Chern Simons condensate from misaligned axions

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We obtain the nonequilibrium condensate of the Chern Simons density induced by a misaligned homogeneous coherent axion field in linear response. The Chern-Simons dynamical susceptibility is simply related to the axion self-energy, a result that is valid to leading order in the axion coupling but to all orders in the couplings of the gauge fields to other fields within or beyond the standard model except the axion. The induced Chern-Simons density requires renormalization which is achieved by vacuum subtraction. For ultralight axions of mass m_a coupled to electromagnetic fields with coupling g, the renormalized high temperature contribution postrecombination is $\langle \vec{E}\cdot\vec{B}\rangle(t) = -\frac{g\pi^2 T^4}{15}\,\bar{a}(t) + \frac{g m_a^2 T}{16\pi}\,\dot{\bar{a}}(t)$ with $\bar{a}(t)$ the dynamical homogeneous axion condensate. We conjecture that emergent axionlike quasiparticle excitations in condensed matter systems may be harnessed to probe cosmological axions and the Chern-Simons condensate. Furthermore, it is argued that a misaligned axion can also induce a non-Abelian Chern-Simons condensate of similar qualitative form, and can also "seed" chiral symmetry breaking and induce a neutral pion condensate after the QCD phase transition.

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I. INTRODUCTION

The strong (CP) problem in quantum chromodynamics (QCD) motivated the proposal of a new pseudoscalar particle beyond the standard model, the axion, as a possible solution [1–3] by elevating a CP violating angle to a dynamical field. Such field may be produced nonthermally in the early Universe, for example by a misalignment mechanism in which an initial axion coherent condensate is produced out of equilibrium and evolves toward the minimum of its (effective) potential. Such axion field has also been recognized as a potentially viable cold dark matter candidate [4–6]. Pseudoscalar particles with properties similar to the QCD axion can also be accommodated within suitable extensions beyond the standard model, collectively referred to as axion-like-particles (ALP), which can also be dark matter candidates [7–11], in particular as compelling candidates for ultralight dark matter [12,13]. Furthermore, a dynamical misaligned axion coherent condensate could also be a dark energy candidate in the form of a quintessence field whose slow dynamical evolution toward an equilibrium minimum would induce an accelerated cosmological expansion phase [14].

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Constraints on the mass and couplings of ultralight ALP [9–11,15] are being established by various observations and experiments ranging from astrophysical phenomena to table-top experiments [16–18]. There are two important features that characterize ALP: (i) a misalignment mechanism results in coherent oscillations of the expectation value of the ALP field which gives rise to its contribution to the energy density as a cold dark matter component [4–6,9–11,19], and (ii) its pseudoscalar nature leads to an interaction between the ALP and photons or gluons via pseudoscalar composite operators of gauge fields, such as $F_{\mu\nu}\tilde{F}^{\mu\nu}$ in the case of the ALP-photon interaction and $G^{\mu\nu;b}\tilde{G}_{\mu\nu;b}$ in the case of gluons. We refer to these operators as Chern-Simons terms which are total surface terms. Such couplings were originally studied in Ref. [20] within the context of parity and Lorentz violating extensions of the standard model and early limits on these couplings were established from birefringence effects, namely different dispersion relations for different polarizations, and the rotation of the plane of polarization from astrophysical sources. A telltale feature of birefringence from the electromagnetic coupling to axions is that the polarization rotation angle is frequency independent [21,22], which differentiates it from the more familiar Faraday effect resulting from the presence of magnetic fields in the astrophysical plasma. Optical properties of axion backgrounds have been discussed in Ref. [23], further electromagnetic signatures of axion electrodynamics were studied in Refs. [24,25], and photon production from parametric amplification of a misaligned axion condensate was studied in Ref. [26]. Analysis of evidence for parity violating effects in the Planck 2018 polarization data of the cosmic microwave background (CMB) anisotropies revealed a nonvanishing cosmic birefringence angle at the 2.4σ level [27], and more recently a combined analysis of WMAP and Planck polarization data revealed hints of isotropic cosmic birefringence at the 3σ level [28–31]. These tantalizing hints may be a signal of cosmological axions.

Axions may also play a role in condensed matter physics, possibly as emergent quasiparticles in topological insulators where magnetic fluctuations couple to electromagnetism just like axions [32–34], as axionic charge density waves in Weyl semimetals [35,36], or as an emergent axion response in multilayered metamaterials with tunable couplings [37] or in multiferroics [38]. The measurement of an emergent dynamic axion field in chromia has been reported in Ref. [39]; therefore, condensed matter systems may very well provide an experimental platform to test the main aspects of axion electrodynamics which may complement and bolster the case for axions in cosmology. Hence, the study of axion (electro) dynamics is of timely interdisciplinary relevance. In this article we suggest that axionlike quasiparticles in condensed matter systems mix with the cosmological axion; therefore topological insulators, Weyl semimetals, or metamaterials may provide experimental platforms to probe the cosmological axion and the Chern-Simons condensate.

A. Motivation and objectives

The possibility of an axion or ALP being the dark matter and/or dark energy candidate with a hallmark signature of frequency independent cosmic birefringence motivates a study of its nonequilibrium evolution when coupled to standard model degrees of freedom. Recently, Refs. [40] implemented methods borrowed from nonequilibrium quantum field theory, namely the in-in Keldysh-Schwinger formulation, and the theory of quantum open systems to study the nonequilibrium dynamics of axionlike particles coupled to a bath in thermal equilibrium. These studies focused on the damping of a coherent misaligned condensate as a consequence of its decay into photons from its coupling to electromagnetic fields via the Chern-Simons density, the concomitant thermalization of the axion fluctuations with the CMB photons yielding a mixed dark matter scenario, and an assessment of the timescales of decoherence and entropy production. The Schwinger-Keldysh formulation of nonequilibrium quantum field theory is suited to obtain the causal equations of motion of the axion field in presence of a heat bath. These were obtained in Ref. [40] and shown to be stochastic, of the Langevin type with a Gaussian noise and include the retarded self-energy. The self-energy describes the damping of axion oscillations as a consequence of decay into the bath degrees of freedom (radiation reaction), and damping and noise are related by the quantum fluctuation dissipation relation, a consequence of which is the thermalization of axion fluctuations with the heat bath. An alternative method based on the quantum master equation confirms these results [40] and unequivocally shows that damping, thermalization, and decoherence are directly related and occur on similar timescales.

Motivated by the confluence of interest on nonequilibrium axion dynamics both in cosmology and in condensed matter physics, in this article we study the emergence of a Chern-Simons (topological) condensate as a consequence of the nonequilibrium dynamics of a misaligned axion macroscopic coherent condensate. A Chern-Simons condensate would be manifest as a nonvanishing expectation value of the Abelian Chern-Simons density $\vec{E}\cdot\vec{B}$ in the nonequilibrium density matrix that describes the heat bath and the dynamical misaligned axion condensate. This is distinctly different from the classical treatments of axion electrodynamics studied in Refs. [21,23–25], and to the best of our knowledge such study has not been previously undertaken. Furthermore, the possibility of testing axion electrodynamics in condensed matter systems, such as topological insulators and Weyl semimetals, bolsters the case for studying the emergence of Chern-Simons condensates as an intrinsic, fundamental nonequilibrium aspect of axion physics with interdisciplinary relevance.

To this aim we implement the theory of linear response, ubiquitous in many body physics [41], to obtain the Chern-Simons condensate *induced* by a misaligned axion condensate. While this study is focused on studying the emergence of a Chern-Simons condensate in Minkowski space-time, as an initial step toward a more complete understanding within the context of an expanding cosmology, the basic concepts are expected to translate to cosmology qualitatively, but with quantitative differences in the time evolution. Such study awaits the consistent extrapolation of the linear response treatment to the realm of an expanding cosmology.

This article is structured as follows: in Sec. II we define various models wherein an axion field is coupled to generic composite pseudoscalar operators $\mathcal{O}$ describing degrees of freedom within (or beyond) the standard model, and obtain an exact relation between the induced condensate $\langle\mathcal{O}\rangle$ and the dynamical expectation value of the axion field. Linear response theory is implemented to obtain the induced nonequilibrium expectation value $\langle\mathcal{O}\rangle$ to leading order in the axion coupling, and introduce the concept of the dynamical susceptibility, namely the response kernel that relates $\langle\mathcal{O}\rangle$ to the coherent misaligned axion condensate. In this section we establish one of the main results: the dynamical susceptibility is directly and simply related to the axion self-energy. In Sec. III, we apply these results to obtain the nonequilibrium expectation value $\langle\vec{E}\cdot\vec{B}\rangle$, namely the Chern-Simons condensate, in axion electrodynamics. In this section we show that this condensate features ultraviolet divergences proportional to the dynamical axion condensate, which acts as an explicit parity-symmetry breaking term; we also obtain the high temperature contributions to the Chern-Simons condensate for ultralight axions. In Sec. IV we argue that axionlike quasiparticles in condensed matter systems such as topological insulators or Weyl semimetals mix with the cosmological axion via correlation functions of the Chern-Simons density, and that a Chern-Simons condensate acts as a nonequilibrium driving term coupled to the emergent axion field. Therefore these experimentally available systems may be harnessed to probe the cosmological axion and the Chern-Simons condensate. In Sec. V we discuss important caveats in the cosmological setting, subtle renormalization aspects of the Chern-Simons condensate, and we argue on corresponding results for the non-Abelian case, albeit with caveats. We also conjecture that a misaligned axion condensate *may* induce a neutral pion condensate and seed chiral symmetry breaking during the QCD phase transition. Section VI summarizes our conclusions.

II. CONDENSATE INDUCED BY AXIONLIKE FIELDS

In this section, we discuss a general composite pseudoscalar operator $\mathcal{O}$ coupled to axionlike fields and show that such coupling implies that a coherent condensate of the axion field induces a macroscopic condensate of the pseudoscalar operator, namely $\langle\mathcal{O}\rangle \neq 0$. We obtain a formal exact relation between the expectation value of the axion field and that of the composite operator $\mathcal{O}$. We then implement linear response to obtain an explicit relation between these condensates to leading order in the axion coupling.

A. Exact relation between the ALP condensate and $\langle\mathcal{O}\rangle$

We consider an axionlike field $a(x)$ coupled to a composite pseudoscalar operator $\mathcal{O}_\chi(x)$ of generic fields $\chi(x)$. The Lagrangian density is

$$\mathcal{L}[a,\chi]=\frac{1}{2}\partial_\mu a(x)\partial^\mu a(x)-\frac{1}{2}m_{0a}^2a^2(x)-ga(x)\mathcal{O}_\chi(x)+\mathcal{L}_\chi, \tag{2.1}$$

where m_{0a} is the bare axion mass and $\mathcal{L}_\chi$ is the Lagrangian density describing the fields χ. Some examples of fields χ are electromagnetic fields with $\mathcal{O}_\chi(x) = \vec{E}(x)\cdot\vec{B}(x)$; gluon fields with $\mathcal{O}_\chi(x) = G^{\mu\nu,b}(x)\tilde{G}_{\mu\nu,b}(x)$, where the tilde stands for the dual of the gauge fields; or fermionic fields with $\mathcal{O}_\chi(x) = i\bar{\Psi}(x)\gamma^5\Psi(x)$. These fields could also be coupled to other degrees of freedom within or beyond the standard model, which are also all included in $\mathcal{L}_\chi$.

For gauge fields, the operators $\mathcal{O}_\chi(x)$ are the Chern-Simons terms and are a total surface term [20]. In $3+1$ dimensions, these operators feature dimension $(mass)^4$, and the axionlike field $a(x)$ features dimension $(mass)$, which means they couple via nonrenormalizable interactions whose coupling strength features dimension $(mass)^{-1}$. Therefore, the axion interacting locally with gauge fields via Chern-Simons terms must be interpreted as an effective field theory, whose validity is restricted to scales below a cutoff Λ. Furthermore, at finite temperature T, the validity of the effective field theory requires that $\Lambda \gg T$ so that high energy degrees of freedom are not thermally excited. This observation will become relevant in the discussion of the induced Chern-Simons condensate in the next section.

The Heisenberg equation of motion for the axion field obtained from the Lagrangian density (2.1) is

$$\frac{\partial^2}{\partial t^2}a(\vec{x},t) - \nabla^2 a(\vec{x},t) + m_{0a}^2 a(\vec{x},t) = -g\mathcal{O}_\chi(\vec{x},t), \tag{2.2}$$

where the time evolution of any operator $\mathcal{A}$ in the full Heisenberg picture is

$$\mathcal{A}(\vec{x},t) = e^{iH(t-t_0)}\mathcal{A}(\vec{x},t_0)e^{-iH(t-t_0)}, \tag{2.3}$$

with $H = H_\chi + H_a + H_i$ being the total Hamiltonian for the χ and axion fields and their interaction. The expectation value of any Heisenberg picture field operator $\mathcal{A}$ is obtained as $\langle\mathcal{A}(\vec{x},t)\rangle = \mathrm{Tr}(\mathcal{A}(\vec{x},t)\rho(t_0))$ where $\rho(t_0)$ is the normalized, initial density matrix. In the Heisenberg picture the density matrix does not depend on time; therefore, taking the expectation value of the Heisenberg field equation (2.2) yields

$$\langle\mathcal{O}_\chi(\vec{x},t)\rangle = -\frac{1}{g}\left[\frac{\partial^2}{\partial t^2}\bar{a}(\vec{x},t) - \nabla^2\bar{a}(\vec{x},t) + m_{0a}^2\bar{a}(\vec{x},t)\right], \tag{2.4}$$

where $\bar{a}(\vec{x},t)$ is the expectation value of axionlike field solutions of the Heisenberg equation of motion (2.2). This is an *exact* relation valid for arbitrary initial conditions and to all orders in the various couplings; however, Eq. (2.4) by itself does not yield a closed expression for $\langle\mathcal{O}_\chi(\vec{x},t)\rangle$. This is because $\bar{a}(\vec{x},t)$ is the solution of the full equation of motion (2.2), which is not known *a priori* and is obtained perturbatively in general. The exact relation (2.4) becomes useful when the solution $\bar{a}(\vec{x},t)$ is obtained.

B. Linear response

We now formulate the general theory of linear response implemented to obtain a nonequilibrium expectation value of composite pseudoscalar operators coupled to axionlike fields as in the Lagrangian (2.1), relegating its specific application to the electromagnetic Chern-Simons term to the next section.

The Lagrangian density (2.1) describes several relevant couplings of axionlike fields to generic fields χ. For $g = 0$ these degrees of freedom are assumed to be described by a parity even thermal equilibrium density matrix ρ_χ, consequently

$$\mathrm{Tr}\mathcal{O}_\chi(x)\rho_\chi = 0. \tag{2.5}$$

A misaligned axionlike condensate is described by a *classical* field $\bar{a}(\vec{x}, t)$ corresponding to the expectation value of the axionlike field in a coherent state density matrix that describes the axion field [40]. Therefore we can decompose $a(x) = \bar{a}(\vec{x}, t) + \tilde{a}(x)$ where $\tilde{a}(x)$ corresponds to the fluctuations of the axion field around the condensate and features a vanishing expectation value in the axion density matrix. As envisaged in cosmology, the misaligned condensate $\bar{a}(\vec{x}, t)$ is a macroscopic field; if it is a quintessence field driving cosmological expansion, it is homogeneous within at least the Hubble scale. Therefore, neglecting the fluctuations $\tilde{a}(x)$ (this is a mean field approximation) the interaction term in (2.1) is $\mathcal{L}_I = -g\bar{a}(\vec{x}, t)\mathcal{O}_\chi(x)$; hence the pseudoscalar coupling to the axion field results in the fields χ being coupled to an *c-number* external source $\bar{a}(\vec{x}, t)$ with an explicit time dependence determined by the time evolution of the misaligned condensate. In the presence of this classical source, the total time dependent Hamiltonian for the χ fields is

$$\tilde{H}_\chi(t) = H_\chi + H_I(t), \tag{2.6}$$

where H_χ is the Schroedinger picture Hamiltonian of the χ fields including coupling to other fields within or beyond the standard model except the axion field, and

$$H_I(t) = g\int d^3x\bar{a}(\vec{x}, t)\mathcal{O}_\chi(\vec{x}) \tag{2.7}$$

is the interaction Hamiltonian in the Schroedinger picture of the χ fields, but with $\bar{a}(\vec{x}, t)$ playing the role of an "external" time dependent source term. In the Schroedinger picture the χ-field density matrix evolves in time as

$$\rho_\chi(t) = U(t, t_0)\rho_\chi(t_0)U^{-1}(t, t_0), \tag{2.8}$$

where the unitary time evolution operator $U(t, t_0)$ obeys

$$i\frac{d}{dt}U(t, t_0) = \tilde{H}_\chi(t)U(t, t_0)\ ;\ U(t_0, t_0) = 1. \tag{2.9}$$

The initial density matrix $\rho_\chi(t_0)$ is *assumed* to describe an ensemble of the χ degrees of freedom in thermal equilibrium at temperature $T = 1/\beta$, namely,

$$\rho_\chi(t_0) = \frac{e^{-\beta H_\chi}}{\mathrm{Tr}e^{-\beta H_\chi}}, \tag{2.10}$$

and therefore

$$e^{iH_\chi t_0}\rho_\chi(t_0)e^{-iH_\chi t_0} = \rho_\chi(t_0); \tag{2.11}$$

since in general $[\mathcal{O}_\chi(\vec{x}), \rho_\chi(t_0)] \neq 0$ it follows that ρ_χ evolves in time out of equilibrium.

Writing

$$U(t, t_0) = e^{-iH_\chi t}\mathcal{U}(t, t_0)e^{iH_\chi t_0}, \tag{2.12}$$

we find that $\mathcal{U}(t, t_0)$ obeys

$$i\frac{d}{dt}\mathcal{U}(t, t_0) = \tilde{H}_I^{(H_\chi)}(t)\mathcal{U}(t, t_0)\ ;\ \mathcal{U}(t_0, t_0) = 1, \tag{2.13}$$

where

$$\tilde{H}_I^{(H_\chi)}(t) = e^{iH_\chi t}H_I(t)e^{-iH_\chi t} = g\int d^3x\bar{a}(\vec{x}, t)\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}, t), \tag{2.14}$$

and

$$\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}, t) = e^{iH_\chi t}\mathcal{O}_\chi(\vec{x})e^{-iH_\chi t} \tag{2.15}$$

is the composite operator in the *Heisenberg* picture in terms of the Hamiltonian H_χ, namely in the absence of the coupling to the axion field. The solution of Eq. (2.13) is

$$\mathcal{U}(t, t_0) = 1 - ig\int_{t_0}^{t}\int \bar{a}(\vec{x}', t')\mathcal{O}_\chi^{(H_\chi)}(\vec{x}', t')dt'd^3x' + \cdots \tag{2.16}$$

The expectation value of the Schroedinger picture operator $\mathcal{O}_\chi(\vec{x})$ in the nonequilibrium density matrix $\rho_\chi(t)$ is

$$\begin{aligned}\langle\mathcal{O}_\chi(\vec{x})\rangle(t) &= \mathrm{Tr}(\mathcal{O}_\chi(\vec{x})\rho_\chi(t))\\ &= \mathrm{Tr}(\mathcal{O}_\chi^{(H_\chi)}(\vec{x}, t)\mathcal{U}(t, t_0)\rho_\chi(t_0)\mathcal{U}^{-1}(t, t_0)),\end{aligned} \tag{2.17}$$

where we have used Eqs. (2.8), (2.12), (2.11), (2.15) and the cyclic property of the trace. Using (2.16) up to first order in g, and using the cyclic property of the trace, we find

$$\begin{aligned}\langle\mathcal{O}_\chi(\vec{x})\rangle(t) &= \langle\mathcal{O}_\chi(\vec{x})\rangle(t_0)\\ &\quad + \int d^3x'\int_{t_0}^{t}\Xi(\vec{x} - \vec{x}', t - t')\bar{a}(\vec{x}', t')dt' + \cdots,\end{aligned} \tag{2.18}$$

where

$$\langle \mathcal{O}_\chi(\vec{x})\rangle(t_0) = \mathrm{Tr}\mathcal{O}_\chi(\vec{x})\rho_\chi(t_0). \tag{2.19}$$

The linear response kernel, namely the dynamical susceptibility, is given by

$$\Xi(\vec{x}-\vec{x}', t-t') = -ig\mathrm{Tr}([\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x},t), \mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}',t')]\rho_\chi(t_0)); \quad t>t', \tag{2.20}$$

and $\bar{a}(\vec{x},t)$ is the solution of the equation of motion for the expectation value of the axion field. We have used that the equilibrium density matrix is space-translational invariant, and because $[H_\chi, \rho_\chi(t_0)] = 0$ it follows that Ξ must be solely a function of $t-t'$, as confirmed by the analysis below. Assuming that $\rho_\chi(t_0)$ is even under parity, it follows that

$$\langle \mathcal{O}_\chi(\vec{x})\rangle(t_0) = 0. \tag{2.21}$$

In Sec. V we discuss the caveats associated with this choice in cosmology.

Therefore, to leading order in the axion coupling we find the *induced* nonequilibrium expectation value for $t > t_0$ in linear response:

$$\langle \mathcal{O}_\chi(\vec{x})\rangle(t) = \int d^3x' \int_{t_0}^{t} \Xi(\vec{x}-\vec{x}', t-t')\bar{a}(\vec{x}',t')dt', \tag{2.22}$$

with the dynamical susceptibility $\Xi(\vec{x}-\vec{x}', t-t')$ given by (2.20).

It is convenient to write the susceptibility Ξ in terms of a Lehmann (spectral) representation. This is achieved by writing

$$\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x},t) = e^{iH_\chi t}e^{-i\vec{P}\cdot\vec{x}}\mathcal{O}_\chi(\vec{0},0)e^{-iH_\chi t}e^{i\vec{P}\cdot\vec{x}}, \tag{2.23}$$

and the density matrix in the basis of simultaneous eigenstates of H_χ and the total momentum operator $\vec{P}$, namely $(H_\chi, \vec{P})|n\rangle = (E_n, \vec{P}_n)|n\rangle$. Introducing the resolution of the identity in this basis $\sum_m |m\rangle\langle m| = 1$, and recognizing that $\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x},t)$ must be a Hermitian operator because the axion field is real and the total Hamiltonian is Hermitian, we find

$$\mathrm{Tr}(\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x},t)\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}',t')\rho_\chi(t_0)) = \frac{1}{\mathrm{Tr}e^{-\beta H_\chi}}\sum_{m,n} e^{-\beta E_n}|\langle n|\mathcal{O}_\chi(\vec{0},0)|m\rangle|^2 e^{i(E_n-E_m)(t-t')}e^{-i(\vec{P}_n-\vec{P}_m)\cdot(\vec{x}-\vec{x}')}$$
$$\mathrm{Tr}(\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}',t')\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x},t)\rho_\chi(t_0)) = \frac{1}{\mathrm{Tr}e^{-\beta H_\chi}}\sum_{m,n} e^{-\beta E_n}|\langle n|\mathcal{O}_\chi(\vec{0},0)|m\rangle|^2 e^{-i(E_n-E_m)(t-t')}e^{i(\vec{P}_n-\vec{P}_m)(\vec{x}-\vec{x}')}. \tag{2.24}$$

In terms of the spectral functions

$$\rho^>(k_0,\vec{k}) = \frac{(2\pi)^4}{\mathrm{Tr}e^{-\beta H_\chi}}\sum_{m,n} e^{-\beta E_n}|\langle n|\mathcal{O}_\chi(\vec{0},0)|m\rangle|^2\delta(k_0-(E_m-E_n))\delta(\vec{k}-(P_m-P_n)) \tag{2.25}$$

$$\rho^<(k_0,\vec{k}) = \frac{(2\pi)^4}{\mathrm{Tr}e^{-\beta H_\chi}}\sum_{m,n} e^{-\beta E_n}|\langle n|\mathcal{O}_\chi(\vec{0},0)|m\rangle|^2\delta(k_0-(E_n-E_m))\delta(\vec{k}-(P_n-P_m)), \tag{2.26}$$

the correlation functions (2.24) can be written as

$$\mathrm{Tr}(\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x},t)\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}',t')\rho_\chi(t_0)) = \int \frac{d^4k}{(2\pi)^4}\rho^>(k_0,\vec{k})e^{-ik_0(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \tag{2.27}$$

$$\mathrm{Tr}(\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}',t')\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x},t)\rho_\chi(t_0)) = \int \frac{d^4k}{(2\pi)^4}\rho^<(k_0,\vec{k})e^{-ik_0(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}. \tag{2.28}$$

Upon relabeling $m \leftrightarrow n$ in the sum in the definition (2.26) and recalling that $\mathcal{O}_\chi$ is a Hermitian operator, we find the Kubo-Martin-Schwinger relation [42]

$$\rho^<(k_0,k) = \rho^>(-k_0,k) = e^{-\beta k_0}\rho^>(k_0,k). \tag{2.29}$$

Introducing the spectral density

$$\rho(k_0,k) = \rho^>(k_0,k) - \rho^<(k_0,k), \tag{2.30}$$

the relation (2.29) implies that

$$\rho(k_0, k) = -\rho(-k_0, k). \tag{2.31}$$

The dynamical susceptibility is now expressed solely in terms of the spectral density $\rho(k_0, k)$ as

$$\Xi(\vec{x}-\vec{x}', t-t') = -ig\int \frac{d^4k}{(2\pi)^4}\rho(k_0, k)e^{-ik_0(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}. \tag{2.32}$$

The Lehmann representations (2.24) and the spectral densities (2.25), (2.26) are *exact* results, valid to all orders in the couplings of the χ fields to degrees of freedom within or beyond the standard model except the axion. Therefore the dynamical susceptibility (2.32) while linear in the coupling g (linear response) is in principle to all orders in all other couplings.

In Ref. [40] it was found that the expectation value of the axion field obeys the equation of motion

$$\frac{\partial^2}{\partial t^2}\bar{a}(\vec{x}, t) - \nabla^2\bar{a}(\vec{x}, t) + m_{0a}^2\bar{a}(\vec{x}, t) + \int\int_{t_0}^{t}\Sigma(\vec{x}-\vec{x}', t-t')\bar{a}(\vec{x}', t')d^3x'dt' = 0, \tag{2.33}$$

where m_{0a}^2 is the bare axion mass, and the retarded self energy $\Sigma(\vec{x}-\vec{x}', t-t')$ is given by [40]

$$\Sigma(\vec{x}-\vec{x}', t-t') = -ig^2\mathrm{Tr}([\mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}, t), \mathcal{O}_\chi{}^{(H_\chi)}(\vec{x}', t')]\rho_\chi(t_0)). \tag{2.34}$$

Remarkably, the dynamical susceptibility is simply related to the axion retarded self-energy to leading order in the axion coupling g but to all orders in the couplings of the fields χ to any other field within or beyond the standard model except the axion [40], namely

$$\Sigma(\vec{x}-\vec{x}', t-t') = g\Xi(\vec{x}-\vec{x}', t-t'). \tag{2.35}$$

This is one of the important results of this study, and applies in general for any of the interactions of the form $ga(\vec{x}, t)\mathcal{O}_\chi(\vec{x}, t)$, with important consequences explored below.

The main result of this section is the nonequilibrium induced expectation value of the composite operator $\mathcal{O}_\chi$, which in linear response is given by

$$\langle\mathcal{O}_\chi(\vec{x})\rangle(t) = -ig\int\frac{d^4k}{(2\pi)^4}\rho(k_0, k)\int d^3x' \times\int_{t_0}^{t}e^{-ik_0(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\bar{a}(\vec{x}', t')dt', \tag{2.36}$$

which is obtained by combining Eq. (2.22) with the spectral representation (2.32). This expression can be written in a more illuminating manner by recognizing that $\bar{a}(\vec{x}, t)$ is the solution of the equation of motion, Eq. (2.33), with the self energy to leading order in the coupling g given by Eq. (2.34). Using the relation (2.35) between the dynamical susceptibility and the self energy, and the equation of motion (2.33), it is straightforward to confirm the result (2.4), which has been obtained as the expectation value of the exact Heisenberg equations of motion, to leading order in the coupling g, namely linear response.

Note that because $\Sigma \propto g^2$, it follows that

$$\frac{\partial^2}{\partial t^2}\bar{a}(\vec{x}, t) - \nabla^2\bar{a}(\vec{x}, t) + m_{0a}^2\bar{a}(\vec{x}, t) \propto g^2, \tag{2.37}$$

and the expectation value $\langle\mathcal{O}_\chi(\vec{x})\rangle(t) \propto g$.

C. $\langle\mathcal{O}\rangle$ for misaligned initial conditions

We now consider the cosmologically relevant case of misaligned initial conditions for the axion condensate, where the axionlike fields are produced nonthermally and undergo a damped oscillations. In Ref. [40] it is shown that in Minkowski space-time the solution of the equation of motion (2.33) is described by exponentially damped oscillations, in which the frequency and decay rate are spatial-momentum and temperature dependent. Therefore, a general form of the axion-field amplitude is

$$\bar{a}(\vec{x}, t) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\bar{a}_k(t),$$
$$\bar{a}_k(t) = [A_k e^{-i\omega_k(t-t_0)} + A_k^* e^{i\omega_k(t-t_0)}]e^{-\frac{\Gamma_k}{2}(t-t_0)} \tag{2.38}$$

where A_k is a classical complex amplitude determined by initial conditions, ω_k is the renormalized frequency with $\omega_k^2 + \delta\omega_k^2 = k^2 + m_{0a}^2$, $\delta\omega_k$ the renormalization counter-term, and Γ_k the renormalized decay rate, both depending on momentum and temperature. In Appendix we provide an alternative method to solve the equation of motion based on multitimescale analysis yielding the same results.

Taking the spatial Fourier transform of (2.36) with the solution (2.38) yields

$$\langle\mathcal{O}_\chi\rangle_k(t) = -ig\int\frac{dk_0}{2\pi}\int_{t_0}^{t}dt'\rho(k_0, k)e^{-ik_0(t-t')}\bar{a}_k(t'). \tag{2.39}$$

The time integrals are straightforwardly evaluated with $\bar{a}_k$ given by Eq. (2.38), yielding

$$\langle\mathcal{O}_\chi\rangle_k(t) = -g\left\{A_k e^{-i\Omega_k t}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\rho(k_0,k)}{k_0-\Omega_k} + A_k^* e^{i\Omega_k^* t}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\rho(k_0,k)}{k_0+\Omega_k^*}\right\}$$
$$+ g\left\{A_k e^{-i\Omega_k t_0}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\rho(k_0,k)}{k_0-\Omega_k}e^{-ik_0(t-t_0)} + A_k^* e^{i\Omega_k^* t_0}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\rho(k_0,k)}{k_0+\Omega_k^*}e^{-ik_0(t-t_0)}\right\}, \quad (2.40)$$

where $\Omega_k = \omega_k - i\Gamma_k/2$.

The second line in Eq. (2.40) features oscillatory integrals that represent transient processes due to the sudden switch on at time $t = t_0$, and vanish fast for $t - t_0 \gg m_a$; therefore this contribution will be neglected.[1]

In the first line, we use the narrow width approximation $\Gamma_k \to 0$ and the identity $1/(x \pm i0^+) = \mathcal{P}(1/x) \mp i\pi\delta(x)$ and obtain to leading order in g,

$$\langle\mathcal{O}_\chi\rangle_k(t) = \frac{1}{g}[\Sigma_R(k,\omega_k)\bar{a}_k(t) + \Gamma_k\dot{\bar{a}}_k(t)], \quad (2.41)$$

where [40] (see also Appendix)

$$\Sigma_R(k,\omega_k) = g^2\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\mathcal{P}\left[\frac{\rho(k_0,k)}{\omega_k-k_0}\right];\ \frac{g^2}{2}\rho(\omega_k,k) = \omega_k\Gamma_k. \quad (2.42)$$

This result is in complete agreement with the result (2.4) as can be seen as follows: taking the spatial Fourier transform of Eq. (2.4), using the expression for $\bar{a}_k(t)$ given by Eq. (2.38), and neglecting terms of order $\Gamma_k^2 \propto g^4$ we find

$$\langle\mathcal{O}_\chi\rangle_k(t) = \frac{1}{g}\left[(\omega_k^2 - (k^2 + m_{0a}^2))\bar{a}_k(t) + \Gamma_k\dot{\bar{a}}_k(t)\right]. \quad (2.43)$$

The exact frequency ω_k corresponds to the real part of the pole in the propagator of the axion field; namely it is the solution of the equation [40]

$$\omega_k^2 = (k^2 + m_{0a}^2) + \Sigma_R(\omega_k, k), \quad (2.44)$$

from which Eq. (2.41) follows, thereby explicitly confirming the equivalence of the results (2.4), (2.41) in linear response.

III. CHERN-SIMONS CONDENSATE

The results above are valid for any generic pseudoscalar composite coupled to the axion field as in the Lagrangian density (2.1). We now focus specifically on the case of the axion field coupled to photons via the Chern-Simons term

$$\mathcal{O}_\chi(\vec{x},t) = \vec{E}(\vec{x},t)\cdot\vec{B}(\vec{x},t) = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu};\ \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}; \quad (3.1)$$

this pseudoscalar density is a total surface term, since

$$F_{\mu\nu}\tilde{F}^{\mu\nu} \propto \partial_\mu(\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta). \quad (3.2)$$

In this case

$$\langle\vec{E}\cdot\vec{B}\rangle(\vec{x},t) = \int d^3x'\int_{t_0}^{t}\Xi(\vec{x}-\vec{x}',t-t')\bar{a}(\vec{x}',t')dt', \quad (3.3)$$

where the susceptibility is the retarded commutator of the Chern-Simons density

$$\Xi(\vec{x}-\vec{x}',t-t') = -ig\mathrm{Tr}([(\vec{E}\cdot\vec{B})^{(H_{em})}(\vec{x},t),(\vec{E}\cdot\vec{B})^{(H_{em})}(\vec{x}',t')] \times\rho_\chi(t_0))\Theta(t-t'), \quad (3.4)$$

to which we refer as the *Chern-Simons susceptibility* in analogy with response functions in many body physics. The superscript (H_{em}) in the operators of the Chern-Simons density refer to the Heisenberg fields in absence of their coupling to the axion, namely to all orders in the electromagnetic interaction with degrees of freedom within or beyond the standard model except for the axion field. The Feynman diagram describing the induced condensate (3.3) in the case of free photons is shown in Fig. 1.

We now obtain the Chern-Simons susceptibility Ξ (3.4) by considering free electromagnetic fields. This approximation is valid within the cosmological setting after recombination for the following reasons: when the temperature ($\simeq$eV) is much smaller than the electron mass the lepton contribution to the renormalized photon self-energy is perturbatively small and thermally suppressed; therefore there is no (gauge invariant) thermal mass for the photon [43,44]. Furthermore the free electron density n vanishes rapidly during recombination; therefore the plasma frequency $\Omega_{pl} = \sqrt{4\pi e^2 n/m}$ is vanishingly small, and the photon bath is described by blackbody radiation, namely

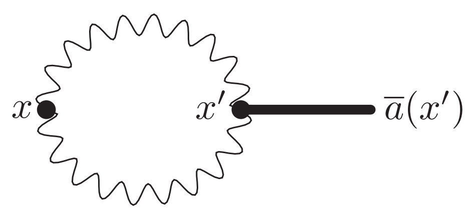

FIG. 1. $\langle\vec{E}\cdot\vec{B}\rangle(x)$ in linear response (3.3) for the case of free photons. The wavy lines correspond to the retarded correlation function (3.4), the heavy solid line to the misaligned axion condensate $\bar{a}(x')$.

[1]Similar contributions are neglected in obtaining the solution (2.38); see Appendix and Ref. [40] for details.

free thermal photons, as evidenced by the (nearly) blackbody spectrum of the cosmic microwave background.

Under these conditions, the spectral density (2.30) has been obtained in Refs. [40], and is given by

$$\rho(k_0,\vec{k}) = \frac{(K^2)^2}{32\pi}\left\{\left(1+\frac{2}{\beta k}\ln\left[\frac{1-e^{-\beta\omega^I_+}}{1-e^{-\beta\omega^I_-}}\right]\right)\Theta(K^2) + \frac{2}{\beta k}\ln\left[\frac{1-e^{-\beta\omega^{II}_+}}{1-e^{-\beta\omega^{II}_-}}\right]\Theta(-K^2)\right\}\mathrm{sign}(k_0),$$
$$K^2 = k_0^2 - k^2\ ;\ \omega^{(I)}_\pm = \frac{|k_0|\pm k}{2}\ ;\ \omega^{(II)}_\pm = \frac{k\pm|k_0|}{2}. \tag{3.5}$$

The terms with $\Theta(k_0^2-k^2)$ arise from the processes $a \leftrightarrow 2\gamma$, namely emission and absorption of photons with the reverse or recombination process $2\gamma \to a$ being a consequence of the radiation bath; these processes feature support on the axion mass shell for massive axions. The contribution proportional to $\Theta(k^2-k_0^2)$ only features support below the light cone; it describes off shell processes $\gamma a \leftrightarrow \gamma$ and vanishes in the $k\to 0$ limit.

Motivated by the cosmological case we now consider a homogeneous misaligned axion condensate depending solely on time by setting

$$A_k = (2\pi)^3\delta^3(\vec{k})a_0, \tag{3.6}$$

namely

$$\bar{a}(t) = e^{-\frac{\Gamma}{2}t}(a_0 e^{-im_a t} + a_0^* e^{im_a t}), \tag{3.7}$$

from which it follows that the induced Chern-Simons condensate is also homogeneous, and from the result (2.39) it is given by

$$\langle\vec{E}\cdot\vec{B}\rangle(t) = -ig\int_{-\infty}^{\infty}\frac{dk_0}{(2\pi)}\rho(k_0,0)e^{-ik_0 t}\int_{t_0}^{t}e^{ik_0 t'}\bar{a}(t')dt'. \tag{3.8}$$

From Eq. (3.5) we find

$$\rho(k_0,0) = \frac{k_0^4}{32\pi}\left(1+2n\left(\frac{k_0}{2}\right)\right)\mathrm{sign}(k_0)\ ;\ n(\omega) = \frac{1}{e^{\beta\omega}-1}. \tag{3.9}$$

Before we study the response to an oscillating coherent misaligned $\bar{a}(t)$, it is illuminating to consider the case wherein such expectation value has relaxed to a time independent equilibrium minimum $\bar{a}_0$ at a time t_0 and remains constant for $t > t_0$. Such a situation emerges from a damped oscillatory expectation value around a minimum away from the origin, if, for example the axion potential features such a minimum. Setting $\bar{a}(t') \to \bar{a}_0$ for $t > t_0$, the time integral in (3.8) becomes $\propto \sin(k_0(t-t_0)/2)/k_0$ yielding a nonvanishing Chern-Simons condensate. However as the time interval $t - t_0 \to \infty$ the time integral $\to 2\pi\delta(k_0)$ and the induced Chern-Simons condensate vanishes. This is consistent with the fact that the Chern-Simons density $\vec{E}\cdot\vec{B}$ is a total surface term (3.2) [20], and its space-time integral vanishes in the infinite time and volume limit. However, the nonequilibrium result within *finite timelike hypersurfaces* is nonvanishing, and as the time integral in (3.8) makes explicit, the induced condensate is proportional to the difference of a function evaluated at the two hypersurfaces at times t and t_0, when the spatial volume has been taken to infinity. Furthermore, $\bar{a}(t)$ is the dynamical axion condensate which is a solution of the equation of motion (2.33) for $k = 0$, and we note that the only space-time constant solution of the equation of motion (2.33) is $\bar{a} = 0$, which obviously yields a vanishing Chern-Simons condensate as suggested in Eq. (2.4).

Within the cosmological setting, the initial time t_0 is approximately the time of the last scattering surface because we are considering free photons in thermal equilibrium in the intermediate state, and t is of theorder of the Hubble time today so that $t \gg t_0$ and $t - t_0 \gg 1/m_a, 1/T$.

Let us now consider a dynamical homogeneous solution of the equation of motion (2.33) given by Eq. (3.7) where m_a is the renormalized axion mass [40], and a_0 is a classical complex amplitude determined by initial conditions (see Appendix). From the general result (2.43) for $k = 0$ with $\omega_{k=0} \equiv m_a$, and consistently keeping terms up to $\mathcal{O}(g^2)$ we find

$$\langle\vec{E}\cdot\vec{B}\rangle(t) = \frac{1}{g}[(m_a^2 - m_{0a}^2)\bar{a}(t) + \Gamma\dot{\bar{a}}(t)]. \tag{3.10}$$

Furthermore, the relations (2.44) and (2.42) yield

$$m_a^2 - m_{0a}^2 = g^2\int_{-\infty}^{\infty}\mathcal{P}\left[\frac{\rho(k_0,0)}{m_a-k_0}\right]\frac{dk_0}{2\pi} = \Sigma_R(0,m_a), \tag{3.11}$$

and [40]

$$\Gamma = g^2\frac{\rho(m_a,0)}{2m_a} = \frac{g^2 m_a^3}{64\pi}\left(1+2n\left(\frac{m_a}{2}\right)\right) \tag{3.12}$$

is the axion decay rate $a \to 2\gamma$ [40] (see Appendix).

This is one of the main results of this study. For the case of an ultralight axion witn $m_a \lesssim \mu$eV and even for temperatures of the order of the CMB temperature today 10^{-4} eV, it follows that $T \gg m_a$. Therefore using the spectral density (3.9) in this high temperature limit we find [40]

$$m_a^2 - m_{0a}^2 = -\frac{g^2\Lambda^4}{128\pi^2}\mathcal{F}_0\left[\frac{m_a^2}{\Lambda^2}\right] - \frac{g^2\pi^2 T^4}{15}\mathcal{F}_T\left[\frac{m_a^2}{T^2}\right], \tag{3.13}$$

where the first term is the zero temperature contribution, for which we carried out the integral in k_0 in (3.11) with an ultraviolet cutoff $\Lambda \gg m_a, T$, delimiting the regime of validity of the effective field theory, and the second term is the finite temperature contribution, with

$$\mathcal{F}_0\left[\frac{m_a^2}{\Lambda^2}\right] = 1 + 4\frac{m_a^2}{\Lambda^2} + \frac{m_a^4}{\Lambda^4}\ln\left(\frac{\Lambda^2}{m_a^2}\right) + \cdots \tag{3.14}$$

$$\mathcal{F}_T\left[\frac{m_a^2}{T^2}\right] = 1 + \frac{15m_a^2}{24\pi^2T^2} - \frac{15m_a^4}{32\pi^2T^4}\ln\left[\frac{T}{m_a}\right] + \cdots. \tag{3.15}$$

The dots stand for higher orders in m_a/Λ and m_a/T respectively, and to leading order in the high temperature limit we find

$$\Gamma = \frac{g^2 m_a^2 T}{16\pi}. \tag{3.16}$$

The ultraviolet divergence of the Chern-Simons condensate resulting from the first term in (3.13) is not unexpected. The Chern-Simons density is an operator of mass dimension four, just like $\vec{E}^2$ and $\vec{B}^2$; however, unlike these operators whose vacuum expectation values yield the zero point energy, the expectation value of $\vec{E}\cdot\vec{B}$ vanishes if the state is invariant under parity. In other words, the expectation value of the Chern-Simons density is protected from ultraviolet divergences by parity. However, an expectation value of the pseudoscalar axion breaks parity, therefore, in the presence of this parity-symmetry breaking term the ultraviolet divergence of $\vec{E}\cdot\vec{B}$ becomes explicit, and as exhibited by Eq. (3.10) is proportional to the symmetry breaking term in linear response. We choose to renormalize the Chern-Simons condensate by subtracting the $T = 0$ (vacuum) contribution; subtle aspects of renormalization are discussed in more detail in Sec. V. Keeping the leading order term in the high temperature expansion, the renormalized expectation value is given by

$$\langle\vec{E}\cdot\vec{B}\rangle^{(R)}(t) = -\frac{g\pi^2T^4}{15}\bar{a}(t) + \frac{gm_a^2T}{16\pi}\dot{\bar{a}}(t) + \mathcal{O}(m_a^2/T^2), \tag{3.17}$$

with $\bar{a}(t)$ the (spatially homogeneous) solution of the equation of motion (2.33). This is another of the main results of this study.

We can obtain an estimate of the energy density stored in the Chern-Simons condensate as compared with that in the cosmic microwave background today, $\rho_{0\gamma} = \pi^2 T_{0\gamma}^4/15$, by using the following estimates:

$$m_a^2 a^2 \simeq \rho_{0DM} = \rho_{0c}\Omega_{\rm DM}\ ;\ g = \frac{\mathcal{C}}{f_a}, \tag{3.18}$$

where $\mathcal{C} < 1$ is a dimensionless constant and f_a the axion decay constant, and using the values, $\Omega_{\rm DM} \simeq 0.23$; $h \simeq 0.7$ and the temperature of the cosmic microwave background today $T_{0\gamma} = 2.37\times 10^{-4}$ eV, we find

$$\left|\frac{\langle\vec{E}\cdot\vec{B}\rangle^{(R)}(t_0)}{\rho_{0\gamma}}\right| \simeq \mathcal{C}\left(\frac{10^{10}\ \mathrm{GeV}}{f_a}\right)\left(\frac{\mu eV}{m_a}\right) \times\left[1 - 2.2\times 10^{-9}\left(\frac{m_a}{\mu eV}\right)^3\right]\times 3\times 10^{-19}. \tag{3.19}$$

This analysis suggests that low mass axions with $m_a \ll \mu eV$ yield larger contributions to the energy density stored in the Chern Simons condensate, perhaps leading to an observational avenue.

IV. PROBING THE CHERN-SIMONS CONDENSATE WITH EMERGENT AXION QUASIPARTICLES

The analysis above unambiguously implies that a macroscopic axion condensate will induce a macroscopic Chern-Simons condensate, leading to the question of what are the observational consequences of such a topological condensate. While it is possible that a cosmological imprint of this condensate may be observable in the polarization signal of the CMB in a manner yet to be understood and studied further, here we *suggest* that the emergent axionlike quasiparticles in topological insulators [33–35], Weyl semimetals [36], multilayered metamaterials [37], or magnetoelectric insulators [39] may be harnessed to probe both the cosmological axion *and* the Chern-Simons condensate. In these materials, an axionlike collective quasiparticle excitation $\Theta(\vec{x},t)$ couples to the electromagnetic fields [32–36,39] with

$$\mathcal{L}_\Theta = \alpha\Theta(\vec{x},t)\vec{E}(\vec{x},t)\cdot\vec{B}(\vec{x},t), \tag{4.1}$$

with α the electromagnetic fine structure constant. In multilayered metamaterials, instead of α the effective coupling can be tuned making these platforms more flexible [37]. This coupling brings *two* important consequences, both relevant to probing cosmological axions:

(i) The emergent axionlike quasiparticles described by the effective field $\Theta(\vec{x},t)$ *mix* with the cosmological axion $a(\vec{x},t)$ via a common two-photon intermediate state. This is depicted in Fig. 2 by the photon loop connecting the external fields $\Theta(\vec{x},t)$ and $a(\vec{x},t)$ resulting in off diagonal components of the propagators in the material. An important aspect of this mixing is that the off diagonal matrix elements of the propagator are of order $g\alpha$. This aspect combined with coherence of the axion field in the form of a macroscopic condensate may yield observational effects at leading order in g, which results in an enhancement in detection efficiency over other possible processes such

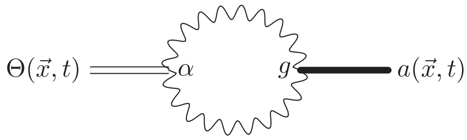

FIG. 2. Mixing between the emergent axion field $\Theta(\vec{x},t)$ and the cosmological axion field $a(\vec{x},t)$ via the Chern-Simons correlation function. The wavy lines correspond to the correlation function (3.4), the heavy solid line to the axion field $a(x')$, and the double solid line to the emergent axion quasiparticle field $\Theta(\vec{x},t)$.

as "axion shining through walls" with a transition probability of order g^4.

(ii) As discussed above, in presence of a misaligned (cosmological) axion condensate, $a(\vec{x},t) = \bar{a}(\vec{x},t) + \tilde{a}(\vec{x},t)$ where $\bar{a}(\vec{x},t)$ is a classical field describing a macroscopic condensate. Replacing $a(\vec{x},t) \to \bar{a}(\vec{x},t)$ in the external leg of the mixed propagator in Fig. 2, the photon loop and the external c-number external leg $\bar{a}(\vec{x},t)$ yield the Chern-Simons condensate (see Fig. 1) $\langle \vec{E}\cdot\vec{B}\rangle(\vec{x},t)$ which acts as an external time dependent c-number source term, namely $\alpha\Theta(\vec{x},t)\langle \vec{E}\cdot\vec{B}\rangle(\vec{x},t) \to \alpha\Theta(\vec{x},t)h(\vec{x},t)$, as depicted in Fig. 3.

Such an external source term, linearly coupled to $\Theta(\vec{x},t)$ results in an effective external time dependent driving term displacing the quasiparticle field off equilibrium with an oscillatory behavior corresponding to the time dependence of the *cosmological* axion. For a homogeneous axion condensate, this driving term induces an oscillatory macroscopic condensate of the emergent axionlike field in the material, namely a coherent state of the quasiparticle degrees of freedom, which in principle could be measured along the lines of the experimental setup in Ref. [39], and perhaps with enhanced tunability of the coupling in the case of multilayered metamaterials [37], thereby directly probing the Chern-Simons condensate and, indirectly, the cosmological axion condensate. However a recent analysis of the signal to noise ratio in multiferroics suggests that the coupling between axion dark matter and ferroic orders in multiferroics [38] may not yield an observable signal of dark matter axions. However, the possibility of harnessing other materials for detection, in particular via the coupling to the Chern Simons condensate remains to be explored.

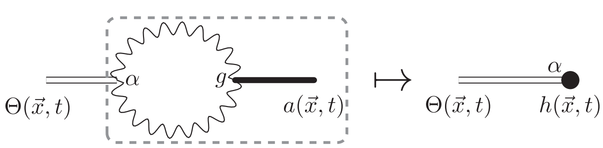

FIG. 3. Replacing the axion field by the misaligned axion condensate $\bar{a}(\vec{x},t)$ the photon loop with $\bar{a}$ as the external leg is identified with the Chern-Simons condensate $\langle \vec{E}\cdot\vec{B}\rangle$ (see Fig. 1), which acts as an external source $h(\vec{x},t) = \langle \vec{E}\cdot\vec{B}\rangle$ linearly coupled to the quasiparticle axion field $\Theta(\vec{x},t)$.

V. DISCUSSION

A. Renormalization of the Chern-Simons condensate

As discussed above, for a nonvanishing dynamical expectation value of the axion field, the induced Chern-Simons condensate features ultraviolet divergences and it must be renormalized. The first term in (3.10) and the function $\mathcal{F}_0$ given by (3.14) are obtained by imposing an ultraviolet cutoff Λ in the k_0 integral of the spectral density in Eq. (3.11). This cutoff is interpreted as the scale below which the effective field theory described by the local Lagrangian density (2.1) is valid. Even when this scale is finite, the result (3.10) implies a strong sensitivity to this scale. There does not seem to be an obvious manner to renormalize the Chern-Simons condensate since it depends explicitly on time through the dynamical expectation value of the axion field. Therefore, we proceed to renormalize it simply by subtracting the vacuum contribution, yielding the leading order renormalized condensate in the high temperature limit $T \gg m_a$ given by the result (3.17). The vacuum subtraction is motivated by the subtraction of the zero point contributions to $\langle \vec{E}^2\rangle$ and $\langle \vec{B}^2\rangle$, namely the subtraction of the zero point energy, since these operators are also of mass dimension four and feature the same type of ultraviolet divergences $\propto \Lambda^4$ as the Chern-Simons condensate. However, unlike the vacuum subtraction for $\langle \vec{E}^2\rangle$, $\langle \vec{B}^2\rangle$ the subtraction for $\langle \vec{E}\cdot\vec{B}\rangle$ is proportional to the misaligned axion condensate which depends explicitly on time. It remains to be explored further if there is a suitable and more rigorous renormalization scheme for $\langle \vec{E}\cdot\vec{B}\rangle(t)$, beyond a vacuum subtraction or subtracting solely the ultraviolet sensitive terms.

B. Cosmological caveats

Our ultimate objective is to understand the cosmological implications of the nonequilibrium dynamics of axions (or axionlike particles) in cosmology. To this aim, the results obtained above in Minkowski space-time serve as a prelude, and a "proof of principle" of the application of the concepts behind linear response to extract the induced parity violating Chern-Simons condensate. There are several obvious differences between the dynamics in Minkowski space-time, and in cosmology: Hubble expansion modifies the time evolution of the axion condensate including a damping term in the equation of motion proportional to the Hubble rate of expansion; axion decay into photons, or other processes that lead to damping of the condensate will also add to the damping dynamics but through a self-energy correction that must be obtained from field quantization in the expanding cosmology. However, by dimensional analysis, linear response, and under the assumption that the axion condensate undergoes damped oscillations, the general form (3.10) qualitatively describes the Chern-Simons condensate, albeit with a different functional form of Γ and the function $\mathcal{F}_T$ in Eq. (3.15) since the high temperature behavior of $m_a^2 - m_{0a}^2 \propto g^2 T^4$ on dimensional grounds, and both must include the effect of Hubble expansion.

These aspects notwithstanding, the results in Minkowski space-time indicate that the *qualitative* aspects and main conclusion, namely a dynamical misaligned coherent axion condensate will *induce* a macroscopic condensate of the composite operator(s) coupled to the axion as in Eq. (2.1), will remain. Therefore, the calculation in Minkowski space-time with the approximation of free photons in the Chern-Simons susceptibility provides a "proof of principle" of the main concepts and the qualitative form of the condensate.

Furthermore, the potential observational consequences of such a condensate in topological or metamaterials as discussed in the previous section are reliably described by the Chern-Simons susceptibility calculated with free photons as such possible experiments would be carried out today when the radiation bath to which the cosmological axion is coupled is the cosmic microwave background.

C. Neutral pion condensate from misaligned axions

The axion is a quasi-Nambu-Goldstone boson, and as such it couples *directly* to other matter fields via a derivative coupling to a pseudovector current. However, the axion couples *indirectly* to the neutral pion via an intermediate state of two photons as can be understood with the following argument. The neutral pion decays into two photons, with an effective coupling of the form $\frac{\alpha}{8\pi f_\pi}\pi^0 F_{\mu\nu}\tilde{F}^{\mu\nu} \simeq \pi^0 \vec{E}\cdot\vec{B}$ as a consequence of the chiral anomaly, with $\alpha/\pi f_\pi \simeq 0.025\ \mathrm{GeV}^{-1}$, with α the fine structure constant, and f_π the pion decay constant. This implies the process $\pi^0 \leftrightarrow 2\gamma \leftrightarrow a$, described by a Feynman diagram similar to that in Fig. 2 but replacing $\Theta \to \pi^0$. This process entails that the axion and the neutral pion can mix via a common intermediate state of two photons; this is an off diagonal self-energy diagram that is completely determined by the Chern-Simons dynamical susceptibility (3.4); therefore we expect this "mixing" to be of order $g\alpha/f_\pi \times T^4$. We are currently exploring this phenomenon.

D. Non-Abelian Chern-Simons condensate

An axion coupling of the form $ga(x)G_{\mu\nu,b}\tilde{G}^{\mu\nu,b}$ where $G_{\mu\nu,b}$ is the gluon gauge field strength tensor and $\tilde{G}^{\mu\nu,b}$ its dual, would yield a non-Abelian Chern-Simons condensate $\langle G_{\mu\nu,b}\tilde{G}^{\mu\nu,b}\rangle(t)$ in the same way as that for the Abelian gauge theory. In this case the dynamical susceptibility is the retarded commutator (2.20) but with $\mathcal{O}_\chi = G_{\mu\nu,b}\tilde{G}^{\mu\nu,b}$. While in principle a calculation similar to that of the Abelian case yields the dynamical susceptibility (and the quark-gluon contribution to the axion self energy) there are several important differences with the Abelian case that would lead to daunting technical aspects. Not only does the non-Abelian nature of the gauge field introduce new vertices, but also below the QCD temperature gluons and quarks are confined to mesons and baryons involving nonperturbative physics. Furthermore for temperatures above the QCD scale, which is larger than the masses of all but the top quark, these degrees of freedom are ultra-relativistic and yield self-energy corrections to the gluon propagators in the form of hard thermal loops both from gluons and quarks which cannot be neglected [43,44]. On dimensional grounds ($G_{\mu\nu,b}\tilde{G}^{\mu\nu,b}$ has mass dimension four) we expect ultraviolet divergences, or rather sensitivity to a cutoff Λ delimiting the validity of the effective field theory description similar to the Abelian case. Furthermore, we also expect that the non-Abelian Chern-Simons condensate will be proportional to $\bar{a}(t), \dot{\bar{a}}(t)$; hence an ambiguity in the renormalization of this condensate should arise in much the same way as for the Abelian case. From the general results (2.4), (3.7)) and just on dimensional grounds we expect that after subtracting the vacuum term altogether, the high temperature limit of the finite temperature contribution is of the form

$$\langle G_{\mu\nu,b}\tilde{G}^{\mu\nu,b}\rangle(t) = gC_G T^4 \bar{a}(t) + \frac{D_G}{g}\Gamma(T)\dot{\bar{a}}(t), \qquad (5.1)$$

where $\Gamma(T)$ is the relaxation rate of the misaligned axion condensate, and C_G, D_G will be functions of the ratios of the various scales, such as the axion mass and quark masses to the temperature. Notwithstanding these quantitative and technical aspects, the general results (2.4), (3.7) imply that a misaligned axion will induce a condensate of the non-Abelian Chern-Simons density.

VI. CONCLUSIONS

The nonequilibrium dynamics of axions is of an interdisciplinary interest both in cosmology, as a possible candidate for dark matter, and/or dark energy as well as in condensed matter physics where axionlike excitations may emerge as collective quasiparticles in parity violating topological insulators, density waves in Weyl semimetals, or in metamaterials. In this article we study hitherto unexplored nonequilibrium aspects of axions in Minkowski space-time as a stepping stone toward a more comprehensive treatment in cosmology with potential observable consequences harnessing emergent axionic quasiparticles in condensed matter systems.

The main objective is to study how a misaligned coherent axion condensate induces a macroscopic condensate of a composite pseudoscalar operator $\mathcal{O}$ coupled to the axion as $ga(\vec{x},t)\mathcal{O}(\vec{x},t)$. To this aim we implement the method of linear response ubiquitous in many body physics. We obtain the macroscopic condensate of such a pseudoscalar operator to leading order in the coupling g but in principle to all orders of the couplings of the degrees of freedom described by $\mathcal{O}$ to any other degree of freedom within or beyond the standard model, except the axion.

We focused in particular on a macroscopic condensate of the Chern-Simons density $\vec{E}\cdot\vec{B}$ in axion electrodynamics and introduced the dynamical susceptibility, which is a response function that relates the induced Chern-Simons condensate to the axion condensate. This susceptibility is simply related to the axion self-energy a relation that holds to all orders in the couplings of photons to other degrees of freedom within or beyond the standard model except the axion. The induced Chern-Simons condensate is strongly sensitive to the cutoff scale of the effective field theory and requires subtle renormalization. Subtracting the vacuum contribution and in the high temperature limit $T \gg m_a$ we find

$$\langle\vec{E}\cdot\vec{B}\rangle(t) = -\frac{g\pi^2T^4}{15}\bar{a}(t) + \frac{gm_a^2T}{16\pi}\dot{\bar{a}}(t) + \mathcal{O}(m_a^2/T^2) \tag{6.1}$$

with $\bar{a}(t)$ the dynamical homogeneous axion condensate solution of the equations of motion including the self-energy.

We have argued that the Chern-Simons condensate can be probed by harnessing axionlike collective quasiparticles in condensed matter systems such as topological insulators, Weyl semimetals, magnetoelectric insulators, or multilayered metamaterials which provide realizations of axion electrodynamics. A homogeneous cosmological axion condensate induces a macroscopic topological Chern-Simons condensate that acts like an external source driving a nonequilibrium macroscopic emergent axionlike condensate in these systems, which oscillates with the same frequency as the cosmological axion. Furthermore, we also conjectured that a misaligned axion condensate also induces a non-Abelian Chern-Simons condensate and mixes with neutral pion degrees of freedom thereby inducing a neutral pion condensate and chiral symmetry breaking during or after the QCD phase transition.

The next stage of the study will include cosmological expansion which requires extending the linear response formulation to the realm of a Friedmann-Robertson-Walker cosmology; we expect to report on these studies in a future article.

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APPENDIX: SOLUTION OF (2.33) VIA MULTITIMESCALE ANALYSIS

Let us consider the spatial Fourier transform of the equation of motion (2.33), which we write as

$$\ddot{\bar{a}}_k(t) + [\omega_k^2 + \delta\omega_k^2]\bar{a}_k(t) + \int_{t_0}^{t}\Sigma(k,t-t')\bar{a}_k(t')dt' = 0, \tag{A1}$$

where anticipating renormalization, ω_k^2 is the *finite temperature renormalized frequency* describing the in-medium dispersion relation, and the counterterm $\delta\omega_k^2$ accounts for renormalization and is defined so that $k^2 + m_{0a}^2 = \omega_k^2 + \delta\omega_k^2$, and

$$\Sigma(k,t-t') = -ig^2\int\frac{dk_0}{2\pi}\rho(k_0,k)e^{-ik_0(t-t')}. \tag{A2}$$

Since $\Sigma \propto g^2$, we expect that $\delta\omega_k^2 \propto g^2$. Let us write the solution of (A1) as

$$\bar{a}_{\vec{k}}(t) = A_{\vec{k}}(t)e^{-i\omega_k t} + A_{\vec{k}}^*(t)e^{i\omega_k t}, \tag{A3}$$

where $e^{\mp i\omega_k t}$ are *fast varying* and $A_{\vec{k}}(t)$ is a slowly varying amplitude, so that $\dot{A}_{\vec{k}}(t), \ddot{A}_{\vec{k}}(t) \propto g^2, \cdots$. The equation for the slowly varying amplitudes $A_{\vec{k}}(t), A_{\vec{k}}^*$ becomes

$$e^{-i\omega_k t}\left[\ddot{A}_{\vec{k}} - 2i\omega_k\dot{A}_{\vec{k}} + \delta\omega_k^2A_{\vec{k}} + \int_{t_0}^{t}\Sigma(k,t-t')e^{i\omega_k(t-t')}A_{\vec{k}}(t')dt'\right] + e^{i\omega_k t}\left[\ddot{A}_{\vec{k}}^* + 2i\omega_k\dot{A}_{\vec{k}}^* + \delta\omega_k^2A_{\vec{k}}^* + \int_{t_0}^{t}\Sigma(k,t-t')e^{-i\omega_k(t-t')}A_{\vec{k}}^*(t')dt'\right] = 0. \tag{A4}$$

Since the fast varying phases are independent and the brackets are of $\mathcal{O}(g^2)$ or higher, we request that each bracket vanishes individually, yielding

$$\ddot{A}_{\vec{k}} - 2i\omega_k\dot{A}_{\vec{k}} + \delta\omega_k^2A_{\vec{k}} + \int_{t_0}^{t}\Sigma(k,t-t')e^{i\omega_k(t-t')}A_{\vec{k}}(t')dt' = 0, \tag{A5}$$

$$\ddot{A}_{\vec{k}}^* + 2i\omega_k\dot{A}_{\vec{k}}^* + \delta\omega_k^2A_{\vec{k}}^* + \int_{t_0}^{t}\Sigma(k,t-t')e^{-i\omega_k(t-t')}A_{\vec{k}}^*(t')dt' = 0. \tag{A6}$$

Let us focus on Eq. (A5) since (A6) is obtained from it by $\omega_k \to -\omega_k$.

The derivatives $\dot{A}_{\vec{k}}; \ddot{A}_{\vec{k}}$ are written as an expansion in g^2 since $A_{\vec{k}}$ is a slowly varying amplitude. In order to generate an expansion in derivatives proportional to powers of g^2, we write

$$\Sigma(k,t-t')e^{i\omega_k(t-t')}=\frac{d}{dt'}W[t;t'];$$
$$W[t;t']=\int_{t_0}^{t'}\Sigma(k,t-t'')e^{i\omega_k(t-t'')}dt'';$$
$$W[t;t_0]=0, \tag{A7}$$

with

$$W[t,t]=-ig^2\int\frac{dk_0}{2\pi}\rho(k_0,k)\int_0^{t-t_0}e^{-i(k_0-\omega_k)\tau}d\tau. \tag{A8}$$

Integrating by parts the last term in (A5) yields

$$\ddot{A}_{\vec{k}}-2i\omega_k\dot{A}_{\vec{k}}+\delta\omega_k^2A_{\vec{k}}+W[t,t]A_{\vec{k}}(t)-\int_{t_0}^{t}W[t;t']\frac{d}{dt'}A_{\vec{k}}(t')dt'=0, \tag{A9}$$

since $W[t;t']\propto g^2$ and $dA_{\vec{k}}(t)/dt\propto g^2$ the last term in (A9) is of $\mathcal{O}(g^4)$ and will be neglected since we only keep terms up to and including $\mathcal{O}(g^2)$. Hence to $\mathcal{O}(g^2)$ the Eq. (A9) simplifies to

$$\ddot{A}_{\vec{k}}-2i\omega_k\dot{A}_{\vec{k}}+[\delta\omega_k^2+W[t,t]]A_{\vec{k}}(t)=0. \tag{A10}$$

Writing

$$A_{\vec{k}}(t)=e^{I_k(t)}\ ;\ I_k(t)=g^2I_k^{(1)}(t)+g^4I_k^{(2)}(t)+\cdots \tag{A11}$$

and keeping only the leading $\mathcal{O}(g^2)$ terms, $I_k^{(1)}$ obeys the simple equation

$$\ddot{I}_k^{(1)}(t)-2i\omega_k\dot{I}_k^{(1)}(t)=-\frac{1}{g^2}[\delta\omega_k^2+W[t,t]]. \tag{A12}$$

At early times $t-t_0\simeq 1/m_a, 1/T$ transient effects are expected, but we are interested in the intermediate and long time asymptotics $t-t_0\gg 1/m_a, 1/T$; in this limit we replace

$$\int_0^{t-t_0}e^{-i(k_0-\omega_k)\tau}d\tau\to -i\mathcal{P}\left(\frac{1}{k_0-\omega_k}\right)+\pi\delta(k_0-\omega_k), \tag{A13}$$

for which $W[t,t]$ becomes a constant, and choosing the counterterm

$$\delta\omega_k^2=-g^2\int\mathcal{P}\left[\frac{\rho(k_0,k)}{\omega_k-k_0}\right]\frac{dk_0}{2\pi}, \tag{A14}$$

the solution of Eq. (A10) is

$$A_k(t)=A_k(0)e^{-\frac{\Gamma_k}{2}t}, \tag{A15}$$

where

$$\Gamma_k=g^2\frac{\rho(\omega_k,k)}{2\omega_k}. \tag{A16}$$

The finite temperature renormalized mass is defined as the long wavelength limit of the dispersion relation, namely $m_a=\omega_{k=0}$; therefore it follows that $\delta\omega_{k=0}^2=m_{0a}^2-m_a^2$, yielding

$$m_a^2-m_{0a}^2=g^2\int\mathcal{P}\left[\frac{\rho(k_0,0)}{m_a-k_0}\right]\frac{dk_0}{2\pi}, \tag{A17}$$

and the damping rate of the misaligned axion condensate is the long wavelength limit of (A16), i.e.,

$$\Gamma=g^2\frac{\rho(m_a,0)}{2m_a}, \tag{A18}$$

which are the results (3.7)–(3.12).

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