Topological classes of thermodynamics of rotating AdS black holes

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In this paper, we extend our previous work [D. Wu, Phys. Rev. D **107**, 024024 (2023)] to the more general cases with a negative cosmological constant, and investigate the topological numbers for the singly rotating Kerr-AdS black holes in all dimensions and the four-dimensional Kerr-Newman-AdS black hole as well as the three-dimensional Bañados-Teitelboim-Zanelli black hole. We find that the topological numbers of black holes are remarkably influenced by the cosmological constant. In addition, we also demonstrate that the dimension of spacetimes has an important effect on the topological number for rotating anti–de Sitter (AdS) black holes. Furthermore, it is interesting to observe that the difference between the topological number of the AdS black hole and that of its corresponding asymptotically flat black hole is always unity. This new observation leads us to conjure that it might be valid also for other black holes. Of course, this novel conjecture needs to be further verified by examining the topological numbers of many other black holes and their AdS counterparts in the future work.

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I. INTRODUCTION

Recently, topology, as an important mathematical tool that applies to black hole physics, has received considerable interest and enthusiasm. The current research on topology is mainly embodied in two aspects. On the one hand, there is the research on the light rings [1–4] of some black holes, which may provide more footprints for the observation of black holes and has been extended to timelike circular orbits [5,6]; on the other hand, there is the research on the thermodynamic topological classification of various black holes [7–20].

In particular, a new method to investigate the thermodynamic topological properties of black holes is proposed in Ref. [7] by considering black hole solutions as topological thermodynamic defects and constructing topological numbers, and further, dividing all black holes into three categories according to their different topological numbers. Because these topological numbers are universal constants that are independent of the black hole solution parameters, hence they are very important for understanding the nature of black holes and gravity. The topological approach proposed in Ref. [7] quickly gained popularity due to its straightforwardness and adaptability, and subsequently it was effectively used to explore the topological numbers of several well-known black hole solutions [14–19], i.e., the Schwarzschild-AdS black hole [14], the static black holes in Lovelock gravity [15], the static Gauss-Bonnet-AdS black holes [16], the static black hole in nonlinear electrodynamics [17], and the static Born-Infeld AdS black hole [18], as well as some static hairy black holes [19]. However, all of the preceding researches [14–19] are limited to the static cases, leaving the topological numbers of rotating black holes and AdS scalar hairy black holes unexplored. Very recently, we have extended the topological approach to rotating black hole cases and investigated the topological numbers for the cases of rotating Kerr and Kerr-Newman black holes [20].

Since the study of the topological number of black holes is still in its infancy and the topological number of the rotating AdS black holes remains virgin territory, it deserves to be explored deeply. On the other hand, the study of rotating AdS black holes has already shed light on the nature of gravity through gauge-gravity dualities [21–23], so it is very important and remarkable to investigate the topological number of rotating AdS black holes. These two aspects motivate us to conduct the present work. In this paper, we shall investigate the topological number of the d-dimensional singly rotating Kerr-AdS black holes and the four-dimensional Kerr-Newman-AdS black hole, as well as the three-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole [24–26]. Compared with the previous paper [20], the aim of the

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present work is concentrated on investigating the impact of the cosmological constant on the topological number of black holes, which has not been studied in any previous related literature. We will see that the cosmological constant is important in determining the topological number of rotating black holes, and observe that the difference between the topological number of the AdS black hole and that of its corresponding asymptotically flat black hole is always unity, which leads us to conjure that it might also hold true for other black holes.

The remaining part of this paper is organized as follows. In Sec. II, we first give a brief review of the topological approach and investigate the topological number of the four-dimensional Schwarzschild-AdS black hole as a warmup exercise. In Sec. III, we will focus on the topological number of the four-dimensional rotating Kerr-AdS black hole. In Sec. IV, we shall extend these discussions to the cases of the d-dimensional singly rotating Kerr-AdS black holes. In Sec. V, we will investigate the topological number of the three-dimensional rotating BTZ black hole. In Sec. VI, we then turn to discuss the topological number of the four-dimensional Kerr-Newman-AdS black hole. Finally, we present our conclusions in Sec. VII. In the Appendix, the topological number of the three-dimensional charged BTZ black hole is also investigated.

II. SCHWARZSCHILD-AdS$_4$ BLACK HOLE

In this section, we first present a brief review of the topological approach proposed in Ref. [7], then investigate the topological number of the four-dimensional Schwarzschild-AdS black hole as a warmup exercise. There are two reasons for us to do so. On the one hand, it can be used to show that the charge parameter has a significant effect on the topological numbers of the static AdS$_4$ black holes. On the other hand, it is also convenient for us to make a comparison with the corresponding results of the rotating Kerr-AdS$_4$ black hole, so as to observe the influence of the rotation parameter on the topological number of the Schwarzschild-AdS$_4$ black hole.

As shown in Ref. [7], one can introduce the generalized off-shell Helmholtz free energy

$$\mathcal{F} = M - \frac{S}{\tau}, \tag{1}$$

for a black hole thermodynamic system with mass M and entropy S, where τ is an extra variable that can be thought of as the inverse temperature of the cavity enclosing the black hole. Only when $\tau = 1/T$ does the generalized Helmholtz free energy become on-shell.

In Ref. [7], a core vector ϕ is defined as[1]

$$\phi = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -\cot\Theta \csc\Theta\right), \tag{2}$$

in which the two parameters r_h and Θ obey $0 < r_h < +\infty$ and $0 \le \Theta \le \pi$, respectively. The component ϕ^Θ is divergent at $\Theta = 0, \pi$, thus the direction of the vector points outward there.

Using Duan's ϕ-mapping topological current theory [28–30], a topological current can be described as follows:

$$j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho}\varepsilon_{ab}\partial_\nu n^a \partial_\rho n^b, \qquad \mu, \nu, \rho = 0, 1, 2, \tag{3}$$

where $\partial_\nu = \partial/\partial x^\nu$ and $x^\nu = (\tau, r_h, \Theta)$. The unit vector n reads as $n = (n^r, n^\Theta)$, where $n^r = \phi^{r_h}/||\phi||$ and $n^\Theta = \phi^\Theta/||\phi||$. It is simple to demonstrate that the topological current (3) given above is conserved, allowing one to easily deduce $\partial_\mu j^\mu = 0$. It is then established that the topological current j^μ is a δ-function of the field configuration [29,30]:

$$j^\mu = \delta^2(\phi) J^\mu\left(\frac{\phi}{x}\right), \tag{4}$$

where the 3-dimensional Jacobian $J^\mu(\phi/x)$ is defined as: $\varepsilon^{ab}J^\mu(\phi/x) = \varepsilon^{\mu\nu\rho}\partial_\nu\phi^a\partial_\rho\phi^b$. It is simple to indicate that j^μ equals to zero only when $\phi^a(x_i) = 0$, hence the topological number W can be determined as follows:

$$W = \int_\Sigma j^0 d^2x = \sum_{i=1}^{N}\beta_i\eta_i = \sum_{i=1}^{N} w_i, \tag{5}$$

where β_i is the positive Hopf index that counts the number of the loops of the vector ϕ^a in the ϕ-space when x^μ are around the zero point z_i, while $\eta_i = \mathrm{sign}(J^0(\phi/x)_{z_i}) = \pm 1$ is the Brouwer degree, and w_i is the winding number for the ith zero point of ϕ that is contained in Σ. It is important to keep in mind that two loops Σ_1 and Σ_2 have the same winding number if they both enclose the same zero point of ϕ. On the other hand, if there is no zero point in the

[1]One can also construct the vector ϕ in a more general form as

$$\phi = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -C\cot\Theta\csc\Theta\right),$$

where C is an arbitrary positive constant. Changing C to a different value will slightly change the direction of the unit vector n, but will not change the position of the zero point of the vector field or its corresponding winding number. Therefore, one can just set $C = 1$ for the sake of simplicity. However, the authors of a new preprint [27] criticize that this definition of the vector field ϕ in Ref. [7] is not intrinsic.

surrounding region, then one can arrive at the topological number: $W = 0$.

In the following, we shall investigate the topological number of the four-dimensional Schwarzschild-AdS black hole via the above topological approach. For the Schwarzschild-AdS$_4$ black hole, its metric has the form [31]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (6)$$

where

$$f(r) = 1 - \frac{2m}{r} + \frac{r^2}{l^2},$$

in which m is the mass parameter, and l is the cosmological scale associated with the pressure $P = 3/(8\pi l^2)$ of the four-dimensional AdS black holes [32–34]. The mass and entropy associated with the above solution (6) can be computed via the standard method and have the following exquisite forms:

$$M = m, \qquad S = \pi r_h^2, \quad (7)$$

where r_h are the locations of the event and Cauchy horizons that satisfy the equation: $f(r_h) = 0$.

For the Schwarzschild-AdS$_4$ black hole, one can define the generalized Helmholtz free energy as

$$\mathcal{F} = \frac{r_h}{2} + \frac{4\pi}{3}Pr_h^2 - \frac{\pi r_h}{\tau}. \quad (8)$$

The components of the vector ϕ can be easily calculated as:

$$\phi^{r_h} = \frac{1}{2} + 4\pi P r_h^2 - \frac{2\pi r_h}{\tau}, \qquad \phi^{\Theta} = -\cot\Theta\csc\Theta. \quad (9)$$

By solving the equation $\phi^{r_h} = 0$, one can obtain a curve on the $r_h - \tau$ plane. For the four-dimensional Schwarzschild-AdS black hole, one can get

$$\tau = \frac{4\pi r_h}{1 + 8\pi P r_h^2}. \quad (10)$$

Taking the pressure $Pr_0^2 = 0.0022$, where r_0 is an arbitrary length scale set by the size of a cavity enclosing the black hole, we show zero points of ϕ^{r_h} in the $r_h - \tau$ plane in Fig. 1. For small τ, such as $\tau = \tau_1$, there are two intersection points for the Schwarzschild-AdS$_4$ black hole. The intersection points exactly satisfy the condition $\tau = 1/T$, and thus represent the on-shell Schwarzschild-AdS$_4$ black holes with the characteristic temperature $T = 1/\tau$. The two intersection points for the Schwarzschild-AdS$_4$ black hole can coincide with each other when $\tau = \tau_c$, and then vanish when $\tau > \tau_c$, therefore τ_c is an annihilation point and can be found at $\tau_c = 26.72r_0$, which can be seen straightforwardly from Fig. 1. Furthermore, the annihilation point τ_c divides the Schwarzschild-AdS$_4$ black hole into the upper and lower branches with the winding numbers $w = 1$ and $w = -1$, respectively. One can see that, for the Schwarzschild-AdS$_4$ solutions at any given temperature, there can exist one thermodynamically stable black hole and one thermodynamically unstable black hole. It has been shown that the winding number can be used to characterize the local thermodynamic stability, with positive and negative values corresponding to thermodynamically stable and unstable black holes [7], respectively.

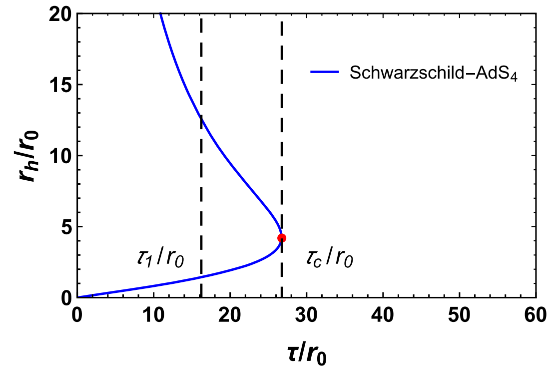


FIG. 1. Zero points of the vector ϕ^{r_h} shown on the $r_h - \tau$ plane with $Pr_0^2 = 0.0022$ for the Schwarzschild-AdS$_4$ black hole. The annihilation point for this black hole is represented by the red dot with τ_c. There are two Schwarzschild-AdS$_4$ black holes when $\tau = \tau_1$. Obviously, the topological number is: $W = 1 - 1 = 0$.

Alternatively, the unit vector field n can also be plotted for any arbitrarily selected typical values (keep in mind that τ must be less than τ_c), for instance, $\tau/r_0 = 26$ and $Pr_0^2 = 0.0022$ in Fig. 2, where we find two zero points: ZP_1 at $r_h = 3.36r_0$ and ZP_2 at $r_h = 5.38r_0$, with the winding numbers $w_1 = 1$ and $w_2 = -1$, respectively, to determine the topological number for the four-dimensional Schwarzschild-AdS black hole. Based upon the local property of the zero point, one can easily find that the topological number is: $W = -1 + 1 = 0$ for the Schwarzschild-AdS$_4$ black hole, which is consistent with the result given in Ref. [14].

In addition, the fact that the topological number of the Schwarzschild-AdS$_4$ black hole is zero while that of the Schwarzschild black hole is -1 [7] suggests that the cosmological constant significantly changes the topological number of the static black holes. Furthermore, since the topological number of the RN-AdS$_4$ black hole is $W = 1$, it is easy to see that the Schwarzschild-AdS$_4$ black hole and the RN-AdS$_4$ black hole belong to two different topological classes according to the topological classification method proposed in Ref. [7], which indicates that the electric charge has an important influence on the topological number of static AdS$_4$ black holes.

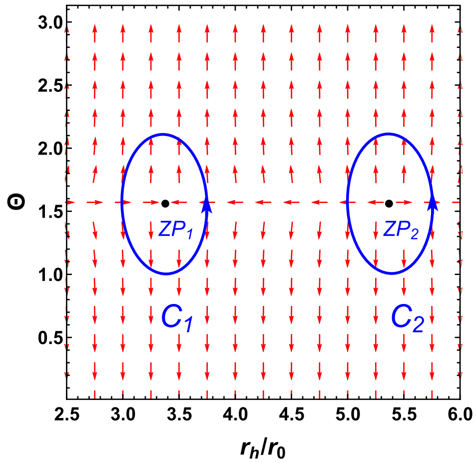


FIG. 2. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 0.0022$ and $\tau/r_0 = 26$ for the Schwarzschild-AdS$_4$ black hole. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (3.36, \pi/2)$, $(5.38, \pi/2)$ for ZP$_1$ and ZP$_2$, respectively. The blue contours C_i are closed loops surrounding the zero points.

III. Kerr-AdS$_4$ BLACK HOLE

From now on, we come to the main subject of this paper, i.e., exploring the topological number of rotating AdS black holes. In this section, we will focus on the topological number of the four-dimensional Kerr-AdS black hole, whose metric in the asymptotically nonrotating frame has the form [31]

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(\frac{\Delta_\theta}{\Xi}dt - \frac{a}{\Xi}\sin^2\theta d\varphi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\left[\frac{a(r^2+l^2)}{l^2\Xi}dt - \frac{r^2+a^2}{\Xi}d\varphi\right]^2 \quad (11)$$

in terms of the Boyer-Lindquist coordinates, where

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr, \qquad \Xi = 1 - \frac{a^2}{l^2},$$
$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Sigma = r^2 + a^2\cos^2\theta,$$

in which a is the rotation parameter, m is the mass parameter, and l is the AdS radius.

The mass M and entropy S associated with the above solution (11) are [35]

$$M = \frac{m}{\Xi^2}, \qquad S = \frac{\pi(r_h^2+a^2)}{\Xi}. \quad (12)$$

Using the definition of the generalized off-shell Helmholtz free energy (1) and $l^2 = 3/(8\pi P)$, one can easily get

$$\mathcal{F} = \frac{3(r_h^2+a^2)[2\pi r_h(8\pi Pa^2 + 4Pr_h\tau - 3) + 3\tau]}{2r_h(8\pi Pa^2-3)^2\tau} \quad (13)$$

for the Kerr-AdS$_4$ black hole. Then the components of the vector ϕ can be computed as

$$\phi^{r_h} = \frac{12\pi(8\pi Pa^2-3)r_h^3 + 3a^2(8\pi Pr_h^2-3)\tau}{2r_h^2(8\pi Pa^2-3)^2\tau} + \frac{9(1+8\pi Pr_h^2)}{2(8\pi Pa^2-3)^2}, \quad (14)$$

$$\phi^\Theta = -\cot\Theta\csc\Theta. \quad (15)$$

By solving the equation $\phi^{r_h} = 0$, one can obtain

$$\tau = \frac{4\pi r_h^3(3-8\pi Pa^2)}{a^2(8\pi Pr_h^2-3) + 3(8\pi Pr_h^2+1)r_h^2} \quad (16)$$

as the zero point of the vector field ϕ.

Taking the pressure $Pr_0^2 = 0.0022$ and the rotation parameter $a = r_0$ for the Kerr-AdS$_4$ black hole, we show zero points of ϕ^{r_h} in the $r_h - \tau$ plane in Fig. 3, and the unit vector field n in Fig. 4 with $\tau = 24r_0$, $26r_0$, and $28r_0$, respectively. From Figs. 3 and 4, one can observe that for these values of Pr_0^2 and a/r_0, one generation point and one annihilation point can be found at $\tau/r_0 = \tau_a/r_0 = 24.90$ and $\tau/r_0 = \tau_b/r_0 = 26.82$, respectively. One can see that there are one large black hole branch for $\tau < \tau_a$, three black hole branches for $\tau_a < \tau < \tau_b$, and one small black hole branch for $\tau > \tau_b$. Calculating the winding number w for these three black hole branches, we find that both the small and large black hole branches have $w = 1$, while the intermediate black hole branch has $w = -1$. The Kerr-AdS$_4$ black hole always has the topological number $W = 1$, unlike the Kerr black hole, which has a topological number

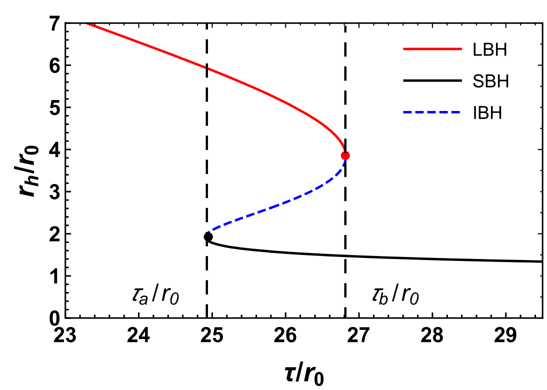
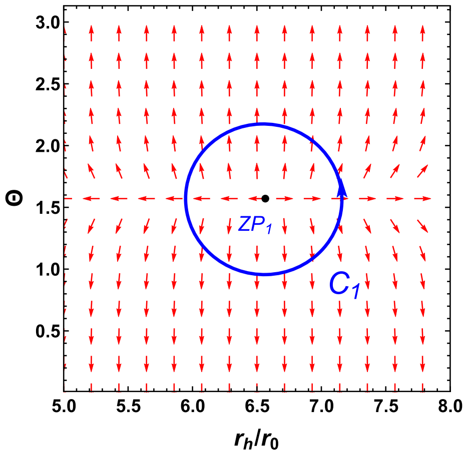
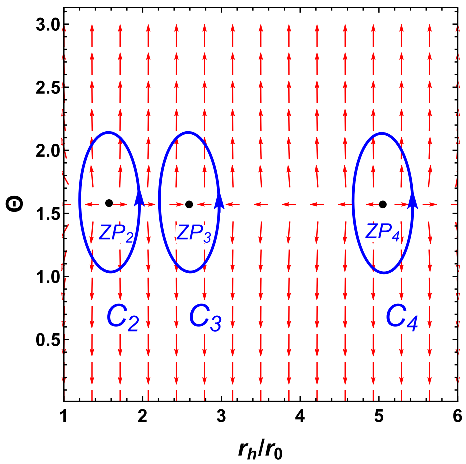


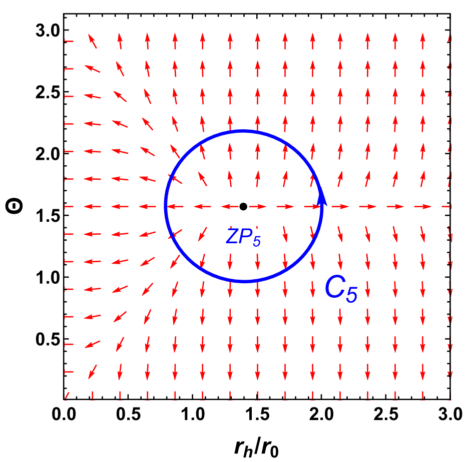
FIG. 3. Zero points of ϕ^{r_h} shown in the $r_h - \tau$ plane with $Pr_0^2 = 0.0022$ and $a = r_0$ for the Kerr-AdS$_4$ black hole. The red solid, blue dashed, and black solid lines are for the large black hole (LBH), intermediate black hole (IBH), and small black hole (SBH), respectively. The annihilation and generation points are represented by red and black dots, respectively.



(a) The unit vector field for the Kerr-AdS$_4$ black hole with $\tau/r_0 = 24$, $a/r_0 = 1$, and $Pr_0^2 = 0.022$.



(b) The unit vector field for the Kerr-AdS$_4$ black hole with $\tau/r_0 = 26$, $a/r_0 = 1$, and $Pr_0^2 = 0.022$.



(c) The unit vector field for the Kerr-AdS$_4$ black hole with $\tau/r_0 = 28$, $a/r_0 = 1$, and $Pr_0^2 = 0.022$.

FIG. 4. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (6.55, \pi/2), (1.55, \pi/2), (2.75, \pi/2), (5.11, \pi/2), (1.40, \pi/2)$, for ZP_1, ZP_2, ZP_3, ZP_4, and ZP_5, respectively. The blue contours C_i are closed loops surrounding the zero points.

of zero [20]. Therefore, from the thermodynamic topological standpoint, the Kerr-AdS$_4$ black hole and the Kerr black hole are different kinds of black hole solutions, and this indicates that the cosmological constant is important in determining the topological number for the rotating black hole. What is more, since the topological number of the Schwarzschild-AdS$_4$ black hole is zero, while that of the Kerr-AdS$_4$ black hole is 1, so it can be inferred that the rotation parameter has a remarkable effect on the topological number for the uncharged AdS$_4$ black hole.

IV. SINGLY ROTATING Kerr-AdS BLACK HOLES IN ARBITRARY DIMENSIONS

In this section, we will extend the above discussions to the cases of higher-dimensional rotating black holes by considering the singly rotating Kerr-AdS solutions in arbitrary dimensions. For d-dimensional singly rotating Kerr-AdS black holes, the metric in the asymptotically nonrotating frame has the form [36,37]

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(\frac{\Delta_\theta}{\Xi}dt - \frac{a}{\Xi}\sin^2\theta d\varphi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\left[\frac{a(r^2+l^2)}{l^2\Xi}dt - \frac{r^2+a^2}{\Xi}d\varphi\right]^2 + r^2\cos^2\theta d\Omega_{d-4}^2, \tag{17}$$

where $d\Omega_{d-4}$ denotes the line element of the $(d-4)$-dimensional unit sphere, and

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr^{5-d}, \qquad \Xi = 1 - \frac{a^2}{l^2},$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Sigma = r^2 + a^2\cos^2\theta.$$

The thermodynamic quantities are [38]

$$M = \frac{\omega_{d-2}m}{4\pi\Xi^2}\left[\frac{(d-4)\Xi}{2}+1\right], \qquad J = \frac{\omega_{d-2}ma}{4\pi\Xi^2},$$
$$\Omega = \frac{a(r_h^2+l^2)}{l^2(r_h^2+a^2)}, \qquad S = \frac{\mathcal{A}}{4} = \frac{\omega_{d-2}}{4\Xi}(r_h^2+a^2)r_h^{d-4},$$
$$T = \frac{r_h}{2\pi}\left(1+\frac{r_h^2}{l^2}\right)\left(\frac{1}{r_h^2+a^2}+\frac{d-3}{2r_h^2}\right) - \frac{1}{2\pi r_h},$$
$$V = \frac{r_h\mathcal{A}}{d-1}\left[1+\frac{a^2(r_h^2+l^2)}{(d-2)\Xi l^2 r_h^2}\right], \qquad P = \frac{(d-1)(d-2)}{16\pi l^2}, \tag{18}$$

where $\omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2]$, and r_h is determined by the horizon equation: $\Delta_r = 0$.

In our previous paper [20], we have uniformly considered the topological numbers of d-dimensional singly rotating Kerr black holes. Here, we will extend that work to the cases of d-dimensional singly rotating Kerr-AdS black holes. Since there is one more additional thermodynamic quantity associated with the cosmological constant to be included, it will be more convenient to separately consider the topological numbers of different dimensions.

A. $d = 5$ case

We first consider the $d = 5$ case. From Eq. (18), one can obtain the expression of the generalized Helmholtz free energy as

$$\mathcal{F} = -\frac{\pi(r_h^2 + a^2)}{8\tau(4\pi Pa^2 - 3)^2}[12\pi r_h(3 - 4\pi Pa^2) + \tau(4\pi Pr_h^2 + 3)(4\pi Pa^2 - 9)], \tag{19}$$

so the components of the vector ϕ can be computed as

$$\phi^{r_h} = \frac{\pi}{4\tau(4\pi Pa^2 - 3)^2}\{6\pi(4\pi Pa^2 - 3)(3r_h^2 + a^2) - r_h\tau(4\pi Pa^2 - 9)[4\pi P(2r_h^2 + a^2) + 3]\}, \tag{20}$$

$$\phi^{\Theta} = -\cot\Theta \csc\Theta. \tag{21}$$

It is simple to obtain

$$\tau = \frac{6\pi(4\pi Pa^2 - 3)(3r_h^2 + a^2)}{r_h(4\pi Pa^2 - 9)[4\pi P(2r_h^2 + a^2) + 3]} \tag{22}$$

as the zero point of the vector field ϕ.

For the singly rotating Kerr-AdS$_5$ black hole, we plot the zero points of the component ϕ^{r_h} with $Pr_0^2 = 0.02$ and $a/r_0 = 1$ in Fig. 5, and the unit vector field n in Fig. 6 with $\tau = 5r_0$, $7r_0$, and $9r_0$, respectively. Note that for these values of Pr_0^2 and a/r_0, one generation point and one

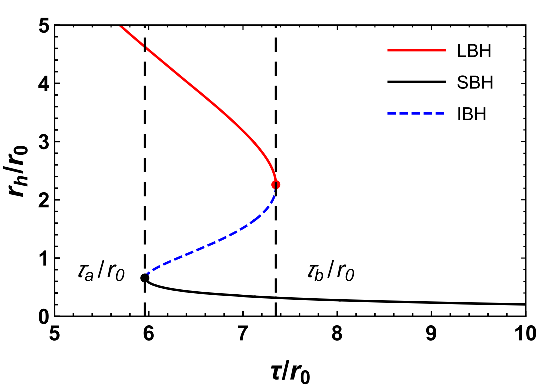
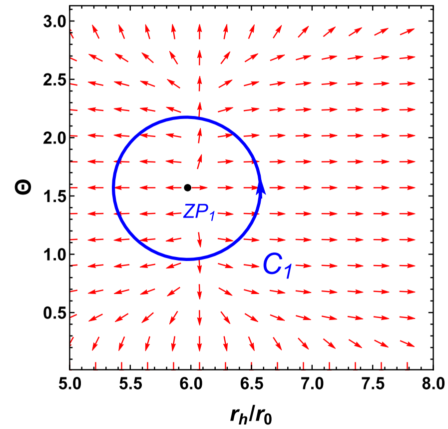
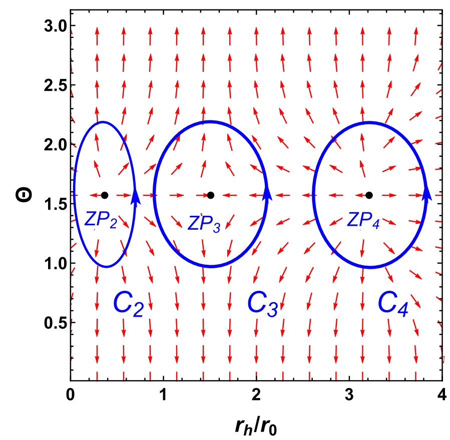


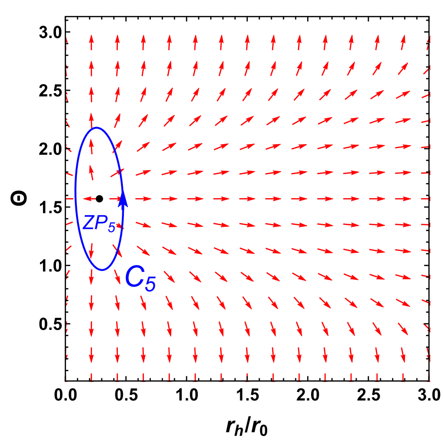
FIG. 5. The zero points of ϕ^{r_h} shown in the $r_h - \tau$ plane with $Pr_0^2 = 0.02$ and $a/r_0 = 1$ for the singly rotating Kerr-AdS$_5$ black hole. The red solid, blue dashed, and black solid lines are for the large black hole (LBH), intermediate black hole (IBH), and small black hole (SBH), respectively. The annihilation and generation points are represented by red and black dots, respectively.



(a) The unit vector field for the five-dimensional singly rotating Kerr-AdS black hole with $\tau/r_0 = 5$, $a/r_0 = 1$, and $Pr_0^2 = 0.02$.



(b) The unit vector field for the five-dimensional singly rotating Kerr-AdS black hole with $\tau/r_0 = 7$, $a/r_0 = 1$, and $Pr_0^2 = 0.02$.



(c) The unit vector field for the five-dimensional singly rotating Kerr-AdS black hole with $\tau/r_0 = 9$, $a/r_0 = 1$, and $Pr_0^2 = 0.02$.

FIG. 6. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (6.07, \pi/2)$, $(0.35, \pi/2)$, $(1.52, \pi/2)$, $(3.18, \pi/2)$, $(0.23, \pi/2)$, for ZP_1, ZP_2, ZP_3, ZP_4, and ZP_5, respectively. The blue contours C_i are closed loops surrounding the zero points.

annihilation point can be found in Fig. 5 at $\tau/r_0 = \tau_a/r_0 = 5.96$ and $\tau/r_0 = \tau_b/r_0 = 7.35$, respectively. From Figs. 5 and 6, one can easily obtain the topological number $W = 1$ for the singly rotating Kerr-AdS$_5$ black hole using the local property of the zero points, which is the same as that of the four-dimensional Kerr-AdS black hole in the previous section but different from that of the five-dimensional singly rotating Kerr black hole, which is $W = 0$ [20]. Therefore, the topological number of the five-dimensional rotating black hole is significantly changed when the cosmological constant is turned on.

B. $d = 6$ case

Next, we consider $d = 6$ case whose generalized Helmholtz free energy is

$$\mathcal{F} = -\frac{2\pi r_h(r_h^2 + a^2)}{3\tau(4\pi Pa^2 - 5)^2}[5\pi r_h(5 - 4\pi Pa^2) + \tau(4\pi Pr_h^2 + 5)(2\pi Pa^2 - 5)]. \tag{23}$$

Thus, the components of the vector ϕ are

$$\phi^{r_h} = \frac{2\pi}{3\tau(4\pi Pa^2 - 5)}\{10\pi r_h(4\pi Pa^2 - 5)(2r_h^2 + a^2) - \tau(2\pi Pa^2 - 5)[20\pi Pr_h^4 + 3(4\pi Pa^2 + 5)r_h^2 + 5a^2]\}, \tag{24}$$

$$\phi^{\Theta} = -\cot\Theta\csc\Theta. \tag{25}$$

So the zero point of the vector field ϕ is

$$\tau = \frac{10\pi r_h(4\pi Pa^2 - 5)(2r_h^2 + a^2)}{(2\pi Pa^2 - 5)[20\pi Pr_h^4 + 3(4\pi Pa^2 + 5)r_h^2 + 5a^2]}. \tag{26}$$

Taking $Pr_0^2 = 0.1$ and $a/r_0 = 1$ for the singly rotating Kerr-AdS$_6$ black hole, we plot zero points of ϕ^{r_h} in the $r_h - \tau$ plane in Fig. 7, and the unit vector field n with $\tau = 2.5r_0$ in Fig. 8, respectively. For small τ, such as $\tau = \tau_1$, there are two intersection points for the singly rotating Kerr-AdS$_6$ black hole. These two intersection points can coincide with each other when $\tau = \tau_c$, and then vanish when $\tau > \tau_c$, therefore τ_c is an annihilation point that can be found at $\tau_c = 2.81r_0$. Based upon the local property of the zero point, one can easily obtain the topological number $W = 0$ for the singly rotating Kerr-AdS$_6$ black hole, which is different from that of the six-dimensional singly rotating Kerr black hole ($W = -1$) [20]. This indicates that the cosmological constant is important in determining the topological number for the six-dimensional rotating black hole.

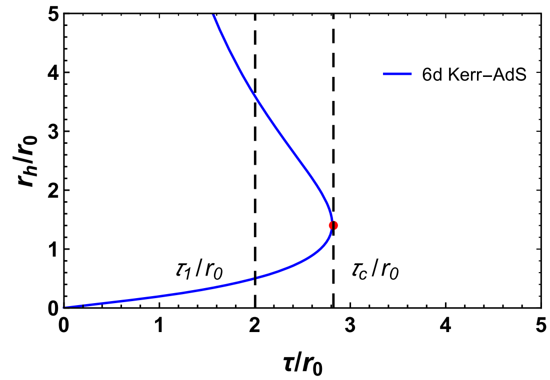


FIG. 7. Zero points of ϕ^{r_h} shown in the $r_h - \tau$ plane with $Pr_0^2 = 0.1$ and $a/r_0 = 1$ for the singly rotating Kerr-AdS$_6$ black hole. The red dot with τ_c represents the annihilation point for the black hole. There are two singly rotating Kerr-AdS$_6$ black holes when $\tau = \tau_1$. It is easy to obtain the topological number: $W = 1 - 1 = 0$.

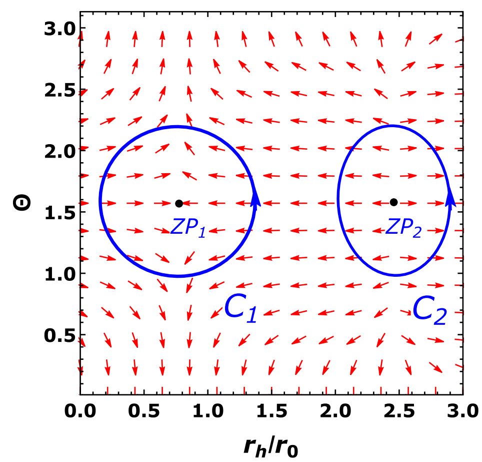


FIG. 8. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 0.1$, $a/r_0 = 1$, and $\tau/r_0 = 2.5$ for the singly rotating Kerr-AdS$_6$ black hole. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.80, \pi/2)$, $(2.43, \pi/2)$ for ZP$_1$ and ZP$_2$, respectively. The blue contours C_i are closed loops surrounding the zero points.

C. $d = 7$ case

Then, we consider $d = 7$ case with its generalized Helmholtz free energy being

$$\mathcal{F} = -\frac{3\pi^2 r_h^2(r_h^2 + a^2)}{16\tau(15 - 8\pi Pa^2)^2}[20\pi r_h(15 - 8\pi Pa^2) + \tau(8\pi Pr_h^2 + 15)(8\pi Pa^2 - 25)]. \tag{27}$$

Therefore, one can straightforwardly obtain

$$\tau = \frac{10\pi r_h(8\pi Pa^2 - 15)(5r_h^2 + 3a^2)}{(8\pi Pa^2 - 25)[6r_h^2(4\pi Pr_h^2 + 5) + a^2(16\pi Pr_h^2 + 15)]} \tag{28}$$

by solving the equation $\phi^{r_h} = 0$.

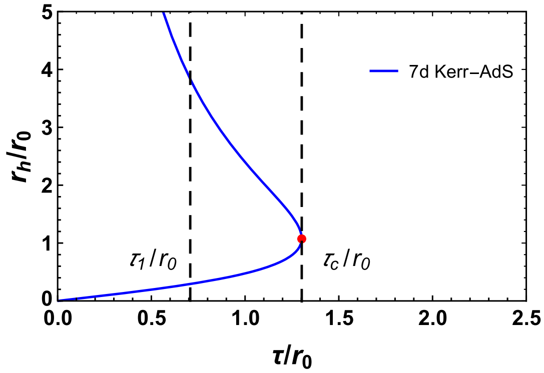


FIG. 9. Zero points of ϕ^{r_h} shown in the $r_h - \tau$ plane with $Pr_0^2 = 0.3$ and $a/r_0 = 1$ for the singly rotating Kerr-AdS$_7$ black hole. The red dot with τ_c denotes the annihilation point for the black hole. There are two singly rotating Kerr-AdS$_7$ black holes for $\tau = \tau_1$. Obviously, the topological number is $W = 0$.

Taking $Pr_0^2 = 0.3$ and $a/r_0 = 1$ for the singly rotating Kerr-AdS$_7$ black hole, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane in Fig. 9, and the unit vector field n with $\tau = r_0$ in Fig. 10, respectively. Note that for the values of $Pr_0^2 = 0.3$ and $a/r_0 = 1$, one annihilation point can be found at $\tau/r_0 = \tau_c/r_0 = 1.30$. Based on the local property of the zero points, we get the topological number $W = 0$ for the singly rotating Kerr-AdS$_7$ black hole, while that of the seven-dimensional singly rotating Kerr black hole is: $W = -1$ [20]. This demonstrates that the cosmological constant is crucial to determine the topological number of the rotating black hole in seven dimensions.

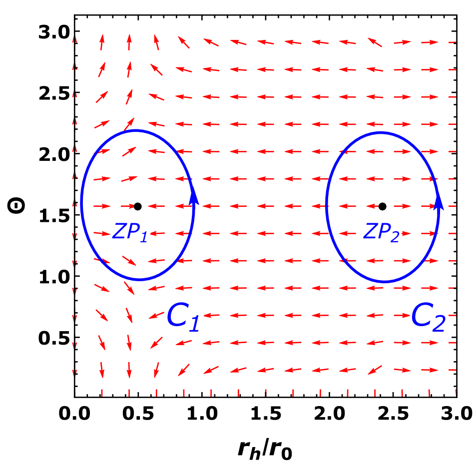


FIG. 10. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 0.3$, $a/r_0 = 1$, and $\tau = r_0$ for the singly rotating Kerr-AdS$_7$ black hole. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.48, \pi/2)$, $(2.40, \pi/2)$ for ZP$_1$ and ZP$_2$, respectively. The blue contours C_i are closed loops surrounding the zero points.

D. $d = 8$ case

Let us turn to the $d = 8$ case. Similar to the procedure done in the previous three subsections, one can get the generalized Helmholtz free energy as follows:

$$\mathcal{F} = -\frac{2\pi^2 r_h^3(r_h^2 + a^2)}{15\tau(8\pi Pa^2 - 21)^2}[42\pi r_h(21 - 8\pi Pa^2) + \tau(8\pi Pr_h^2 + 21)(16\pi Pa^2 - 63)]. \quad (29)$$

As a result, by solving the equation $\phi^{r_h} = 0$, one can easily arrive at

$$\tau = \frac{84\pi r_h(8\pi Pa^2 - 21)(3r_h^2 + 2a^2)}{(16\pi Pa^2 - 63)[56\pi Pr_h^4 + 5(8\pi Pa^2 + 21)r_h^2 + 63a^2]} \quad (30)$$

as the zero point of the vector field.

In Figs. 11 and 12, taking $Pr_0^2 = 0.5$ and $a/r_0 = 1$ for the singly rotating Kerr-AdS$_8$ black hole, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane and the unit vector field n with $\tau/r_0 = 0.8$, respectively. Note that for the values of $Pr_0^2 = 0.5$ and $a/r_0 = 1$, one annihilation point can be found at $\tau/r_0 = \tau_c/r_0 = 0.92$. Based on the local property of the zero points, it is easy to find that the topological number $W = 0$ for the singly rotating Kerr-AdS$_8$ black hole. Combined with the fact that the eight-dimensional singly rotating Kerr black hole has a topological number: $W = -1$ [20], it is evident that the cosmological constant is important in determining the topological number for the rotating black hole in eight dimensions.

E. $d = 9$ case

Finally, we investigate the topological number for the nine-dimensional singly rotating Kerr-AdS black hole whose generalized Helmholtz free energy is

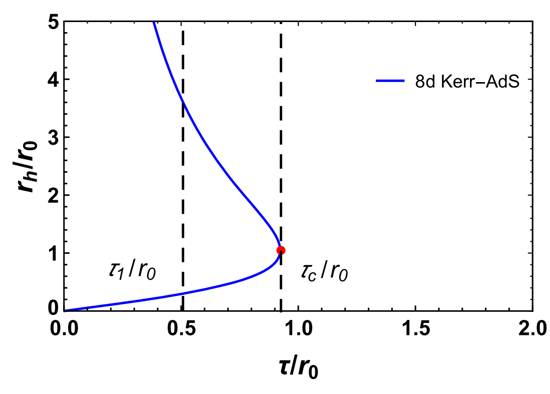


FIG. 11. Zero points of ϕ^{r_h} shown in the $r_h - \tau$ plane with $Pr_0^2 = 0.5$ and $a/r_0 = 1$ for the singly rotating eight-dimensional Kerr-AdS black hole. The red dot with τ_c denotes the annihilation point for the black hole. There are two singly rotating Kerr-AdS$_8$ black holes for $\tau = \tau_1$. It is easy to find that the topological number is $W = 0$.

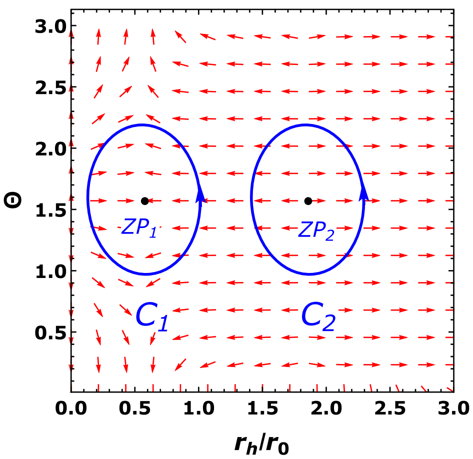


FIG. 12. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 0.5$, $a/r_0 = 1$, and $\tau/r_0 = 0.8$ for the singly rotating Kerr-AdS$_8$ black hole. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.59, \pi/2)$, $(1.85, \pi/2)$ for ZP$_1$ and ZP$_2$, respectively. The blue contours C_i are closed loops surrounding the zero points.

$$\mathcal{F} = -\frac{\pi^3 r_h^4 (r_h^2 + a^2)}{48\tau(2\pi P a^2 - 7)^2}[28\pi r_h(7 - 2\pi P a^2) + \tau(10\pi P a^2 - 49)(2\pi P r_h^2 + 7)]. \tag{31}$$

Thus, the zero point of the vector constructed in the topological approach can be written as

$$\tau = \frac{14\pi r_h(2\pi P a^2 - 7)(7r_h^2 + 5a^2)}{(10\pi P a^2 - 49)[8\pi P r_h^4 + 3(2\pi P a^2 + 7)r_h^2 + 14a^2]}. \tag{32}$$

In Figs. 13 and 14, taking $Pr_0^2 = 1$ and $a/r_0 = 1$ for the nine-dimensional singly rotating Kerr-AdS black hole, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane and the unit vector field n with $\tau/r_0 = 0.25$, respectively. Note that for the values of $Pr_0^2 = 1$ and $a/r_0 = 1$, one annihilation point can be found at $\tau/r_0 = \tau_c/r_0 = 0.27$. Based on the local property of the zero points, the topological number is easily determined as $W = 0$, which is different from the topological number of the nine-dimensional singly rotating Kerr black hole ($W = -1$) [20]. Therefore, this fact indicates that the cosmological constant plays a significant role in determining the topological number for the nine-dimensional rotating black hole.

F. Summary: The impact of dimension of the spacetime

Summarizing our results from Secs. IV A–IV E, we can find that the topological number of the five-dimensional singly rotating Kerr-AdS$_5$ black hole is $W = 0$, while the six- to nine-dimensional singly rotating Kerr-AdS black holes have both $W = -1$. Thus it indicates the dimension of spacetime has an important effect on the topological number for rotating AdS black holes.

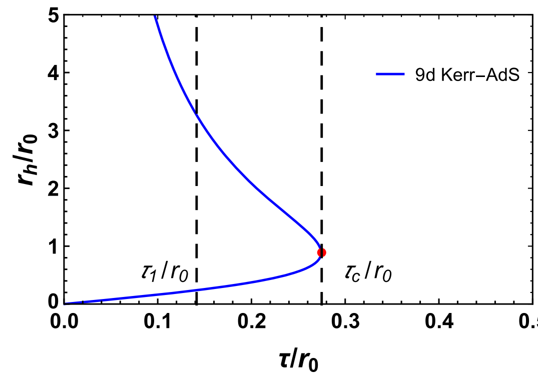


FIG. 13. Zero points of ϕ^{r_h} shown in the $r_h - \tau$ plane with $Pr_0^2 = 1$ and $a/r_0 = 1$ for the singly rotating Kerr-AdS$_9$ black hole. The red dot with τ_c represents the annihilation point for the black hole. There are two singly rotating Kerr-AdS$_9$ black holes when $\tau = \tau_1$. Furthermore, the topological number is $W = 0$.

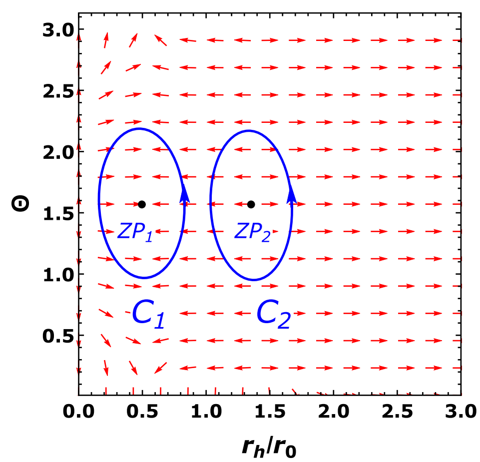


FIG. 14. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 1$, $a/r_0 = 1$, and $\tau/r_0 = 0.25$ for the singly rotating Kerr-AdS$_9$ black hole. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.56, \pi/2)$, $(1.39, \pi/2)$ for ZP$_1$ and ZP$_2$, respectively. The blue contours C_i are closed loops surrounding the zero points.

V. THREE-DIMENSIONAL ROTATING BTZ BLACK HOLE

Because the BTZ black hole [24,25] is a first nontrivial exact solution to the three-dimensional gravity theory, it is important to study the topological number of the rotating BTZ black hole. Therefore, in this section, we turn our attention to the three-dimensional BTZ black hole solution, whose metric is given by [24–26,36]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\varphi - \frac{J}{2r^2}dt\right)^2, \quad (33)$$

where

$$f(r) = -2m + \frac{r^2}{l^2} + \frac{J^2}{4r^2},$$

in which m is the mass parameter, l is the AdS radius, J is the angular momentum that must satisfy $|J| \leq ml$.

The mass and entropy associated with the above solution (33) are given by [39]

$$M = \frac{m}{4} = \frac{r_h^2}{8l^2} + \frac{J^2}{32r_h^2}, \qquad S = \frac{1}{2}\pi r_h, \quad (34)$$

where r_h is the location of the event horizon. Utilizing the definition of the generalized Helmholtz free energy (1) and substituting $l^2 = 1/(8\pi P)$, one can obtain

$$\mathcal{F} = \pi P r_h^2 + \frac{J^2}{32r_h^2} - \frac{\pi r_h}{2\tau}. \quad (35)$$

Thus, the components of the vector ϕ are

$$\phi^{r_h} = 2\pi P r_h - \frac{J^2}{16r_h^3} - \frac{\pi}{2\tau}, \qquad \phi^{\Theta} = -\cot\Theta\csc\Theta. \quad (36)$$

By solving the equation $\phi^{r_h} = 0$, one can obtain

$$\tau = \frac{8\pi r_h^3}{32\pi P r_h^4 - J^2} \quad (37)$$

as the zero point of the vector field ϕ.

For the three-dimensional rotating BTZ black hole, we take $Pr_0^2 = 0.02$ and $J/r_0 = 0.5$, and then plot the zero points of the component ϕ^{r_h} in Fig. 15, and the unit vector field n with $\tau/r_0 = 10$ in Fig. 16, respectively. Obviously, there is only one thermodynamically stable rotating BTZ black hole for any value of τ, which is also consistent with the conclusion given in Ref. [40] via the Joule-Thomson expansion. Based on the local property of the zero points, the topological number $W = 1$ can be found for the three-dimensional rotating BTZ black hole. In the Appendix, we will also investigate the topological number of the three dimensional charged BTZ black hole, and find that its value is $W = 0$.

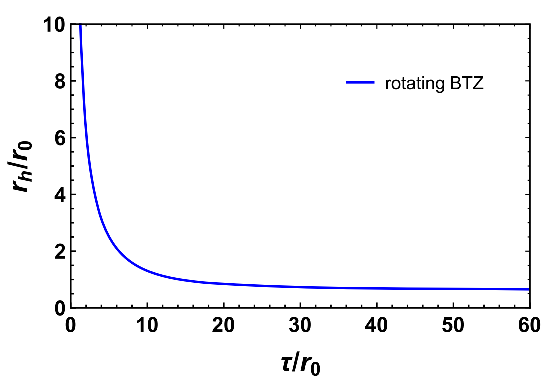


FIG. 15. Zero point of the vector ϕ^{r_h} shown on the $r_h - \tau$ plane with $Pr_0^2 = 0.02$ and $J/r_0 = 0.5$ for the rotating BTZ black hole. There is only one thermodynamically stable rotating BTZ black hole for any value of τ.

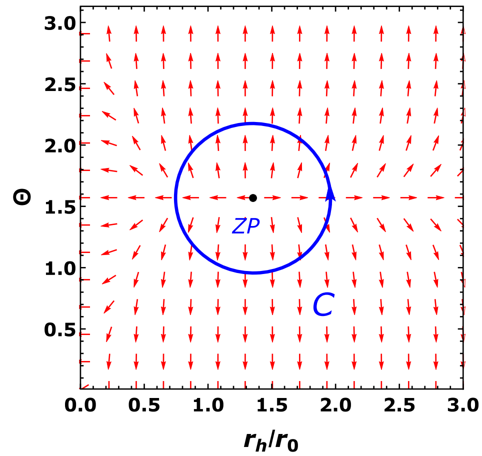


FIG. 16. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 0.02$, $J/r_0 = 0.5$, and $\tau/r_0 = 10$ for the rotating BTZ black hole. The zero point (ZP) marked with black dot is at $(r_h/r_0, \Theta) = (1.31, \pi/2)$. The blue contour C is closed loop surrounding the zero point.

VI. Kerr-NEWMAN-AdS$_4$ BLACK HOLE

Finally, we would like to investigate the topological number of the four-dimensional Kerr-Newman-AdS black hole [31], whose metric and Abelian gauge potential are [35,36]

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(\frac{\Delta_\theta}{\Xi}dt - \frac{a}{\Xi}\sin^2\theta d\varphi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\left[\frac{a(r^2+l^2)}{l^2\Xi}dt - \frac{r^2+a^2}{\Xi}d\varphi\right]^2, \quad (38)$$

$$A = \frac{qr}{\Sigma}\left(\frac{\Delta_\theta}{\Xi}dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right), \quad (39)$$

where

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr + q^2, \qquad \Xi = 1 - \frac{a^2}{l^2},$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Sigma = r^2 + a^2\cos^2\theta,$$

in which a, m, and q are the rotation, the mass and electric charge parameters, respectively, and l is the AdS radius. The horizon radius r_h is determined by equation: $\Delta_r = 0$.

The mass and entropy associated with the above metric (38) can be calculated via the standard method and their results are

$$M = \frac{m}{\Xi^2}, \qquad S = \frac{\pi(r_h^2 + a^2)}{\Xi}. \tag{40}$$

Then, one can straightforwardly obtain the generalized Helmholtz free energy of this black hole as

$$\mathcal{F} = \frac{24\pi P r_h^2(r_h^2 + a^2) + a^2[16\pi P Q^2(4\pi P a^2 - 3) + 9]}{2r_h(8\pi P a^2 - 3)^2} + \frac{9(r_h^2 + Q^2)}{2r_h(8\pi P a^2 - 3)^2} + \frac{6\pi(r_h^2 + a^2)}{2\tau(8\pi P a^2 - 3)} \tag{41}$$

with $Q = q/\Xi$ being the electric charge of the black hole. Therefore, the zero point of the vector can be easily given as

$$\tau = \frac{12\pi r_h^3(8\pi P a^2 - 3)}{X - 24\pi P(3r_h^2 + a^2)r_h^2}, \tag{42}$$

where $X = a^2[16\pi P Q^2(4\pi P a^2 - 3) + 9] + 9(Q^2 - r_h^2)$.

For the four-dimensional Kerr-Newman-AdS black hole, we take $Pr_0^2 = 0.02$, $a/r_0 = 1$, $Q/r_0 = 1$, and plot the zero points of the component ϕ^{r_h} in Fig. 17, and the unit vector field n with $\tau/r_0 = 10$ in Fig. 18, respectively. Obviously, there is only one thermodynamically stable Kerr-Newman-AdS$_4$ black hole for any value of τ. Based upon the local property of the zero points, one can get the topological number $W = 1$ for the four-dimensional Kerr-Newman-AdS black hole, which is identical to that of the four-dimensional Kerr-AdS black hole. This fact indicates that the electric charge parameter has no effect on the topological number of rotating AdS black holes. Compared with the four-dimensional Kerr-Newman black hole which has a topological number of zero, it can be inferred that the cosmological constant plays an crucial role in determining the topological number for the rotating charged black hole.

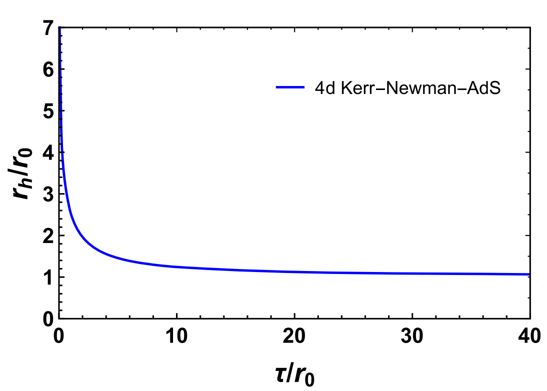


FIG. 17. Zero point of the vector ϕ^{r_h} shown on the $r_h - \tau$ plane with $Pr_0^2 = 0.02$, $a/r_0 = 1$, and $Q/r_0 = 1$ for the Kerr-Newman-AdS$_4$ black hole. There is only one stable Kerr-Newman-AdS$_4$ black hole for any value of τ. The topological number of this black hole is $W = 1$.

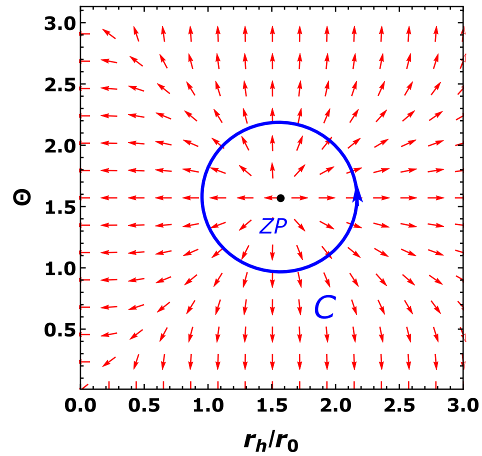


FIG. 18. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 0.02$, $a/r_0 = 1$, $Q/r_0 = 1$, and $\tau/r_0 = 10$ for the Kerr-Newman-AdS$_4$ black hole. The zero point (ZP) marked with black dot is at $(r_h/r_0, \Theta) = (1.52, \pi/2)$. The blue contour C is closed loop surrounding the zero point.

VII. CONCLUSIONS

In this paper, we have extended our previous work [20] to the more general rotating AdS black hole cases and investigated the topological numbers of the singly rotating Kerr-AdS black holes in arbitrary dimensions and the four-dimensional Kerr-Newman-AdS black hole as well as the three-dimensional rotating or charged BTZ black hole. Table I summarizes some interesting results found in the present work. We find that the $d \geq 6$ singly rotating Kerr-AdS black holes, the Schwarzschild-AdS black hole, and the charged BTZ black hole belong to the same kind of topological classes because of their topological numbers being $W = 0$, while the Kerr-Newman-AdS black hole, the $d = 4$, 5 singly rotating Kerr-AdS black holes, and the rotating BTZ black hole belong to another kind of topological class due to their topological number being $W = 1$.

As a new consequence, we have discovered that the topological number of the rotating black holes is significantly influenced by the cosmological constant. Furthermore, combining our results with those in Refs. [7,14,20], we have tabulated Table II, from which we have also observed a new interesting phenomenon: the

TABLE I. The topological number W, numbers of generation and annihilation points for various AdS black holes.

BH solution	W	Generation point	Annihilation point
Schwarzschild-AdS_4 BH [14]	0	0	1
$d \geq 6$ singly rotating Kerr-AdS BH	0	0	1
Charged BTZ BH	0	0	1
$d = 5$ singly rotating Kerr-AdS BH	1	1 or 0	1 or 0
RN-AdS_4 BH [7]	1	1 or 0	1 or 0
Kerr-AdS_4 BH	1	1 or 0	1 or 0
Kerr-Newman-AdS_4 BH	1	0	0
Rotating BTZ BH	1	0	0

TABLE II. The topological number W, numbers of generation and annihilation points for various black holes and their AdS extensions.

BH solution	W	Generation point	Annihilation point
Schwarzschild BH [7]	−1	0	0
Schwarzschild-AdS_4 BH [14]	0	0	1
RN BH [7]	0	1	0
RN-AdS_4 BH [7]	1	1 or 0	1 or 0
Kerr BH [20]	0	1	0
Kerr-AdS_4 BH	1	1 or 0	1 or 0
Kerr-Newman BH [20]	0	1	0
Kerr-Newman-AdS_4 BH	1	0	0
$d = 5$ singly rotating Kerr BH [20]	0	1	0
$d = 5$ singly rotating Kerr-AdS BH	1	1 or 0	1 or 0
$d \geq 6$ singly rotating Kerr BH [20]	−1	0	0
$d \geq 6$ singly rotating Kerr-AdS BH	0	0	1

difference between the topological number of the AdS black hole and that of its corresponding asymptotically flat black hole is always unity. We conjecture that this might also be true for other kinds of black holes. However, this needs to be tested by further investigating the topological numbers of many other black holes and their AdS counterparts.

As far as the impact of the electric charge parameter on the topological number of the four-dimensional black holes is concerned, one can infer from Table I that the electric charge parameter can change the topological number of the static AdS_4 black holes, while from Tables I and II that it has no impact on the topological number of the four-dimensional rotating AdS black holes because the four-dimensional Kerr-AdS and Kerr-Newman-AdS black holes have the same topological number.

Finally, it should be mentioned that all rotating AdS black holes studied in the present paper are underrotating, namely, their rotation angular velocities are less than the speed of light (i.e., $a < l$). Therefore, a most related issue is to investigate their ultraspinning and overrotating cases, and to test our guess by checking the topological numbers of the ultraspinning AdS black holes (i.e., $a = l$) [41–48] and the over-rotating Kerr-AdS black holes (i.e., $a > l$) [49,50].

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APPENDIX: THREE-DIMENSIONAL CHARGED BTZ BLACK HOLE

In this appendix, we will investigate the topological number of the three-dimensional charged BTZ black hole, whose metric reads [51]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \tag{A1}$$

$$A = -q \ln\left(\frac{r}{l}\right) dt, \tag{A2}$$

where

$$f(r) = -2m - \frac{q^2}{2} \ln\left(\frac{r}{l}\right) + \frac{r^2}{l^2}, \tag{A3}$$

in which m and q are the mass parameter and electric charge, respectively, and l is the AdS radius. The event horizon is determined by: $f(r_h) = 0$.

For the three-dimensional charged BTZ black hole, the mass and the entropy are [39]

$$M = \frac{m}{4} = \frac{r_h^2}{8l^2} - \frac{q^2}{16} \ln\left(\frac{r_h}{l}\right), \qquad S = \frac{1}{2}\pi r_h. \tag{A4}$$

Substituting $l^2 = 1/(8\pi P)$ into the definition of the generalized Helmholtz free energy (1), one can arrive at

$$\mathcal{F} = \pi P r_h^2 + \frac{q^2}{16} \ln\left(2r_h\sqrt{2\pi P}\right) - \frac{\pi r_h}{2\tau}, \tag{A5}$$

Thus, the components of the vector ϕ are

$$\phi^{r_h} = 2\pi P r_h - \frac{q^2}{16 r_h^3} - \frac{\pi}{2\tau}, \qquad \phi^{\Theta} = -\cot\Theta \csc\Theta. \tag{A6}$$

By solving the equation $\phi^{r_h} = 0$, one can obtain

$$\tau = \frac{8\pi r_h}{32\pi P r_h^2 + q^2} \tag{A7}$$

as the zero point of the vector field ϕ.

In Figs. 19 and 20, we take $P = 0.002$ and $q/r_0 = 1$ for the three-dimensional charged BTZ black hole, and plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane and the unit vector field n with $\tau/r_0 = 25$, respectively. Note that for the values of $P = 0.002$ and $q/r_0 = 1$, one annihilation point can be found at $\tau/r_0 = \tau_c/r_0 = 28.03$. Based on the local property of the zero points, one can get its topological number $W = 0$.

Note that by simply eliminating the charge q in Eq. (A7), the zero point of the vector field ϕ of the static neutral BTZ black hole can be directly expressed as $\tau = 1/(4Pr_h)$, and its topological number can be easily obtained as $W = 1$, which implies that the electric charge has a remarkable impact on the topological number of static AdS_3 black holes.

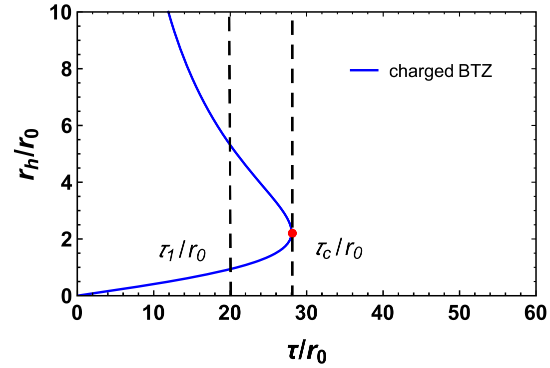

FIG. 19. Zero points of ϕ^{r_h} shown in the $r_h - \tau$ plane with $P = 0.002$ and $q/r_0 = 1$ for the charged BTZ black hole. The red dot with τ_c represents the annihilation point for the black hole. There are two charged BTZ black holes when $\tau = \tau_1$. Furthermore, its topological number is: $W = 0$.

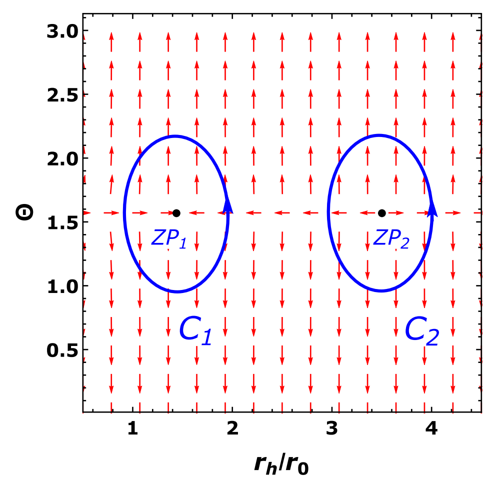

FIG. 20. The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $P = 0.002$, $q/r_0 = 1$, and $\tau/r_0 = 25$ for the charged BTZ black hole. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (1.37, \pi/2)$, $(3.63, \pi/2)$ for ZP_1 and ZP_2, respectively. The blue contours C_i are closed loops surrounding the zero points.

[1] P. V. P. Cunha, E. Berti, and C. A. R. Herdeiro, Light Ring Stability in Ultra-Compact Objects, Phys. Rev. Lett. **119**, 251102 (2017).
[2] P. V. P. Cunha and C. A. R. Herdeiro, Stationary Black Holes and Light Rings, Phys. Rev. Lett. **124**, 181101 (2020).
[3] S.-W. Wei, Topological charge and black hole photon spheres, Phys. Rev. D **102**, 064039 (2020).
[4] M. Guo and S. Gao, Universal properties of light rings for stationary axisymmetric spacetimes, Phys. Rev. D **103**, 104031 (2021).
[5] S.-W. Wei and Y.-X. Liu, Topology of equatorial timelike circular orbits around stationary black holes, Phys. Rev. D **107**, 064006 (2023).
[6] X. Ye and S.-W. Wei, Topological study of equatorial timelike circular orbit for spherically symmetric (hairy) black holes, arXiv:2301.04786.
[7] S.-W. Wei, Y.-X. Liu, and R. B. Mann, Black Hole Solutions as Topological Thermodynamic Defects, Phys. Rev. Lett. **129**, 191101 (2022).
[8] S.-W. Wei and Y.-X. Liu, Topology of black hole thermodynamics, Phys. Rev. D **105**, 104003 (2022).
[9] P. K. Yerra and C. Bhamidipati, Topology of black hole thermodynamics in Gauss-Bonnet gravity, Phys. Rev. D **105**, 104053 (2022).
[10] M. B. Ahmed, D. Kubiznak, and R. B. Mann, Vortex/anti-vortex pair creation in black hole thermodynamics, Phys. Rev. D **107**, 046013 (2023).
[11] P. K. Yerra and C. Bhamidipati, Topology of Born-Infeld AdS black holes in 4D novel Einstein-Gauss-Bonnet gravity, Phys. Lett. B **835**, 137591 (2022).
[12] Z.-Y. Fan, Topological interpretation for phase transitions of black holes, Phys. Rev. D **107**, 044026 (2023).
[13] D. Wu, Classifying topology of consistent thermodynamics of four-dimensional Lorentzian NUT-charged spacetimes, arXiv:2302.01100.
[14] P. K. Yerra, C. Bhamidipati, and S. Mukherji, Topology of critical points and Hawking-Page transition, Phys. Rev. D **106**, 064059 (2022).
[15] N.-C. Bai, L. Li, and J. Tao, Topology of black hole thermodynamics in Lovelock gravity, Phys. Rev. D **107**, 064015 (2023).
[16] C. H. Liu and J. Wang, The topological natures of the Gauss-Bonnet black hole in AdS space, Phys. Rev. D **107**, 064023 (2023).
[17] C. X. Fang, J. Jiang, and M. Zhang, Revisiting thermodynamic topologies of black holes, J. High Energy Phys. 01 (2023) 102.
[18] N.-C. Bai, L. Song, and J. Tao, Reentrant phase transition in holographic thermodynamics of Born-Infeld AdS black hole, arXiv:2212.04341.
[19] N. Chatzifotis, P. Dorlis, N. E. Mavromatos, and E. Papantonopoulos, On the thermal stability of hairy black holes, arXiv:2302.03980.
[20] D. Wu, Topological classes of rotating black holes, Phys. Rev. D **107**, 024024 (2023).
[21] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[22] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).
[23] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).
[24] M. Bañados, C. Teitelboim, and J. Zanelli, Black Hole in Three-Dimensional Spacetime, Phys. Rev. Lett. **69**, 1849 (1992).
[25] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Geometry of the $2+1$ black hole, Phys. Rev. D **48**, 1506 (1993).
[26] S. Carlip, The $(2+1)$-dimensional black hole, Classical Quantum Gravity **12**, 2853 (1995).
[27] R. Li, C. H. Liu, K. Zhang, and J. Wang, Topology of the landscape and dominant kinetic path for the thermodynamic phase transition of the charged Gauss-Bonnet AdS black holes, arXiv:2302.06201.
[28] Y.-S. Duan and M.-L. Ge, $SU(2)$ gauge theory and electrodynamics of N moving magnetic monopoles, Sci. Sin. **9**, 1072 (1979).
[29] Y.-S. Duan, S. Li, and G.-H. Yang, The bifurcation theory of the Gauss-Bonnet-Chern topological current and Morse function, Nucl. Phys. **B514**, 705 (1998).
[30] L.-B. Fu, Y.-S. Duan, and H. Zhang, Evolution of the Chern-Simons vortices, Phys. Rev. D **61**, 045004 (2000).
[31] B. Carter, Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations, Commun. Math. Phys. **10**, 280 (1968).
[32] S. Wang, S. Q. Wu, F. Xie, and L. Dan, The first laws of thermodynamics of the $(2+1)$-dimensional BTZ black holes and Kerr-de Sitter spacetimes, Chin. Phys. Lett. **23**, 1096 (2006).
[33] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the mechanics of AdS black holes, Classical Quantum Gravity **26**, 195011 (2009).
[34] M. Cvetič, G. W. Gibbons, D. Kubizňák, and C. N. Pope, Black hole enthalpy and an entropy inequality for the thermodynamic volume, Phys. Rev. D **84**, 024037 (2011).
[35] M. M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Classical Quantum Gravity **17**, 399 (2000).
[36] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Rotation and the AdS-CFT correspondence, Phys. Rev. D **59**, 064005 (1999).
[37] G. W. Gibbons, H. Lü, Don N. Page, and C. N. Pope, Rotating Black Holes in Higher Dimensions with a Cosmological Constant, Phys. Rev. Lett. **93**, 171102 (2004).
[38] S.-W. Wei, P. Cheng, and Y.-X. Liu, Analytical and exact critical phenomena of d-dimensional singly spinning Kerr-AdS black holes, Phys. Rev. D **93**, 084015 (2016).
[39] A. M. Frassino, R. B. Mann, and J. R. Mureika, Lower-dimensional black hole chemistry, Phys. Rev. D **92**, 124069 (2015).
[40] J. Liang, B. R. Mu, and P. Wang, Joule-Thomson expansion of lower-dimensional black holes, Phys. Rev. D **104**, 124003 (2021).
[41] D. Klemm, Four-dimensional black holes with unusual horizons, Phys. Rev. D **89**, 084007 (2014).

[42] R. A. Hennigar, R. B. Mann, and D. Kubizňák, Entropy Inequality Violations from Ultraspinning Black Holes, Phys. Rev. Lett. **115**, 031101 (2015).
[43] A. Gnecchi, K. Hristov, D. Klemm, C. Toldo, and O. Vaughan, Rotating black holes in 4d gauged supergravity, J. High Energy Phys. 01 (2014) 127.
[44] D. Wu and P. Wu, Null hypersurface caustics for high-dimensional superentropic black holes, Phys. Rev. D **103**, 104020 (2021).
[45] D. Wu, P. Wu, H. Yu, and S.-Q. Wu, Notes on the thermodynamics of superentropic AdS black holes, Phys. Rev. D **101**, 024057 (2020).
[46] D. Wu, P. Wu, H. Yu, and S.-Q. Wu, Are ultraspinning Kerr-Sen-AdS_4 black holes always superentropic?, Phys. Rev. D **102**, 044007 (2020).
[47] D. Wu, S.-Q. Wu, P. Wu, and H. Yu, Aspects of the dyonic Kerr-Sen-AdS_4 black hole and its ultraspinning version, Phys. Rev. D **103**, 044014 (2021).
[48] D. Wu and S.-Q. Wu, Ultra-spinning Chow's black holes in six-dimensional gauged supergravity and their thermodynamical properties, J. High Energy Phys. 11 (2021) 031.
[49] B. McInnes, Characterising the most rapidly rotating AdS_5-Kerr black holes, Classical Quantum Gravity **38**, 095001 (2021).
[50] S. Noda and Y. C. Ong, Null hypersurface caustics, closed null curves, and super entropy, Phys. Rev. D **103**, 024053 (2021).
[51] C. Martínez, C. Teitelboim, and J. Zanell, Charged rotating black hole in three spacetime dimensions, Phys. Rev. D **61**, 104013 (2000).