New accretion constraint on the evaporation of primordial black holes

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In this paper, we investigate the processes of evaporation and accretion of primordial black holes during the radiation-dominated era and the matter-dominated era. This subject is very important since usually these two processes are considered independent of each other. In other words, previous works consider them in such a way that they do not have a direct effect on each other, and as a result, their effects on the mass of primordial black holes are calculated separately. The calculations of this paper indicate that assuming these two processes independent of each other will lead to wrong results that only give correct answers within certain limits. In fact, in general, it is a mistake to consider the static state for the event horizon of primordial black holes and perform calculations related to their evaporation, while the radius of primordial black holes is constantly changing due to accretion. In addition, we show that considering the dynamic event horizon in some masses and in some times can lead to the shutdown of the Hawking evaporation process. This study is much more accurate and detailed than our previous study. These calculations show well the mass evolution of primordial black holes from the time of formation to the end of the matter-dominated era, taking into account both the main processes governing black holes: evaporation and accretion.

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I. INTRODUCTION

The detection of gravitational waves generated by the mergers of two black holes [1,2] has led to renewed interest in primordial black holes (PBHs) [3–5], as they could be part of a fraction of the events observed by the LIGO/Virgo/KAGRA Collaboration [6–8].

PBHs may be formed through the gravitational collapse of rare overdense regions upon horizon entry in the early stages of the Universe's evolution. The collapse could take place during the radiation-dominated era when PBHs are generated only if the initial amplitude of the density perturbation is on the far side of a large threshold (see, e.g., [9–12]).

There are two main features of PBH dynamics: first, their evaporation by Hawking radiation and, second, their accretion, which is due to the nature of the black hole's significant gravity. PBH Hawking radiation flux is not independent of its accretion flux [13–15]. Since all stationary BHs evaporate due to Hawking radiation [16], losing their mass in a time related to their initial mass by equation $\tau \sim M^3$, then the PBHs with initial mass less than 10^{15} g have entirely evaporated until now. With respect to the fact that the accretion could overcome Hawking radiation during the radiation-dominated era and cause the PBH radius to grow, the constraint from evaporation for PBHs is reduced from 10^{15} to 10^{14} g [17]. Therefore, they safely show the constraints down to $M \geq 10^{14}$ g, which leads to the remaining possible PBH mass range windows to be extended, explaining dark matter.

These two dynamical features help us to know the abundance of PBHs that share in detected gravitational waves and dark matter mass fraction. The abundance of PBHs is constrained by observations in different mass ranges (for a comprehensive review, see [4]).

For example, Ricotti *et al.* [18] derived strong constraints from the cosmic microwave background (CMB) frequency spectrum and temperature and polarization anisotropies for PBHs more massive than one solar mass. The basic idea about these constraints is that PBHs accrete primordial gas in the early Universe and then convert a fraction of the accreted mass to radiation that affects the CMB. To proceed, first one has to model the PBH accretion to quantify their mass value in time. Second, the type of the

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accretion flux (gas) and the era of the Universe in which the PBHs evolve in it determine PBH mass spectrum.

The rest of this paper is organized as follows. In Sec. II, we give an overview of some general cosmological equations. Then, we explain the evaporation process and the equations leading it, and we continue the same process for the accretion. In Sec. III, the equations of accretion of matter and radiation are analyzed. Considering the significance of the two eras, radiation and matter dominated, we examine the evolution of mass due to the accretion of matter and radiation in each era separately in Secs. IV and V. The mass graph is drawn in terms of time, and the effects of accretion of radiation and matter are discussed. In Sec. VI, we talk about the presence or absence of evaporation by examining the rate of increase in the radius of the PBH due to accretion.

II. GENERAL EQUATIONS

The study of PBH mass gives much information about its evolutionary process and effects on the surrounding environment. In this work, we suppose all PBHs have been formed in the radiation-dominated era. The mass of PBHs that formed at the time t after the big bang is equivalent to or less than the Hubble mass [19],

$$M_{\rm PBH} \sim \frac{c^3 t}{G} \sim 10^{15}\left(\frac{t}{10^{-23}\ {\rm s}}\right)\ {\rm g}, \tag{1}$$

where $c \simeq 3\times 10^8\ {\rm m/s}$ is the speed of light and $G \simeq 6.67\times 10^{-11}\ {\rm m^3/kg\,s^2}$ is the gravitational constant.

The cosmological evolution of PBHs, such as accretion, evaporation, and merging, can significantly impact PBH mass and release radiation, injecting energy into the surrounding medium, strongly affecting its thermal state, and leaving influential observable signatures [20].

To study PBHs, we need to survey the Universe's evolution. Friedmann equations describe the homogeneous and isotropic universe as [21]

$$\left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho, \tag{2}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right), \tag{3}$$

and the total energy conservation equation is

$$\dot\rho + 3H(\rho + p) = 0. \tag{4}$$

The general equation of state is $p = \omega\rho$, where ω for matter, radiation, and cosmological constant are 0, $1/3$, and -1; thus we can rewrite Eq. (4) as [22]

$$\rho = \rho_{cr}\left(\frac{a}{a_0}\right)^{-3(1+\omega)},$$

$$\rho(a) \propto \begin{cases} a^{-4} & \text{Radiation} \\ a^{-3} & \text{Matter} \\ \text{constant} & \text{Vacuum} \end{cases}, \tag{5}$$

and by substituting Eq. (3) in Eq. (5) we have

$$a(t) \propto \begin{cases} t^{\frac{1}{2}} & \text{Radiation} \\ t^{\frac{2}{3}} & \text{Matter} \\ e^{H_0 t} & \text{Vacuum} \end{cases}. \tag{6}$$

Now we want to calculate the rate of mass change of PBHs through evaporation and accretion processes.

A. Evaporation

After inspecting quantum properties for black holes, Hawking indicated that black holes emit particles with a thermal spectrum [23]. The properties of the emitted particles depend on mass, angular momentum, and charge of BHs [24]. Because of this process, called Hawking evaporation, PBHs lose mass at the rate given by [25]

$$\frac{dM_{\rm PBH}}{dt} = -\frac{2\pi^3}{15}\frac{f_{\rm eva} g_*(T_{\rm PBH}) M_{\rm PBH}^2 (k_B T_{\rm PBH})^4}{c^5 \hbar c M_{\rm Pl}^4}, \tag{7}$$

where $f_{\rm eva}$ is the evaporation efficiency factor, $M_{\rm PBH}$ and $M_{\rm Pl}$ are the PBH mass and Planck mass; k_B, $\hbar$, and c are the Boltzmann constant, reduced Planck constant, and light speed. g_* is the number of relativistic particle degrees of freedom, which is obtained by

$$g_*(T_{\rm PBH}) = \sum_i (\omega_i g_i). \tag{8}$$

In order to get a numerical value for $g_*(T_{\rm PBH})$, we need to have values of ω_i and g_i [26]

$$\omega_i = \begin{cases} 2s_i + 1 & \text{massive particles} \\ 2 & \text{masslesss pecies} \\ 1 & s_i = 0 \end{cases}, \tag{9}$$

$$g_i(T_{\rm PBH}) = \begin{cases} 1.82 & s = 0 \\ 1.0 & s = \frac{1}{2} \\ 0.41 & s = 1 \\ 0.05 & s = 2 \end{cases}, \tag{10}$$

and, obviously, s_i is the particle spin. Hence, if $M_{\rm PBH} \ll 10^{11}$ g for standard model particles $g_*(T_{\rm PBH}) \simeq 108$. As a substitute, the minimal supersymmetric standard model approximates $g_*(T_{\rm PBH}) \simeq 316$.

$T_{\rm PBH}$ is the temperature of the radiation particles from PBHs, which is equal to the temperature of PBHs. As we will demonstrate, PBH temperature is critical in the accretion and evaporation processes that are given as [19]

$$T_{\rm PBH} = \frac{\hbar c^3}{8\pi G k_B M_{\rm PBH}} \simeq 10^{-7}\left(\frac{M_{\rm PBH}}{M_\odot}\right)^{-1}. \quad (11)$$

This process slowly reduces the PBH mass, so if the dominant process is evaporation, the lifetime of a PBH with initial mass M derives from the following equation [27]:

$$\tau(M) \simeq (10^{-26}\ {\rm s})\left(\frac{M}{1\ {\rm g}}\right)^3. \quad (12)$$

B. Accretion

As mentioned, accretion has a significant effect on the evolution of PBHs. Infalling matter and photons onto PBHs increase the mass and other observable parameters.

The physical parameter of a cosmological fluid determines the accretion rate in each cosmic epoch [21]. In this study, we focus on accretion equations, and all calculations are performed by considering the spherical symmetric condition. The Bondi-Hoyle accretion model is used for this goal [28],

$$\frac{dM_{\rm PBH}}{dt} = 4\pi R_{\rm PBH}^2 \rho v. \quad (13)$$

In accretion of radiation $v = c/\sqrt{3}$ and $R_{\rm PBH} = R_s = 2GM_{\rm PBH}/c^2$. Therefore, we can rewrite Eq. (13) as follows:

$$\frac{dM_{\rm PBH}}{dt} = 16\pi G^2 M_{\rm PBH}^2 \rho_r \left(\frac{c}{\sqrt{3}}\right)^{-3} f_{\rm acc}, \quad (14)$$

where $f_{\rm acc}$ is the accretion efficiency. Now we consider conditions under which a PBH acquires matter in the accretion process. This case is more complex, and we need more information about the environment. To obtain the rate of mass increases by baryonic matter, we use the following equation:

$$\frac{dM_b}{dt} = \lambda 4\pi m_H n_{\rm gas} v_{\rm eff} r_B^2. \quad (15)$$

Here, m_H and $n_{\rm gas}$ are the mass of hydrogen and its number density, $r_B = GM_{\rm PBH} v_{\rm eff}^{-2}$ is the Bondi-Hoyle radius, and $v_{\rm eff} = (v_{\rm rel}^2 + c_s^2)^{\frac{1}{2}}$ is the effective velocity of the PBH, expressed in terms of the PBH relative velocity $v_{\rm rel}$ with regard to the gas with sound speed c_s [18]. The gas viscosity, Compton drag, Compton cooling by CMB photons, and free electron fraction are factors that determine the value of dimensionless accretion rate λ, which is effective in obtaining the final mass value. Provided both Compton drag and Compton cooling are negligible, the classic Bondi problem can be solved for an adiabatic gas [29].

III. ACCRETION OF THE UNIVERSE'S COMPONENTS

As we know, the Universe is made up of baryonic matter (gas), dark matter, radiation, and dark energy. In this section, we will examine the accretion of these components. However, due to the fact that, in this paper, we study equations until the end of the matter-dominated era, and we expect that, in these two eras, mass gain by matter and radiation will be dominant, our focus will be on the accretion of matter and radiation, so we neglect accretion of dark energy. In the following section, we peruse these two regimes of accretion individually.

A. Accretion of radiation

The presence of CMB anisotropies and fluctuations on scales larger than the Hubble radius in the recombination era points strongly to the early inflationary epoch [30]. The thermal bath result from reheating is an essential aspect of inflation. Thereupon, we can consider the Universe is the precise blackbody [31]. Considering that equations associated with accretion of radiation are distinct in the radiation- and matter-dominated eras, we discuss them separately. In the radiation-dominated era, photons from thermal bath fall into PBHs and increase their mass. As mentioned, we consider spherical symmetrical accretion and use Eq. (15). We need to have this equation in terms of time or redshift to examine the evolution of PBHs. Therefore, by using Eq. (5), we know $\rho_r = \rho_{cr}(a/a_0)^{-4}$, then Eq. (6) is used to enter the time parameter, and Eq. (15) is rewritten as follows [22]:

$$\frac{dM_{\rm PBH}}{dt} = 16\pi G^2 \rho_{cr} \Omega_r^0 \left(\frac{c}{\sqrt{3}}\right)^{-3} f_{\rm acc} \times \left(t_1^{-\frac{2}{3}} t_2^{\frac{8}{3}} e^{-4H_0(t_2 - t_0)}\right)\left(\frac{M_{\rm PBH}}{t}\right)^2, \quad (16)$$

where $\rho_{cr} = 9.2 \times 10^{-30}\ {\rm g/cm^3}$ is the critical energy density, $\Omega_r^0 = 4.2 \times 10^{-5}$ is the relative contribution of relativistic particles, $H_0 = 72 \pm 8\ {\rm km\,s^{-1}\,Mpc^{-1}} \simeq 2.3 \times 10^{-18}$ is the Hubble parameter, $t_1 = 2.1 \times 10^{12}$ s is the time of the end of the radiation-dominated era, $t_2 = 2.4 \times 10^{17}$ s is the time of the end of the matter-dominated era, and $t_0 = 4.4 \times 10^{17}$ s is the present time [32]. By solving the differential equation of Eq. (16), the final mass of PBHs due to accretion of radiation in the radiation-dominated era is obtained in terms of time,

$$M_{R\text{-RD}}(t) = \left(\frac{1}{M_i} + 1.3 \times 10^{-35} f_{\rm acc}\left(\frac{1}{t} - \frac{1}{t_i}\right)\right)^{-1} \quad \text{for} \quad t_i < t < t_1, \tag{17}$$

where M_i is the initial mass, and t_i is the formation time of the PBH. Explicitly, Eq. (17) determines the mass resulting from the accretion of radiation at any time during the radiation-dominated era and, specifically, the final mass of the PBH at the end of this era. Correspondingly, in the matter-dominated era $\rho_r = \rho_{cr}(a/a_0)^{-3}$ and $a(t) \propto t^{2/3}$, so we have

$$\frac{dM_{\rm PBH}}{dt} = 16\pi G^2 \rho_{cr} \Omega_r^0 \left(\frac{c}{\sqrt{3}}\right)^{-3} f_{\rm acc} \times \left(t_2^{\frac{8}{3}} e^{-4H_0(t_2 - t_0)}\right) \frac{M_{\rm PBH}^2}{t^{\frac{8}{3}}}. \tag{18}$$

Now, by solving the differential equation of Eq. (18), we can have mass evolution through the accretion of radiation in the matter-dominated era. The final mass because of accretion of radiation in the matter-dominated era is obtained as

$$M_{R\text{-MD}}(t) = \left(\frac{1}{M_{i\text{-MD}}} + 3.5 \times 10^{-27} f_{\rm acc}\left(\frac{1}{t^{\frac{5}{3}}} - \frac{1}{t_1^{\frac{5}{3}}}\right)\right)^{-1} \quad \text{for} \quad t_1 < t < t_2, \tag{19}$$

where $M_{i\text{-MD}}$ is the initial mass of PBHs in the matter-dominated era. In Ref. [17], the importance and consequences of correctly determining the value for the accretion efficiency factor have been well studied. However, in the literature, values between 0.05 and 0.2 are usually attributed to it. In all the calculations of this paper, we have considered the value of 0.1 for it.

B. Accretion of matter

Throughout this paper, we assume that the PBH with point mass M is immersed in the hydrogen gas. In order to continue, we need to refer to Eq. (15) and investigate each term of the equation. The numerical value of the mean cosmic gas density is

$$n_{\rm gas} \simeq 200 \text{ cm}^{-3}\left(\frac{1+z}{1000}\right)^3. \tag{20}$$

As aforementioned, $v_{\rm eff}$ is a variety of c_s and the $v_{\rm rel}$ between the PBH and the medium averaged with a Gaussian distribution. From [33], we have

$$\sqrt{\langle v_L^2 \rangle} \simeq \min\left[1, \frac{1+z}{1000}\right] \times 30 \text{ km/s}. \tag{21}$$

Given the equation $c_s = (5.7 \text{ km s}^{-1})(T_{\rm gas}/2730)^{1/2}$ to compute the speed of sound, we need the gas temperature. Before decoupling, we can consider the gas temperature was roughly equal to the CMB temperature. After that, $T_{\rm gas}$ started to decrease adiabatically due to the Hubble parameter. Therefore, the value of c_s can be written approximately as follows [29]:

$$c_s \simeq \begin{cases} (5.7 \text{ km s}^{-1})\left(\frac{1+z}{1000}\right)^{\frac{1}{2}} & z \gg 132 \\ 1800 \text{ km s}^{-1} & z \ll 132 \end{cases}. \tag{22}$$

Finally, we should introduce

$$v_{\rm eff} \simeq \begin{cases} c_s \mathcal{M}^{\frac{1}{2}}\left[3\sqrt{\frac{2}{2\pi} B\left(\frac{3}{2}, \frac{3}{2}\right)}\right]^{-\frac{1}{6}} & \mathcal{M} \gg 1 \\ c_s & \mathcal{M} \ll 1 \end{cases}, \tag{23}$$

where $B(x, y)$ is the beta function, and $\mathcal{M}$ is defined as $\mathcal{M} \equiv \sqrt{\langle v_L^2 \rangle}/c_s$ [34].

The value of λ must be determined in terms of redshift. We assume the constant free electron fraction x_e is equal to the free electron fraction of background $\overline{x_e} = 1$; also, we need the characteristic dimensionless Compton drag rate β and Compton cooling rate γ as a function of redshift. We can get [35]

$$\beta = \left(\frac{M}{10^4 M_\odot}\right)\left(\frac{z+1}{1000}\right)^{\frac{3}{2}}\left(\frac{v_{\rm eff}}{5700}\right)^{-3} \times \left[0.275 + 1.45\left(\frac{x_e}{0.01}\right)\left(\frac{1+z}{1000}\right)^{\frac{5}{2}}\right], \tag{24}$$

$$\gamma = \frac{2m_p}{m_e(1+x_e)}\beta. \tag{25}$$

Although λ can vary according to how γ and β relate to each other, in general, the following relationship applies to all redshifts [29]:

$$\lambda(\beta, \gamma) \approx \frac{\lambda(\gamma; \beta \ll 1)\lambda(\gamma \gg 1; \beta)}{\lambda_{\rm iso}}. \tag{26}$$

In this equation, $\lambda_{\rm iso} = 1.12$ in the isothermal case and $\lambda_{ad} = 0.12$ is the adiabatic case. Additionally, $\lambda(\gamma; \beta \ll 1)$ is the accretion rate numerical solution for $\beta \ll 1$ and arbitrary γ. Similarly, $\lambda(\gamma \gg 1; \beta)$ is the numerical solution for $\gamma \gg 1$ and arbitrary β. Equations (27) and (28) show equations of these special λ,

$$\lambda(\gamma; \beta \ll 1) \approx \lambda_{\rm ad} + (\lambda_{\rm iso} - \lambda_{\rm ad})\left(\frac{\gamma^2}{88 + \gamma^2}\right)^{0.22}, \tag{27}$$

$$\lambda(\beta;\gamma\gg 1)\approx \exp\left[\frac{4.5}{3+\beta^{\frac{3}{4}}}\right]\times\frac{1}{(\sqrt{1+\beta}+1)^2}. \quad (28)$$

Now we have all the parameters of Eq. (15) in terms of redshift, and we can substitute them for getting the mass rate equation. As in the previous section, with placing $1+z=(a_0/a)=e^{H_0(t_0-t_2)}(t_2/t_1)^{2/3}(t_1/t)^{1/2}$ according to Eq. (6), we can solve Eq. (15) in terms of time and obtain the mass evolution of PBHs in the radiation-dominated era,

$$M_{M\text{-RD}}(t)=M_i\left(1+\frac{123}{25\times 10^{39}}M_i(\sqrt[4]{t_i}-\sqrt[4]{t})\right)^{-1} \quad \text{for} \quad t_i<t<t_1. \quad (29)$$

In addition, the mass evolution equation in terms of time in the matter-dominated era by using $1+z=(a_0/a)=e^{H_0(t_0-t_2)}(t_2/t)^{2/3}$ is

$$M_{M\text{-MD}}(t)=M_{i\text{-MD}}\left(1+1.5\times 10^{-36}M_{i\text{-MD}}\ln\frac{t_1}{t}\right)^{-1} \quad \text{for} \quad t_1<t<t_2. \quad (30)$$

IV. ACCRETION DURING THE RADIATION-DOMINATED ERA

Radiation and matter fall into PBHs all the time and increase their mass. In the last section, we discussed the evolution of mass for accretion of radiation and accretion of matter. In this section, we want to establish whether the assumptions that radiation has a more serious effect on increasing the mass of the PBH in the radiation-dominated era or the matter is responsible for increasing the mass in the matter-dominated era are correct and not. Because of the importance of observing PBHs, many studies have been conducted on limiting the possible masses for the existence of PBHs and for explaining dark matter. After applying all constraints, including evaporation [36], lensing [37], gravitational waves [38], cosmic microwave background distortions [39], four mass windows 10^{16}–10^{17}, 10^{20}–10^{24}, and 1–$10^4 M_\odot$ remain [40,41]. We select four initial masses to study our idea. The first mass is 10^{17} g from the first window, which can explain the whole or much of dark matter. The second mass is 10^{27} g, which can cover about 10% of dark matter. The two other masses we select are in the last window. We consider this window slightly wider to cover all the previous studies, so the selected masses are 10^{33} and 10^{37} g. Nevertheless, we should mention that some recent studies have stated that this last window should be closed, such as [42]; however, it is a controversial matter. We have examined the graph of mass over time in two eras separately. Figure 1 demonstrates the growth of PBHs mass during the radiation-dominated era and compares the effect of matter and radiation on the increase of PBHs mass. As we expected, radiation during the radiation-dominated era significantly increases the mass of PBHs, and we can neglect the accretion of matter in this epoch. However, we should note that from the mass larger than $\sim 10^{36}$ g, the accretion of matter is not negligible though.

V. ACCRETION DURING THE MATTER-DOMINATED ERA

Now we should investigate the accretion of PBHs during the matter-dominated era. Equations (19) and (30) illustrate the mass of PBHs that has started to devour matter and radiation in this era. Figure 2 has satisfied our expectations in the matter-dominated era; the growth of PBH mass is mainly done because of matter. Nevertheless, we should notice that, for PBHs with masses less than $\sim 10^{22}$ g, neither the accretion of radiation nor matter can change the mass of PBHs significantly.

To examine the accuracy of our work, we compared our results with the previous works, in particular, with papers of Kamionkowski *et al.* [29] and Ricotti *et al.* [18]. For this goal, it is necessary to define the dimensionless Bondi-Hoyle accretion rate that shows the evolution of the accretion rate normalized to the Eddington rate as $\dot{m}\equiv \dot{M}_b/\dot{M}_{\rm Ed}$, where $\dot{M}_{\rm Ed}=1.44\times 10^{17}(M_{\rm PBH}/M_\odot)$ erg s^{-1} is the Eddington accretion rate. Figure 3 gives us practical information about the mass evolution only by gas accretion. This paper uses an analytical solution to calculate the equations as much as possible.

Regarding our semianalytical approach, Fig. 3 depicts a slight difference between the mentioned approach and fully numerical methods. Although in low redshifts, Kamionkowski *et al.* [29] have considered the adiabatic accretion in this era because of the neglectable Compton cooling effect, Ricotti *et al.* [18] implicitly have assumed that $\gamma\gg 1$ at all times when accounting for Compton drag in the analysis [29].

VI. EVAPORATION VS ACCRETION

In previous sections, we investigated the process of PBH mass increase during the radiation- and matter-dominated eras. One of the most meaningful results obtained from a PBH mass is the calculation of its radius. According to the Schwarzschild radius relationship, $R_s=2GM_{\rm PBH}/c^2$, if we substitute $\dot{M}_b$ obtained from the previous parts, we can study Hawking evaporation by comparing the growth rate of the event horizon and Planck length. This should be done because in [17] the importance of this comparison has been elaborated.

The concept of BH complementarity is one of the potential answers. The membrane paradigm for describing the BH horizon serves as the foundation for the concept of complementarity. For observers who are positioned outside the BH and close to the horizon, the horizon is shown in

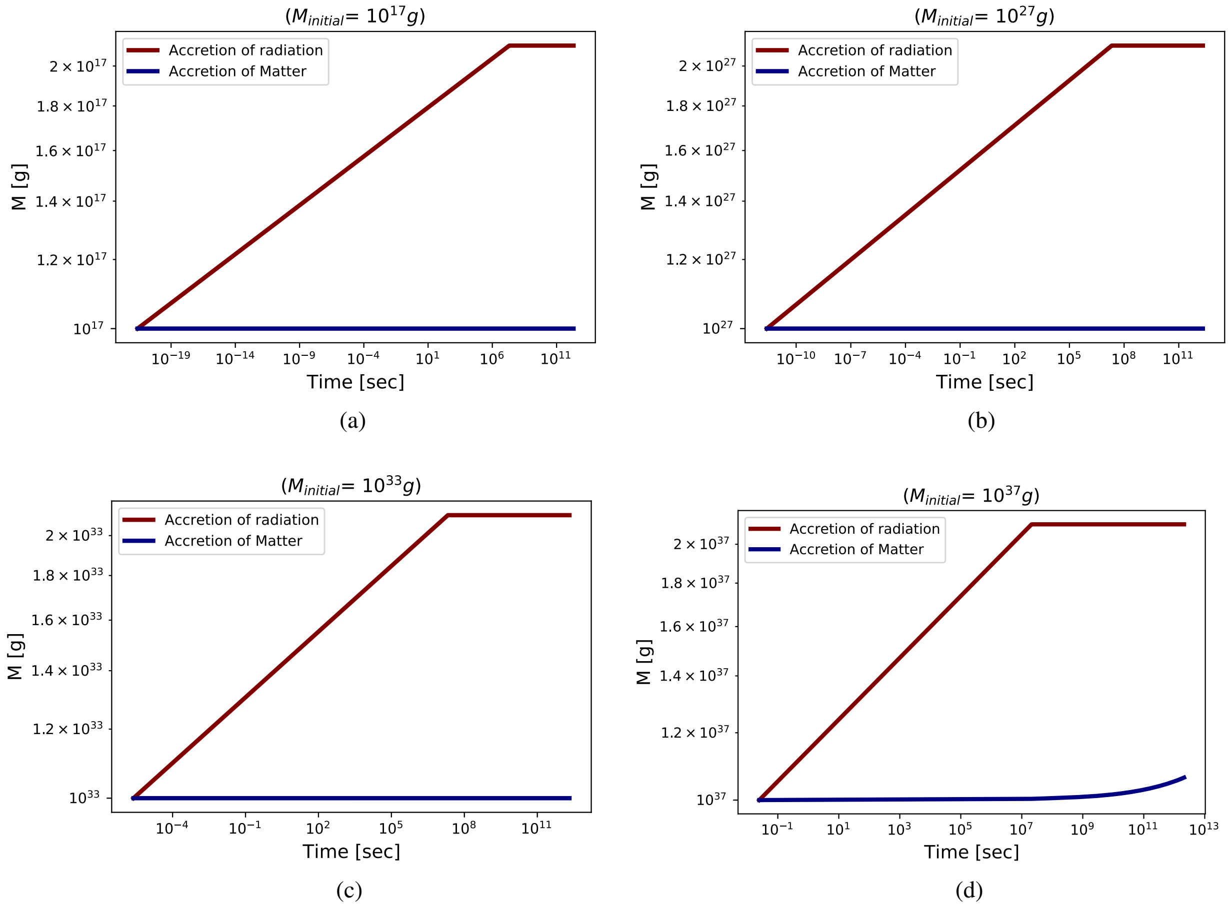

FIG. 1. This figure shows how the mass grows during the radiation-dominated era by accretion of radiation (red line) and accretion of matter (blue line). As expected, during this epoch, mass growth occurs mainly due to accreting of radiation. However, around the mass range of $10^4 M_\odot$, the accretion of matter also starts to affect the mass increase. The graphs are plotted for different initial masses (a) 10^{17}, (b) 10^{27}, (c) 10^{33}, and (d) 10^{37} g. These masses were chosen because there are no strict constraints on these masses to explain for at least part of the dark matter.

this model as a heated membrane, commonly known as a stretched horizon [43].

Locally, the Hawking radiation is experienced by the fiducial observers to becoming warmer as it is tracked back toward the BH horizon, until one reaches around a Planck distance (l_p beyond the Schwarzschild radius). From the perspective of the fiducial observers, this expanded horizon is thought to be the source of the Hawking radiation emission [44].

Another approach for determination of the source of Hawking radiation is based on using the Beckenstein-Hawking entropy $S = A/4l_p^2$ and setting characteristics of thermal fluctuations about equilibrium $\delta S \sim 1$. We can estimate the scale of quantum fluctuations of the horizon. If we consider the horizon as independent fluctuating area elements $N \equiv A/l_p^2$, the equation $\delta A \sim \sqrt{N}\delta a \sim l_p \delta r$ holds for each era and radius r. Therefore, according to equation $\delta a \sim l_p^2$, we have $\delta r \sim l_p$ [45]. This derivation shows that particles that escape the black hole start their journey from about a Planck length farther than the event horizon. Therefore, in this work we consider that the source of Hawking radiation is the stretched horizon.

However, other studies recently have questioned the starting point of Hawking radiation and they believe that escaping particles are created from the quantum atmosphere as an alternative to Hawking radiation's origin [46]. Contrary to popular belief, which holds that the Hawking radiation comes from ultrahigh energy excitations very close to the horizon, they have demonstrated that the Hawking radiation originates from what could be called a quantum atmosphere around the black hole with energy density and fluxes of particles peaking at about 4 GM/c^2 [47,48]. It should be noted that some other works like [49] believe in other distances for the Hawking radiation flux peak.

Although Giddings's arguments [47,50] have been given based on physical tests that are more physical to describe the Hawking radiation as produced in a quantum atmosphere region of size $\Delta r \sim R_s$ near the Schwarzschild horizon, this quantum atmosphere formation depends on

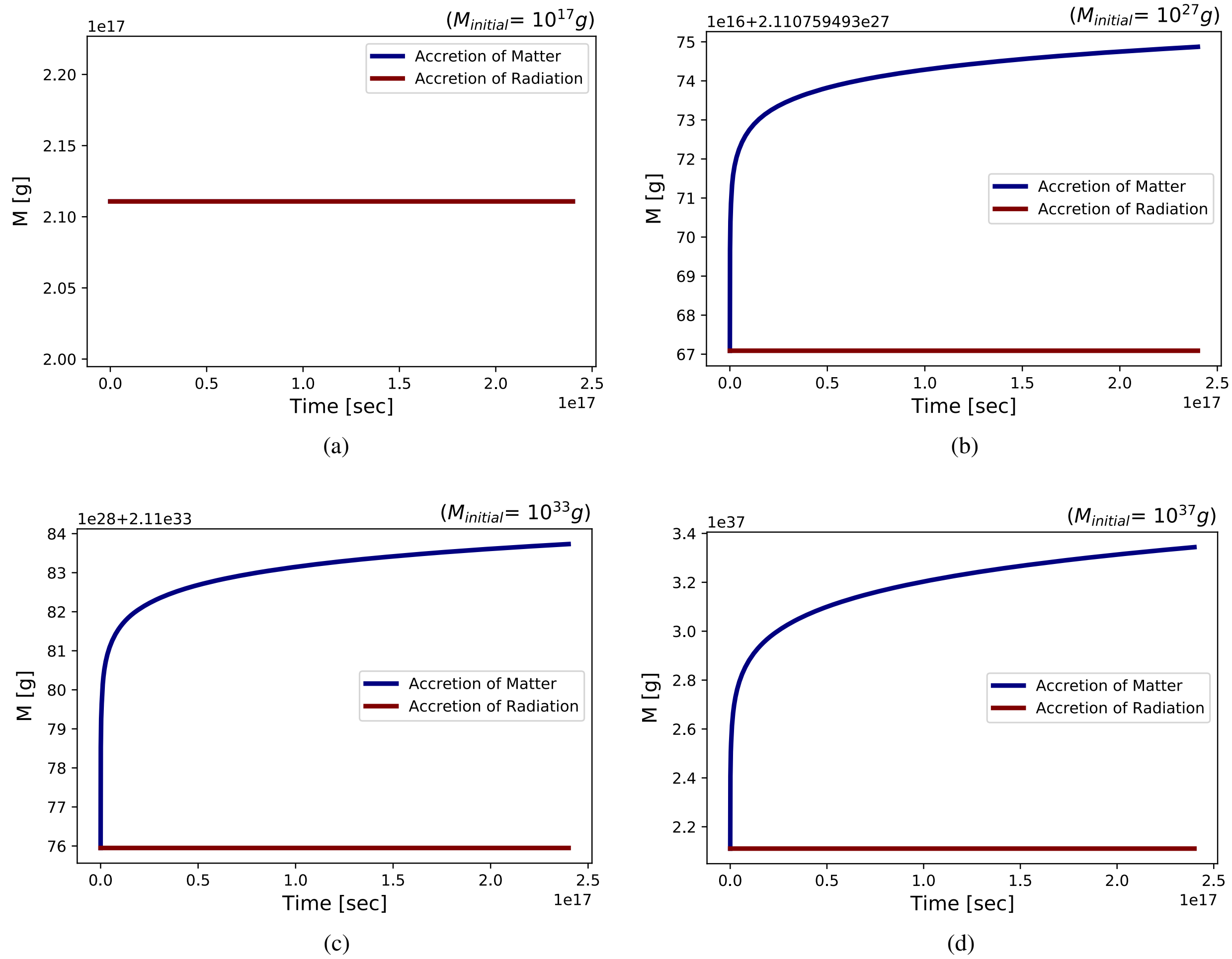

FIG. 2. In this figure, we can see the accretion of matter is dominant over the accretion of radiation in the matter-dominated era. In (a), changing the mass is so slight and neither accretion of radiation nor accretion of matter has any significant impact. Nevertheless, in (b)–(d), accreting of matter evidently changes the mass of PBHs.

the known definition of the Hartle-Hawking vacuum and Unruh vacuum for the spherical stationary metric [48], i.e.,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2. \tag{31}$$

In contrast to the dynamical metric, this metric has some special characteristics that discriminate it from the dynamical one. First, the geometric optic approximation can be written for the lights near the horizon, $f(r) = 0$. In other words, when the modes are traced back, they become highly blueshifted near the horizon and we are not well aware of the laws of physics in such a high trans-Planckian domain.

Second, the absence of a unique notion of a vacuum state is the main difference between the quantum theory in stationary space and the quantization in a general curved space-time. However, there is a situation in which one can naturally select a space of positive frequency solutions for the curved space-time that is stationary. When space-time is not stationary, as it happens, for instance, in the process of gravitational collapse, one loses the natural criterium to define positive frequency modes. Therefore, the unambiguous concept of particle states of stationary space-time space disappears [51]. This causes one to not be able to calculate the expectation value of the stress tensor for Hawking radiation via the conformal anomaly [47]. After the radiation-dominated era, when the black hole gets close to its equilibrium state, one can use stationary metrics like Eq. (31) to calculate Hawking thermal flux.

In the cosmological context, a spherically symmetrical black hole with a dynamical horizon cannot create pairs of particles and antiparticles, as this violates the principle of conservation of energy and would make the apparent horizon spacelike. In other words, the apparent horizon of any dynamical space-time must be inside the event horizon; thus, any virtual pair particles created by the vacuum cannot escape and should fall back into PBHs.

In the case of a fully dynamical black hole, we can not apply Hawking's quantum field theory approach to black

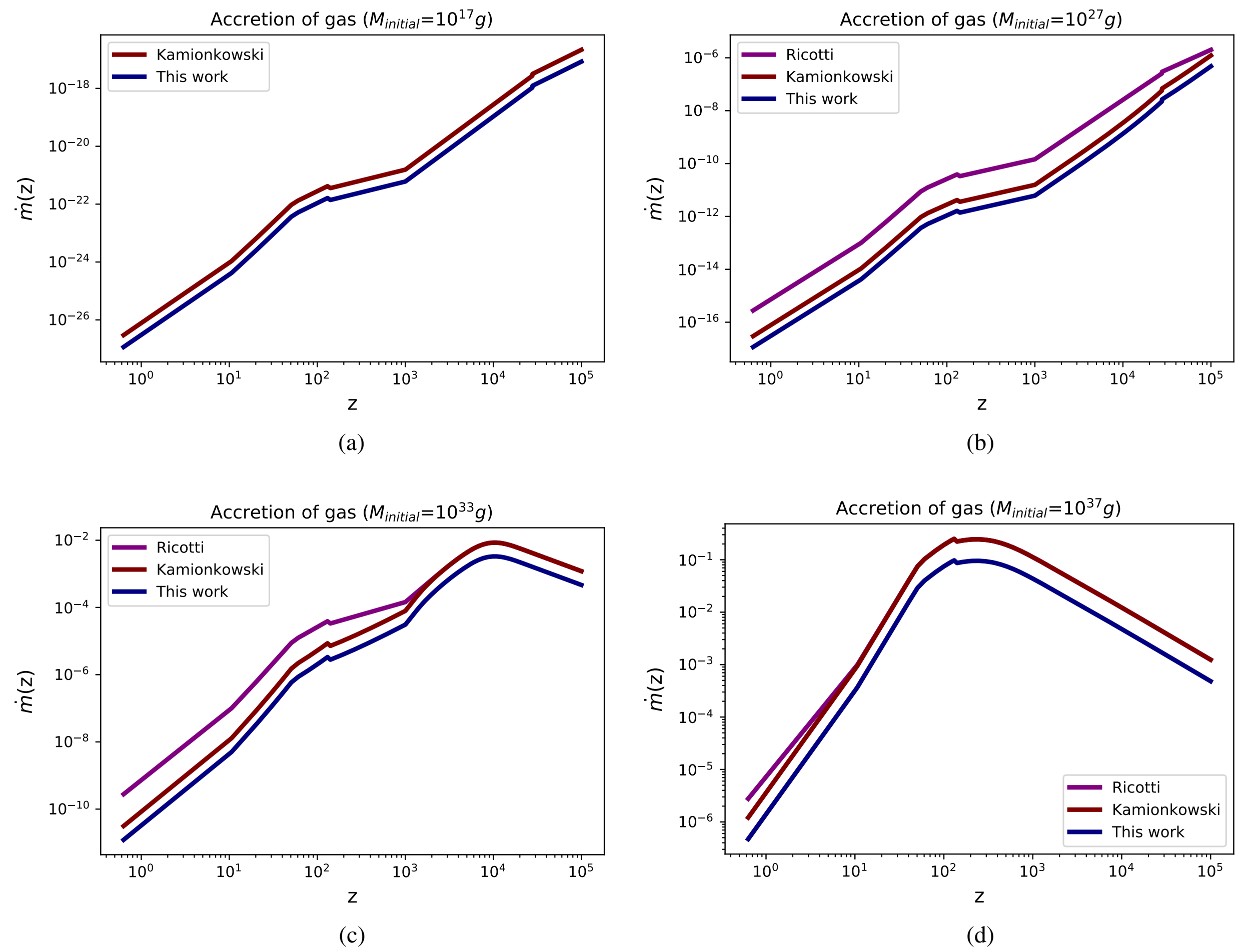

FIG. 3. We have compared the accretion of gas for selected PBH masses (a) 10^{17}, (b) 10^{27}, (c) 10^{33}, and (d) 10^{37} g with results of the papers of Ricotti *et al.* [18] and Kamionkowski *et al.* [29] by using the dimensionless accretion rate as $\dot{m} \equiv \dot{M}_b/\dot{M}_{\rm Ed}$. It should be noted that the Ricotti equation does not behave correctly in low masses, such as mass 10^{17} g. However, it is clear that all three models are close to each other, and we see similar behavior. Afterward, we can state that our model works correctly.

hole radiation [16], which applies to late-time stationary black holes and is not a suitable method for calculating the Hawking radiation thermal aspect. In such cases, new approaches [52–54] were developed to calculate Hawking radiation in a dynamical background [55–58]. In these approaches, which are based on the semiclassical approach based on the adiabatic vacuum in quantum field theory in curved space-time, the radiation is plausibly emitted from the vicinity of apparent horizons rather than near the event horizon [59]. In this approach, consider a null curve that comes from past null infinity parametrized by u and reflects off the center at $r=0$ and goes to the future null infinity, which is parametrized by U. Around the null curve labeled by u^* that passes near the horizon, we can write

$$U = U^* + C^* \int \exp\left(-\int \kappa(\tilde{u})d\tilde{u}\right) d\overline{u}, \tag{32}$$

for some constant U^* and C^*.

For the Planckian emission of the Hawking radiation at U^*, the following adiabatic condition has to be satisfied:

$$\frac{|\dot{\kappa_*}|}{\kappa_*^2} \ll \epsilon \ll 1. \tag{33}$$

For the stationary space-times, κ_* is the surface gravity, and for dynamical space-time, this condition cannot be satisfied.

For the stationary space-times (like Schwarzschild), having the exponential factor $u \sim (-1/\kappa_H)\ln(U_H - U)$ between the past and future infinity parameter guarantees the adiabatic condition and the WKB condition for the wave equation around the horizon [60]. If we verify this condition for PBHs that formed in the early radiation-dominated era, one can see that the accretion of the cosmic fluid can grow PBH mass up to $\Delta M/M \sim 400\%$. This dynamics clearly breaks the adiabatic condition for these PBHs in the radiation-dominated era.

Note that, in space-times with a slowly varying geometry, the known adiabatic vacuum allows one to define a meaningful notion of particles that can be applied to an evolving geometry [61]. The adiabatic vacuum concept relies on the WKB (eikonal) estimation for solutions of the wave equation. As mentioned in Ref. [59], physically the adiabaticity constraint (eikonal approximation) is equivalent to the assertion that a photon emitted near the peak of the Planckian spectrum should not see a significant fractional change in the peak energy of the spectrum over one oscillation of the electromagnetic field (whereas, the change in space-time geometry is adiabatic as recognized by a photon near the peak of the Hawking spectrum). The canonic point is that if a black hole is in the dynamical stage, for example, due to the accretion [62], the essential conditions such as WKB approximation of our adiabatic condition around the apparent horizon for Hawking radiation cannot be applied. This leads to extinguishing the black hole radiation in the dynamical stage [13,15].

As we stated, due to accretion, a PBH is in the dynamical phase, so it cannot have adiabatic conditions around the apparent horizon for Hawking radiation [14]. Some works have shown that a dynamical black hole with a very slowly evolving horizon is more likely to emit Hawking radiation as usual. Furthermore, calculating the probability rate for a dynamical black hole allows us to determine when an incoming flux of matter or radiation can turn off Hawking evaporation [15]. As mentioned, the length of quantum fluctuations on the event horizon are on the order of Planck length, so if the growth rate of the event horizon is too high, particles that try to escape should fall back into the black hole. If one adds the backscattering effect and all non-s-wave contributions in the Hawking radiation gray factor, there is no significant change relative to the s-wave in total radiation luminosity [51].

In this context, we are interested in following changing rate of radius for the mentioned masses. We would like to know if, for these different masses, there are time periods where evaporation is turned off. In Figs. 4 and 5, radius growth rates are plotted in terms of time for selected masses. In order to facilitate conclusions, the radiation- and matter-dominated eras are separately shown, and the

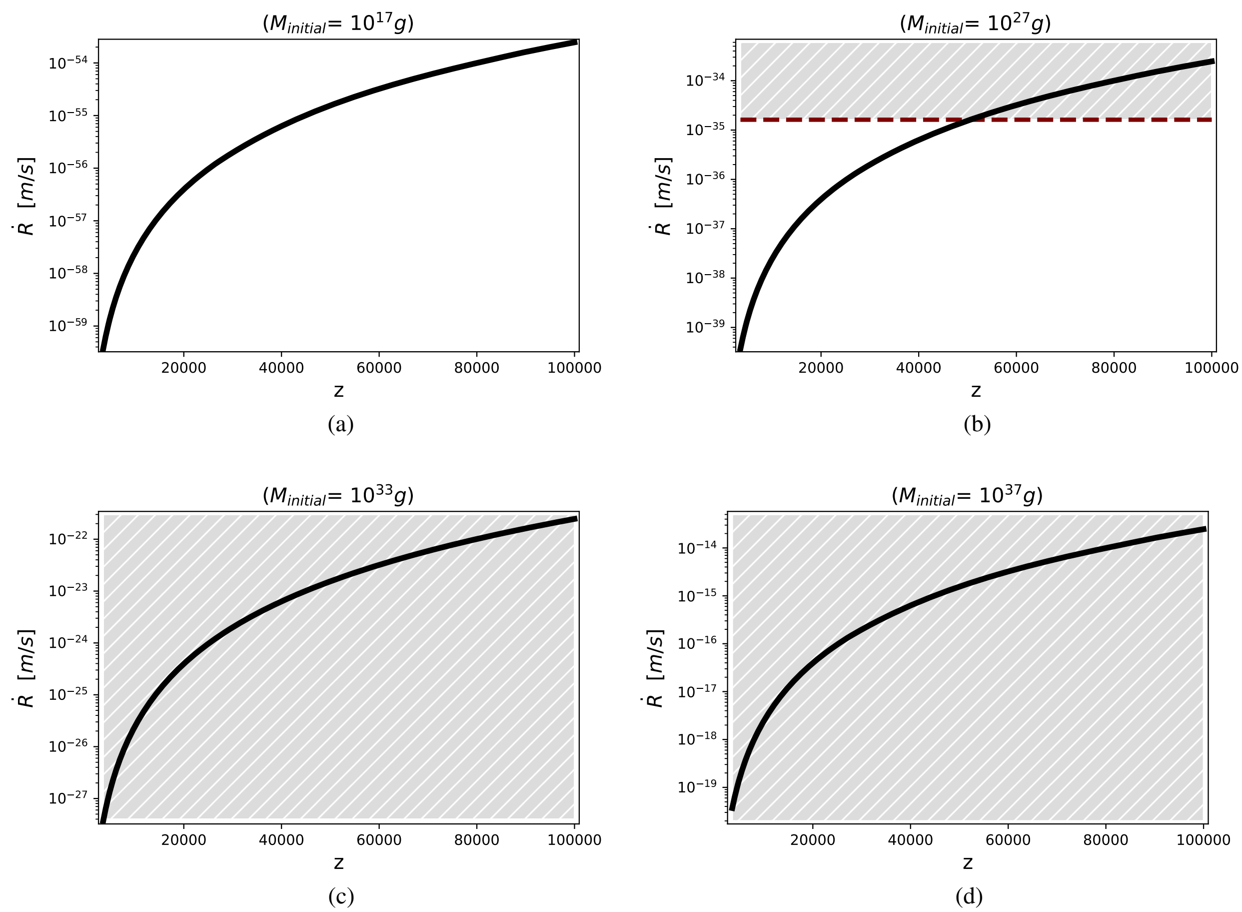

FIG. 4. We have compared the evaporation and the accretion processes in the radiation-dominated era. In (a), the PBH with the initial mass of 10^{17} g has low accretion effect, as expected, and the evaporation process remains powerful during this period. (b) In the case of PBHs with the initial mass of 10^{27} g, evaporation stops early in the radiation-dominated era. (c),(d) Evaporation does not occur in PBHs with high initial mass due to the high growth rate of radius and event horizon.

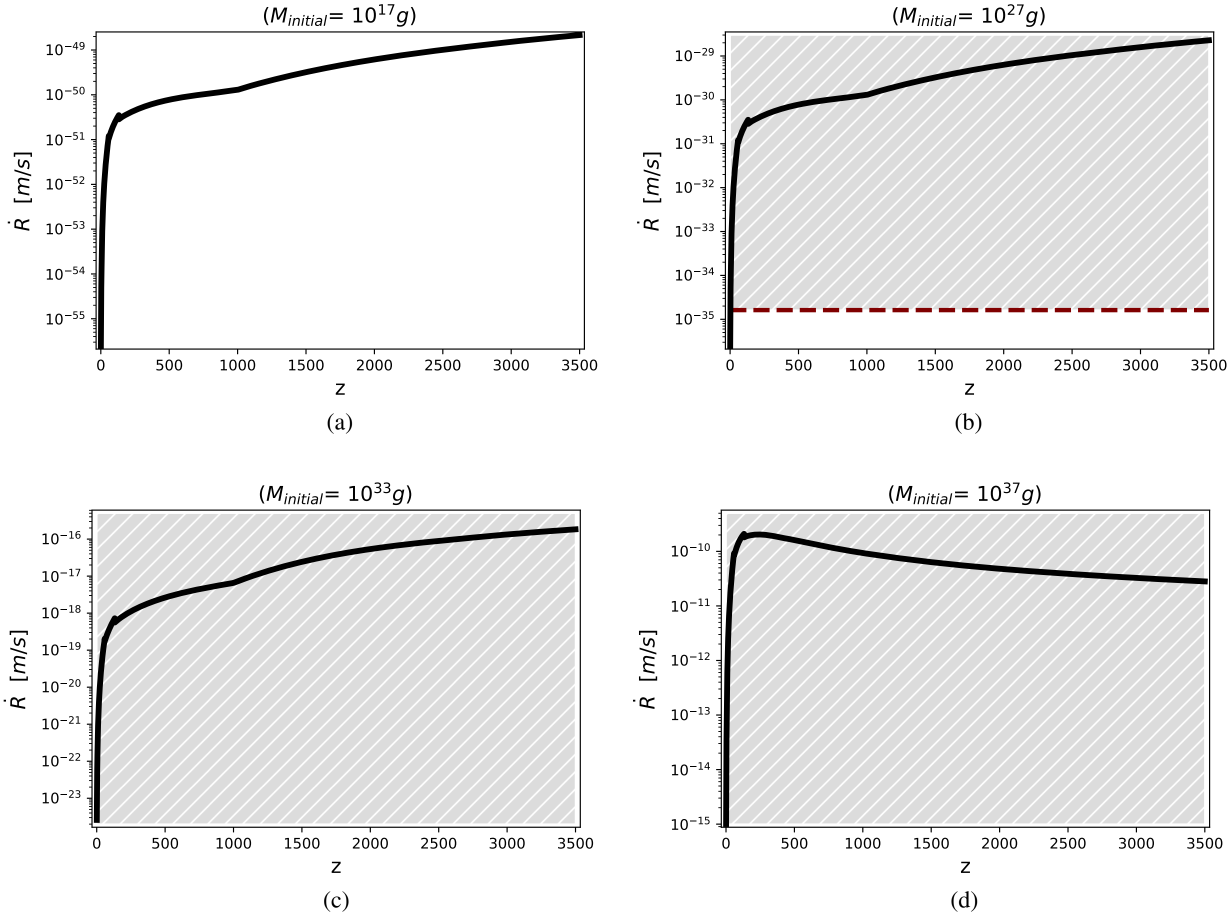

FIG. 5. This figure points out the competition between evaporation and accretion in the matter-dominated era. (a) Shows that the accretion of PBHs with initial mass 10^{17} g does not overcome evaporation until the end of the matter-dominated era. PBHs with an initial mass of 10^{27} g in (b), after the evaporation turns of; it resumes the evaporation process at the end of the matter-dominated epoch when the change of the radius rate decreases sharply. (c),(d) Indicate that the radii of PBHs with these masses grows fast so that still no evaporation until the end of the matter-dominated era can be seen.

regions where the radius changes are more than the Planck length are crosshatched.

As we can see in Fig. 4, PBHs with the initial mass 10^{17} g constantly evaporate during the radiation-dominated era. The situation is a bit more complicated for PBHs with the initial mass of 10^{27} g. Because, according to Fig. 4(b), these PBHs first evaporate during the radiation-dominated era, the radius increase rate shortly exceeds the Planck length and the evaporation process stops. In the case of the other two selected masses, these PBHs do not evaporate at all during the radiation-dominated era.

We apply the same calculations on PBHs during the matter-dominated era. In Fig. 5, we can see that masses where the evaporation process was turned off during the radiation-dominated era do not evaporate in the matter-dominated era as well due to the increase in the radius change rate. At the end of the matter-dominated era, the radius change rate drops sharply, whereby quenched evaporation may be reactivated in some masses; for example, in Fig. 5, we see this condition in the PBH with an initial mass of 10^{27} g. For the initial mass of 10^{17} g, radius changes are less than the Planck length and continue to evaporate during this era. Calculations show that evaporation will immediately dominate the accretion process in the case of PBHs with an initial mass of less than $\sim 10^{26}$ g.

Since it is usually more appropriate to work with dimensionless parameters for comparison, in this paper, we define a new parameter $\chi = \dot{R}/v_{\rm eff}$. In addition to the fact that this parameter is dimensionless and this makes it suitable for comparing different models, there is another reason for defining it. This parameter is dependent on $v_{\rm eff}$ and, as a result, it is related to sound speed and relative velocity of PBHs. This dependence makes the effects of the cosmic environment, which is diverse in various models as well as the relative velocity of the initial PBHs for which there are different estimations to be seen in the changes of this parameter. On the other hand, the type of accretion that is chosen, whether it is spherical symmetrical accretion or disk accretion, also has a serious effect on this parameter.

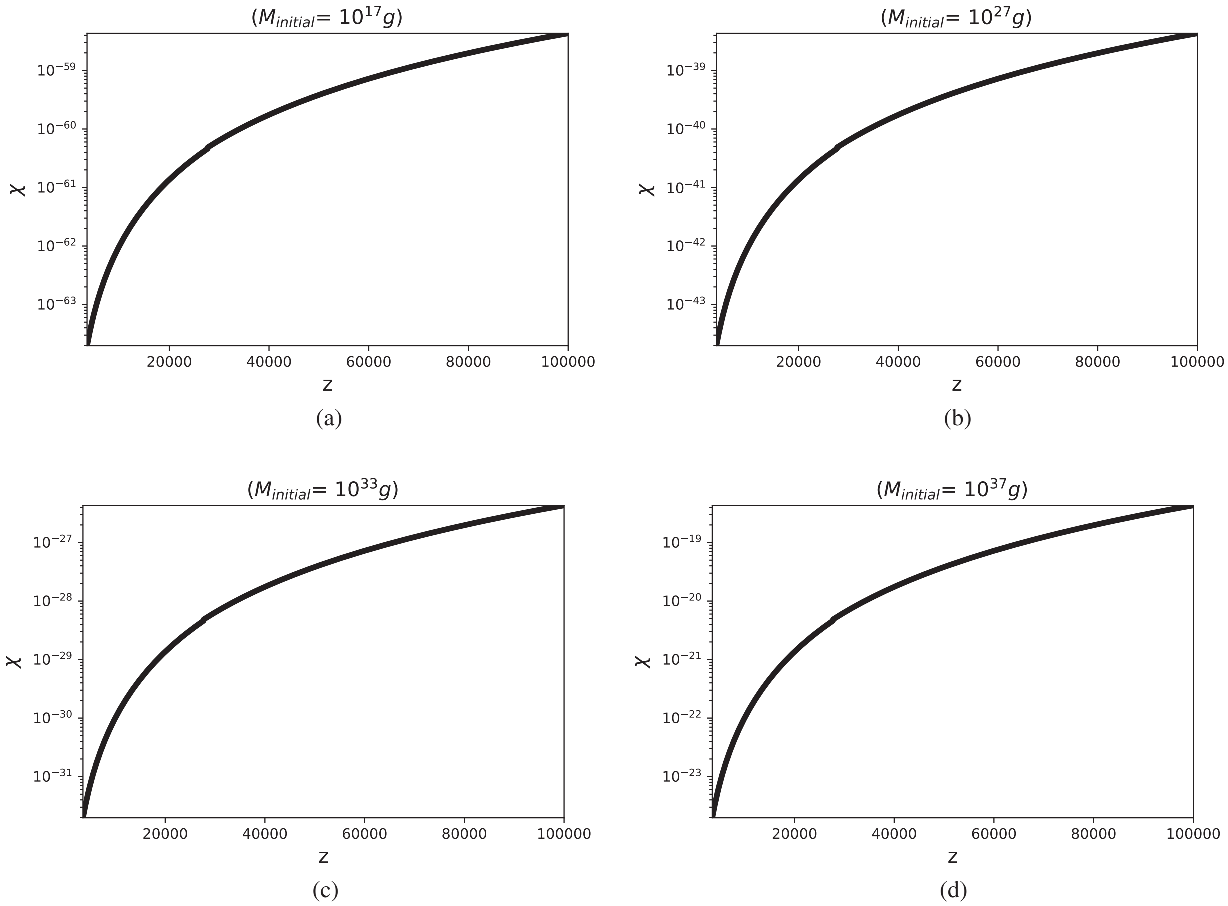

FIG. 6. The changes of dimensionless parameter χ in the radiation-dominated era for four initial masses (a) 10^{17} g, (b) 10^{27} g, (c) 10^{33} g, and (d) 10^{37} g are plotted. To compare this work with future works or other models of accretion, it is very important to pay attention to these plots because calculations related to the type of the accretion, the properties of the cosmic environment and how PBHs were formed are considered in this parameter.

Therefore, the definition of this parameter is necessary. χ as a function of z for four masses is plotted in Figs. 6 and 7.

In our opinion, all models in which the evaporation of PBHs has been proposed to justify a phenomenon in the history of the Universe should be reexamined. Since the starting and stopping times of Hawking evaporation are different for PBHs with different masses, these calculations must be done first to ensure that PBHs with the proposed masses will evaporate at all at that time or not. This issue is much more important for primordial black holes with low masses. Thus, the calculations related to this work must be checked for them first. We also suggest that the χ parameter should be used seriously in all future works, because this very important parameter contains many features of a model related to PBHs, like the model of their formation, their accretion model, cosmic environment situations, etc.

VII. CONCLUSIONS

Since PBHs are one of the most important candidates for dark matter, their evolution in time is also very important. As we know, the two main processes that can change the mass of black holes are Hawking radiation and accretion. Therefore, the behavior of these processes must be well understood in order to be able to calculate the evolution of PBHs. The accretion equations can be well represented by the Bondi-Hoyle model. Of course, this model is a well-defined model with the condition of spherical symmetry. A disk model can also be considered, which will provide more accurate answers. However, for simplicity, the Bondi-Hoyle model is used in this paper.

On the other hand, Hawking's approach to considering black holes as blackbodies and trying to investigate the thermodynamic properties of black holes is very attractive and practical. Although no such radiation has been observed so far, the logic of its existence is so convincing that we cannot deny its existence. Nevertheless, the main question is whether a black hole can always swallow particles through accretion and emit particles from itself through Hawking evaporation. This question becomes even more important when we realize that any of these processes, when applied to PBHs, can have important

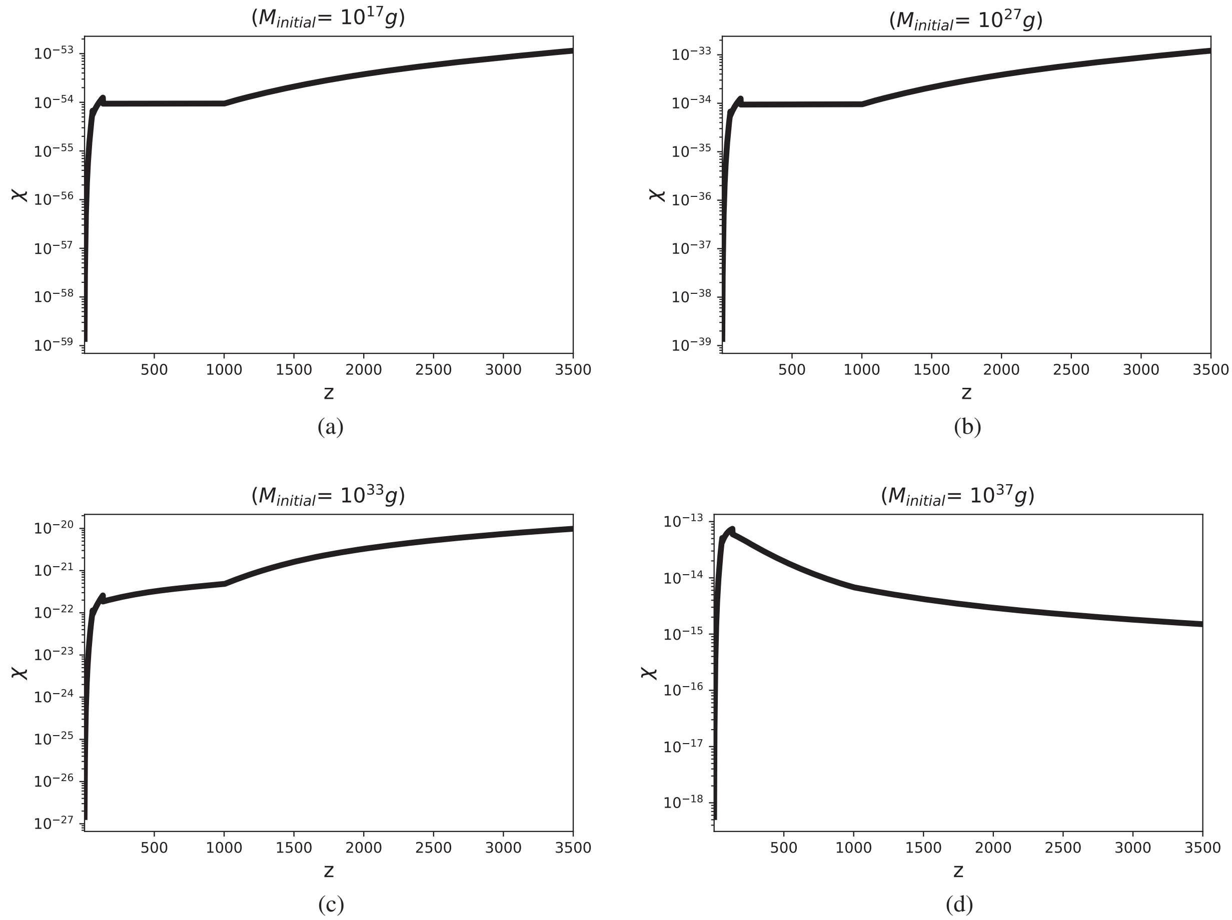

FIG. 7. Similar to the radiation-dominated era, the changes of dimensionless parameter χ in the matter-dominated era for four initial masses (a) 10^{17}, (b) 10^{27}, (c) 10^{33}, and (d) 10^{37} g are plotted. Clearly, the behavior of PBHs with the mass 10^{34} g is completely different.

cosmological and astrophysical consequences. Thus, without a doubt, this question must be answered.

In this paper, we first showed that in the radiation-dominated era, the rate of mass increase of PBHs due to the swallowing radiation is much higher than the rate of increase of mass due to swallowing matter. It should be noted that, for masses greater than $\sim 10^{36}$ g, the accretion of matter is effective in this era. Despite this, it is the opposite in the matter-dominated era. That is, matter accretion is much more effective than radiation accretion in the mass accretion of PBHs. Such a thing was to be expected and was consistent with our imaginations. Furthermore, we compared the model we obtained for augmentation with the works of Ricotti *et al.* [18] and Kamionkowski *et al.* [29] in Fig. 3 to ensure its accuracy.

The question of precisely where the Hawking radiation-emitting particles begin their motion has been debated in the literature. In Sec. VI, we talked extensively about this crucial topic. Although if we use Bekenstein-Hawking entropy, we will find that Hawking particles start to escape from the black hole at a distance of around a Planck length from the Schwarzschild radius, this approach has been challenged in recent years. According to a recently developed alternative approach, Hawking particles may be created at various distances from black holes, with their production peak occurring at a distance of $4\,\mathrm{GM}/c^2$. However, in other works that we have mentioned, this length has also been discussed and other values have been imagined for it in different works.

In this work, we briefly stated the main method of finding the starting point of Hawking particle motion. We also investigated this issue from different angles, the most important of which was that we stated why assuming a static horizon for these black holes due to the presence of accretion could be wrong and to what limit this assumption would be correct. In other words, in various papers, the apparent horizon of black holes was considered static, and calculations related to Hawking radiation were performed with this assumption. However, it is obvious that PBHs cannot be isolated and there is matter and radiation around them, especially when we consider them as the constituents of dark matter. We know that the proportionality between the radius and the mass of the Schwarzschild black hole is established, so considering the PBHs as Schwarzschild black holes, it is clear that, with the increase in mass, the radius will definitely go out of the static state and become dynamic. Particularly if the increase is continuous, the radius also changes continuously. As a result, there is a competition between mass reduction due to evaporation and mass increase due to accretion.

Nonetheless, we have to be very careful about evaporation calculations. Hawking radiation is the result of tunneling in the horizon potential barrier. Now, if the horizon is growing, this potential barrier is no longer the same as the static horizon potential barrier. Knowing that the quantum fluctuations on the horizon are related to the Planck length, it is enough to check the graphs related to the rate of change of the radius of black holes over time in order to know in which cases the accretion can cause rapid growth of the radius and, as a result, for what mass and at what times accretion can prevent particles from escaping from the black hole's gravity. In this paper, we considered four masses: 10^{16}–10^{17}, 10^{20}–10^{24}, and 1–$10^4 M_{\odot}$. The gray hatching in Figs. 4 and 5, means that the growth apparent horizon is so great that it actually forces escaping particles to fall back into PBHs, thus stopping the evaporation of them. Our calculations show that PBHs with mass more than $\sim 10^{27}$ g in the radiation-dominated era do not have Hawking radiation. On the other hand, PBHs with the mass 10^{26} g start radiating again at the end of the matter-dominated era due to the reduction of the accretion rate, despite the evaporation turning off during the radiation-dominated era.

This paper is a very interesting start to investigating the models that claim that the evaporation of PBHs creates cosmological effects or that they want to explain a phenomenon with the help of the evaporation of PBHs. It seems that, before any calculation to explain a phenomenon with the help of Hawking radiation, it should be checked whether the PBH with a specific mass could have Hawking radiation at all or not.

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