Conformally coupled scalar in Lovelock theory

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In arbitrary higher dimension, we consider the combination of Lovelock gravity alongside a scalar-tensor action built out of higher order operators and Euler densities. The latter action is constructed in such a way as to ensure conformal invariance for the scalar field. For the combined version of these theories, we show the existence of black hole solutions interpreted as stealth configurations within Lovelock gravity theory. The scalar field solutions are endowed with an integration constant that may be identified as a scalar charge. In particular, we show that these stealth solutions can be extended to include a time-dependent scalar field despite the underlying theory being non shift symmetric. Finally, we present a procedure to obtain a nonconformally invariant action in even dimensions from the considered theory. For the target theory, the scalar field is not conformally coupled to gravity although the scalar field equation itself is conformally invariant. By means of this procedure, the black hole stealth configurations are converted into nonstealth black hole solutions, as discovered recently in four dimensions.

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I. INTRODUCTION

With the given precision of observational data, the theory of general relativity remains unchallenged. However, given that general relativity fails to give a self-consistent quantum gravity theory and the yet unknown nature of dark energy and dark matter, quite naturally, the scientific community scrutinizes modified theories of gravity. One of the simplest, nontrivial yet robust modifications consists in introducing a scalar field, (non)minimally coupled to the metric, yielding the so-called scalar tensor theory. The search for black holes for such theories finds its origin with the pioneering work of Bocharova, Bronnikov, Melnikov [1] and Bekenstein (BBMB) [2] who were the first to exhibit a nontrivial, asymptotically flat four-dimensional hairy black hole with a conformally coupled scalar field. In the Jordan frame, the action for the scalar field is given by the standard kinetic term together with a coupling between the scalar field φ and the scalar curvature R,

$$S = b_1 \int d^D x \sqrt{-g}\left(-\frac{1}{2}(\partial\varphi)^2 - \frac{(D-2)}{8(D-1)} R\varphi^2\right), \quad (1)$$

where D stands for the dimension and b_1 is a coupling constant. The solution to the theory (1) in $D=4$, coupled to the Einstein-Hilbert action, is known as the BBMB solution, with a metric corresponding to an extremal Reissner-Nordström spacetime while the scalar field is shown to blow up at the horizon. Note that this pathology can be cured by adding a cosmological constant with a conformally invariant self-interacting potential [3,4] and by adding axionic fields in the case where the $(D-2)$-orthogonal Euclidean space is a plane [5]. To be more complete, we mention that black hole solutions with a self-interacting potential breaking the conformal invariance also exist in four dimensions [6], and these latter can be generated by a certain mapping from the conformal solutions [7] (see also [8]).

In higher dimensions $D > 4$, the extension of the BBMB solution is known, but unfortunately the metric has a naked singularity, which cannot be removed, unlike the four-dimensional case [9]. More recently, a conformal action generalizing (1) was proposed in higher D, via nonminimal couplings of the scalar field with a four-rank tensor built out of the Riemann tensor and the scalar field [10]. Such a generalized conformal scalar field coupled to the Einstein or Lovelock gravity gives rise to an analog of the BBMB black hole solutions with (anti–)de Sitter [(A)dS] asymptotics [11], while the couplings introduced in [10] were used in [12] as a counterterm for spaces with AdS asymptotics. The lesson that can be drawn from these studies is that the conformal invariance of the action of the scalar field plays an important role, and notably in order to obtain analytical solutions of the black hole type. One should note, however, that the full action giving rise to the BBMB solution and to its extension in higher dimensions is not conformally invariant, since, apart from the conformally invariant part, the full action contains the Einstein-Hilbert or Lovelock terms. Strictly speaking, the conformal symmetry only holds at the level of the equation of motion for the scalar field. It is then natural to ask whether the conformal invariance of a part of the action is crucial.

Indeed, recently it has been shown in $D = 4$ that this assumption can be relaxed by requiring only the conformal invariance of the scalar field equation of motion [13]. In this case, two classes of black hole solutions with a regular scalar field (even in the absence of the cosmological constant) were found for different fine-tuning of the coupling constants of the theory [13].

In the present paper, we show that the theories constructed in [11] admit in addition to the presented solution there, stealth black hole solutions.[1] These are of the Schwarzschild-(A)dS type for pure Einstein gravity and Boulware-Deser spacetimes [14] in the Einstein-Gauss-Bonnet theory: see [15] for the topological case and [16,17] for general Lovelock theory. In all these cases, our solutions have a nontrivial profile for the scalar field with an additional constant of integration that may be interpreted as an independent scalar charge. This scalar charge nevertheless does not appear in the metric. In other words we have, apart from the mass of the black hole, an additional independent charge (not modifying the metric); therefore the solutions we will describe have neither primary nor secondary hair. We will refer to this constant simply as scalar hair. In addition, introducing extra assumptions on the parameters of the action, stealth configurations defined on the same black hole spacetimes, albeit with a time-dependent scalar field, can also be constructed. This result is all the more surprising since the theories under consideration are not even shift symmetric. We will see how such a construction is possible even in the absence of symmetry. Last but not least, we will present a procedure yielding a nonconformally invariant 4D action for the scalar field from the generalized conformal scalar of [11] by performing a singular limit. The resulting action will lead to a conformally invariant scalar field equation of Ref. [13]. We show that the singular limit is also compatible at the level of the black hole solutions and allows us to map the stealth black hole solutions in higher dimensions to the four-dimensional nonstealth black hole solutions of Ref. [13]. Similarly, in higher even dimensions, (nonstealth) black holes in a generalization of the theory [13] are obtained by means of this singular limit from the stealth black holes.

The plan of the paper is organized as follows. In the next section, we present stealth black hole solutions of the theory [11]. The extension of this solution to a time-dependent scalar field is given in Sec. III. The singular limit that allows to construct nonstealth black hole solutions in a theory with conformal scalar equation of motion, from stealth solutions of the theory [11] is explained in Sec. IV. A last section is devoted to our conclusions.

[1]In a scalar-tensor theory, a solution is called stealth if its spacetime coincides with the one from a pure metric theory, while having a nontrivial scalar field, which means that the energy-momentum tensor of the scalar field vanishes on shell.

II. STEALTH BLACK HOLES WITH A CONFORMALLY COUPLED SCALAR IN LOVELOCK THEORY

In order to be self-contained, we recall the useful formalism and notations of [10,11] used for the construction of the most general theory of gravity conformally coupled to a single scalar field and yielding second-order field equations. This construction is aimed to generalize the action (1). Indeed, consider a four-rank tensor $S_{\mu\nu}{}^{\gamma\delta}$ constructed out of the Riemann curvature tensor $R_{\mu\nu}{}^{\gamma\delta}$ and the scalar field, ϕ,

$$S_{\mu\nu}{}^{\gamma\delta} = \phi^2 R_{\mu\nu}{}^{\gamma\delta} - 4\phi\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\nabla^{\delta]}\phi + 8\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\phi\nabla^{\delta]}\phi - 2\delta^{[\gamma}_{[\mu}\delta^{\delta]}_{\nu]}\nabla_\rho\phi\nabla^\rho\phi, \qquad (2)$$

where brackets stand for antisymmetrization. One can check to see that this tensor, under a conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\phi \to \Omega^{-1}\phi$, transforms covariantly, i.e. $S_{\mu\nu}{}^{\gamma\delta} \to \Omega^{-4} S_{\mu\nu}{}^{\gamma\delta}$. In arbitrary dimension D, the action we will consider is given by

$$S = \int d^D x\sqrt{-g}\sum_{k=0}^{[\frac{D-1}{2}]}\frac{1}{2^k}\delta^{(k)}(a_k R^{(k)} + b_k\phi^{D-4k}S^{(k)}), \qquad (3)$$

where a_k and b_k are *a priori* arbitrary coupling constants,[2] where $\delta^{(k)}$ is defined as

$$\delta^{(k)} = (2k)!\delta^{\mu_1}_{[\alpha_1}\delta^{\nu_1}_{\beta_1}\ldots\delta^{\mu_k}_{\alpha_k}\delta^{\nu_k}_{\beta_k]}$$

and where $R^{(k)}$ and $S^{(k)}$ are given by

$$R^{(k)} = \prod_{r=1}^{k} R^{\alpha_r\beta_r}{}_{\mu_r\nu_r}, \qquad S^{(k)} = \prod_{r=1}^{k} S^{\alpha_r\beta_r}{}_{\mu_r\nu_r}. \qquad (4)$$

The $R^{(k)}$ mark Lovelock scalars of increasing rank k ($k = 0$ cosmological constant, $k = 1$ Einstein-Hilbert, $k = 2$ Gauss-Bonnet, etc.) while $S^{(k)}$, the specific scalar tensor combinations obtained from (2). It is then easy to see that, due to the covariant transformation of the $4k$-rank $S_{\mu\nu}{}^{\gamma\delta}$, the different b_k parts of the action (3) will independently acquire conformal invariance. Following [10,11], the metric field equations can be written as $G_{\mu\nu} = T_{\mu\nu}$, with

$$G^\nu_\mu = -\sum_{k=0}^{[\frac{D-1}{2}]}\frac{a_k}{2^{k+1}}\delta^{\nu\lambda_1\cdots\lambda_{2k}}_{\mu\rho_1\cdots\rho_{2k}} R^{\rho_1\rho_2}{}_{\lambda_1\lambda_2}\cdots R^{\rho_{2k-1}\rho_{2k}}{}_{\lambda_{2k-1}\lambda_{2k}}, \qquad (5)$$

[2]In order to simplify the notations, we will fix the coupling $a_0 = -2\Lambda$ and $a_1 = 1$.

$$T^{\nu}_{\mu} = \sum_{k=0}^{[\frac{D-1}{2}]} \frac{b_k}{2^{k+1}} \phi^{D-4k} \delta^{\nu\lambda_1\cdots\lambda_{2k}}_{\mu\rho_1\cdots\rho_{2k}} S^{\rho_1\rho_2}{}_{\lambda_1\lambda_2} \cdots S^{\rho_{2k-1}\rho_{2k}}{}_{\lambda_{2k-1}\lambda_{2k}}, \quad (6)$$

while the scalar field equation reads

$$\sum_{k=0}^{[\frac{D-1}{2}]} \frac{(D-2k)b_k}{2^k} \phi^{D-4k-1} \delta^{(k)} S^{(k)} = 0. \quad (7)$$

Black hole solutions with secondary hair have been obtained for this theory in Ref. [11]. We now proceed to show that the theory (3) admits another class of black hole solutions with scalar hair, with an ansatz of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{D-2,\gamma}, \qquad \phi = \phi(r), \quad (8)$$

where $d\Sigma^2_{D-2,\gamma}$ is the metric of a $(D-2)$-dimensional Euclidean space of constant curvature $\gamma(D-2)(D-3)$ with $\gamma = (0, \pm 1)$.

When the a_k part of the action only contains the Einstein-Hilbert term with (potentially) a cosmological constant, that is $a_k = 0$ for $k > 1$, two different analytic classes of solutions can be found for the ansatz (8). These two classes correspond to two different relations between the coupling constants of the action. The solutions can be generically given in terms of the metric functions

$$f^{(i)}(r) = \gamma - \frac{M}{r^{D-3}} - \frac{2\Lambda}{(D-1)(D-2)} r^2 + \frac{q^{(i)}}{r^{D-2}}, \quad (9)$$

dressed with a scalar field given by

$$\phi^{(1)}(r) = \frac{N}{r}, \quad (10)$$

$$\phi^{(2)}(r) = \frac{N}{r\sigma_\gamma(c \pm \int \frac{dr}{\sqrt{f^{(2)}(r)}})}, \quad (11)$$

where the index (i) denotes the first and the second class of the solution, and the function σ_γ depends on the topology of the base manifold

$$\sigma_1(X) = \cosh(X), \qquad \sigma_{-1}(X) = \cos(X), \qquad \sigma_0(X) = X.$$

In the above expressions, M is an integration constant proportional to the mass, while the constant c appearing in the scalar field, for the second class of solutions (11), is the scalar hair. The constant N of both scalar fields (10)–(11) is fixed in terms of the coupling constants of the theory through the relation

$$\sum_{k=1}^{[\frac{D-1}{2}]} k \frac{b_k}{(D-2k-1)!} \tilde{\gamma}^{k-1}_{(i)} N^{2-2k} = 0, \quad (12)$$

while the coupling of the conformal potential $b_0^{(i)}$ is fixed in terms of other couplings as

$$\frac{D(D-1)}{(D-1)!} b_0^{(i)} + \sum_{k=1}^{[\frac{D-1}{2}]} \frac{(D(D-1) + 4\epsilon_k^{(i)}) b_k \tilde{\gamma}^k_{(i)}}{N^{2k}(D-2k-1)!} = 0, \quad (13)$$

with $\epsilon_k^{(1)} = k^2$, $\epsilon_k^{(2)} = k$, $\tilde{\gamma}_{(1)} = \gamma$, and $\tilde{\gamma}_{(2)} = \gamma - \delta_{\gamma,0}$. Finally, for both solutions the constant $q^{(i)}$ appearing in the metric function (9) is fixed in terms of the coupling constants as

$$q^{(i)} = -\frac{b_0^{(i)}}{(D-2)} N^D - \sum_{k=1}^{[\frac{D-1}{2}]} \frac{b_k (D-3)! \tilde{\gamma}^k_{(i)}}{(D-2k-2)!} N^{D-2k}. \quad (14)$$

The first class of solutions with $i = 1$ has $q^{(1)} \neq 0$ for $\gamma \neq 0$ and corresponds to the black hole with secondary hair found in [11]. For the second class of solutions for $i = 2$, we have $q^{(2)} = 0$, and hence the scalar hair solution can be interpreted as a stealth solution on the Schwarzschild-(A) dS spacetime, see Eq. (9). Importantly, the two classes of spacetimes $i = 1, 2$ are solutions of distinct theories since $b_0^{(1)} \neq b_0^{(2)}$ as shown by (13).

In the general Lovelock case, where $a_k \neq 0$ for at least one $k > 1$, similar classes of solutions exist. The scalar field profiles keep the same form (10)–(11) and are subjected to the same conditions (12) and (13), while the metric functions $f^{(i)}$ have a different form and are now given by a polynomial equation of order $[\frac{D-1}{2}]$,

$$\sum_{k=0}^{[\frac{D-1}{2}]} \frac{a_k (D-1)!}{(D-2k-1)!} \left(\frac{\gamma - f^{(i)}(r)}{r^2}\right)^k = \frac{M(D-1)(D-2)}{r^{D-1}} - \frac{q^{(i)}(D-1)(D-2)}{r^D}, \quad (15)$$

where M is an arbitrary constant related to the mass, and $q^{(i)}$ are given again by (14), meaning in particular that $q^{(2)} = 0$. It follows then that the second class of solutions can be interpreted as stealth black holes of Lovelock theory (see [16,17]). In the quadratic case $a_k = 0$ for $k > 2$, the real roots of this polynomial can be easily written down and we have a Boulware-Deser black hole [14] (see [18] for a review) while for the other cases, the expression for f is quite cumbersome, except the case when the polynomial equation (15) has a single root. This occurs for the particular choice of the coupling constants

$$a_k = C_k^{[\frac{D-1}{2}]}\frac{(D-2k-1)!}{(D-1)!},$$

which in an odd number of dimensions corresponds to the Chern-Simons point. For this particular choice, one can easily express the solution for metric function in an odd dimension as

$$f^{(i)}(r) = \gamma + r^2 - \left(\tilde{M} - \frac{\tilde{q}^{(i)}}{r}\right)^{\frac{2}{D-1}}, \tag{16}$$

while in an even dimension we have

$$f^{(i)}(r) = \gamma + r^2 - \left(\frac{\tilde{M}}{r} - \frac{\tilde{q}^{(i)}}{r^2}\right)^{\frac{2}{D-2}}, \tag{17}$$

where we have defined $\tilde{M} = M(D-1)(D-2)$ and $\tilde{q}^{(i)} = q^{(i)}(D-1)(D-2)$. For the second solution $\tilde{q}^{(2)} = 0$, the spacetime metrics correspond to the black hole solutions obtained in [19].

III. TIME-DEPENDENT SOLUTIONS IN THEORIES WITH NO SHIFT SYMMETRY

As it was originally shown in [20], scalar tensor theories with shift symmetry $\phi \to \phi + \mathrm{const}$ may accommodate black hole solutions with a scalar field that depends linearly on time. The underlying idea of this feature is that the field equations only involve derivatives of the scalar field, and hence its explicit time dependence does not appear at the level of the field equations. Here, the action (3) is not shift symmetric, nevertheless, if $b_0 = b_1 = 0$ in the action (3), the stealth metric function $f^{(2)}(r)$ with $q^{(2)} = 0$ can be dressed with a time-dependent scalar field given by

$$\phi(t,r) = \exp\left(c + \zeta t + \int \frac{\pm\sqrt{\gamma f^{(2)}(r) + \zeta^2 r^2/f^{(2)}(r) - 1}}{r}dr\right), \tag{18}$$

where c and ζ are arbitrary constants. The emergence of such stealth solutions in spite of the absence of shift symmetry in the theory under consideration can be understood as follows. The vanishing condition of the energy-momentum tensor of the scalar field can be schematically written as

$$\sum_{k\geq 2} b_k \phi^{D-2k}\mathcal{A}_{\mu\nu}^{(k)} = 0, \tag{19}$$

where the $\mathcal{A}_{\mu\nu}^{(k)}$ for $k \geq 2$ only depend on the derivatives of $\Phi \equiv \log\phi$. One can clearly see that the above expression is not shift symmetric, since it involves explicit dependence on the scalar field, in accord with the fact that the action is not shift symmetric. One can verify however that for the stealth configuration described by the metric function $f^{(2)}(r)$ and the time-dependent scalar field (18), each $\mathcal{A}_{\mu\nu}^{(k)}$ vanishes identically, and one gets a solution which is effectively shift symmetric for $\Phi = \ln\phi$, as highlighted by the form of (18).

IV. FROM CONFORMAL ACTION TO CONFORMAL EQUATION

Here, we present a limiting process in even dimensions which breaks the conformal symmetry of the scalar field action (3) but still preserving the conformal symmetry of the scalar field equation. Such an action has been recently proposed in four dimensions [13] and is given in the present notations by

$$S_{\mathcal{F}} = \int d^4x\sqrt{-g}\left[R - 2\Lambda + b_0\phi^4 + b_1\phi^2\left(R + 6\frac{(\partial\phi)^2}{\phi^2}\right) + b_2\left(\log(\phi)\mathcal{G} - 4\frac{G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{\phi^2} - 4\frac{\Box\phi(\partial\phi)^2}{\phi^3} + 2\frac{(\partial\phi)^4}{\phi^4}\right)\right], \tag{20}$$

where $\mathcal{G}$ is the Gauss-Bonnet density $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\delta}R^{\mu\nu\lambda\delta}$. In order to make apparent this limiting process, let us consider the action (3) in arbitrary dimension D and rewrite it in a similar way, assuming also $a_k = b_k = 0$ for $k > 2$,

$$S = \int d^Dx\sqrt{-g}\left[R - 2\Lambda + a_2\mathcal{G} + b_0\phi^D + b_1\phi^{D-2}\left(R + (D-1)(D-2)\frac{(\partial\phi)^2}{\phi^2}\right) + b_2\phi^{D-4}\left(\mathcal{G} - 4(D-3)(D-4)\frac{G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{\phi^2} - 2(D-2)(D-3)(D-4)\frac{\Box\phi(\partial\phi)^2}{\phi^3} - (D-2)(D-3)(D-4)(D-5)\frac{(\partial\phi)^4}{\phi^4}\right)\right], \tag{21}$$

and let us show how the action (20) can be obtained from (21) by a singular limit. This is done by rescaling the coupling constant $b_2 \to \frac{b_2}{D-4}$, fixing the Gauss-Bonnet coupling $a_2 = -\frac{b_2}{D-4}$, performing a Taylor expansion of ϕ^{D-4} at the neighborhood of $D = 4$, i.e. $\phi^{D-4} = 1 + (D-4)\log(\phi) + o(D-4)$, and finally taking the limit $D \to 4$. This procedure only works for a nonvanishing Gauss-Bonnet coupling a_2, and hence at the level of the solutions, the limit makes sense only for the two classes of solutions presented before in the Lovelock case, and not for the pure Einstein case. One can verify that following the above prescription, one recovers two classes of four-dimensional solutions of the action (20) discovered in [13], from the solutions (10)–(15). In a similar way, the higher-dimensional time-dependent stealth solution of (21) with $b_0 = b_1 = 0$, with the scalar given by (18), projects to the time-dependent nonstealth solution of (20) with $b_0 = b_1 = 0$ as given in [21].

The same procedure can be easily extended to any even dimension $D = 2p$ with $p \geq 2$ where the Euler density $\delta^{(p)} R^{(p)}$ is a boundary term. Indeed, starting from the action given by (3) with $a_k = b_k = 0$ for $k > p$, one should rescale $b_p \to \frac{b_p}{D-2p}$, fix the Euler coupling $a_p = -\frac{b_p}{D-2p}$ and perform a Taylor expansion around $D = 2p$, and finally take the limit $D \to 2p$. The result of this procedure is an action of a nonconformal scalar field coupled to Lovelock gravity, yielding however a scalar field equation *which is* conformally invariant. Some details of this limiting procedure are given in the Appendix. Two time-independent solutions of the resulting action can be read off from Eqs. (10)–(15) by applying them the described limit. It is worth noting how the procedure works for the time-dependent solutions in the general Lovelock theory in even dimension $D = 2p > 4$. Indeed, the energy-momentum tensor (19) must not vanish since the projected solution yields a nonstealth solution and must not depend on the time coordinate t since the metric solution is time independent. In fact, one can see that, in the considered limit and after rescaling the couplings, all the $\mathcal{A}_{\mu\nu}^{(k)}$ of (19) do vanish on the projected solution, except $\mathcal{A}_{\mu\nu}^{(p)}$, whose time-dependent factor ϕ^{D-2p} disappears precisely in the limit $D \to 2p$.

V. CONCLUDING REMARKS

In this paper we presented three main results. First, we showed that the conformally coupled scalar field in Lovelock theory admits a class of black hole stealth configurations, Eqs. (9), (11), and (14) with the subscript $i = 2$. The metric in this case is nothing but the Boulware-Deser spacetime [14] in the quadratic case, or its extension for higher Lovelock theory [16]. The expression of the scalar field contains the metric function and a constant of integration that may be interpreted as the scalar charge of the field.

We then demonstrated that in the particular case of the coupling constants $b_0 = b_1 = 0$ in the action (3), these stealth configurations can be endowed with a time-dependent scalar field, Eq. (18).

Finally, a singular limit in even dimension $D = 2p$ was presented, which allows to obtain a nonconformally coupled scalar field, starting from a conformally coupled invariant scalar field in the Lovelock gravity. The obtained action includes a direct coupling between the scalar field and the Euler density of order p, which breaks the conformal symmetry at the level of the action. Nevertheless for such an action a conformal invariance is kept at the level of the scalar equation of motion. Within this singular limit, the black hole stealth configurations (both static and with a time-dependent scalar) in even dimensions are converted to solution for black holes with a nonvanishing energy-momentum tensor of the scalar field. In [22], it was already mentioned that the nonconformal action (20) can be obtained from an alternative Kaluza-Klein compactification of the D-dimensional Einstein-Gauss-Bonnet theory, while more recently [23] studied a Kaluza-Klein compactification yielding the same $D = 6$ action coupling the scalar field with the cubic Euler density.

There are yet open questions left for future work concerning dimensional reduction procedures. In particular, in Ref. [24], it was shown that eternal wormholelike solutions can also be generated from the four-dimensional black hole configurations of (20) by means of a disformal transformation. An interesting question in the context of our present work is whether these solutions correspond to some higher dimensional solutions.

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APPENDIX: SINGULAR LIMIT

The action (3) can be decomposed as

$$S = S^{(a)} + S^{(b)}, \tag{A1}$$

where $S^{(a)}$ is the pure metric part with coefficients a_k, and $S^{(b)}$ is the scalar-tensor part with coefficients b_k and enjoying the conformal invariance,

$$S^{(b)}[\Omega^2 g_{\mu\nu}, \Omega^{-1}\phi] = S^{(b)}[g_{\mu\nu}, \phi]. \tag{A2}$$

Writing down the vanishing of the variation $\delta S^{(b)}$ under an infinitesimal transformation $\Omega = 1 + \epsilon$, one gets the identity

$$2g^{\mu\nu}\mathcal{E}^{(b)}_{\mu\nu} + \phi\mathcal{E}^{(b)}_{\phi} = 0, \tag{A3}$$

where

$$\mathcal{E}^{(i)}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S^{(i)}}{\delta g^{\mu\nu}}, \qquad \mathcal{E}^{(i)}_{\phi} = \frac{1}{\sqrt{-g}}\frac{\delta S^{(i)}}{\delta \phi}. \tag{A4}$$

Using (A3) and taking into account that $\mathcal{E}^{(a)}_{\phi} = 0$, one gets that the following combination of the equations of the full action (noted $\mathcal{E}_{\mu\nu}$, $\mathcal{E}_{\phi}$ with obvious notations) yields a pure geometric constraint:

$$2g^{\mu\nu}\mathcal{E}_{\mu\nu} + \phi\mathcal{E}_{\phi} = 2g^{\mu\nu}\mathcal{E}^{(a)}_{\mu\nu} = 0. \tag{A5}$$

Conversely, it was shown in [13,25] that an action, such that the combination of field equations given by the left-hand side of (A5) is a pure geometric equation, is not necessarily conformally invariant, but has a scalar field equation which is conformally invariant. Let us therefore show that the procedure described in Sec. IV transforms the geometric equation (A5) for $D \geq 2p + 1$ in another geometric equation in the singular limit $D \to 2p$. Note that the singular limit procedure does not affect the conformal symmetry of the Lagrangians $b_k\phi^{D-4k}\delta^{(k)}S^{(k)}$ for $k < p$. Thus, in order to simplify the presentation, we only focus on actions defined for $D \geq 2p + 1$ with $a_k = b_k = 0$ for $k \neq p$. We therefore have $\mathcal{E}^{(a)}_{\mu\nu} = G_{\mu\nu}$ and $\mathcal{E}^{(b)}_{\mu\nu} = -T_{\mu\nu}$ with

$$G^{\nu}_{\mu} = -\frac{a_p}{2^{p+1}}\delta^{\nu\lambda_1\cdots\lambda_{2p}}_{\mu\rho_1\cdots\rho_{2p}} R^{\rho_1\rho_2}{}_{\lambda_1\lambda_2}\cdots R^{\rho_{2p-1}\rho_{2p}}{}_{\lambda_{2p-1}\lambda_{2p}},$$
$$T^{\nu}_{\mu} = \frac{b_p}{2^{p+1}}\phi^{D-4p}\delta^{\nu\lambda_1\cdots\lambda_{2p}}_{\mu\rho_1\cdots\rho_{2p}} S^{\rho_1\rho_2}{}_{\lambda_1\lambda_2}\cdots S^{\rho_{2p-1}\rho_{2p}}{}_{\lambda_{2p-1}\lambda_{2p}}, \tag{A6}$$

while

$$\mathcal{E}^{(b)}_{\phi} = \frac{(D-2p)b_p}{2^p}\phi^{D-4p-1}\delta^{(p)}S^{(p)}, \tag{A7}$$

see Eqs. (5)–(7). On the other hand, the traces yield

$$G^{\nu}_{\nu} = \frac{(2p-D)a_p}{2^{p+1}}\delta^{(p)}R^{(p)},$$
$$T^{\nu}_{\nu} = -\frac{(2p-D)b_p}{2^{p+1}}\phi^{D-4p}\delta^{(p)}S^{(p)},$$

and hence one gets

$$2g^{\mu\nu}\mathcal{E}_{\mu\nu} + \phi\mathcal{E}_{\phi} = \frac{(2p-D)a_p}{2^p}\delta^{(p)}R^{(p)}. \tag{A8}$$

It is then easy to see that, under the redefinitions $b_p \to \frac{b_p}{D-2p}$ and $a_p \to -\frac{b_p}{D-2p}$, Eqs. (A7) and (A8) have a regular limit as $D \to 2p$ and that the right-hand side of (A8) is a pure geometric quantity, thus ensuring the conformal symmetry of the scalar field equation. As for the metric field equations (A6), they display a generalized δ-Kronecker symbol with $2p + 1$ indices, which vanish in $D = 2p$ and therefore gives rise to a vanishing factor $(D - 2p)$ in dimensional continuation. This vanishing factor compensates the infinite factor $(D - 2p)^{-1}$ from the rescaling of a_p and b_p, giving rise to finite metric field equations. Here, we have focused on the field equations and proved that the limiting scalar field equation is conformally invariant. Let us now show that the singular limit is also well defined at the level of the action.

Up to a global factor $2^{-p}\sqrt{-g}$, the considered Lagrangian density is

$$\mathcal{L}_p \equiv a_p\delta^{(p)}R^{(p)} + b_p\phi^{D-4p}\delta^{(p)}S^{(p)}, \tag{A9}$$

and for clarity, we define a function

$$W \equiv \phi^{D-4p}\delta^{(p)}S^{(p)} - \phi^{D-2p}\delta^{(p)}R^{(p)}$$
$$= \phi^{D-4p}\delta^{(p)}S^{(p)} - \delta^{(p)}R^{(p)} - (D-2p)(\log\phi)\delta^{(p)}R^{(p)}$$
$$+ o(D-2p).$$

Here and in what follows, the notations $o(\cdots)$ and $\mathcal{O}(\cdots)$ have to be understood in the limit $D \to 2p$. The variation of the first two terms in the last expression with respect to the metric are proportional to $T_{\mu\nu}$ and $G_{\mu\nu}$, respectively, see (A6). The resulting expressions contain a generalized δ Kronecker with $2p + 1$ indices, which vanish in $D = 2p$. This means that these first two terms are a boundary term in dimension $D = 2p$. Therefore, up to integration by parts, one has

$$\phi^{D-4p}\delta^{(p)}S^{(p)} - \delta^{(p)}R^{(p)} = \mathcal{O}(D-2p)$$

and

$$W = (D-2p)\tilde{W},$$

with $\tilde{W}$ being regular as $D \to 2p$. The Lagrangian $\mathcal{L}_p$ can thus be written as

$$\mathcal{L}_p = (a_p + b_p)\delta^{(p)}R^{(p)} + b_p(D-2p)$$
$$\times [\tilde{W} + (\log\phi)\delta^{(p)}R^{(p)} + o(1)].$$

As a consequence, the limiting procedure, namely the rescaling $b_p \to \frac{b_p}{D-2p}$, $a_p \to -\frac{b_p}{D-2p}$ followed by the limit $D \to 2p$ indeed yields a well-defined Lagrangian density,

$$\tilde{L}_p = b_p[\tilde{W} + (\log\phi)\delta^{(p)}R^{(p)}].$$

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