Superradiant energy extraction from rotating hairy Horndeski black holes

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Adopting the manifestation of low frequency and low mass for the scalar perturbation, we perform a semiclassical analysis of the superradiance phenomenon for a rotating hairy Horndeski black hole (BH). For the spacetime under study enriched by the hairy Horndeski parameter h, in addition to the mass M and spin a, we compute the amplification factor of scalar wave scattering indicating the energy extraction from the BH. We find that due to the addition of the hairy parameter h in the geometry, the superradiance scattering and its frequency range enhance compared to the Kerr BH. This implies that Horndeski's gravity belongs to those alternative theories of gravity that make the amplification factor larger than the Kerr BH so that the energy extraction in its framework is more efficient than general relativity. Calculating the outgoing energy flux measured by an observer at infinity verifies the role of the hairy parameter h in the increase in energy extraction efficiency from the rotating BH. By implementing the BH bomb mechanism, we present an analysis of the superradiant instability of the underlying BH spacetime against massive scalar fields. Our analysis indicates that the hairy Horndeski parameter leaves no imprint on the standard superradiant instability regime.

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I. INTRODUCTION

The exciting idea of extracting energy from a black hole (BH) through the amplification process dates back to the pioneering work of Penrose [1,2], who proposed energy extraction for a particle falling and decaying into the ergoregion of a Kerr BH.[1] Indeed, the ergoregion is a region in which a timelike captured particle can have negative energy, as perceived by an observer at infinity. According to the energy-conservation law, the swallowing of the particles with negative energy by a BH means extracting energy from the BH. This process also can be generalized to wave scattering off BHs. Using the massless scalar field, Misner derived the essential inequality, $\omega < m\Omega$, between the frequency of the incitant wave ω and the rotational frequency of the BH Ω [10]. This, in essence, is known as superradiant scattering i.e., the amplification of waves when scattering off a dissipative rotating body. By analyzing a dissipative system such as an absorber rotating cylinder subject to scattering of waves, Zeldovich derived successfully the mentioned condition of rotational superradiance ($\omega < m\Omega$) [11,12]. Indeed, it is well-known that an incident wave in case of scattering off any dissipative object is prone to experiencing superradiance. In this regard, Teukolsky [13] has shown that the Misner/Zeldovich amplification process also occurs for other bosonic fields (electromagnetic and gravitational waves) provided that satisfies the inequality above. Further, by analyzing BH superradiance from the perspective of thermodynamics, Bekenstein derived [14] results in agreement with previous ones (see also [15]). The milestone of Bekenstein's study was the discovery of a fundamental origin for the energy extraction via connecting it to Hawking's theorem, expressing that the surface area of a classical BH cannot decrease [16]. Historically, these seminal papers on the BH superradiance were the first steps that later resulted in discovering BH evaporation by Hawking[2] [17]. This, along with some of the newer studies

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[1]It can be interesting to point out that energy extraction via the Penrose process is not affordable astrophysical and instead by regarding the magnetic field around the BH its efficiency improves [3–5]. Recently, Ref. [6], has shown that the magnetic field enclosing a rotating BH, via a theoretical mechanism known as magnetic reconnection, has the potential for energy extraction with more efficiency, see also followup studies [7–9].

[2]For this reason, by taking the quantum effects into account, the rotational superradiance would become a spontaneous process and BH would slow down by spontaneous emission of photons satisfying $\omega < m\Omega$.

on this subject such as [18,19], may support the idea that superradiant scattering strongly depends on the boundary conditions at the horizon. This implies that superradiant scattering is nothing but a boundary condition problem. However, several evidences reveal that the ergoregion is an essential component for occurring superradiance phenomena, since it provides required friction as a form of dissipation [20,21] (see also [22] for review). Based on this, horizonless compact objects such as stars are also prone to trigger superradiance [23]. Indeed, due to the probable existence of viscous matter, stars can be regarded as a model that may provide requisite dissipation for superradiance phenomena to occur [24] (see also [25]).

Apart from the fact that superradiant amplification can be utilized to explain energy extraction from a BH, it may cause several important instabilities in BH spacetimes, as well. If these instabilities occur, due to rotational energy extraction, the spin of BH slows down and can lead to a hairy BH solution. Therefore, these instabilities may open a new window to investigate the no-hair theorem for any new BH solution [26]. The idea behind superradiant instability is not complicated and comes from the fact that confinement of the system subject to perturbation causes unstable environment against superradiant modes. By amplifying any incoming pulse near the ergoregion by superradiance, and then confining the pulse via a perfect reflecting surface at some distance, the amplitude of the pulse exponentially increases through numerous interactions. It means making instability in the background subject to some perturbation. Commonly, a reflecting surface is supplied in two different ways; natural and artificial. The former can be achieved if there is any provision for anti–de Sitter (AdS) spacetime[3] [28–32] and the massive scalar field [33–39], while the latter can be reached by placing a mirrorlike surface around BH or confining the BH into a box with Dirichlet boundary conditions [40–43]. Press and Teukolsky [44] explained the mechanism of the instability in this way that the initial fluctuation arising from superradiant modes grows exponentially, leading to an ever-increasing field density and pressure inside the confinement region such that finally disrupts the confining surface, resulting in an explosion. This system is known as the BH bomb, see Ref. [45] for more details. Another motivation for studying superradiant instability of BH comes from the fact that constraining the mass of ultra-light degrees of freedom may shed light on the dark matter puzzle [46–49].

In recent years, this subject has received a considerable attention from different perspectives, including astrophysics, higher-dimension spacetimes, and also alternative theories of general relativity. The importance of superradiant instability in the framework of astrophysics originates from the fact that its development due to the extraction of energy and angular momentum from the BH results in the formation of a nonspherical bosonic cloud near the BH and subsequently gravitational-wave emission [50–52] (see also recent papers [53,54]). The application of the gravitational wave emitted by the cloud is that creating specific quasimonochromatic signals would address the existence of ultralight bosons. In other words, the superradiant instabilities allow us to use astrophysical BHs as effective detectors [55] to look for new particles [56]. It was argued that the superradiant instability at interplay with the BH shadow can potentially be used to constrain the ultralight boson candidates [57–60]. Other studies [61–65] propose the possibility of probing ultralight bosons using the superradiant clouds around the supermassive BHs recorded recently by the Event Horizon Telescope. The investigations on the superradiant phenomenon for higher-dimensional models are well-motivated since according to these models in particle accelerators such as LHC (Large Hadron Collider) there is a chance to produce micro BHs [66]. By serving the scalar and vector perturbations, the efficiency of superradiant amplification as well as its instabilities for higher-dimensional rotating BHs have been addressed in [67–73]. Similar investigations have been performed for nonrotating charged BHs [74–76].

Another domain of interest for studying the superradiance phenomena of BHs is modified gravity, which we shall address by utilizing one of the well-known models beyond Einstein's gravity. In other words, BH superradiance is not a prerogative of general relativity, rather any relativistic gravitational theory that admits BH solutions, is prone potentially to it. Despite the agreement of the recent gravity experiments in the strong field regime [77–79] with the standard Kerr BH, due to the statistical error in current observations, the possibility of admitting the Kerr BH solutions modified by some alternative theories of gravity is still not ruled out. In general, the details of the superradiant energy extraction are affected by two factors; BH geometry and the wave dynamics in the alternative theory of gravity. The motivation for the study of superradiance within the modified gravities is twofold. First, Einstein's gravity, despite all its admirable achievements, cannot be a reliable theory at all scales due to some shortcomings, and it is expected that it needs some modifications [80]. Second, finding the imprint of corrections imposed to the standard model of gravity on the superradiance efficiency and its instabilities is also potentially interesting. In recent years, we are seeing a variety of research on this topic (see e.g., [81–94]) which compared to standard general relativity enhances or subsidizes. Such investigations allow us to separate those modified gravities which are in favor of superradiance from those that are not. Despite these studies, there is still room for some classes of BH solutions. In the present work, we focus on hairy rotating BHs that arise from Horndeski theory of gravity.

Introducing the scalar field is one of the well-known ways to extend gravity to overcome cosmological issues, including dark energy, dark matter, and the evolution of the

[3]In Ref. [27] it has shown that the asymptotically Godel spacetime is also similar to AdS which can play the natural role of a timelike boundary. Note that no timelike particle can reach spatial infinity, therefore any background similar to AdS can be considered as a confining system.

Universe in early- and late-time epochs. There are some modified gravity theories that are mathematically equivalent to a gravitational theory with degrees of freedom containing the metric $g_{\mu\nu}$ and one or more scalar fields ϕ [95–98]. The scalar-tensor theories are likely the simplest, most consistent, and nontrivial extensions of general relativity [99]. One of the most famous four-dimensional scalar-tensor theory is the Horndeski gravity proposed in 1970 [100]. In the light of some leading research [101,102], it was argued that Horndeski gravity is equivalent to the Galilean theories which, in essence, are the scalar-tensor theories with Galilean symmetry in flat spacetime [103,104]. Indeed, due to propagating one scalar degree of freedom by general second-order field equations in the Horndeski gravity, it is free of Ostrogradski instabilities. In Refs. [105,106], authors have shown that gravitational waves are an efficient and robust tool for distinguishing the models of the Horndeski theory describing the cosmic accelerating expansion. One of the freedoms that appear in Horndeski's theories is that, unlike the standard general relativity, there is no requirement that gravitational waves (tensor speed c_T) travel at the speed of light in the vacuum i.e., $c_T = c$. Apart from the key physical reason giving rise to such anomalous propagation, gravitational waves when analyzed in the framework of modified gravity no longer travel on null geodesics of the background metric as photons do [107]. The release of the observational data of GW170817 and whose optical counterpart GRB170817A, have placed very tight constraints on the deviation from the speed of light [108]. In general, the results of GW170817 and GRB170817A indicate that the deviation of the tensor speed (gravitational waves) from the speed of light is no more than one part in 10^{15} i.e., $|c_T - 1| \lesssim 10^{15}$. Thanks to this tight constraint, it is possible to test the validity of Horndeski's descriptions of late-times cosmological evolution (e.g., see Refs. [109–111]). Cosmological consequences of this theory such as alleviating the cosmological constant problem [112] and other interesting features [113–118] have been studied. While, observational constraints on the parameters of Horndeski theories have been carried out in [119–124], thermodynamics of BH solutions of this theory were explored in [125–130]. Taking hairy BH solution[4] [138] derived from the quartic scalar field version of the Horndeski gravity, the role of adding a hairy parameter on the strong gravitational effects such as lensing [139], deflection of light [140], and BH shadow [141] have been explored.

In line with what is mentioned above, in the present work, we investigate the influence of the hairy parameter, admitted by the quartic scalar field Horndeski gravity, on massive scalar field superradiant amplification, and the relevant stability linked with it. This study is well-motivated in the sense that it allows us to address the phenomenological imprint induced by the correction in the geometric structure of spacetime. Throughout this paper, we use the signature convention $(-,+,+,+)$ and work in the units where $c = 1 = \kappa = 8\pi G_N$.

This paper is structured as follows. In Sec. II, by serving the quartic Horndeski scalar field model, we present the rotating counterpart for the spherically symmetry hairy BH solution [138], as already was released in [140]. Section III consists of three parts. We first address the conditions of massive scalar superradiant scattering, and then by serving a semiclassical technique, we compute analytically the amplification factor to look for the role of hairy Horndeski in strengthening/weakening of scalar wave. In the third part of Sec. III, we investigate the energy extraction from the rotating Horndeski BH. In Sec. IV, by analyzing the effective potential in the framework of the BH bomb mechanism, we discuss the superradiant instability of the dynamics of the massive scalar fields. We finish with closing remarks in Sec. V.

II. ROTATING HORNDESKI BLACK HOLE METRIC

The action of Horndeski gravity consists of the metric $g_{\mu\nu}$ and the scalar field ϕ include four arbitrary functions $Q_{i=2,..5}(\chi)$ of kinetic term $\chi = -1/2\partial_\mu\phi\partial^\mu\phi$ [142]. Adopting the special case in which $Q_5 = 0$, the quartic action of Horndeski gravity can be expressed as

$$S = \int d^4x\sqrt{-g}\{Q_2(\chi) + Q_3(\chi)\Box\phi + Q_4(\chi)R + Q_{4,\chi}((\Box\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi))\}, \tag{1}$$

where g, R, $\Box$, and ∇ denote the determinant of the metric tensor, the Ricci scalar, the d'Alembert operator, and the covariant derivative, respectively.

The spherically symmetric hairy Horndeski BH spacetime has the following line elements [138]

$$ds^2 = -A(r)dt^2 + B^{-1}(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$C(r) = r^2. \tag{2}$$

By varying the action (1) with respect to $\phi_{,\mu}$, and $g^{\mu\nu}$, we obtain

[4]One can imagine the Kerr hypothesis as a consequence of the no-hair theorem, meaning that the endpoint of any gravitational collapse will be nothing but a Kerr BH. However, this is different in the framework of modified theories and it is expected to there exist classes of these theories that predict the hairy BH solutions. Note that the existence of hairy BH solutions always is not meaning that there is an additional new conserved charge (for a comprehensive overview refer to Refs. [131,132]). Already several static and spherically symmetric hairy BHs in the framework of scalar-tensor theories were obtained which as the simplest case can mention those solutions with a radial dependency scalar field [133–135]. Apart from the theoretical aspects, some of these hairy solutions have also been subjected to observations [136,137].

$$j^{\nu} = -Q_{2,\chi}\,\phi^{,\nu} - Q_{3,\chi}\,(\phi^{,\nu}\Box\phi + \chi^{,\nu}) - Q_{4,\chi}\,(\phi^{,\nu}R - 2R^{\nu\sigma}\phi_{,\sigma}) - Q_{4,\chi,\chi}\,(\phi^{,\nu}[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)(\nabla^\alpha\nabla^\beta\phi)] + 2(\chi^{,\nu}\Box\phi - \chi_{,\mu}\nabla^\mu\nabla^\nu\phi)), \tag{3}$$

and

$$Q_4 G_{\mu\nu} = T_{\mu\nu}, \tag{4}$$

which are respectively the four-current and the field equations with

$$\begin{aligned} T_{\mu\nu} = {}& \frac{1}{2}(Q_{2,\chi}\,\phi_{,\mu}\,\phi_{,\nu} + Q_2 g_{\mu\nu}) + \frac{1}{2}Q_{3,\chi}\,(\phi_{,\mu}\,\phi_{,\nu}\,\Box\phi - g_{\mu\nu}\chi_{,\alpha}\,\phi^{,\alpha} + \chi_{,\mu}\,\phi_{,\nu} + \chi_{,\nu}\,\phi_{,\mu}) \\ & - Q_{4,\chi}\left(\frac{1}{2}g_{\mu\nu}[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)(\nabla^\alpha\nabla^\beta\phi) - 2R_{\sigma\gamma}\phi^{,\sigma}\phi^{,\gamma}] - \nabla_\mu\nabla_\nu\phi\Box\phi \right. \\ & \left. + \nabla_\gamma\nabla_\mu\phi\nabla^\gamma\nabla_\nu\phi - \frac{1}{2}\phi_{,\mu}\,\phi_{,\nu}\,R + R_{\sigma\mu}\phi^{,\sigma}\phi_{,\nu} + R_{\sigma\nu}\phi^{,\sigma}\phi_{,\mu} + R_{\sigma\nu\gamma\mu}\phi^{,\sigma}\phi^{,\gamma}\right) \\ & - Q_{4,\chi,\chi}\,(g_{\mu\nu}(\chi_{,\alpha}\,\phi^{,\alpha}\Box\phi + \chi_{,\alpha}\chi^{,\alpha}) + \frac{1}{2}\phi_{,\mu}\,\phi_{,\nu}\,(\nabla_\alpha\nabla_\beta\phi\nabla^\alpha\nabla^\beta\phi - (\Box\phi)^2) \\ & - \chi_{,\mu}\chi_{,\nu} - \Box\phi(\chi_{,\mu}\,\phi_{,\nu} + \chi_{,\nu}\,\phi_{,\mu}) - \chi_{,\gamma}\,[\phi^{,\gamma}\nabla_\mu\nabla_\nu\phi - (\nabla^\gamma\nabla_\mu\phi)\phi_{,\nu} - (\nabla^\gamma\nabla_\nu\phi)\phi_{,\mu}]). \end{aligned} \tag{5}$$

To have spherically symmetric spacetime, as addressed by the metric (2), it is essential one set a scalar field $\phi \equiv \phi(r)$. Without going into the details of [138], by introducing simple forms for $Q_2 = \alpha_{22}(-\chi)^{3/2}$, $Q_3 = 0$ and $Q_4 = \kappa^{-2} + \alpha_{42}(-\chi)^{1/2}$,[5] and satisfying the conditions; a vanishing radial four-current at infinity $j^r = 0$, finiteness of the energy of ϕ i.e., $\int_V \sqrt{-g}T^0_0 d^3x$, and utilizing the field equations (4), the metric components of (2) and the derivative of the scalar field background take the following forms, respectively

$$A(r) = B(r) = 1 - \frac{2M}{r} + \frac{h}{r}\ln\left(\frac{r}{2M}\right), \tag{6}$$

$$\phi'(r) = \pm\frac{2}{r}\sqrt{\frac{-\alpha_{42}}{3A(r)\alpha_{22}}}. \tag{7}$$

If $h \to 0$ (with the mass dimension $[h] = M$), the Schwarzschild solution is recovered. It, in essence, is related to parameters α_{22} and α_{42} respectively in the functions Q_2, and Q_4 in the action (1). More exactly, it reads as $h = (2/3)^{3/2}\kappa^2\alpha_{42}\sqrt{-\alpha_{42}/\alpha_{22}}$ [138]. Recently, the range of the hairy Horndeski parameter h in the scale of the supermassive black hole horizon located in the center of our Galaxy has been scanned [144]. As it is clear from (7) the derivative of the scalar field is finite and real if $A(r) > 0$ (i.e., everywhere outside of the black hole) and $\alpha_{42}/\alpha_{22} < 0$. Namely, the current j^r (as only nonzero component of four-current j^μ), is not finite exactly on the horizon, but in the vicinity of it $[A(r) \to 0^+]$ the behavior of both $\phi'(r)$, and j^r are regular. Providing a more details discussion on this can be helpful to understand such a behavior of the derivative of the ϕ on the horizon. For the gravity-Galileon field coupled systems, it is shown in Ref. [145] by Hui and Nicolis that a no-hair result of black holes hold under some assumptions: (a) asymptotic flatness, (b) vanishing derivative of the scalar field at infinity, (c) the finiteness of norm of the Noether four-current $j_\mu j^\mu$ down to the horizon, (d) the presence of canonical kinetic term χ in the action, (e) the χ-derivatives of $Q_{2\ldots5}$ contain only positive or zero powers of χ. In other words, bypassing one or more of these assumptions in the no-go theorem above may result in a hairy black hole solution. In [145] have shown that satisfying the assumption (c) results in a time-independent but constant profile for scalar field i.e. $\phi(r) =$ constant. The hairy black hole solution in [138] that we are interested in is, in essence, obtained from leaving the assumptions (c) and/or (e), along with taking a time-independent profile but nonconstant (r-dependent) into account of the scalar field. Indeed, assumption (c) is replaced with the finiteness of the Noether four-current j^μ at infinity, meaning that the energy of the scalar field in a volume outside the event horizon, is finite. Note that the derivative of scalar field (7) is direct result of imposing condition $j^r = 0$ [as $r \to \infty$ i.e., break of assumption (c)], which comes from (3)

[5]It means that the Horndeski model under our attention throughout the paper just includes Q_2 and Q_4. Namely, it belong a subclass of Horndeski theories that at same time has both proprieties; shift symmetric (i.e., symmetric under $\phi \to \phi +$ constant) and reflection symmetric (i.e., symmetric under $\phi \to -\phi$). The presence of Q_2 is vital for justify the cosmic-accelerating expansion and the gravitational-wave propagation, simultaneously. More exactly, constraining the Horndeski theory via simultaneous confronting with GW170817 and GRB170817A [108], explicitly exclude the subclass models that contain Q_4 without Q_2 [143].

$$j^r = -Q_{2,\chi}\phi' - \frac{2Q_{4,\chi\chi}}{r^2}\phi'^3. \tag{8}$$

Because it is expected that the profile of the underlying time-independent scalar field obeys the metric symmetries, only the nonzero component of the four-current is radial j^r. The divergence of the derivative of scalar-field profile (7) on the horizon merely does not restrict to the hairy BH solution under our attention rather for a wide class of Horndeski gravity theories, one can find slowly-rotating black hole solutions with such behavior, see Ref. [146]. In this direction also by setting coupling functions $Q_2 \propto \chi$, $Q_3 = 0$ and $Q_4 \propto (-\chi)^{1/2}$ in the quartic Horndeski theory, authors in [142] constructed spherically symmetric and static BHs with the nonconstant profile for the scalar field that its derivative diverges on the horizon. As another example in Ref. [142] one can mention also a special subclass of Horndeski theories violating assumption (d) and admitting the standard Kerr metric with a nontrivial profile for the scalar field that its derivative is regular everywhere outside of the Kerr BH except at the event horizon. Here it is required to comment on the irregular behavior of $\phi'(r)$ on the horizon. Although $\phi'(r)$ is not regular on the horizon, in Ref. [138] authors have discussed that the components of the energy-momentum tensor (5) calculated using scalar field profile (7) in agreement with the components of the Riemann tensor, on the horizon and outside of it are finite. In this way, the metric of background solution (6), effectively lets us take a primary step in direction of finding an intuition of the footprint of the Horndeski field on the phenomenology of BHs.

Recently, in Ref. [140] provided the following rotational version for the spherically symmetric spacetime (2) with laps function (6)

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{\tilde{\Delta} - a^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{\Sigma}{\tilde{\Delta}}dr^2 - 2a\sin^2\theta\left(\frac{\tilde{\Delta} - (r^2+a^2)}{\Sigma}\right)dtd\varphi + \Sigma d\theta^2 + \sin^2\theta\left(\frac{(r^2+a^2) - a^2\tilde{\Delta}\sin^2\theta}{\Sigma}\right)d\varphi^2, \tag{9}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \tilde{\Delta} = r^2 - 2Mr + a^2 + hr\ln\left(\frac{r}{2M}\right). \tag{10}$$

The metric (9), in essence, has been derived using the modified Newman-Janis algorithm (i.e., Azreg-Aïnou's noncomplexification procedure [147]), without showing the relevant Horndeski scalar field profile in the rotating background. As stated above, the scalar field must respect the symmetries of the metric. This lets us evaluate the regularity of the Horndeski scalar field on the horizon, infinity, and along the symmetry axis. Already Ref. [142] has shown that some subclasses of Horndeski gravity admit the standard Kerr BH with a nontrivial scalar field profile as an exact solution. Despite the lack of such studies for the modified Kerr metric, it is well-known from Ref. [133] that the scalar-field profile does not affect by the rotation parameter at first order i.e., $\mathcal{O}(a^2)$. Besides, since in the linear order of rotation, the spherical symmetry still holds, thereby, the only nonzero component of the four-current for the time-independent scalar field is still radial. As a result, the expression (8) for the current j^r, and subsequently the derivative of the scalar field in (7), remain unchanged in slowly rotating BH solutions. However, this argument works just for moderate BHs, and doing some studies aimed at deriving the nontrivial scalar field profile for full rotating BH solutions in Horndeski's theories is essential. This deserves comprehensive and separate research in the future. In any case, the lack of an analytical expression for the profile of scalar field with axial symmetry does not prohibit the study of gravitational lensing in [140], and superradiance scattering here. Since in both one, just the rotating background metric (9) is essential, not the profile of the Horndeski scalar field. In particular, concerning the latter, we will consider a test massive scalar field Φ, propagating in the background deviated from the standard Kerr, i.e., (9).

In the limiting case where $h = 0$, the metric (9) coincides with the Kerr spacetime. The positions of the event horizon (r_{eh}) and the Cauchy horizon (r_{ch}) are given by the solution of the equation $\tilde{\Delta} = 0$ which can only be solved numerically due to the presence of the logarithmic term. Outside the event horizon, there is a region known as the ergosphere, covering the area from the event horizon to the outer ergoregion i.e., the largest positive real root of $g_{tt} = 0$ (its smallest root, addresses the inner ergoregion which is located behind the event horizon).

In Fig. 1 by fixing some different values for the free parameter h, depicted the location of horizons and ergoregion of the rotating Horndeski BH, respectively. One can see the displacements of the location of horizons relative to the standard Kerr ($h = 0$). We observe that for some values of h, the compact object at hand is no longer BH, since the event horizon disappears. It also depends on the value of the spin setting so that for instance by setting $a = 0.9M$ and $a = 0.95M$, respectively for values beyond $h = -0.27M$ and $h = -0.15M$, we indeed deal with a rotating naked singularity. Given our interest in the BH case,[6] in the following, we have to take care in setting values of the free parameter of the model.

[6]Although, the event horizon as the defining property of BH, in essence, is not directly observable [148–150], nowadays rotating BHs are widely admitted as astrophysical objects [151,152].

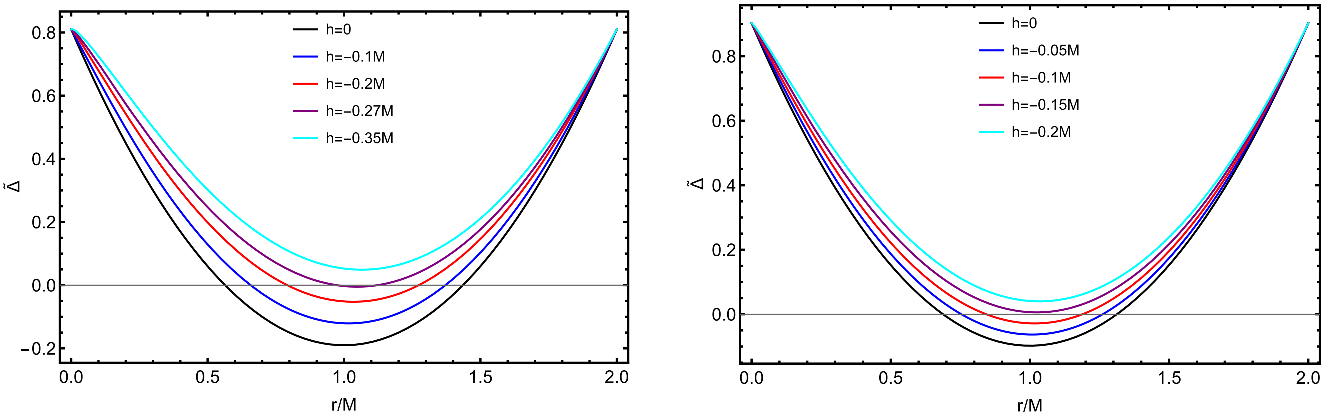

FIG. 1. The location of horizons of the rotating Horndeski BH with various values of hairy parameter h, and spin parameters: $a = 0.9M$ (left panel) and $a = 0.95M$ (right panel).

III. SCALAR SUPERRADIANCE SCATTERING

The propagation of the test scalar field Φ on a curved spacetime is described by the following Klein-Gordon equation [153,154]

$$(\nabla_\alpha\nabla^\alpha + \mu_s^2)\Phi(t,r,\theta,\varphi) = \left[\frac{-1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\mu) + \mu_s^2\right] \times \Phi(t,r,\theta,\varphi) = 0, \qquad (11)$$

where μ_s and $g^{\mu\nu}$, respectively, denote the mass of the scalar field, and the inverse spacetime metric. By adopting the standard separation of variables method, we use the following ansatz with the standard Boyer-Lindquist coordinates (t,r,θ,φ)

$$\Phi(t,r,\theta,\varphi) = R_{\omega jm}(r)\Theta(\theta)e^{-i\omega t}e^{im\varphi}, \quad j \geq 0, \quad -j \leq m \leq j, \quad \omega > 0, \qquad (12)$$

to separate Eq. (11) into radial and angular parts. Here $R_{\omega jm}(r)$ is the radial part of the wave function and $\Theta(\theta)$ is the oblate-spheroidal wave function. The symbol j is the angular eigenfunction, m is the angular quantum number, and ω is the positive frequency of the field under investigation as measured by a faraway observer. The ansatz (12), cause the differential equation (11) to yield two ordinary differential equations with the following radial and angular parts

$$\frac{d}{dr}\left(\tilde{\Delta}\frac{dR_{\omega jm}(r)}{dr}\right) + \left(\frac{((r^2+a^2)\omega - am)^2}{\tilde{\Delta}}\right)R_{\omega lm}(r) - (\mu_s^2 r^2 + j(j+1) + a^2\omega^2 - 2m\omega a)R_{\omega lm}(r) = 0, \qquad (13)$$

and

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\omega jm}(\theta)}{d\theta}\right) + (j(j+1)\sin^2\theta - ((a\omega\sin^2\theta - m)^2))\Theta_{\omega jm}(\theta) + a^2\mu_s^2\sin^2\theta\cos^2\theta\,\Theta_{\omega jm}(\theta) = 0, \qquad (14)$$

respectively. Given that we intend to study the scattering of the field Φ, just Eq. (13) is under our attention until the end of this paper. Following the earlier investigations (e.g., [153,154]), we may find a general solution of the radial equation (13). We now apply a Regge-Wheeler-like coordinate r_* which is given by

$$r_* \equiv \int dr\frac{r^2+a^2}{\tilde{\Delta}}, \quad (r_* \to -\infty \quad \text{at event horizon}, \qquad r_* \to \infty \quad \text{at infinity}). \qquad (15)$$

To have the equation into the desired shape, we consider a new radial function $\mathcal{S}_{\omega jm}(r_*) = \sqrt{r^2+a^2}R_{\omega jm}(r)$. A few steps of algebra yields

$$\frac{d^2\mathcal{S}_{\omega lm}(r_*)}{dr_*^2} + V_{\omega jm}(r)\mathcal{S}_{\omega jm}(r_*) = 0, \qquad (16)$$

where the effective potential $V_{\omega jm}(r)$ reads as

$$V_{\omega jm}(r) = \left(\omega - \frac{ma}{r^2+a^2}\right)^2 - \frac{\tilde{\Delta}}{(r^2+a^2)^2} \times \left[j(j+1) + a^2\omega^2 - 2ma\omega + \mu_s^2 r^2 + \sqrt{r^2+a^2}\frac{d}{dr}\left(\frac{r\tilde{\Delta}}{(r^2+a^2)^{\frac{3}{2}}}\right)\right]. \qquad (17)$$

So, what results now is the study of the scattering of the scalar field Φ under the effective potential (17). For this purpose, it is usually studied the asymptotic behavior of the scattering potential at the event horizon and spatial infinity. The potential at the limit of the event horizon is

$$\lim_{r\to r_{eh}} V_{\omega jm}(r) = (\omega - m\Omega_{eh})^2 \equiv k_{eh}^2, \qquad \Omega_{eh} \equiv \frac{a}{r_{eh}^2 + a^2}, \tag{18}$$

and the same at spatial infinity gives

$$\lim_{r\to\infty} V_{\omega jm}(r) = \omega^2 - \lim_{r\to\infty} \frac{\mu_s^2 r^2 \tilde{\Delta}}{(r^2+a^2)^2} = \omega^2 - \mu_s^2 \equiv k_\infty^2. \tag{19}$$

It is important to observe that the potential shows constant behavior at both extremal points, though the values of the constants are different at the two extremal points. As we now know the asymptotic behavior of the potential at two extremal points, we can proceed to observe the asymptotic behavior of the radial solution. A few steps of algebra yield the following asymptotic solutions of the radial equation (16)

$$R_{\omega jm}(r) \to \begin{cases} \mathcal{P}_{\rm in}^{eh} \frac{e^{-ik_{eh}r_*}}{\sqrt{r_{eh}^2+a^2}} & \text{for } r \to r_{eh} \\ \mathcal{P}_{\rm in}^{\infty} \frac{e^{-ik_\infty r_*}}{r} + \mathcal{P}_{\rm ref}^{\infty} \frac{e^{ik_\infty r_*}}{r} & \text{for } r \to \infty. \end{cases} \tag{20}$$

Here, $\mathcal{P}_{\rm in}^{eh}$ corresponds to the amplitude of the incoming scalar wave at the event horizon (r_{eh}) and $\mathcal{P}_{\rm in}^{\infty}$ is the corresponding quantity of the incoming scalar wave at infinity (∞) whereas the amplitude of the reflected part of scalar wave at infinity (∞) is $\mathcal{P}_{\rm ref}^{\infty}$. Finally, by computing and equating the Wronskian at the event horizon (W_{eh}) and spatial infinity (W_∞) we obtain the following relation

$$|\mathcal{P}_{\rm ref}^{\infty}|^2 = |\mathcal{P}_{\rm in}^{\infty}|^2 - \frac{k_{eh}}{k_\infty} |\mathcal{P}_{\rm in}^{eh}|^2. \tag{21}$$

The featured point in the above relation is that it is free of the details of the potential $V_{\omega jm}(r)$ in the Schrödinger-like differential equation (16). The relation (21) tells us that the scalar wave is superradiantly amplified, if $\frac{k_{eh}}{k_\infty} < 0$ i.e., $\omega < m\Omega_{eh}$.

A. Amplification factor Z_{jm} for superradiance

For our purpose i.e., the study of the cross section of the scalar field Φ scattering from the Horndeski-based rotating BH, we employ the asymptotic matching procedure proposed in the seminal papers [155,156]. Despite the lack of an exact solution for the singularly-perturbed differential equation (13), providing an approximation solution via asymptotic expansions in relevant extremal points, is still possible. More precisely, in this method, one indeed finds two approximate solutions each one valid for part of the range of the independent variable so that eventually by their matching, one acquires a reliable single approximate solution. It is important to point out that matching is possible just provided that the relevant expansions have a domain of overlap, meaning that the exact solutions derived for two asymptotic regions are matched in an overlapping region. The mentioned procedure is semianalytical and based on two assumptions; the slowly rotating $a\omega \ll 1$ and the gravitational size of the BH is much smaller than the Compton wavelength of the scalar field Φ i.e., $M\omega \ll 1$ (or $\mu_s M \ll 1$). By rewriting the radial equation (13) in the following form

$$\tilde{\Delta}^2 \frac{d^2 R_{\omega jm}(r)}{dr^2} + \tilde{\Delta} \frac{d\tilde{\Delta}}{dr} \cdot \frac{dR_{\omega jm}(r)}{dr} + (((r^2+a^2)\omega - am)^2 - \tilde{\Delta}(\mu_s^2 r^2 + j(j+1) + a^2\omega^2 - 2ma\omega))R_{\omega jm}(r) = 0, \tag{22}$$

we derive separate solutions related to the two overlapping regions, namely the near-region $r - r_{eh} \ll \omega^{-1}$, and the far-region $r - r_{eh} \gg M$, and finally, by using the matching procedure we get a single solution.

With the change of variable $z = \frac{r - r_{eh}}{r_{eh} - r_{ch}}$ and taking the approximation $a\omega \ll 1$ into account, the equation (22) for the near-region turns into

$$z^2(z+1)^2 \frac{\mathrm{d}^2 R_{\omega jm}(z)}{\mathrm{d}z^2} + z(z+1)(2z+1)\frac{\mathrm{d}R_{\omega jm}(z)}{\mathrm{d}z} + (B^2 - j(j+1)z(z+1))R_{\omega jm}(r) = 0, \tag{23}$$

where $B = \frac{(\omega - m\Omega_{eh})}{r_{eh} - r_{ch}} r_{eh}^2$. To get the equation above we used the approximations $\mathcal{Q}z \ll 1$ and $\mu_s^2 r_{eh}^2 \ll 1$, with $\mathcal{Q} = (r_{eh} - r_{ch})\omega$. The latter comes from the consideration that the Compton wavelength of the scattered scalar field is much bigger than the size of the BH. The general solution of the above equation in terms of ordinary hypergeometric function ${}_2F_1(a, b; c; z)$ is written as

$$R_{\omega jm}(z) = d\left(\frac{z}{z+1}\right)^{-iB} {}_2F_1\left(\frac{1-\sqrt{1+4j(j+1)}}{2}, \frac{1+\sqrt{1+4j(j+1)}}{2}; 1-2iB; -z\right), \tag{24}$$

where d is a coefficient. Given that in the matching procedure we need to observe the large z behavior of the above expression so the Eq. (24) for case $z \to \infty$ turns into

$$R_{\text{near-large}\,z} \sim d\left(\frac{\Gamma(\sqrt{1+4j(j+1)})\Gamma(1-2iB)}{\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}-2iB\right)\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}\right)}z^{\frac{\sqrt{1+4j(j+1)}-1}{2}}+\frac{\Gamma(-\sqrt{1+4j(j+1)})\Gamma(1-2iB)}{\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}\right)\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}-2iB\right)}z^{-\frac{\sqrt{1+4j(j+1)}+1}{2}}\right). \tag{25}$$

Concerning the far-region, by taking approximations $z+1 \approx z$ and $\mu_s^2 r_{eh}^2 \ll 1$ into account and dropping all the terms except those which describe the free motion with momentum j, we get from Eq. (22)

$$\frac{\mathrm{d}^2 R_{\omega jm}(z)}{\mathrm{d}z^2}+\frac{2}{z}\frac{\mathrm{d}R_{\omega jm}(z)}{\mathrm{d}z}+\left(k_{qh}^2-\frac{j(j+1)}{z^2}\right)R_{\omega jm}(z)=0, \tag{26}$$

where $k_{qh} \equiv \frac{\Omega}{\omega}\sqrt{\omega^2-\mu_s^2}$. The general solution of Eq. (26) is

$$R_{\omega jm,\text{far}} = e^{-ik_{qh}z}\left(f_1 z^{\frac{\sqrt{1+4j(j+1)}-1}{2}}M\left(\frac{1+\sqrt{1+4j(j+1)}}{2}, 1+\sqrt{1+4j(j+1)}, 2ik_{qh}z\right)\right.$$
$$\left.+f_2 z^{-\frac{\sqrt{1+4j(j+1)}+1}{2}}M\left(\frac{1-\sqrt{1+4j(j+1)}}{2}, 1-\sqrt{1+4j(j+1)}, 2ik_{qh}z\right)\right), \tag{27}$$

where $M(a,b,z)$ is the confluent hypergeometric Kummer function of the first kind. To match the solution above with (25), we have to find the small z behavior of the solution (25) which within the limit $z \to 0$ results in

$$R_{\omega jm,\text{far-small z}} \sim f_1 z^{\frac{\sqrt{1+4j(j+1)}-1}{2}} + f_2 z^{-\frac{1+\sqrt{1+4j(j+1)}}{2}}. \tag{28}$$

Now, by matching of the two asymptotic solutions (25) and (28) (as they have a common region of interest), we can determine coefficients $f_{1,2}$ as follows:

$$f_1 = d\frac{\Gamma(\sqrt{1+4j(j+1)})\Gamma(1-2iB)}{\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}-2iB\right)\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}\right)},$$
$$f_2 = d\frac{\Gamma(-\sqrt{1+4j(j+1)})\Gamma(1-2iB)}{\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}-2iB\right)\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}\right)}. \tag{29}$$

At this stage, by performing several consecutive analytical steps we will come to $|\mathcal{P}_{\text{in}}^{\infty}|$ and $|\mathcal{P}_{\text{ref}}^{\infty}|$, as two essential components involved in the extraction of scattering amplification factor Z_{jm} or cross section

$$Z_{jm} \equiv \frac{|\mathcal{P}_{\text{ref}}^{\infty}|^2}{|\mathcal{P}_{\text{in}}^{\infty}|^2}-1. \tag{30}$$

Expanding Eq. (27) around infinity together with the approximations $\frac{1}{z} \sim \frac{\Omega}{\omega}\cdot\frac{1}{r}$ and $e^{\pm ik_{qh}z} \sim e^{\pm i\sqrt{(\omega^2-\mu_s^2)}r}$ and matching it with the radial solution (20), after inserting the expressions of f_1 and f_2 from Eq. (29), we finally get

$$\mathcal{P}_{\text{in}}^{\infty} = \frac{b(-2i)^{-\frac{1+\sqrt{1+4j(j+1)}}{2}}}{\sqrt{\omega^2-\mu_s^2}}\cdot\frac{\Gamma(\sqrt{1+4j(j+1)})\Gamma(1+\sqrt{1+4j(j+1)})}{\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}-2iB\right)\left(\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}\right)\right)^2}\Gamma(1-2iB)k_{qh}^{\frac{1-\sqrt{1+4j(j+1)}}{2}}+\frac{b(-2i)^{\frac{\sqrt{1+4j(j+1)}-1}{2}}}{\sqrt{\omega^2-\mu_s^2}}$$
$$\times\frac{\Gamma(1-\sqrt{1+4j(j+1)})\Gamma(-\sqrt{1+4j(j+1)})}{\left(\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}\right)\right)^2\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}-2iB\right)}\Gamma(1-2iB)k_{qh}^{\frac{1+\sqrt{1+4j(j+1)}}{2}}, \tag{31}$$

and

$$\mathcal{P}^{\infty}_{\rm ref} = \frac{b(2i)^{-\frac{1+\sqrt{1+4j(j+1)}}{2}}}{\sqrt{\omega^2-\mu_s^2}} \cdot \frac{\Gamma(\sqrt{1+4j(j+1)})\Gamma(1+\sqrt{1+4j(j+1)})}{\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}-2iB\right)\left(\Gamma\left(\frac{1+\sqrt{1+4j(j+1)}}{2}\right)\right)^2}\Gamma(1-2iB)k_{qh}^{\frac{1-\sqrt{1+4j(j+1)}}{2}} + \frac{b(2i)^{\frac{\sqrt{1+4j(j+1)}-1}{2}}}{\sqrt{\omega^2-\mu_s^2}}$$
$$\times \frac{\Gamma(1-\sqrt{1+4j(j+1)})\Gamma(-\sqrt{1+4j(j+1)})}{\left(\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}\right)\right)^2\Gamma\left(\frac{1-\sqrt{1+4j(j+1)}}{2}-2iB\right)}\Gamma(1-2iB)k_{qh}^{\frac{1+\sqrt{1+4j(j+1)}}{2}}. \quad (32)$$

To find more details of the trend of the above calculations, we recommend referring to previous works [86,88,90] (also, review paper [22]). Concerning Eq. (30), there will be superradiance phenomena, precisely in the case $\frac{|\mathcal{P}^{\infty}_{\rm ref}|^2}{|\mathcal{P}^{\infty}_{\rm in}|^2} > 1$ i.e., when $Z_{jm} > 0$. Since we are interested in the occurrence of the superradiant phenomenon, in this paper we ignore cases where $m \leq 0$ as they are nonsuperradiant. In what follows we will see how the hairy Horndeski parameter h favorably affects the scalar superradiant scattering.

Through the display of the plots $Z_{11,22} - M\omega$ we evaluate the role of Horndeski parameter h on the amplification factor of BH superradiance. Figure 2 clearly shows that in the presence of hairy parameter the amplification factor of superradiance scattering and whose frequency range becomes larger and wider than the standard Kerr case, respectively. It means that the hairy Horndeski parameter h acts as an amplifier of the scalar wave and enhances the chance of occurrence of superradiance phenomena.

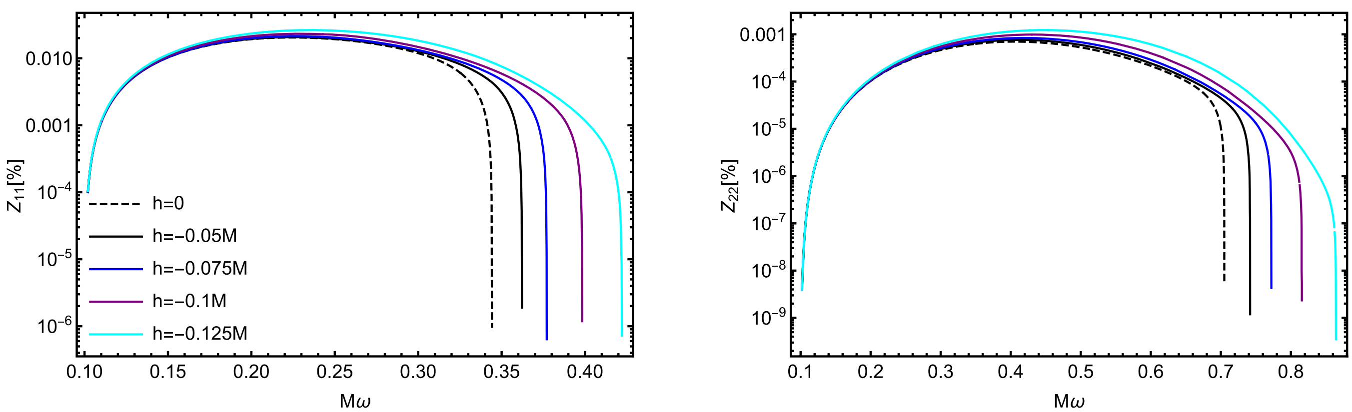

FIG. 2. Percentage amplification factors Z_{11} and Z_{22} in terms of frequency for the rotating hairy Horndeski BH with variable values of hairy parameter h. Here and in the latter figures we take numerical values $\mu_s = 0.1$ and $a = 0.95M$ for the mass of the scalar-bosonic field and the rotation-parameter ratio of angular momentum to BH mass, respectively. Values fixed for h in the right panel are the same in the left panel.

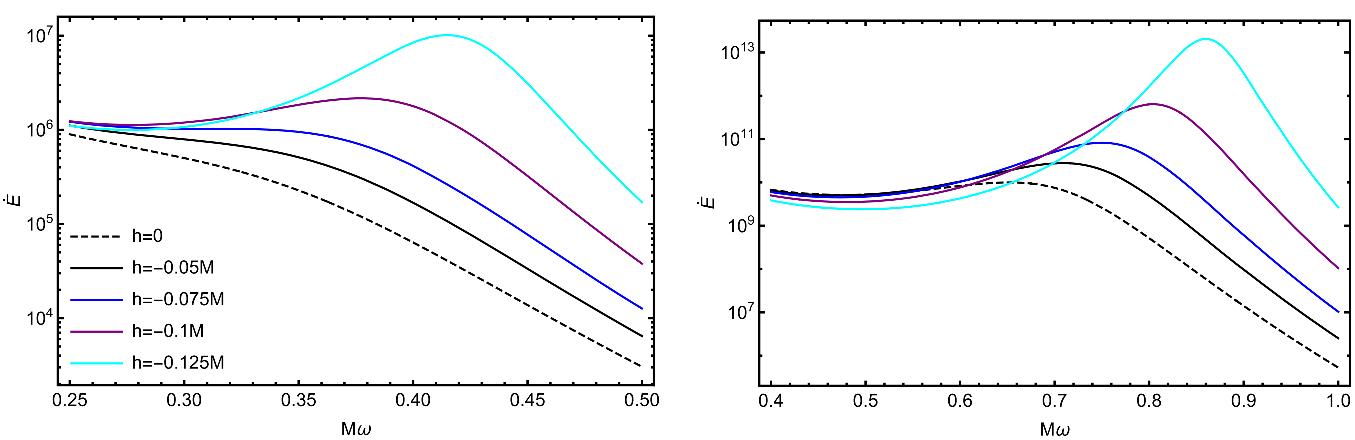

FIG. 3. Outgoing energy flux at infinity in terms of $M\omega$ for superradiant modes $l = 1 = m$ (left panel) and $l = 2 = m$ (right panel). Values fixed for h in the right panel are the same as the left panel.

TABLE I. Numerical values of $\dot{E}_{\rm tot}$ in terms of hairy parameter h/M for a BH with rotation parameter $a = 0.95M$, and superradiant modes, $l = 1 = m$ and $l = 2 = m$.

h/M	$\dot{E}_{\rm tot}(l = 1 = m)$	$\dot{E}_{\rm tot}(l = 2 = m)$
0	3.5×10^4	3.65×10^7
−0.05	7.3×10^4	8.45×10^7
−0.075	1.3×10^5	2.01×10^8
−0.1	4×10^5	1.11×10^9
−0.125	1.2×10^6	2.13×10^{10}

B. Energy extraction

Above revealed the role of hairy Horndeski parameter h on amplifying the scalar wave scattered off a rotating hairy Horndeski BH. This amplifying scenario is indeed equivalent to energy extraction from a BH. Here, we plan to show this via the investigation of the impact of the hairy Horndeski parameter on luminosity. The outgoing energy flux $\dot{E}$ measured by a observer at infinity, namely for $r_{eh} \ll r \ll \mu_s^{-1}$, can be calculated from the energy-momentum tensor of the field. If the massive scalar field is monochromatic, one can derive it as follows [22,85]:

$$\dot{E} = \frac{\omega k_\infty}{2} |\mathcal{P}^\infty_{\rm ref}|^2 |\mathcal{P}^\infty_{\rm in}|^2. \tag{33}$$

By taking the superradiant modes into account of Eq. (33), we display plot $\dot{E} - M\omega$ in Fig. 3. One can see the effect of the enhancement of the extraction of energy from the BH in comparison with the standard Kerr. Setting different values for h which ensure the nature of the compact object as a BH, the energy extraction in peak frequencies that differ from case $h = 0$ may boosted up by a few orders of magnitude. One effective technique to more clearly distinguish Horndeski's and Einstein's theories of gravity is to integrate from Eq. (33), over the flux contribution of each mode weighted by the normalized initial mode distribution $n(\omega)$ i.e.,

$$\dot{E}_{\rm tot} = \int d\omega \frac{\omega k_\infty}{2} |\mathcal{P}^\infty_{\rm ref}|^2 |\mathcal{P}^\infty_{\rm in}|^2 n(\omega). \tag{34}$$

To determine the function related to $n(\omega)$, we consider a thermal spectrum at temperature T for the incident spectrum which obeys from

$$n(\omega) = \frac{\omega^2}{2\zeta(3) k_B T^3 (\exp[\omega/k_B T] - 1)}, \tag{35}$$

where is the normalized black body mode spectrum of a massless scalar field ($\mu_s = 0$). Here, $\zeta(x)$ and k_B, are the Riemann zeta function ($\zeta(3) \simeq 1.2$), and Boltzmann constant, respectively. By setting the temperature of the thermal spectrum around CMB ($\simeq 2.7$ K) as a background source, in Table I, we list values derived of numerically solving of Eq. (34) in terms of the hairy parameter h/M for superradiant modes at hand.[7] The main message of the trend of numbers in the Table I, is that superradiant energy flux is amplified by parameter h/M, as shown earlier.

IV. SUPERRADIANT INSTABILITY REGIME

In this section, we investigate the effects of the hairy Horndeski parameter on the stability of rotating BH via a phenomenon known as the BH bomb [44]. The basic idea behind this phenomenon is to use and enclose the extracted rotational energy, by a mirrorlike surface whether natural (massive scalar field and AdS spacetime) or artificial (any reflecting surface) outside of BH for gradually growing and amplifying waves via frequent round trips. More technically, here superradiant instability is a consequence of enclosing the massive modes of a system composed of the Kerr background enriched with hairy Horndeski parameter (9) and the massive scalar perturbations Φ, inside the effective potential well placed outside the BH. In other words, to trigger superradiant instability, the existence of a potential well outside the BH, apart from the ergoregion, is essential.

Beginning from the radial equation (13) we get

$$\tilde{\Delta} \frac{d}{dr}\left(\tilde{\Delta} \frac{dR_{\omega jm}}{dr}\right) + \mathcal{G} R_{\omega jm} = 0, \tag{36}$$

where for a slowly-rotating BH ($a\omega \ll 1$)

$$\mathcal{G} \equiv ((r^2 + a^2)\omega - ma)^2 + \tilde{\Delta}(2ma\omega - j(j+1) - \mu_s^2 r^2).$$

Following the BH bomb mechanism, we get the following solutions for the radial equation (36)

[7] The one-dimensional integral equation (34) just like other analyses done in this paper is evaluated by the Mathematica software with version number (12.2.0.0) [157]. An integration strategy prescribes how to manage and create new elements of a set of separate subregions of the initial integral region. Each subregion might have its own integration rule related to it so that the integral estimation indeed is the sum of the integral estimates of all subregions. The integration rules to calculate the subregion integral estimates, in essence, are created by sampling the integrand by a set of points so-called sampling points. To improve an integral estimate it should be sampled at additional points. There are two main approaches; adaptive and nonadaptive strategies. In the former is identifies the problematic integration areas and concentrate the computational efforts (i.e., sampling points) on them, while in the latter increase the number of sampling points over the whole region. The default strategy for numerical solving the integral in the Mathematica software i.e., an algorithm to compute integral estimates according to the precision specified by the user is the so-called "Global Adaptive' algorithm'. This method, in general, by recursive bisecting the subregion with the largest error estimate into two halves reaches the required precision goal of the integral estimate and thereby, calculates the integral estimate for each half.

$$R_{\omega jm} \sim \begin{cases} e^{-i(\omega - m\Omega_{eh})} r_* & \text{as } r \to r_{eh} \quad (r_* \to -\infty) \\ \frac{e^{-\sqrt{\mu_s^2 - \omega^2 r_*}}}{r} & \text{as } r \to \infty \quad (r_* \to \infty). \end{cases}$$

The above solution represents the physical boundary conditions that the scalar wave at the BH horizon is purely ingoing while at spatial infinity it is a decaying exponential (bounded) solution, provided that $\omega^2 < \mu_s^2$. With the new radial function

$$\psi_{\omega jm} \equiv \sqrt{\tilde{\Delta}} R_{\omega jm},$$

the radial equation (36) yields the Regge-Wheel equation

$$\left(\frac{d^2}{dr^2} + \omega^2 - V\right)\psi_{\omega jm} = 0,$$

with

$$\omega^2 - V = \frac{\mathcal{G} + \mathcal{H}}{\tilde{\Delta}^2},$$

where

$$\mathcal{H} = \frac{-2a^2(h+2r) + r(h^2 + 2hr + 4M^2) + h^2 r \log^2(\frac{r}{2M}) - 4hMr \log(\frac{r}{2M})}{4r}. \quad (37)$$

By discarding the terms $\mathcal{O}(1/r^2)$, the asymptotic form of the effective potential $V(r)$ takes the following form:

$$V(r) = \mu_s^2 - \frac{(2\omega^2 - \mu_s^2)(2M - h\log(\frac{r}{2M}))}{r}. \quad (38)$$

As a cross-check, one can see that if $h = 0$, the above expression recovers its standard form [34]. The potential represents trapping well when its asymptotic derivative is positive i.e., $V' \to 0^+$ as $r \to \infty$ [34]. By taking derivative of potential (38), we have

$$\frac{dV}{dr} = \frac{(2\omega^2 - \mu_s^2)(2M + h(1 - \log(\frac{r}{2M}))}{r^2}. \quad (39)$$

Given that by default the hairy parameter is negative ($h < 0$), then by demanding $\log(\frac{r}{2M}) \geq 1$ i.e., $r \gtrsim 5.5M$, the expression above satisfies condition $V' \to 0^+$, if

$$\mu_s < \sqrt{2}\omega, \quad (40)$$

By taking this fact into account that the superradiance amplification occurs when $\omega < m\Omega_{eh}$, so the integrated system of Horndeski BH and massive scalar fields may experience superradiant instability within the following regime

$$\mu_s < \sqrt{2} m\Omega_{eh}, \quad (41)$$

where is nothing but the standard superradiant instability regime is expected from a Kerr BH. Generally, for the system at hand, the hairy Horndeski parameter h has no deterministic role in the superradiant instability regime.

V. CLOSING REMARKS

In this paper, we first took into account a spherically symmetric spacetime metric associated with the nonrotating Horndeski BH, and then to study the superradiance phenomenon we used the rotational version developed in [140]. This metric is characterized by a hairy Horndeski parameter h, addressing a spacetime beyond Einstein's gravity. We have studied the superradiant scattering of the massive scalar test field in the background of the underlying spacetime and the extraction of energy from it. The motivation to apply such a modified geometry as a toy model for evaluating superradiant energy extraction comes from the fact that the statistical error reported in strong gravity tests potentially indicates small but detectable deviations from the standard Kerr BH.

Figure 1 reveals that depending on the suitable choice for the hairy Horndeski parameter $h < 0$ in interplay with spin parameter a, the compact object arising from the metric at hand is a BH. As has been seen, by fixing the value of a, for the case of $h < 0$, the spacing between the Cauchy horizon and the event horizon reduces, as the value of h becomes more negative so that after a certain value of h the horizon disappears i.e., there will be no longer a BH. Note that the critical value h beyond which the spacetime can not be BH substantially depends on the rotation of the BH.

Given our interest in superradiant energy extraction from the BH, until the end of our analysis, we have taken care of the aforementioned point for choices of the values of h. It is clear from Fig. 2 that for the rotating Horndeski BH, if the value of the hairy parameter h becomes more negative, thereby, the superradiance amplification factor and whose frequency range rises and widens, respectively, relative to the standard Kerr case that corresponds to case $h = 0$. It means that in an extended framework of gravity as the Horndeski model, scalar wave-based superradiance scattering is strengthened. In this direction, by deriving outgoing energy flux for the faraway observer, we have addressed explicitly the role of hairy parameter h on energy extraction from BH. In agreement with the enhancement of the superradiance amplification factor by hairy parameter h, we have demonstrated that it causes the increase of energy extraction from the rotation BH. It can be easily verified through Fig. 3 and Table I. As a consequence, the hairy parameter h amplifies the scalar wave and enhances the chance of the superradiance. Adding some comments here to understand the physics behind this phenomenon can be helpful. Recall that the friction and some negative-energy states for energy extraction via superradiance are essential. Concerning rotating BHs both, in essence, are supplied by the ergoregion as a region near the event horizon in which the energy of timelike particles is negative [22]. Besides, it is well-known that the background geometry plays role in increasing/decreasing the amplification factor of the wave so that the strengthening/weakening of the scattered waves from the rotating BHs, understand in terms of the increase/decrease of the proper volume of the ergoregion. Namely, between the ergoregion proper volume and the superradiant amplification factor, there is a correlation, meaning that the larger the former, the more time the wave spends in the ergoregion and the more energy it extracts from BH [158]. As a result, the modifications induced on the background geometry by alternative theories of gravity affect the proper volume, and the amplification factor subsequently. More precisely, as the proper volume of the ergoregion increases/decreases, more/less energy compared to Kerr is extracted through superradiance scattering. To support this statement, it is enough one calculate the proper volume via $V = 4\pi\int_{\theta_i}^{\pi/2} d\theta \int_{r_i}^{r_f} dr\sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}$ which for case of $a < M$ ergoregion extends from r_i to r_f i.e., between the location of event horizon and outer ergosphere radius [159]. In the Table II, by setting $\theta_i = 0$ for two cases $a = 0.9M$ and $0.95M$, we release the numerical values of the proper volume of ergoregion in terms of different values of hairy parameter h. As can be seen in the presence of the hairy parameter h correction, the proper volume of the ergoregion increases compared to standard Kerr BH, meaning that it behaves as an amplifier of the scalar wave.

TABLE II. Numerical values of the ergoregion proper volume in terms of hairy parameter h/M for a BH with the rotation parameters: $a = 0.9M$ and $a = 0.95M$.

h/M	Volume/M^3 ($a = 0.9M$)	Volume/M^3 ($a = 0.95M$)
0	84.5	96.4
−0.05	89.3	104.8
−0.075	91.94	110.5
−0.1	94.83	118.45
−0.125	97.4	128.8

The importance and worth of these results become especially clear when we contrast them with earlier findings from scalar-tensor theories utilizing Kerr BH surrounded by the matter profile [81,82]. It was demonstrated that the amplification factor in scalar-tensor gravity can be higher than in the typical situation because of scalar-matter interactions. For Horndeski gravity, as the most general four-dimensional scalar-tensor theory, this enhancement in the amplification factor occurs even in the absence of enclosing the BH by a matter profile.

By analyzing the effective potential within the context of BH bomb mechanism, we have surveyed the superradiant instability of the rotating Horndeski BH, subjected to massive scalar perturbation. We found out that the hairy Horndeski parameter h leaves no effect on the standard superradiant instability regime.

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