, the d dimensional Lagrangian in case (a) then looks more similar to (1) with respect to the gauge couplings, which is useful in the presence of mixing between the gauge fields as in the SM.

C. Abelian theories: QED + dilaton

In (scale invariant) gauge theories, one encounters not only UV but also IR poles which remains true in our SIR scheme. We consider here the simpler case of Abelian gauge theories⁵ to illustrate how both UV and IR poles are handled in this scheme in applications (Sec. III). Consider then quantum electrodynamics in $d = 4 - 2\epsilon$ dimensions, analytically continued in a scale invariant way. This means the action is “upgraded” to include the additional dilaton field, as already discussed. In this Abelian case, the Faddeev-Popov ghosts completely decouple and are not shown. Using approach (a) discussed earlier, the Lagrangian becomes⁶

$$\begin{aligned}
 \mathcal{L}^{(d)} &= -\frac{1}{4} \mu^{-2\epsilon}(\sigma) F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}_f (\not{\partial} - ieQ_f \not{A}) \psi_f \\
 &\quad + \frac{1}{2} (\partial_\mu \sigma)(\partial^\mu \sigma) - y_f \mu^\epsilon(\sigma) \sigma \bar{\psi}_f \psi_f \\
 &\quad - \frac{1}{2\xi} \mu^{-2\epsilon}(\sigma) (\partial^\mu A_\mu)^2.
 \end{aligned} \tag{26}$$

The Lagrangian is scale invariant in d dimensions. Since scale invariance is broken spontaneously, this does not affect the ultraviolet behavior of the theory. In principle, one can work either in the symmetric phase or in the spontaneously broken phase obtained by an expansion about the VEV of the dilaton but keeping all terms in such an expansion. Then the counterterms and thus the quantum corrections are not affected.

⁴In the next section we use approach (a) in terms of \bar{G}_μ^a but the “overbar” is not displayed, for simplicity.

⁵The case of IR divergences in the non-Abelian case is significantly more difficult.

⁶Note that we include an additional tree-level dilaton Yukawa interaction term. This term is gauge and scale invariant, but it was not considered in the discussion above in Sec. II A. Furthermore, one can in principle also have a $\lambda\sigma^4$ coupling, which is, however, not considered here.

For practical purposes and for Feynman diagrams derivation, $\mathcal{L}^{(d)}$ is expanded first in powers of ϵ with every order in ϵ (and loop order) being manifestly scale invariant. Subsequently, one can express the dilaton in terms of its fluctuations about its nonzero VEV $\langle\sigma\rangle \equiv w$, $\sigma = w + \mathfrak{D}$ and expand in powers of $\eta = \mathfrak{D}/w$ [11]; scale invariance is maintained by including all terms of this second expansion. Accordingly, for the function $\mu(\sigma) = z\sigma^{\frac{1}{1-\epsilon}}$ in $\mathcal{L}^{(d)}$, one has

$$\mu^{ke}(\sigma) = \mu_0^{ke} \left[1 + \epsilon k \left(\eta - \frac{1}{2}\eta^2 + \frac{1}{3}\eta^3 - \frac{1}{4}\eta^4 + \mathcal{O}(\eta^5) \right) + \epsilon^2 k \left(\eta - \frac{1-k}{2}\eta^2 + \frac{2-3k}{6}\eta^3 - \frac{6-11k}{24}\eta^4 + \mathcal{O}(\eta^5) \right) + \mathcal{O}(\epsilon^3) \right], \quad (27)$$

for integer k . With this, the Lagrangian becomes⁷

$$\begin{aligned} \mathcal{L}^{(d)} = & -\frac{1}{4}\mu_0^{-2\epsilon} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}_f (\not{\partial} - ieQ_f \not{A}) \psi_f + \frac{1}{2}(\partial_\mu \mathfrak{D})(\partial^\mu \mathfrak{D}) - \mu_0^\epsilon y_f w \bar{\psi}_f \psi_f - \mu_0^\epsilon [1 + \epsilon + \epsilon^2 + \mathcal{O}(\epsilon^3)] y_f \mathfrak{D} \bar{\psi}_f \psi_f \\ & - \frac{1}{2\xi} \mu_0^{-2\epsilon} (\partial^\mu A_\mu)^2 - \mu_0^\epsilon [\epsilon(1 + 2\epsilon) + \mathcal{O}(\epsilon^3)] \frac{y_f}{2w} \mathfrak{D}^2 \bar{\psi}_f \psi_f + \mu_0^\epsilon [\epsilon(1 + \epsilon) + \mathcal{O}(\epsilon^3)] \frac{y_f}{6w^2} \mathfrak{D}^3 \bar{\psi}_f \psi_f \\ & + \mu_0^{-2\epsilon} [\epsilon(1 + \epsilon) + \mathcal{O}(\epsilon^3)] \frac{\mathfrak{D}}{w} \left[\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{\xi} (\partial^\mu A_\mu)^2 \right] + \dots \end{aligned} \quad (28)$$

While every loop order (encoded in powers of ϵ) keeps manifest scale invariance, any truncation of the subsequent expansion in powers of η breaks this symmetry. Nevertheless, the Lagrangian as shown in (28) is useful for deriving the Feynman rules that formally keep all terms in the expansion about w . These are presented in Appendix A and used in Sec. III.

For completeness, the renormalization of (26) at one-loop using (28) is shown in Appendix B. Notice that, unlike in previous literature, σ undergoes a wave function renormalization already at one-loop, due to Yukawa interactions.

III. APPLICATION: MUON PRODUCTION AT ONE-LOOP

In this section we use the formalism of the previous section for the Abelian case and discuss the effect of both UV and IR poles in the SIR scheme. We illustrate this for the muon production at the one-loop level based on Lagrangian (26), for three fermion flavors, i.e., $f \in \{e^-, \mu^-, \tau^-\}$, $N_f = 3$ and $Q_f = -1$.⁸

A. General considerations

The scattering process $e^- e^+ \rightarrow \mu^- \mu^+$ is computed in the $\overline{\text{MS}}$ scheme and Feynman gauge $\xi = 1$. We work in the approximation $m_f^2 \ll s$, where s is the center-of-mass

⁷Note that the counterterm Lagrangian (see Appendix B) must be expanded accordingly.

⁸In principle, the corrections that we find can be used to set bounds on the scale of new physics represented by $\langle\sigma\rangle$, provided one considered a full scale invariant SM (see Refs. [6,10] for the Lagrangian and V_{eff}).

energy, i.e., the fermions are essentially massless.⁹ This approximation displays a more interesting IR-divergence structure than the massive case as it contains not only a simple pole but also a second order pole in ϵ_{IR} . Hence, this leads to both new finite and also new divergent quantum corrections induced by evanescent interactions; the latter emerge when a term $\sim\epsilon$ “meets” a second order pole in ϵ_{IR} .

For simplicity, only the one-loop muon vertex corrections contributing to the above scattering process are considered, and thus only final state real emission needs to be taken into account in order to cancel the IR divergences at the level of cross sections. Despite this simplification, the result admits nonetheless the same general structure as the full one-loop result, meaning that it contains a simple and a second order pole in ϵ_{IR} as well as new finite and divergent quantum corrections; see below. Consequently, for the present conceptual study, it is sufficient to restrict oneself to one-loop muon vertex corrections. All necessary Feynman diagrams, contributing to virtual and real muon corrections up to the one-loop level, are found in Appendix C.

B. The cross section at one-loop

The scattering amplitude and the cross section were calculated in the above setup by using standard methods and *Mathematica* [20]. In particular, all Feynman diagrams have been generated using FeynArts [21], with FeynArts model files generated by FeynRules [22,23]. The generated

⁹Practically, this scenario is realized by setting $m_f = 0$ in the free Lagrangian, i.e., in (A1), but keeping y_f and w nonzero in the interaction Lagrangian, i.e., in (A2) and (A3), despite $m_f = \mu_0^\epsilon y_f w$.

Feynman diagrams and their amplitudes have been computed using FeynCalc [24–26] and Package-X [27], connected to FeynCalc using FeynHelpers [28].

The results for the considered one-loop contributions to the cross section as well as for the $2 \rightarrow 3$ cross sections have the general structure

$$\sigma_k = \sigma_k^{1/\epsilon_{\text{IR}}^2} + \sigma_k^{1/\epsilon_{\text{IR}}} + \Delta_{\text{IR}}\sigma_k^{1/\epsilon_{\text{IR}}} + \sigma_k^{\text{fin}} + \Delta_{\text{UV}}\sigma_k^{\text{fin}} + \Delta_{\text{IR}}\sigma_k^{\text{fin}} + \mathcal{O}(\epsilon), \quad (29)$$

where Δ_{UV} and Δ_{IR} denote new quantum corrections arising from evanescent interactions canceling UV and IR divergences,¹⁰ respectively.

The tree-level cross section (in four dimensions) is

$$\sigma_{\text{tree}}(ee \rightarrow \mu\mu) = \frac{e^4}{12\pi s} + \frac{y_e^2 y_\mu^2}{16\pi s}. \quad (30)$$

The one-loop contribution to the cross section, containing only muon vertex corrections, is given in (D2)–(D7) in the $\overline{\text{MS}}$ scheme and having used decomposition (29).

The tree-level cross sections of muon production with real photon and real dilaton emission, i.e., for $e^-e^+ \rightarrow \mu^- \mu^+ \gamma$ and $e^-e^+ \rightarrow \mu^- \mu^+ \mathfrak{D}$, respectively, arranged as in (29), are provided in Appendix D as well.

The total cross section, considering virtual and real corrections only to the muon vertex, has a form given in (D20). The corresponding one-loop contribution to this total cross section in the $\overline{\text{MS}}$ scheme may be written as

$$\sigma_{\text{total,1L}\mu}(ee \rightarrow \mu\mu) = \sigma_{\text{total,1L}\mu}^{\text{old}}(ee \rightarrow \mu\mu) + \Delta_{\text{UV}}\sigma_{\text{total,1L}\mu}(ee \rightarrow \mu\mu) \quad (31)$$

with the standard¹¹ one-loop contribution

$$\begin{aligned} & \sigma_{\text{total,1L}\mu}^{\text{old}}(ee \rightarrow \mu\mu) \\ &= \frac{1}{512\pi^3 s} (8e^6 - 4e^4 y_\mu^2 + 34e^2 y_e^2 y_\mu^2 - 17y_e^2 y_\mu^4) \\ & \quad - \frac{1}{256\pi^3 s} [6e^2 y_e^2 y_\mu^2 - 3y_e^2 y_\mu^4] \log\left(\frac{s}{\mu_0^2}\right) \end{aligned} \quad (32)$$

and the new, finite quantum correction that emerged from UV divergences,

$$\Delta_{\text{UV}}\sigma_{\text{total,1L}\mu}(ee \rightarrow \mu\mu) = -\frac{1}{128\pi^3 s} y_e^2 y_\mu^2 (y_e^2 + 4y_\mu^2 + y_\tau^2), \quad (33)$$

¹⁰The IR divergences are regularized dimensionally, not with a small mass as IR regulator.

¹¹This is the usual result found without imposing scale invariance at quantum level (QED with dilaton Yukawa couplings but employing DR instead of SIR).

with s being the center-of-mass energy. Thus, the considered one-loop contribution to cross section (31), that contains virtual and real corrections, is UV and IR finite, as expected. Therefore, the IR poles do cancel in the SIR scheme. We can summarize the main results of this section as follows:

- (i) The one-loop contribution (31) contains the regular contribution (32) that would also be obtained in the usual massless QED with an additional scalar field coupling to the fermions, and an additional contribution (33) which is a new quantum correction (from the UV sector) due to evanescent interactions.
- (ii) The one-loop contribution (31) to the cross section of $e^-e^+ \rightarrow \mu^- \mu^+$ scattering is IR finite after summing over the contributing final state real emissions. Thus, all IR divergences, even the new ones, cf. (D4), emerging as a result of evanescent interactions (due to quantum scale invariance), cancel after summing over the virtual and real corrections to the cross section. This is an important consistency check of theories with quantum scale invariance.
- (iii) All new, *finite* quantum corrections to the cross section, arising from evanescent interactions ($\propto \epsilon^n$) that cancel IR poles, see Eq. (D7), cancel as well. Therefore, quantum corrections due to the UV poles given in Eq. (D6), are the only new finite contributions to the cross section that remain, as shown in (33). The essential reason for this cancellation is that the standard IR poles cancel between real and virtual diagrams of the same orders in all coupling constants. The new IR quantum corrections then arise from the same evanescent interactions $\propto \epsilon^n$ in both real and virtual contributions hitting $1/e^m$ IR poles; hence they cancel simultaneously with the IR poles.

Results (ii) and (iii) are the main results of this section.
- (iv) The remaining new, finite quantum correction to the cross section $\Delta_{\text{UV}}\sigma_{\text{total,1L}\mu}$, given in (33), is an effect from the Yukawa sector. Hence, the gauge sector does not give rise to new quantum corrections at one-loop, but these are expected at the two-loop level.
- (v) Note that the new quantum correction (33) is suppressed by Yukawa couplings to power 6, i.e., it is suppressed by w^6 , where w is the dilaton VEV (using $y_f = m_f/w$).
- (vi) Since $\mu_0 = \mu(\langle\sigma\rangle) \equiv \mu(w) = zw^{1-\epsilon}$ and supposing that the VEV of the dilaton $\langle\sigma\rangle \equiv w$ scales in the same way as \sqrt{s} , it can be seen that the tree-level cross section (30) and the one-loop contribution (31) scale as $\sim 1/s$, as expected on dimensional arguments for the cross section.

IV. CONCLUSIONS

We studied scale invariance at the quantum level in (scale invariant) gauge theories that have both UV and IR divergences. The interest in quantum (global) scale invariant theories is that they can naturally preserve an initial, classical hierarchy of scales generated by field VEVs; this hierarchy could be arranged e.g., by one initial classical fine-tuning of dimensionless coupling constants. More generally, quantum scale symmetry becomes necessary in gauge theories of scale invariance that include Einstein gravity. Quantum scale invariance is achieved by implementing a scale invariant regularization and renormalization, in which the traditional subtraction scale is generated (dynamically) by the VEV of the dilaton, the Goldstone of the (global) scale symmetry. The VEV of this field can be regarded, in a sense, as the scale of new physics. In the limit of a large dilaton VEV, when this field decouples, the quantum scale invariant theory recovers the traditional quantum theory obtained without respecting scale invariance at the quantum level (anomalous breaking). Above this scale, however, the two quantum theories can differ in their UV completion, UV spectrum and quantum symmetry.

We first checked the BRST invariance of quantum scale invariant theories. This can be done in two apparently different formulations of scale invariant regularizations (SIR); we showed that the two approaches are equivalent and that the BRST symmetry is maintained.

Further, gauge theories have not only UV poles but also IR ones and one should check their fate in quantum scale invariant theories using the SIR scheme. We illustrated their analysis in an Abelian gauge theory (quantum electrodynamics extended by a dilaton field). While the UV poles do bring new quantum corrections (counterterms), finite or divergent, beyond those of the traditional approach, and which we computed, it was shown that the IR poles do not bring any new physics. The infrared quantum corrections, *both finite and divergent*, cancel out similar to the way the IR poles themselves cancel in the traditional approach. The cancellation is at the cross section level, after summing over the virtual and real corrections. Therefore, the evanescent interactions due to analytical continuation to d dimensions (SIR approach) do not affect the IR physics. This was

illustrated for the muon production cross section $e^-e^+ \rightarrow \mu^-\mu^+$ for the aforementioned Abelian theory. This result is a strong consistency check of quantum scale invariance in gauge theories.

ACKNOWLEDGMENTS

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APPENDIX A: FEYNMAN RULES

The Feynman rules for Lagrangian (28) are shown below, for the propagators:

$$\begin{aligned}
 \bullet \text{---} \frac{\mathcal{D}}{p} \text{---} \bullet &= \frac{i}{p^2} \\
 \bullet \text{---} \frac{\psi_f}{p} \text{---} \bullet &= \frac{i(\not{p} + m_f)}{p^2 - m_f^2} \\
 \bullet^\mu \text{---} \frac{A}{p} \text{---} \bullet^\nu &= \mu_0^{2\epsilon} \frac{-i}{p^2} \left(\eta^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right),
 \end{aligned} \tag{A1}$$

vertices:

$$\begin{aligned}
 A_\mu \text{---} \text{---} \begin{array}{l} \nearrow \psi_f \\ \searrow \bar{\psi}_f \end{array} &= i e Q_f \gamma^\mu \\
 \mathcal{D} \text{---} \text{---} \begin{array}{l} \nearrow \psi_f \\ \searrow \bar{\psi}_f \end{array} &= -i \mu_0^\epsilon \left[1 + \epsilon + \epsilon^2 + \mathcal{O}(\epsilon^3) \right] y_f
 \end{aligned} \tag{A2}$$

and evanescent vertices:

$$\begin{aligned}
 & \text{Diagram 1} = -i \mu_0^\epsilon \left[\epsilon(1 + 2\epsilon) + \mathcal{O}(\epsilon^3) \right] \frac{y_f}{w} \\
 & \text{Diagram 2} = i \mu_0^\epsilon \left[\epsilon(1 + \epsilon) + \mathcal{O}(\epsilon^3) \right] \frac{y_f}{w^2} \\
 & \text{Diagram 3} = i \mu_0^{-2\epsilon} \left[2\epsilon(1 + \epsilon) + \mathcal{O}(\epsilon^3) \right] \frac{1}{w} \left[(p_1 \cdot p_2) \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) p_1^\mu p_2^\nu \right]
 \end{aligned} \tag{A3}$$

APPENDIX B: RENORMALIZATION OF THE ABELIAN CASE

The quantum scale invariant Abelian case (QED + dilaton) of (26) is renormalized as follows:

$$\begin{aligned}
 \mathcal{L} &\rightarrow \mathcal{L}_0^{(d)} = \mathcal{L}_{\text{ren}}^{(d)} + \mathcal{L}_{\text{ct}}^{(d)} \\
 A &\rightarrow A_0 = \sqrt{Z_A} A \\
 \psi_f &\rightarrow \psi_{f,0} = \sqrt{Z_{\psi_f}} \psi_f \\
 \sigma &\rightarrow \sigma_0 = \sqrt{Z_\sigma} \sigma \\
 e &\rightarrow e_0 = \mu^\epsilon (\sqrt{Z_\sigma} \sigma) Z_e e \\
 y_f &\rightarrow y_{f,0} = \mu^\epsilon (\sqrt{Z_\sigma} \sigma) Z_{y_f} y_f \\
 \lambda &\rightarrow \lambda_0 = \mu^{2\epsilon} (\sqrt{Z_\sigma} \sigma) Z_\lambda \lambda \\
 \xi &\rightarrow \xi_0 = Z_\xi \xi,
 \end{aligned} \tag{B1}$$

where $Z_\xi = Z_A$ due to the Ward identity. Despite setting $\lambda \equiv 0$ at tree level, as done in Eq. (26), the associated counterterm $\delta\lambda$ must be taken into account for the complete one-loop renormalization of the considered model.¹²

The one-loop counterterms¹³ of (26) in the $\overline{\text{MS}}$ scheme are generated by the poles below:

$$\begin{aligned}
 \delta Z_{\psi_f} &= -\frac{1}{16\pi^2} \left[e^2 + \frac{y_f}{2} \right] \frac{1}{\epsilon}, & \delta Z_{y_f} &= \frac{1}{16\pi^2} \left[\frac{3}{2} y_f^2 + \sum_l y_l^2 - 3e^2 \right] \frac{1}{\epsilon}, \\
 \delta Z_A &= -\frac{1}{16\pi^2} \frac{4N_f e^2}{3} \frac{1}{\epsilon}, & \delta Z_e &= -\frac{1}{2} \delta Z_A = \frac{1}{16\pi^2} \frac{2N_f e^2}{3} \frac{1}{\epsilon}, \\
 \delta Z_\sigma &= -\frac{1}{16\pi^2} 2 \sum_l y_l^2 \frac{1}{\epsilon}, & \delta \lambda &= -\frac{1}{16\pi^2} 24 \sum_l y_l^4 \frac{1}{\epsilon},
 \end{aligned} \tag{B2}$$

where $f, l \in \{e^-, \mu^-, \tau^-\}$.

¹²Note that $\delta\lambda$ does not contribute to the one-loop muon production considered in Sec. III.

¹³These counterterms are scale invariant.

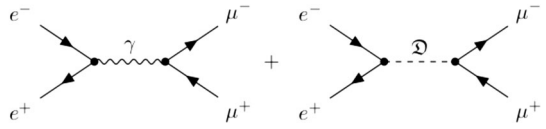
Note the nonvanishing wave function renormalization δZ_σ . In contrast to the two-scalar model, discussed in [5,7,9,10,13,29], where the scalar field wave function renormalization vanishes at the one-loop level, here the renormalization of σ in $\mu(\sigma)$ needs to be taken into account as it can give rise to new finite quantum corrections of the form $\epsilon\delta Z_\sigma$.¹⁴ Indeed, such a correction contributes to the new finite quantum correction of the one-loop cross section given in (33). In particular,

$$\begin{aligned} \Delta_{\text{UV}}\sigma_{\text{total,1L}\mu}(ee \rightarrow \mu\mu) &= -\frac{3}{128\pi^3 s} y_e^2 y_\mu^4 + \sigma_{\text{tree}}^{\text{Yuk}}(ee \rightarrow \mu\mu)\epsilon\delta Z_\sigma \\ &= -\frac{3}{128\pi^3 s} y_e^2 y_\mu^4 + \frac{y_e^2 y_\mu^2}{16\pi s} \epsilon\delta Z_\sigma \\ &= -\frac{1}{128\pi^3 s} y_e^2 y_\mu^2 (y_e^2 + 4y_\mu^2 + y_\tau^2), \end{aligned} \quad (\text{B3})$$

with $\sigma_{\text{tree}}^{\text{Yuk}}(ee \rightarrow \mu\mu) = y_e^2 y_\mu^2 / (16\pi s)$ being the tree-level contribution of the Yukawa sector.

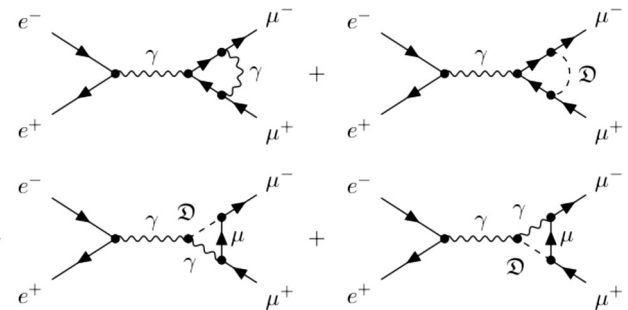
APPENDIX C: FEYNMAN DIAGRAMS FOR MUON PRODUCTION

In the theory considered in (28), there are two Feynman diagrams contributing to the scattering process $e^-e^+ \rightarrow \mu^-\mu^+$ at tree level:



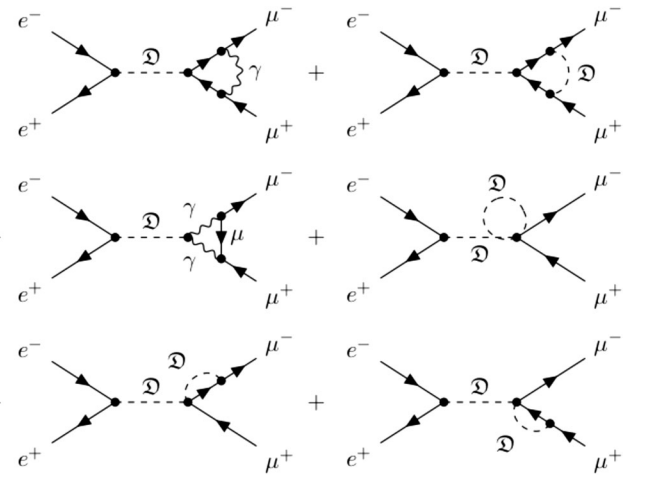
$$i\mathcal{M}_{\text{tree}} = \text{[Photon Mediator]} + \text{[Dilaton Mediator]} \quad (\text{C1})$$

At the one-loop level, there are ten Feynman diagrams containing a one-loop muon vertex correction, four of them with a photon mediator as illustrated in (C2),



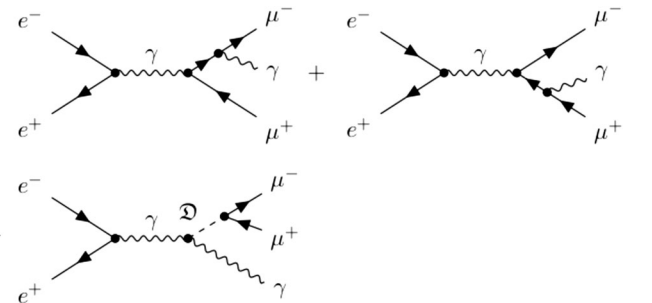
$$\text{[Four diagrams]} \quad (\text{C2})$$

and the other six with a dilaton mediator as shown below in (C3):



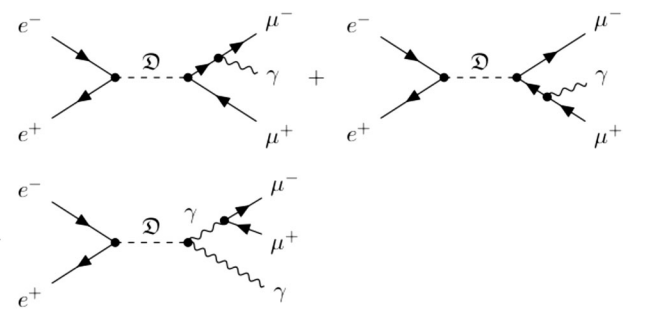
$$\text{[Six diagrams]} \quad (\text{C3})$$

For the scattering process $e^-e^+ \rightarrow \mu^-\mu^+\gamma$ there are six tree-level Feynman diagrams of which three are photon mediated as in (C4):



$$\text{[Three diagrams]} \quad (\text{C4})$$

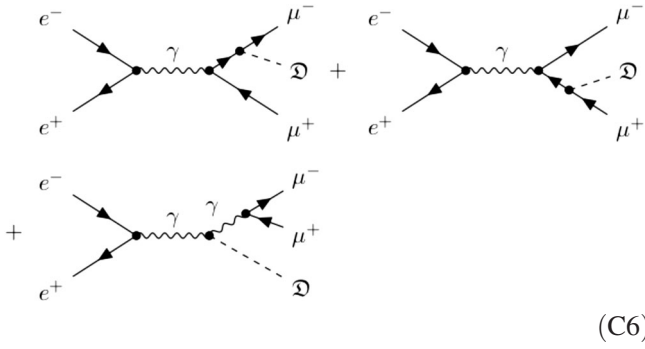
and the remaining three are dilaton mediated, as illustrated in (C5):



$$\text{[Three diagrams]} \quad (\text{C5})$$

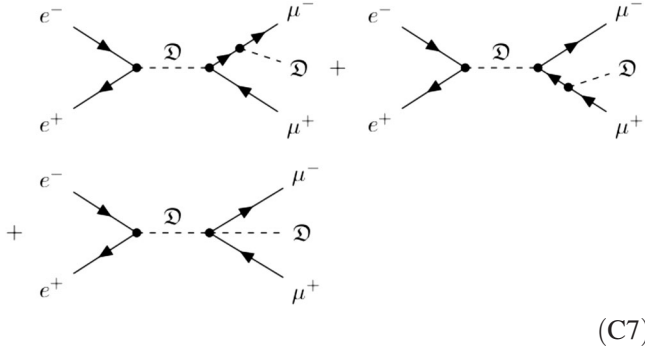
Additionally, for the scattering process $e^-e^+ \rightarrow \mu^-\mu^+\mathcal{D}$ there are six tree-level Feynman diagrams. Three of them are photon mediated as shown in (C6),

¹⁴This is seen after expanding the Lagrangian with respect to ϵ , w and \hbar for a given loop order.



$$(C6)$$

whereas the remaining three of them are dilaton mediated, shown below in (C7):



$$(C7)$$

APPENDIX D: MUON PRODUCTION: CROSS SECTIONS

In the following, explicit results for the cross section of muon production are provided at the one-loop level including virtual and real corrections to the muon vertex only.

(i) *Tree level.*—The tree-level cross section, in four dimensions, is provided by

$$\sigma_{\text{tree}}(ee \rightarrow \mu\mu) = \frac{e^4}{12\pi s} + \frac{y_e^2 y_\mu^2}{16\pi s}. \quad (D1)$$

(ii) *One-loop.*—Using (29), the one-loop cross section for $e^-e^+ \rightarrow \mu^-\mu^+$, containing only one-loop muon vertex corrections, in the $\overline{\text{MS}}$ scheme, is the sum of the following contributions:

$$\sigma_{\text{1L},\mu}^{1/\epsilon_{\text{IR}}^2}(ee \rightarrow \mu\mu) = -\frac{\mu_0^{2\epsilon}}{192\pi^3 s} (4e^6 + 3e^2 y_e^2 y_\mu^2) \frac{1}{\epsilon_{\text{IR}}^2} \quad (D2)$$

$$\sigma_{\text{1L},\mu}^{1/\epsilon_{\text{IR}}}(ee \rightarrow \mu\mu) = -\frac{\mu_0^{2\epsilon}}{2304\pi^3 s} \left[104e^6 - 12e^4 y_\mu^2 + 126e^2 y_e^2 y_\mu^2 - 9y_e^2 y_\mu^4 - 24(4e^6 + 3e^2 y_e^2 y_\mu^2) \log\left(\frac{s}{\mu_0^2}\right) \right] \frac{1}{\epsilon_{\text{IR}}} \quad (D3)$$

$$\Delta_{\text{IR}} \sigma_{\text{1L},\mu}^{1/\epsilon_{\text{IR}}}(ee \rightarrow \mu\mu) = -\frac{\mu_0^{2\epsilon}}{16\pi^3 s} e^2 y_e^2 y_\mu^2 \frac{1}{\epsilon_{\text{IR}}} \quad (D4)$$

$$\begin{aligned} \sigma_{\text{1L},\mu}^{\text{fin}}(ee \rightarrow \mu\mu) &= \frac{\mu_0^{2\epsilon}}{3456\pi^3 s} [(60\pi^2 - 464)e^6 + 30e^4 y_\mu^2 + 9(5\pi^2 - 48)e^2 y_e^2 y_\mu^2 - 27y_e^2 y_\mu^4] \\ &+ \frac{\mu_0^{2\epsilon}}{2304\pi^3 s} [208e^6 - 24e^4 y_\mu^2 + 198e^2 y_e^2 y_\mu^2 + 9y_e^2 y_\mu^4] \log\left(\frac{s}{\mu_0^2}\right) - \frac{\mu_0^{2\epsilon}}{96\pi^3 s} [4e^6 + 3e^2 y_e^2 y_\mu^2] \log^2\left(\frac{s}{\mu_0^2}\right) \end{aligned} \quad (D5)$$

$$\Delta_{\text{UV}} \sigma_{\text{1L},\mu}^{\text{fin}}(ee \rightarrow \mu\mu) = -\frac{\mu_0^{2\epsilon}}{128\pi^3 s} y_e^2 y_\mu^2 (y_e^2 + 4y_\mu^2 + y_\tau^2) \quad (D6)$$

$$\Delta_{\text{IR}} \sigma_{\text{1L},\mu}^{\text{fin}}(ee \rightarrow \mu\mu) = \frac{\mu_0^{2\epsilon}}{384\pi^3 s} [4e^4 y_\mu^2 - 84e^2 y_e^2 y_\mu^2 + 9y_e^2 y_\mu^4] + \frac{\mu_0^{2\epsilon}}{8\pi^3 s} e^2 y_e^2 y_\mu^2 \log\left(\frac{s}{\mu_0^2}\right) - \frac{5\mu_0^{2\epsilon}}{32\pi^3 s} e^2 y_e^2 y_\mu^2. \quad (D7)$$

(iii) *Real photon emission.*—The contributions to the tree-level cross section for $e^-e^+ \rightarrow \mu^-\mu^+\gamma$, arranged as in (29), are

$$\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}^2}(ee \rightarrow \mu\mu\gamma) = \frac{\mu_0^{2\epsilon}}{192\pi^3 s} (4e^6 + 3e^2 y_e^2 y_\mu^2) \frac{1}{\epsilon_{\text{IR}}^2} \quad (D8)$$

$$\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}(ee \rightarrow \mu\mu\gamma) = \frac{\mu_0^{2\epsilon}}{1152\pi^3 s} \left[52e^6 + 63e^2 y_e^2 y_\mu^2 - 12(4e^6 + 3e^2 y_e^2 y_\mu^2) \log\left(\frac{s}{\mu_0^2}\right) \right] \frac{1}{\epsilon_{\text{IR}}} \quad (D9)$$

$$\Delta_{\text{IR}}\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}(ee \rightarrow \mu\mu\gamma) = \frac{\mu_0^{2\epsilon}}{16\pi^3 s} e^2 y_e^2 y_\mu^2 \frac{1}{\epsilon_{\text{IR}}} \quad (\text{D10})$$

$$\begin{aligned} \sigma_{\text{tree}}^{\text{fin}}(ee \rightarrow \mu\mu\gamma) &= -\frac{\mu_0^{2\epsilon}}{6912\pi^3 s} [4(30\pi^2 - 259)e^6 + 9(10\pi^2 - 147)e^2 y_e^2 y_\mu^2] \\ &\quad - \frac{\mu_0^{2\epsilon}}{576\pi^3 s} [52e^6 + 63e^2 y_e^2 y_\mu^2] \log\left(\frac{s}{\mu_0^2}\right) + \frac{\mu_0^{2\epsilon}}{96\pi^3 s} [4e^6 + 3e^2 y_e^2 y_\mu^2] \log^2\left(\frac{s}{\mu_0^2}\right) \end{aligned} \quad (\text{D11})$$

$$\Delta_{\text{UV}}\sigma_{\text{tree}}^{\text{fin}}(ee \rightarrow \mu\mu\gamma) = 0 \quad (\text{D12})$$

$$\begin{aligned} \Delta_{\text{IR}}\sigma_{\text{tree}}^{\text{fin}}(ee \rightarrow \mu\mu\gamma) &= \frac{7\mu_0^{2\epsilon}}{32\pi^3 s} e^2 y_e^2 y_\mu^2 - \frac{\mu_0^{2\epsilon}}{8\pi^3 s} e^2 y_e^2 y_\mu^2 \log\left(\frac{s}{\mu_0^2}\right) \\ &\quad + \frac{5\mu_0^{2\epsilon}}{32\pi^3 s} e^2 y_e^2 y_\mu^2. \end{aligned} \quad (\text{D13})$$

(iv) *Real dilaton emission.*—Similarly, the tree-level cross section for $e^-e^+ \rightarrow \mu^- \mu^+ \mathfrak{D}$ is the sum of the following terms:

$$\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}^{\text{fin}}(ee \rightarrow \mu\mu\mathfrak{D}) = 0 \quad (\text{D14})$$

$$\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}(ee \rightarrow \mu\mu\mathfrak{D}) = -\frac{\mu_0^{2\epsilon}}{768\pi^3 s} (4e^4 y_\mu^2 + 3y_e^2 y_\mu^4) \frac{1}{\epsilon_{\text{IR}}} \quad (\text{D15})$$

$$\Delta_{\text{IR}}\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}(ee \rightarrow \mu\mu\mathfrak{D}) = 0 \quad (\text{D16})$$

$$\begin{aligned} \sigma_{\text{tree}}^{\text{fin}}(ee \rightarrow \mu\mu\mathfrak{D}) &= -\frac{\mu_0^{2\epsilon}}{4608\pi^3 s} (76e^4 y_\mu^2 + 117y_e^2 y_\mu^4) \\ &\quad + \frac{\mu_0^{2\epsilon}}{384\pi^3 s} [4e^4 y_\mu^2 + 3y_e^2 y_\mu^4] \log\left(\frac{s}{\mu_0^2}\right) \end{aligned} \quad (\text{D17})$$

$$\Delta_{\text{UV}}\sigma_{\text{tree}}^{\text{fin}}(ee \rightarrow \mu\mu\mathfrak{D}) = 0 \quad (\text{D18})$$

$$\Delta_{\text{IR}}\sigma_{\text{tree}}^{\text{fin}}(ee \rightarrow \mu\mu\mathfrak{D}) = -\frac{\mu_0^{2\epsilon}}{384\pi^3 s} (4e^4 y_\mu^2 + 9y_e^2 y_\mu^4) \quad (\text{D19})$$

(v) *Total cross section.*—Finally, the total cross section, considering only virtual and real corrections to the muon vertex, is given by

$$\begin{aligned} \sigma_{\text{total},\mu}(ee \rightarrow \mu\mu) &= \sigma_{\text{tree}}(ee \rightarrow \mu\mu) + \sigma_{1L,\mu}(ee \rightarrow \mu\mu) + \sigma_{\text{tree}}(ee \rightarrow \mu\mu\gamma) + \sigma_{\text{tree}}(ee \rightarrow \mu\mu\mathfrak{D}) + \mathcal{O}(\alpha_i^4) \\ &= \sigma_{\text{tree}}(ee \rightarrow \mu\mu) + \sigma_{\text{total},1L\mu}(ee \rightarrow \mu\mu) + \mathcal{O}(\alpha_i^4), \end{aligned} \quad (\text{D20})$$

where α_i are the fine structure constants for e and y_i . The full tree-level cross section is to be found in (D1), whereas the considered one-loop contribution in the $\overline{\text{MS}}$ -scheme and in four dimensions is given by

$$\sigma_{\text{total},1L\mu}(ee \rightarrow \mu\mu) = \sigma_{\text{total},1L\mu}^{\text{old}}(ee \rightarrow \mu\mu) + \Delta_{\text{UV}}\sigma_{\text{total},1L\mu}(ee \rightarrow \mu\mu) \quad (\text{D21})$$

with the standard one-loop contribution

$$\sigma_{\text{total},1L\mu}^{\text{old}}(ee \rightarrow \mu\mu) = \frac{1}{512\pi^3 s} (8e^6 - 4e^4 y_\mu^2 + 34e^2 y_e^2 y_\mu^2 - 17y_e^2 y_\mu^4) - \frac{1}{256\pi^3 s} [6e^2 y_e^2 y_\mu^2 - 3y_e^2 y_\mu^4] \log\left(\frac{s}{\mu_0^2}\right) \quad (\text{D22})$$

and the new finite quantum correction that emerged from UV divergences, quoted in Eq. (33):

$$\Delta_{\text{UV}}\sigma_{\text{total},1L\mu}(ee \rightarrow \mu\mu) = -\frac{1}{128\pi^3 s} y_e^2 y_\mu^2 (y_e^2 + 4y_\mu^2 + y_\tau^2). \quad (\text{D23})$$

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