Nonlocal-in-time effective one body Hamiltonian in scalar-tensor gravity at third post-Newtonian order

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We complete the nonlocal-in-time effective-one-body (EOB) formalism of the conservative dynamics for massless scalar-tensor (ST) theories at third post-Newtonian (PN) order. The nonlocal-in-time EOB Hamiltonian is obtained by mapping the order-reduced Hamiltonian corresponding to the nonlocal-in-time Lagrangian derived in [Phys. Rev. D **99**, 044047 (2019)]. To transcribe the dynamics within the EOB formalism, we use a strategy of order-reduction of nonlocal dynamics to a local ordinary action-angle Hamiltonian. We then map this onto the EOB Hamiltonian to determine the nonlocal-in-time ST corrections to the EOB potentials (A, B, Q_e) at 3PN order.

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I. INTRODUCTION

The direct observation of gravitational waves by the LIGO-Virgo Collaboration in 2015 [1] emitted by inspiralling compact binary, opened new avenues for probing the dynamics in the strong gravity regime [2–8] and led to the first bounds on high-order post-Newtonian coefficients [9]. In the future, the next generation of detectors, such as the Einstein Telescope [10] and Cosmic Explorer [11] will allow a better understanding of the strong-field dynamics, by constraining the parameters of the gravitational theories alternative to Einstein's general relativity (GR).

The simplest well-posed theory among the alternative theories of gravity is the addition of a massless scalar field to GR, scalar-tensor (ST) theories, which are thoroughly studied and tested [12–17]. Besides the motivation to explain the accelerated expansion of the universe as $f(R)$-theories [18], the additional gravitational scalar field also naturally arises in UV complete alternate theories of GR.

The detection of gravitational waves requires a bank of highly accurate waveform templates, which are match filtered against the data observed in detectors. Therefore, to conduct the tests of GR, accurate waveform templates are required for the alternative theories of gravity. Currently, the vast majority of tests conducted on GW signals detected from the coalescing black-hole and neutron-star binaries is based on theory-independent tests such as searches for generic GR-violating features (dispersion, nontensorial polarizations, etc.) [19–22]. Here, we focus on a specific model of massless scalar-tensor theory.

The analytical knowledge of the two-body problem in ST theories, both for the dynamics [23–29] and the waveform generation [30,31] through post-Newtonian (PN) theory is widely increasing. The presence of the finite-size (tidal) effects in ST theories further modify the dynamics at the 3PN order [27], and the tidal Love numbers (hence the tidal polarizability) in ST theories might be very different from their GR counterparts [32]. This can then further impact the constraints put on the equation of state of cold matter at extreme densities [33–37].

The important violations for ST theories arises through the nonperturbative strong field effects in neutron stars such as spontaneous scalarization [14]. The most stringent constraints for ST theories come from binary pulsar observations. There is hope that future gravitational wave (GW) detections of coalescing compact binaries will complement current studies on constraints using genuine strong-field information and additional terms in the radiation, i.e., dipolar radiation which is not present in GR, due to the scalar extension of GR [16,38,39].

This present work is a part of a series of articles to construct the waveform templates for ST theories by recasting the PN results within the effective-one-body (EOB) description of the two-body problem [40–46]. The generalization of the EOB method to ST theories at the 2PN level has been recently worked out [47,48]. The aim of this paper is to determine the complete nonlocal-in-time EOB potentials following our results of [49] at 3PN order starting from the 3PN nonlocal-in-time Lagrangian in ST of [25,26]. Hereafter, the companion paper [49] will be referred as Paper I.

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The paper is organized as follows. In Sec. II, we give a summary of results obtained in Paper I. Then, in Sec. III we derive the conserved energy for nonlocal-in-time part using two methods, (i) non-order-reduced nonlocal Hamiltonian using nonlocal phase shift, and (ii) order-reduction of nonlocal dynamics to local ordinary action-angle Hamiltonian. Finally, in Sec. IV we map the nonlocal-in-time ordinary Hamiltonian into an EOB Hamiltonian at 3PN order.

II. SUMMARY OF PREVIOUS RESULTS

We consider monoscalar massless ST theories described by the following action in the Einstein frame (the scalar field minimally couples to the metric),

$$S = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi) + S_m[\Psi, \mathcal{A}(\varphi)^2 g_{\mu\nu}], \tag{2.1}$$

where $g_{\mu\nu}$ is the Einstein metric, R is the Ricci scalar, φ is the scalar field, Ψ collectively denotes the matter fields, $g \equiv \det(g_{\mu\nu})$, and G is the bare Newton's constant. As in Paper I (see Table I therein), we adopt the conventions and notations of Refs. [12,14]. In the Einstein frame, the dynamics of the scalar field arises from its coupling to the matter fields Ψ, and the field equations can be found in Ref. [12] where the parameter

$$\alpha(\varphi) = \frac{\partial \ln \mathcal{A}}{\partial\varphi}, \tag{2.2}$$

arising in the equations of motion measures the coupling between the matter and the scalar field. The scalar field is nonminimally coupled to the metric in the Jordan frame (physical frame)

$$\tilde{g}_{\mu\nu} = \mathcal{A}(\varphi)^2 g_{\mu\nu}, \tag{2.3}$$

where $\tilde{g}_{\mu\nu}$ is the metric in the Jordan frame.

We follow the approach suggested by [50] to "skeletonize" the compact, self-gravitating objects in ST theories as point particles, i.e., the total mass of each body is dependent on the local value of the scalar field. The skeletonized matter action with the scalar field dependent mass $\tilde{m}_I(\varphi)$ is then given by

$$S_m = -\sum_{J=A,B}\int\sqrt{-\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\tilde{m}_J(\varphi), \tag{2.4}$$

where λ is the affine parameter. Since $\tilde{g}_{\mu\nu} = \mathcal{A}(\varphi)^2 g_{\mu\nu}$, the Einstein-frame mass is defined as

$$m(\varphi) = \mathcal{A}(\varphi)\tilde{m}(\varphi). \tag{2.5}$$

In Paper I, we first derive the ordinary Hamiltonian (dependent only on the positions and momenta) using the contact transformation at 3PN order starting from the Lagrangian of Ref. [26] only for the local-in-time part of the dynamics. The Jordan-frame parameters of Ref. [26] that encompass the scalar field effect are converted to the dimensionless Einstein-frame parameters (see Table I). The mass function $m(\varphi)$ is used to define these dimensionless body-dependent parameters following Refs. [12,14,48] i.e.

$$\alpha_I = \frac{d\ln m(\varphi)_I}{d\varphi}, \tag{2.6}$$

$$\beta_I = \frac{d\alpha_I}{d\varphi}, \tag{2.7}$$

$$\beta'_I = \frac{d\beta_I}{d\varphi}, \tag{2.8}$$

$$\beta''_I = \frac{d\beta'_I}{d\varphi}. \tag{2.9}$$

Here, we follow the notations of Paper I for the binary parameters and use the same notation as [25,26] to denote weak-field and strong-field parameters.

Finally, we then determine the ST corrections to the EOB metric potential (A, B, Q_e) at 3PN order for the local in time (instantaneous) part of the dynamics by mapping the EOB Hamiltonian in Damour–Jaranowski–Schäfer (DJS hereafter) gauge, first mentioned in [42]

$$\hat{H}_{\rm eff} = \frac{H_{\rm eff}}{\mu} = \sqrt{A(\hat{r})\left(1 + \frac{\hat{p}_r^2}{B(\hat{r})} + \frac{\hat{p}_\phi^2}{\hat{r}^2} + \hat{Q}_e\right)}, \tag{2.10}$$

where $\hat{p}_r, \hat{p}_\phi$ are the dimensionless radial and angular momenta, and $\hat{r}(= r/(G_{AB}M)$ is the dimensionless radial separation, to the ordinary two-body Hamiltonian (here, and after the superscript *hat* is used to denote the dimensionless variables).

The three EOB potentials at 3PN formally read

$$A(\hat{r}) = 1 - \frac{2}{\hat{r}} + \frac{a_2}{\hat{r}^2} + \frac{a_3}{\hat{r}^3} + \frac{a_4}{\hat{r}^4}, \tag{2.11}$$

$$B(r) = 1 + \frac{b_1}{\hat{r}} + \frac{b_2}{\hat{r}^2} + \frac{b_3}{\hat{r}^3}, \tag{2.12}$$

$$\hat{Q}_e(\hat{r}) = q_3\frac{\hat{p}_r^4}{\hat{r}^2}. \tag{2.13}$$

The GR and ST corrections in the ν-dependent coefficients $(a_i,\ b_i)$ are then separated as

$$a_i = a_i^{\rm GR} + \delta a_i^{\rm ST}, \tag{2.14}$$

$$b_i = b_i^{\rm GR} + \delta b_i^{\rm ST}, \tag{2.15}$$

$$q_3 = q_3^{\mathrm{GR}} + \delta q_3^{\mathrm{ST}}. \tag{2.16}$$

Since there are also nonlocal-in-time and tidal contributions at 3PN order in ST theory, all the 3PN ST coefficients can thus be decomposed as Eq. (5.23) of Paper I. The complete expressions of local-in-time ST corrections at 3PN can be found in Eqs. (5.14)–(5.16) of Paper I.

In Paper I, we also derive the nonlocal-in-time (tail) and tidal corrections only for the circular orbits using the gauge invariant energy for circular orbits given in Refs. [26,27]. The complete expression for these coefficients can be found in Eqs. (5.25)–(5.27) of Paper I.

III. TAIL CONTRIBUTION TO THE 3PN DYNAMICS

The nonlocal-in-time two-body 3PN Lagrangian for massless ST theory obtained in Ref. [26] is in harmonic coordinates, i.e., it depends (linearly) on the acceleration of the two bodies. In this section, we will use two different methods to derive the *Noetherian* conserved energy for the tail contributions. First, we will remove the acceleration dependence from the Lagrangian (hence, the Hamiltonian) and stay within the non-order-reduced nonlocal framework (as done in Refs. [51,52] for GR). Second, we will derive the order-reduced, local Hamiltonian using the action-angle variables (see, Ref. [45] for GR).

A. Non-order-reduced ordinary Hamiltonian

In Paper I, we derived the ordinary (dependent only on positions and momenta) Hamiltonian for local-in-time contribution using contact transformation (see, Appendix A of Paper I for the contact transformation). Now, due to presence of nonlocal piece in the nonlocal-in-time part of the Hamiltonian we need to find the nonlocal coordinate shift that removes the acceleration dependence from the tail part of the Lagrangian of Ref. [26] (see, Refs. [51,52] for GR). Corresponding to this ordinary Lagrangian, we can then derive the ordinary Hamiltonian.

The tail part of the Lagrangian at 3PN order reads [26],

$$L^{\mathrm{tail}} = \frac{2G^2M}{3c^6}(3+2\omega_0)\mathrm{Pf}_{2r_{AB}/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}I_{s,i}^{(2)}(t)I_{s,i}^{(2)}(t+\tau), \tag{3.1}$$

where Pf is the Hadamard *partie finie* function, Hadamard scale $r_{AB}(=r)$ is the relative separation of two bodies, and $I_{s,i}^{(2)}$ is the second time derivative of the dipole moment. Here, we find the coordinate shift that transforms this Lagrangian into the same expression but with the derivatives of the dipole moment evaluated using the Newtonian equations of motion. In the center-of-mass (COM) frame in notations of Ref. [25,26] it is,

$$\acute{I}_{s,i}^{(2)} = \frac{2M\nu(s_A-s_B)}{\phi_0(3+2w_0)}\left(-\frac{G_{AB}M}{r^2}n_{AB}^i\right), \tag{3.2}$$

where s_A, s_B are the sensitivity of two bodies.

As the nonlocal contribution starts at 3PN order, the ordinary Lagrangian is

$$L_{\mathrm{ord}}^{\mathrm{tail}} = L^{\mathrm{tail}} + \sum_{J=A,B} m_J\left(-a_J^i - \sum_{J\neq K}\frac{G_{AB}m_K}{r^2}n_{JK}^i\right)\xi_{J,i}, \tag{3.3}$$

where $L_{\mathrm{ord}}^{\mathrm{tail}}$ is given by the same expression as Eq. (3.1) but with the second time derivative of the dipole moment replaced by its on-shell value given in Eq. (3.2), and the nonlocal coordinate shift, $\xi_{J,j}$,

$$\xi_{J,j} = \frac{1}{m_J}\frac{2G^2M}{3c^6}(3+2w_0)\left[-\frac{m_J(1-2s_J)}{\phi_0(3+2w_0)}\right]\delta_j^i \times \mathrm{Pf}_{2r/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}\acute{I}_{s,i}^{(2)}(t+\tau). \tag{3.4}$$

The ordinary Hamiltonian is then derived using the ordinary Legendre transformation, $H_{\mathrm{ord}} = \sum_A p_A v_A - L_{\mathrm{ord}}$ which reads $H_{\mathrm{ord}} = H_{\mathrm{ord}}^{\mathrm{loc}} + H_{\mathrm{ord}}^{\mathrm{tail}}$, where the local contribution $H_{\mathrm{ord}}^{\mathrm{loc}}$ is derived in Paper I (see Appendix C) and the tail contribution is

$$H_{\mathrm{ord}}^{\mathrm{tail}} = -\frac{2G^2M}{3c^6}(3+2w_0)\mathrm{Pf}_{2r/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}\acute{I}_{s,i}^{(2)}(t)\acute{I}_{s,i}^{(2)}(t+\tau). \tag{3.5}$$

The tail part of the Hamiltonian is just opposite to the tail part of Lagrangian.

As shown in Ref. [46,52] for the non-order-reduced, nonlocal framework the *Noetherian* conserved energy (E_{cons}) is not given by the Hamiltonian but is given by, $E_{\mathrm{cons}} = H_{\mathrm{ord}}^{\mathrm{tail}} + \delta H$. This additional term δH consists of purely a constant term (DC type) and time oscillating term with zero average value (AC type) and is same as given in Eq. (4.10) of Ref. [26].

B. Order-reduced ordinary Hamiltonian

The second method to derive the conserved energy for tail part is to work in the order-reduced, local framework as given in Refs. [45,46] for GR.

The tail part of the Hamiltonian in ST theory is,

$$H^{\mathrm{tail}} = -\frac{2G^2M}{3c^6}(3+2\omega_0)\left[\mathrm{Pf}_{2r/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}I_{s,i}^{(2)}(t)I_{s,i}^{(2)}(t+\tau) - 2\ln\left(\frac{\hat{r}}{a}\right)I_{s,i}^{(2)}(t)^2\right]. \tag{3.6}$$

As mentioned in Ref. [52], in the action-angle form there should be an additional term [second term in Eq. (3.6)] which is local and accounts for dependence of Hadamard *partie finie* function on the radial separation (r) at time t i.e., $\hat{r} = a(1 - e\cos(u))$ in action-angle variables.

The basic methodology we use to order-reduce the nonlocal dynamics of the above form is based on Refs. [45,46] for GR, and consists of four main steps: (i) Reexpress the Hamiltonian in terms of action angle variables, (ii) "order-reduce" the nonlocal dependence on action angle variable, (iii) expand it in powers of eccentricity, and (iv) eliminate the periodic terms in order-reduced Hamiltonian by a canonical transformation. All of these steps lead to the order-reduced ordinary local Hamiltonian for the tail part in terms of action-angle variables.

Let us consider the expression of nonlocal-in-time piece of Eq. (3.6), i.e.

$$\mathcal{K}(t,\tau) = \ddot{I}_{s,i}(t)\ddot{I}_{s,i}(t+\tau). \tag{3.7}$$

To order reduce the nonlocal piece, we use the equations of motion to express the phase-space variables at shifted time $t+\tau$ in terms of the phase-space variables at time t. As the zeroth order equations are Newtonian equations, it will be convenient to use the action-angle form of the Newtonian equations of motion,

$$\frac{\partial l}{\partial \hat{t}} = \frac{\partial H_0}{\partial \mathcal{L}} = \frac{1}{\mathcal{L}^3} = \Omega(\mathcal{L}), \qquad \frac{\partial \mathcal{L}}{\partial \hat{t}} = \frac{\partial H_0}{\partial l} = 0,$$
$$\frac{\partial \mathcal{G}}{\partial \hat{t}} = \frac{\partial H_0}{\partial g} = 0, \qquad \frac{\partial g}{\partial \hat{t}} = \frac{\partial H_0}{\partial \mathcal{G}} = 0, \tag{3.8}$$

where $\hat{t} = t/(G_{AB}M)$ is the dimensionless time variable, $(\mathcal{L}, l, \mathcal{G}, g)$ are the action-angle variables. The zeroth-order (Newtonian) Hamiltonian in action-angle variable is $H_0 = -1/(2\mathcal{L}^2)$.

Here, the variable $\mathcal{L}$ is conjugate to the "mean anomaly" l and $\mathcal{G}$ is conjugate to argument of periastron g. In terms of the Keplerian variables, semimajor axis a, and eccentricity e, these are

$$\mathcal{L} = \sqrt{a}, \qquad \mathcal{G} = \sqrt{a(1-e^2)}. \tag{3.9}$$

From Eq. (3.8), the variables $\mathcal{L}$, $\mathcal{G}$ and g are independent of time, and l varies linearly with time, hence it will be sufficient to use

$$l(t+\tau) = l(t) + \Omega\hat{\tau}, \tag{3.10}$$

where $\hat{\tau} = \tau/(G_{AB}M)$. The order-reduced nonlocal in time expression of Eq. (3.7) becomes

$$\begin{aligned}\mathcal{K}(t,\tau) &= \left(\frac{1}{G_{AB}M}\right)^4 \mathcal{K}(\hat{t},\hat{\tau}) \\ &= \left(\frac{\Omega}{G_{AB}M}\right)^4 \frac{d^2}{dl^2} I_{s,i}(l) \frac{d^2}{dl^2} I_{s,i}(l+\Omega\hat{\tau}).\end{aligned} \tag{3.11}$$

Using the Fourier decomposition of dipole moment given in Eq. (A11), we find the structure of nonlocal-in-time expression $\mathcal{K}(t,\tau)$ and hence the Hamiltonian. As shown in [45] for GR, all the periodically varying terms can be eliminated by a suitable canonical transformation. Hence, the order-reduced Hamiltonian can be further simplified by replacing $H^{\rm tail}$ with its l-average value

$$\overline{H}_{\rm tail} = \int_0^{2\pi} dl H^{\rm tail}. \tag{3.12}$$

Using the result

$$\mathrm{Pf}_T \int_0^\infty \frac{dv}{v} \cos(\omega v) = -(\gamma_{\rm E} + \ln(\omega T)) \quad \forall\ (\omega > 0), \tag{3.13}$$

where $\gamma_{\rm E}$ is the Euler's constant, and inserting the expression of r from Eq. (A7), the Hamiltonian, Eq. (3.12), reads

$$\begin{aligned}\bar{H}_{\rm tail} &= \frac{8G^2M}{3}\left(\frac{\Omega}{G_{AB}M}\right)^4 (3+2w_0) \sum_{p=1}^{\infty} p^4 |I_{s,i}(p)|^2 \\ &\quad \times \ln\left(\mathrm{e}^{\gamma_{\rm E}} \frac{2pa\Omega}{c}\right).\end{aligned} \tag{3.14}$$

Now, inserting the Fourier-Bessel expansion of scalar dipole moment from Eqs. (A13)–(A14) (see, Appendix for derivation) in Eq. (3.14), the real two-body nonlocal-in-time Hamiltonian in order-reduced, local framework is (in notations of Paper I)

$$\begin{aligned}\hat{\bar{H}}_{\rm tail} \equiv \frac{\bar{H}_{\rm tail}}{\mu} &= \frac{2\nu}{3a^4}\left(2\delta_+ + \frac{\bar{\gamma}_{AB}(\bar{\gamma}_{AB}+2)}{2}\right) \sum_{p=1}^{\infty} \frac{p^2}{e^2} \{4e^2 J_{p-1}^2(pe) + (8-4e^2) J_p^2(pe) - 8eJ_{p-1}(pe)J_p(pe)\} \\ &\quad \times \left[\gamma_{\rm E} + \ln\left(\frac{2pa^{-1/2}}{c}\right)\right].\end{aligned} \tag{3.15}$$

Expanding the result in powers of eccentricity, the Hamiltonian as an expansion in eccentricity up to order of e^4 reads

$$\hat{H}_{\rm tail} = \frac{2\nu}{3a^4}\left(2\delta_+ + \frac{\bar{\gamma}_{AB}(\bar{\gamma}_{AB}+2)}{2}\right)\Big\{2\ln(2) - \ln(a) + 2\gamma_{\rm E} + e^2(14\ln(2) + 6\gamma_{\rm E} - 3\ln(a)) + e^4\left(\frac{45}{4}\gamma_{\rm E} - \frac{3}{4}\ln(2) + \frac{729}{32}\ln(3) - \frac{45}{8}\ln(a)\right) + \mathcal{O}(e^6)\Big\}. \tag{3.16}$$

IV. SCALAR TENSOR CORRECTIONS TO EFFECTIVE ONE BODY AT 3PN:TAIL

In this section, we will derive the complete tail corrections to the EOB metric potentials (A, B, Q_e) for ST theories at 3PN order.

Similar to the decomposition of complete 3PN coefficient $\delta a_4^{\rm ST}$ in Eq. (5.23) of Paper I, we decompose the complete 3PN ST coefficients $\delta b_3^{\rm ST}, \delta q_3^{\rm ST}$ as

$$\delta b_3^{\rm ST} = \delta b_{3,\rm loc}^{\rm ST} + \delta b_{3,\rm nonloc}^{\rm ST} + \delta b_{3,\rm tidal}^{\rm ST}, \tag{4.1}$$

$$\delta q_3^{\rm ST} = \delta q_{3,\rm loc}^{\rm ST} + \delta q_{3,\rm nonloc}^{\rm ST} + \delta q_{3,\rm tidal}^{\rm ST}, \tag{4.2}$$

where the local contributions $(\delta b_{3,\rm loc}^{\rm ST}, \delta q_{3,\rm loc}^{\rm ST})$ are derived in Paper I [see, Eqs. (5.14)–(5.15)], $(\delta b_{3,\rm nonloc}^{\rm ST}, \delta q_{3,\rm nonloc}^{\rm ST})$ are the nonlocal contributions, and $(\delta b_{3,\rm tidal}^{\rm ST}, \delta q_{3,\rm tidal}^{\rm ST})$ are the tidal contributions. The nonlocal contributions can be further decomposed similar to Eq. (5.24) of Paper I as

$$\delta a_{4,\rm nonloc}^{\rm ST} = \delta a_{4,\rm nonloc,0}^{\rm ST} + \delta a_{4,\rm nonloc,log}^{\rm ST}\ln(\hat{r}), \tag{4.3}$$

$$\delta b_{3,\rm nonloc}^{\rm ST} = \delta b_{3,\rm nonloc,0}^{\rm ST} + \delta b_{3,\rm nonloc,log}^{\rm ST}\ln(\hat{r}), \tag{4.4}$$

$$\delta q_{3,\rm nonloc}^{\rm ST} = \delta q_{3,\rm nonloc,0}^{\rm ST} + \delta q_{3,\rm nonloc,log}^{\rm ST}\ln(\hat{r}). \tag{4.5}$$

Inserting the split of the EOB functions (A, B, q_3) using Eqs. (4.1)–(4.2) and Eq. (5.23) of Paper I in the effective Hamiltonian of Eq. (2.10), and then after expanding the right-side into a Taylor series of $1/c^2$, we obtain

$$\hat{H}_{\rm eff} = \hat{H}_{\rm eff}^{\rm loc} + \hat{H}_{\rm eff}^{\rm nonloc}, \tag{4.6}$$

where $\hat{H}_{\rm eff}^{\rm loc}$ is computed only by the local contributions $(\delta a_{4,\rm loc}^{\rm ST}, \delta b_{3,\rm loc}^{\rm ST}, \delta q_{3,\rm loc}^{\rm ST})$ and $\hat{H}_{\rm eff}^{\rm nonloc}$ is the nonlocal contribution of Hamiltonian computed by $(\delta a_{4,\rm nonloc}^{\rm ST}, \delta b_{3,\rm nonloc}^{\rm ST}, \delta q_{3,\rm nonloc}^{\rm ST})$. The nonlocal contribution $\hat{H}_{\rm eff}^{\rm nonloc}$ reads

$$\hat{H}_{\rm eff}^{\rm nonloc} = \frac{1}{2}\left(\delta a_{4,\rm nonloc}^{\rm ST}\frac{1}{\hat{r}^4} - \delta b_{3,\rm nonloc}^{\rm ST}\frac{\hat{p}_r^2}{\hat{r}^3} + \delta q_{3,\rm nonloc}^{\rm ST}\frac{\hat{p}_r^4}{\hat{r}^2}\right). \tag{4.7}$$

To map the real two-body dynamics to EOB, we express the nonlocal effective Hamiltonian, $\hat{H}_{\rm eff}^{\rm nonloc}$, in action-angle variables $\mathcal{L}$, l, $\mathcal{G}$, and g (hence the Keplerian variables a and e) and compute its l-averaged value,

$$\bar{\hat{H}}_{\rm eff}^{\rm nonloc} = \frac{1}{2\pi}\int_0^{2\pi} dl \hat{H}_{\rm eff}^{\rm nonloc}. \tag{4.8}$$

The explicit expression of $\bar{\hat{H}}_{\rm eff}^{\rm nonloc}$ depends on l-average monomials involving powers of $1/\hat{r}$ and $\hat{p}_r$ [and also $\ln(\hat{r})$ from Eqs. (4.3), (4.4), and (4.5)]. These computations can be performed by expanding Eq. (4.7) in terms of eccentricity up to e^5 using the Newtonian equations of motion in action-angle form recalled in Sec. III B. The l-averaged value we obtain is

$$\begin{aligned}\bar{\hat{H}}_{\rm eff}^{\rm nonloc} = \frac{1}{2a^4}\Big\{&\delta a_{4,\rm nonloc,0}^{\rm ST} + \delta a_{4,\rm nonloc,log}^{\rm ST}\ln(a) + \Big(3\delta a_{4,\rm nonloc,0}^{\rm ST} - \frac{7}{4}\delta a_{4,\rm nonloc,log}^{\rm ST} - \frac{1}{2}\delta b_{3,\rm nonloc,0}^{\rm ST} + 3\delta a_{4,\rm nonloc,log}^{\rm ST}\ln(a)\\ &- \frac{1}{2}\delta b_{3,\rm nonloc,log}^{\rm ST}\ln(a)\Big)e^2 + \Big(\frac{45}{8}[\delta a_{4,\rm nonloc,0}^{\rm ST} + \delta a_{4,\rm nonloc,log}^{\rm ST}\ln(a)] - \frac{5}{4}[\delta b_{3,\rm nonloc,0}^{\rm ST} + \delta b_{3,\rm nonloc,log}^{\rm ST}\ln(a)]\\ &+ \frac{3}{8}[\delta q_{3,\rm nonloc,0}^{\rm ST} + \delta q_{3,\rm nonloc,log}^{\rm ST}\ln(a)] - \frac{171}{32}\delta a_{4,\rm nonloc,log}^{\rm ST} + \frac{9}{16}\delta b_{3,\rm nonloc,log}^{\rm ST}\Big)e^4 + \mathcal{O}(e^6)\Big\}.\end{aligned} \tag{4.9}$$

The final step is then to map the real two-body dynamics to EOB metric by the *nontrivial* map,

$$\hat{H}_{\rm real} = \frac{H_{\rm real}}{\mu} = \frac{1}{\nu}\sqrt{1 + 2\nu(\hat{H}_{\rm eff} - 1)}, \tag{4.10}$$

between the EOB Hamiltonian $(\hat{H}_{\rm eff})$ and real two-body Hamiltonian $(\hat{H}_{\rm real})$. The quadratic map relating the two Hamiltonians is proven at *all* PN orders in GR and ST within the post-Minkowskian scheme in Ref. [53]. However, it can be seen that only for the nonlocal contributions at 3PN order, the map relating the two nonlocal Hamiltonians is

$$\hat{\hat{H}}_{\rm eff}^{\rm nonloc} = \hat{\hat{H}}_{\rm real,nonloc}^{\rm II}, \tag{4.11}$$

where $\hat{\hat{H}}_{\rm real,nonloc}^{\rm II} = \hat{\hat{H}}_{\rm tail}$. The *unique* nonlocal ST contributions at 3PN from this matching are

$$\delta a_{4,\rm nonloc,0}^{\rm ST} = \frac{4}{3}\nu\left[2\delta_+ + \frac{\bar{\gamma}_{AB}(\bar{\gamma}_{AB}+2)}{2}\right](2\ln 2 + 2\gamma_{\rm E}), \tag{4.12}$$

$$\delta a_{4,\rm nonloc,log}^{\rm ST} = -\frac{4}{3}\nu\left[2\delta_+ + \frac{\bar{\gamma}_{AB}(\bar{\gamma}_{AB}+2)}{2}\right], \tag{4.13}$$

$$\delta b_{3,\rm nonloc,0}^{\rm ST} = \frac{4}{3}\nu\left[2\delta_+ + \frac{\bar{\gamma}_{AB}(\bar{\gamma}_{AB}+2)}{2}\right]\left(\frac{21}{2} - 16\ln 2\right), \tag{4.14}$$

$$\delta b_{3,\rm nonloc,log}^{\rm ST} = 0, \tag{4.15}$$

$$\delta q_{3,\rm nonloc,0}^{\rm ST} = \frac{4}{3}\nu\left[2\delta_+ + \frac{\bar{\gamma}_{AB}(\bar{\gamma}_{AB}+2)}{2}\right] \times\left(-\frac{31}{4} - \frac{256}{3}\ln 2 + \frac{243}{4}\ln 3\right), \tag{4.16}$$

$$\delta q_{3,\rm nonloc,log}^{\rm ST} = 0. \tag{4.17}$$

The ST tensor correction $\delta a_{4,\rm nonloc}^{\rm ST}$ for the circular orbit case, Eqs. (4.12)–(4.13), matches with the results obtained in Paper I [see, Eqs. (5.25)–(5.26)] using the results of Ref. [26] except a negative sign in Eq. (4.13). The negative sign is due to the difference in the definition of $\delta a_{4,\rm nonloc}^{\rm ST}$ in Eq. (4.3) used in this work with the Eq. (5.24) of Paper I.

V. CONCLUSIONS

In Paper I, building upon the results of [26] for massless scalar-tensor theory, we determined the EOB coefficients at 3PN order though restricting ourselves to local-in-time part of the dynamics and nonlocal-in-time and tail contributions only for the circular case. In the present paper, we derived the complete nonlocal-in-time EOB coefficients starting from the nonlocal-in-time Lagrangian of Ref. [26]. First, we derived the two-body *conserved* ordinary Hamiltonian (dependent only on positions and momenta) for nonlocal-in-time part by two methods: (i) non-order-reduced nonlocal Hamiltonian using nonlocal shift (see, Ref. [51,52] for GR), and (ii) order-reduction of nonlocal dynamics to local ordinary action-angle Hamiltonian [45]. We then expressed the effective Hamiltonian in Delaunay variables to recast the order-reduced ordinary action-angle Hamiltonian into equivalent, 3PN-accurate, nonlocal part of EOB potentials (A, B, Q_e), see Eqs. (4.12)–(4.17).

By combining the results of Paper I and the present work, we could transcribe the two-body Hamiltonian into equivalent 3PN-accurate EOB potentials (A, B, Q_e) for both local-in-time and nonlocal-in-time part of dynamics. During the preparation of the final manuscript of this work, we became aware of an independent effort computing similar contributions [54]. We have cross-checked our results with the results presented in [54], specifically the complete (including both local and nonlocal) correction in A, B, and Q_e EOB potentials. Although computed following different steps, we found that the two results are consistent.

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APPENDIX: FOURIER COEFFICIENTS OF DIPOLE MOMENT IN ST THEORY

In this appendix, we will determine the explicit expressions of *Newtonian* dipole moment in ST theory using the known Fourier decomposition of the Keplerian motion (see, Refs. [55,56] for GR).

The dipole moment, $I_{s,i}(t)$, in COM frame is

$$I_{s,i}(t) = \frac{2M\nu(s_A - s_B)}{\phi_0(3+2w_0)}{\rm x}_i, \tag{A1}$$

where ${\rm x}_i = (Z_A - Z_B)_i$ is the relative separation vector and $Z_{A,B}$ indicate the positions of the two bodies.

Since the motion is planar, we can choose the coordinate system (x, y, z) such that it coincides with the xy-plane. Using the polar coordinates $(\hat{r}, \phi_a)$,

$$x = \hat{r}\cos(\phi_a), \qquad y = \hat{r}\sin(\phi_a). \tag{A2}$$

The coordinates (x, y) are the coordinates of the dimensionless relative separation, $\hat{r} = x_A - x_B$ with $x_J = {\rm x}_J/(G_{AB}M)$ denoting the position of two bodies.

As mentioned in Ref. [45,55,56] for GR, for leading order contributions it is convenient to use the Delaunay (action-angle) form of the Newtonian equations of motion. In terms of the action-angle variables $(\mathcal{L}, l, \mathcal{G}, g)$, the Cartesian coordinates (x, y) are given by (Here, we follow the notations of [57])

$$x = x_0\cos(g) - y_0\sin(g), \tag{A3}$$

$$y = x_0\cos(g) + y_0\sin(g), \tag{A4}$$

$$x_0 = \hat{r}\cos(f) = a(\cos(u) - e), \tag{A5}$$

$$y_0 = \hat{r}\sin(f) = a\sqrt{1-e^2}\sin(u), \tag{A6}$$

$$\hat{r} = a(1 - e\cos(u)), \tag{A7}$$

where a is the semimajor axis, e is the eccentricity, f is the "true anomaly" and the "eccentric anomaly" u in terms of Bessels functions is given by

$$u = l + \sum_{n=1}^{\infty} \frac{2}{n} \mathrm{J}_n(ne)\sin(nl). \tag{A8}$$

The Bessel-Fourier expansion of $\cos(u)$ and $\sin(u)$, which directly enters x_0, y_0 are:

$$\cos(u) = -\frac{e}{2} + \sum_{n=1}^{\infty} \frac{1}{n}[\mathrm{J}_{n-1}(ne) - \mathrm{J}_{n+1}(ne)]\cos(nl), \tag{A9}$$

$$\sin(u) = \sum_{n=1}^{\infty} \frac{1}{n}[\mathrm{J}_{n-1}(ne) + \mathrm{J}_{n+1}(ne)]\sin(nl). \tag{A10}$$

From Eqs. (A3)–(A8), the dipole moment $I_{s,i}$ is a periodic function of l (and hence time) at the Newtonian order. Thus it can be decomposed into Fourier series

$$I_{s,i}(l) = \sum_{p=-\infty}^{\infty} I_{i,s}(p)\mathrm{e}^{ipl}, \tag{A11}$$

with

$$I_{s,i}(p) = \frac{1}{2\pi}\int_0^{2\pi} dl I_{s,i}\mathrm{e}^{-ipl}. \tag{A12}$$

The Fourier coefficients of the scalar dipole moment at the Newtonian order are derived using Eq. (A12) in terms of combinations of Bessel functions.

Inserting the expression of Cartesian coordinates in terms of action-angle variables using Eqs. (A3)–(A10), we find the Fourier-Bessel coefficients of the scalar dipole moment are

$$I_{s,x}(p) = G_{AB}M\left[\frac{2M\nu(s_A - s_B)}{\phi_0(3+2w_0)}\frac{a}{2p}\{[J_{p-1}(pe) - J_{p+1}(pe)]\cos(g) + i\sqrt{1-e^2}[J_{p-1}(pe) + J_{p+1}(pe)]\sin(g)\}\right], \tag{A13}$$

$$I_{s,y}(p) = G_{AB}M\left[\frac{2M\nu(s_A - s_B)}{\phi_0(3+2w_0)}\frac{a}{2p}\{[J_{p-1}(pe) - J_{p+1}(pe)]\sin(g) + i\sqrt{1-e^2}[J_{p-1}(pe) + J_{p+1}(pe)]\cos(g)\}\right]. \tag{A14}$$

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