Axion cosmology with post-Newtonian corrections

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We present first-order post-Newtonian (1PN) approximations of a general imperfect fluid and of an axion as a coherently oscillating massive scalar field, both in the cosmological context. For the axion, using the Klein transformation and Madelung transformation we derive the Schrödinger and Madelung hydrodynamic formulations, respectively, in an exact covariant way and to 1PN order. Complete sets of equations for the 1PN formulations are derived without fixing the temporal gauge condition. We study the linear instability in cosmology and a static limit for both fluid and axion; these are presented independently of the gauge condition to 1PN order, thus are naturally gauge invariant.

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I. INTRODUCTION

Post-Newtonian (PN) hydrodynamics is a consistent approximation of Einstein's gravity with Newtonian hydrodynamics appearing as the zeroth-order PN (0PN) in the $c \to \infty$ limit in some dimensionless combinations of the metric and energy-momentum variables [1,2]. Cosmological extension is possible with different temporal gauge conditions readily available [3]. The background Friedmann equations are subtracted in cosmology, and the remaining deviations from the background can be consistently expanded to the PN orders.

The scalar field with various forms of potential is popularly used in physical cosmology as tools for variety of essential cosmological roles. For example, the inflation, dark matter, and dark energy are often modeled by using the scalar field. However, with general forms of potential the scalar field does not allow the PN approximation. This is understandable as the general scalar field does not necessarily have the Newtonian (0PN) limit. The situation changes as we consider an axion.

As an axion, we consider a coherently oscillating stage of a massive scalar field; we may include a self-interaction term *assuming* that due to small coupling it does not interfere with the coherent oscillation of the field. The cosmological axion is known to behave as a zero-pressure fluid, thus nonrelativistic in both background and perturbations [4–7], in fact, even to fully nonlinear and exact perturbations [8]. The axion, being an oscillating scalar field, actually has a characteristic stress (both isotropic and anisotropic) reflecting the wave nature and uncertainty principle [9,10]. For an extremely small axion mass with macroscopic Compton wavelength its effect becomes cosmologically important [11,12].

Here we study the first-order PN (1PN) approximation of the cosmological axion. The PN expansion differs from the relativistic perturbation theory. In the latter, all deviations from the Friedmann background in the metric and energy-momentum tensor are regarded as perturbation, and a consistent expansion is made for the perturbations order by order assuming that the perturbations are small. In the former, the remaining deviations after subtracting the Friedmann background are expanded in PN expansion by identifying dimensionless PN variables with c^{-1} involved. The lowest expansion gives the Newtonian limit, and the next order involving c^{-2}, like GM/Rc^2, Φ/c^2, v^2/c^2, $p/\varrho c^2$, etc., gives the 1PN expansion; M, R, Φ, v, p, and ϱ are characteristic mass, length, gravitational potential, velocity, pressure, and density, respectively. The perturbation theory is fully relativistic but applicable for small deviations (i.e., weakly nonlinear) from the background, whereas, the PN expansion is weakly relativistic but fully nonlinear. Thus, the two approximations (if available) are complementary to each other.

The weak gravity limit is yet another complementary approximation where the gravity is assumed to be weak (near 0PN) while considering fully relativistic and nonlinear energy-momentum (thus, ∞PN) [13]. This approximation is relevant in many astrophysical situations including cosmology; in the observable Universe, except for nearby compact objects like neutron stars and black holes, the gravity represented by a dimensionless metric parameter Φ/c^2 is extremely small with a typical value less than 10^{-5} [14]. The weak gravity limit of a general scalar field (including an axion) in cosmology is presented in [15] in the context of electrodynamics and magnetohydrodynamics.

The PN approximation is available for an axion only after a transformation which was introduced by Klein in his way to derive the Schrödinger equation as the nonrelativistic limit of the Klein-Gordon equation [16–18]. By further applying a transformation by Madelung the Schrödinger equation leads to Madelung's hydrodynamic formulation of the system [9]. Although the Klein transformation is suitable to derive the nonrelativistic limit and the Madelung transformation originally applied to the Schrödinger equation, here we apply these to the relativistic Klein-Gordon equation in our way to derive the PN corrections.

As our 1PN fluid formulation is valid for a general imperfect fluid, we can also derive the axion hydrodynamic equations from the fluid equations using the PN order fluid quantities for the axion. In this work, the 1PN fluid and axion formulations are applied to the gravitational instability and to a static limit. Both are available *without* imposing the temporal gauge condition. Thus, we derive the Jeans scale and the equilibrium scale for an axion fluid in a naturally gauge-invariant manner.

In Sec. II we summarize the complete set of 1PN equations for the three (hydrodynamic, Schrödinger, and Madelung) formulations without imposing temporal gauge condition, thus in gauge-ready forms. The equations are derived in the appendixes. In Secs. III and IV we apply the formulations to the gravitational instability and the static limit. Section V is a discussion. Appendix A presents covariant forms of fluid quantities under the Klein and Madelung transformations. The complete PN corrections for these three (fluid, Schrödinger, and Madelung) formulations are derived in Appendix B.

II. COSMOLOGICAL 1PN EQUATIONS

Here we summarize the 1PN equations of the three formulations. Derivations are presented in the appendixes.

The 1PN metric convention is [1,3]

$$ds^2 = -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}(2U^2 - 4\Upsilon)\right]c^2dt^2 - \frac{1}{c^3}2aP_i c dt dx^i + a^2\left(1 + \frac{1}{c^2}2V\right)\delta_{ij}dx^i dx^j, \quad (1)$$

where $a(t)$ is the cosmic scalar factor, the index of P_i is raised and lowered using δ_{ij} and its inverse, and to 1PN order we have $V = U$. The energy-momentum (thus fluid quantities) convention can be found in Eqs. (B7)–(B11). In the above metric convention we ignored the transverse-trace-free tensor-type metric, and imposed the spatial gauge conditions (without losing any generality and convenience) to make the spatial part of the metric simple, but we have not imposed the temporal gauge condition yet [3]. Together with the temporal gauge condition to be introduced below, all remaining 1PN variables are spatially and temporally gauge invariant [3].

The 1PN order equations will be presented without imposing the temporal gauge condition. The general temporal gauge condition can be written as [3]

$$\frac{1}{a}P^i{}_{,i} + n\dot{U} + m\frac{\dot{a}}{a}U = 0, \quad (2)$$

with arbitrary real numbers n and m. As the gauge condition we can choose any number for n and m; $n = 3$ (Chandrasekhar, standard PN gauge, or maximal slicing), $n = 3 = m$ (uniform-expansion gauge), $n = 4$ (harmonic gauge), $n = 0 = m$ (transverse-shear gauge), etc. As these gauge conditions, together with the spatial gauge conditions we already have imposed in the metric, completely remove the gauge degrees of freedom, all remaining 1PN variables after imposing the gauge condition can be equivalently regarded as gauge invariant, see [3]. Later in Secs. III and IV, we will show that for the gravitational instability and in the static equilibrium limit, analyses are possible without imposing the gauge condition. This implies that these two analyses are naturally gauge invariant.

A. 1PN hydrodynamic equations

The hydrodynamic conservation equations and Einstein equation to 1PN order give

$$\dot{\varrho} + 3H\varrho + \frac{1}{a}(\varrho v^i)_{,i} + \frac{1}{c^2}\left\{(\varrho\Pi + \varrho v^2)^{\cdot} + 3H(\varrho\Pi + p) + \frac{1}{a}\left[\varrho v^i(\Pi + v^2 - 3U) + \varrho P^i + pv^i + Q^i + \Pi^i_j v^j\right]_{,i} + \varrho\left(3\dot{U} + \frac{2}{a}U_{,i}v^i + 4Hv^2\right)\right\} = 0, \quad (3)$$

$$
\begin{aligned}
&\frac{1}{a^4}(a^4 \varrho v_i)^\cdot + \frac{1}{a}(\varrho v^j v_i + p\delta_i^j + \Pi_i^j)_{,j} - \frac{1}{a}\varrho U_{,i} + \frac{1}{c^2}\bigg\{\frac{1}{a^4}\{a^4[\varrho v_i(\Pi + v^2 + U) + p v_i + Q_i + \Pi_{ij}v^j]\}^\cdot \\
&\quad + \frac{1}{a}[\varrho v^j v_i(\Pi + v^2 - 2U) + \varrho P^j v_i + p v^j v_i + Q^j v_i + Q_i v^j - 2U\Pi_i^j]_{,j} \\
&\quad + 2\dot{U}\varrho v_i + \frac{2}{a}U_{,j}(\varrho v^j v_i + \Pi_i^j) - \frac{2}{a}\varrho \Upsilon_{,i} - \frac{1}{a}U_{,i}[\varrho(\Pi + 2v^2) + p] + \varrho v^j P_{j,i}\bigg\} = 0,
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&\frac{\Delta}{a^2}U + 4\pi G(\varrho - \varrho_b) + \frac{1}{c^2}\bigg\{2\frac{\Delta}{a^2}\Upsilon + 3\left(\ddot{U} + 3H\dot{U} + 2\frac{\ddot{a}}{a}U\right) - \frac{2}{a^2}U\Delta U \\
&\quad + \frac{1}{a^2}(aP^i{}_{,i})^\cdot + 8\pi G\left[\frac{1}{2}(\varrho\Pi - \varrho_b\Pi_b) + \varrho v^2 + \frac{3}{2}(p - p_b)\right]\bigg\} = 0,
\end{aligned} \tag{5}
$$

$$
(\dot{U} + HU)_{,i} + \frac{1}{4a}(P^k{}_{,ki} - \Delta P_i) = 4\pi G\varrho a v_i, \tag{6}
$$

where $H \equiv \dot{a}/a$. To the background order, we have Eqs. (B16) and (B25).

B. 1PN Schrödinger equations

The Klein transformation is [16]

$$
\phi \equiv \frac{\hbar}{\sqrt{2m}}(\psi e^{-i\omega_c t} + \psi^* e^{i\omega_c t}), \tag{7}
$$

where ϕ is a real scalar field and ψ is a complex wave function; $\omega_c \equiv mc^2/\hbar$ is the Compton frequency. The Schrödinger-Einstein equations to 1PN order give

$$
\begin{aligned}
&\left(\frac{\Delta}{a^2} - 8\pi\ell_s|\psi|^2\right)\psi + \frac{2im}{\hbar}\left(\dot{\psi} + \frac{3}{2}H\psi\right) + \frac{2m^2}{\hbar^2}U\psi + \frac{1}{c^2}\bigg[-\ddot{\psi} - 3H\dot{\psi} - 2U\frac{\Delta}{a^2}\psi \\
&\quad + \frac{2im}{\hbar}\left(2U\dot{\psi} + \frac{1}{a}P^i\psi_{,i}\right) + \frac{im}{\hbar}\left(\frac{1}{a}P^i{}_{,i} + 4\dot{U} + 6HU\right)\psi + \frac{2m^2}{\hbar^2}(2\Upsilon + U^2)\psi\bigg] = 0,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&\frac{\Delta}{a^2}U + 4\pi Gm(|\psi|^2 - |\psi_b|^2) + \frac{1}{c^2}\bigg\{2\frac{\Delta}{a^2}\Upsilon + 3\left(\ddot{U} + 3H\dot{U} + 2\frac{\ddot{a}}{a}U\right) - \frac{2}{a^2}U\Delta U + \frac{1}{a^2}(aP^i{}_{,i})^\cdot \\
&\quad + 8\pi G\frac{\hbar^2}{m}\left[-\frac{1}{2a^2}(\psi\Delta\psi^* + \psi^*\Delta\psi) + 6\pi\ell_s(|\psi|^4 - |\psi_b|^4)\right]\bigg\} = 0,
\end{aligned} \tag{9}
$$

$$
(\dot{U} + HU)_{,i} + \frac{1}{4a}(P^k{}_{,ki} - \Delta P_i) = 2\pi Gi\hbar(\psi\psi^*_{,i} - \psi^*\psi_{,i}). \tag{10}
$$

To the background order, we have Eq. (B16) with

$$
\varrho_b = m|\psi_b|^2, \qquad \varrho_b\Pi_b = 3p_b = \frac{6\pi\ell_s\hbar^2}{m}|\psi_b|^4, \qquad \dot{\psi}_b\psi_b^* + \psi_b\dot{\psi}_b^* + 3\frac{\dot{a}}{a}|\psi_b|^2\left(1 - \frac{4\pi\ell_s\hbar^2}{m^2c^2}|\psi_b|^2\right) = 0. \tag{11}
$$

C. 1PN Madelung equations

The Madelung transformation is [9]

$$
\psi \equiv \sqrt{\frac{\varrho}{m}}e^{imu/\hbar}. \tag{12}
$$

The Madelung-Einstein equations to 1PN order give

$$\dot{\varrho} + 3H\varrho + \frac{1}{a}\nabla\cdot(\varrho\mathbf{u}) + \frac{1}{c^2}\left[-(\varrho\dot{u})^{\cdot} - 3H\varrho\dot{u} - 4U\frac{1}{a}\nabla\cdot(\varrho\mathbf{u}) + 4\varrho\dot{U} + \frac{1}{a}\nabla\cdot(\varrho\mathbf{P})\right] = 0, \tag{13}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{a}\nabla U - \frac{\hbar^2}{2m^2}\frac{1}{a}\nabla\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right)$$
$$+ \frac{1}{c^2}\frac{1}{a}\nabla\left[-\frac{1}{2}\dot{u}^2 - 2\Upsilon + U^2 - 2U\mathbf{u}^2 + \mathbf{P}\cdot\mathbf{u} + \frac{\hbar^2}{2m^2}\left(\frac{\sqrt{\varrho}^{\cdot\cdot}}{\sqrt{\varrho}} + 3H\frac{\sqrt{\varrho}^{\cdot}}{\sqrt{\varrho}} + 4U\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - 2U\frac{8\pi\ell_s}{m}\varrho\right)\right] = 0, \tag{14}$$

$$\frac{\Delta}{a^2}U + 4\pi G(\varrho - \varrho_b) + \frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Upsilon + 3\left(\ddot{U} + 3H\dot{U} + 2\frac{\ddot{a}}{a}U\right) - \frac{2}{a^2}U\Delta U + \frac{1}{a^2}(a\nabla\cdot\mathbf{P})^{\cdot}\right.$$
$$\left. + 8\pi G\varrho\mathbf{u}^2 + 8\pi G\frac{\hbar^2}{m^2}\left[-\frac{1}{a^2}\sqrt{\varrho}\Delta\sqrt{\varrho} + \frac{6\pi\ell_s}{m}(\varrho^2 - \varrho_b^2)\right]\right\} = 0, \tag{15}$$

$$\nabla(\dot{U} + HU) + \frac{1}{4a}(\nabla\nabla\cdot\mathbf{P} - \Delta\mathbf{P}) = 4\pi G\varrho a\mathbf{u}, \tag{16}$$

where we defined $\mathbf{u} \equiv \frac{1}{a}\nabla u$. The original equation of Eq. (14) is

$$\dot{u} - U + \frac{1}{2}\mathbf{u}^2 - \frac{\hbar^2}{2m^2}\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right)$$
$$+ \frac{1}{c^2}\left[-\frac{1}{2}\dot{u}^2 - 2\Upsilon + U^2 - 2U\mathbf{u}^2 + \mathbf{P}\cdot\mathbf{u} + \frac{\hbar^2}{2m^2}\left(\frac{\sqrt{\varrho}^{\cdot\cdot}}{\sqrt{\varrho}} + 3H\frac{\sqrt{\varrho}^{\cdot}}{\sqrt{\varrho}} + 4U\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - 2U\frac{8\pi\ell_s}{m}\varrho\right)\right] = 0. \tag{17}$$

The relation between u and fluid velocity v_i, introduced in the four-vector in Eq. (B7), is given in Eq. (B43). To the background order, we have Eq. (B16) with

$$\varrho_b\Pi_b = 3p_b = \frac{6\pi\ell_s\hbar^2}{m^3}\varrho_b^2, \qquad \dot{\varrho}_b + 3\frac{\dot{a}}{a}\varrho_b\left(1 - \frac{4\pi\ell_s\hbar^2}{m^3c^2}\varrho_b\right) = 0. \tag{18}$$

D. 0PN equations

To 0PN order, we take the $c \to \infty$ limit. For a hydrodynamic fluid, the mass and momentum conservation equations and Poisson's equation are

$$\dot{\varrho} + 3H\varrho + \frac{1}{a}\nabla\cdot(\varrho\mathbf{v}) = 0, \tag{19}$$

$$\frac{1}{a^4}(a^4\varrho\mathbf{v})^{\cdot} + \frac{1}{a}\left(\varrho v^j v_i + p\delta_i^j + \Pi_i^j\right)_{,j} - \frac{1}{a}\varrho\nabla U = 0, \tag{20}$$

$$\frac{\Delta}{a^2}U = -4\pi G(\varrho - \varrho_b). \tag{21}$$

For the axion, the Schrödinger-Poisson's equations are

$$i\hbar\left(\dot{\psi} + \frac{3}{2}H\psi\right) = -\frac{\hbar^2}{2m}\left(\frac{\Delta}{a^2} - 8\pi\ell_s|\psi|^2\right)\psi - mU\psi, \tag{22}$$

$$\frac{\Delta}{a^2}U = -4\pi Gm(|\psi|^2 - |\psi_b|^2). \tag{23}$$

For the axion, the Madelung conservation equations with Poisson's equation are

$$\dot{\varrho} + 3H\varrho + \frac{1}{a}\nabla\cdot(\varrho\mathbf{u}) = 0, \tag{24}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{a}\nabla U$$
$$- \frac{\hbar^2}{2m^2}\frac{1}{a}\nabla\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right) = 0, \tag{25}$$

$$\frac{\Delta}{a^2}U = -4\pi G(\varrho - \varrho_b). \tag{26}$$

These equations are valid to fully nonlinear order.

III. GRAVITATIONAL INSTABILITY

We consider linear perturbations in the cosmological context. We set

$$\varrho \to \varrho + \delta\varrho = \varrho(1+\delta), \qquad \Pi \to \Pi + \delta\Pi,$$
$$p \to p + \delta p, \qquad \psi \to \psi + \delta\psi. \tag{27}$$

The other variables are already perturbed order; we ignore the self-interaction term. Here we keep only to linear order in perturbation variables, and in case clarification is needed we indicate the background order variables with a subindex b, like ϱ_b, etc.

A. Hydrodynamics

To the linear order perturbation, Eqs. (3)–(5) give

$$\left[\varrho\left(1+\frac{\Pi}{c^2}\right)\right]^{\cdot} + 3\frac{\dot{a}}{a}\left[\varrho\left(1+\frac{\Pi}{c^2}\right)+\frac{p}{c^2}\right] + \frac{1}{a}\nabla\cdot\left\{\left[\varrho\left(1+\frac{\Pi}{c^2}\right)+\frac{p}{c^2}\right]\mathbf{v}+\frac{1}{c^2}\varrho\mathbf{P}\right\}+\frac{3}{c^2}\varrho\dot{U} = 0, \tag{28}$$

$$\frac{1}{a^4}\left\{a^4\left\{\left[\varrho\left(1+\frac{\Pi}{c^2}\right)+\frac{p}{c^2}\right]\mathbf{v}+\frac{1}{c^2}\mathbf{Q}\right\}\right\}^{\cdot}+\frac{1}{a}\nabla p + \frac{1}{a}\Pi^j_{i,j} - \frac{1}{a}\nabla\left\{\left[\varrho\left(1+\frac{\Pi}{c^2}\right)+\frac{p}{c^2}\right]U+\frac{2}{c^2}\varrho\Upsilon\right\} = 0, \tag{29}$$

$$\frac{\Delta}{a^2}\left(U+\frac{2}{c^2}\Upsilon\right) + 4\pi G\left[\varrho\left(1+\frac{\Pi}{c^2}\right)+\frac{3p}{c^2}-\varrho_b\left(1+\frac{\Pi_b}{c^2}\right)-\frac{3p_b}{c^2}\right] + \frac{1}{c^2}\left[3\ddot{U}+9\frac{\dot{a}}{a}\dot{U}+6\frac{\ddot{a}}{a}U+\frac{1}{a^2}(a\nabla\cdot\mathbf{P})^{\cdot}\right] = 0. \tag{30}$$

Subtracting the background equation, we can derive density perturbation equation to 1PN order

$$\ddot{\delta}_\mu + 2\frac{\dot{a}}{a}\dot{\delta}_\mu - \frac{4\pi G\mu}{c^2}\delta_\mu - \frac{c^2}{a^2\mu}(\Delta\delta p + \Pi^{ij}{}_{,ij}) + \frac{1}{c^2}\left\{\frac{3}{a^2}\left[a^2H\left(\frac{\delta p}{\varrho}-\frac{p}{\varrho}\delta\right)\right]^{\cdot} - 3\frac{\dot{a}}{a}\frac{p}{\varrho}\dot{\delta} - 4\pi G\varrho\left(3\frac{\delta p}{\varrho}-\frac{p}{\varrho}\delta\right) + \frac{12\pi G\varrho a^2}{\Delta}\left[\frac{\dot{a}}{a}\dot{\delta}+\left(2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)\delta\right] - \frac{1}{a^5\varrho}(a^4\nabla\cdot\mathbf{Q})^{\cdot}\right\} = 0, \tag{31}$$

where we used $\delta \equiv \delta\varrho/\varrho$ and $\delta_\mu \equiv \delta\mu/\mu = \delta + \delta\Pi/c^2$ which follows from $\mu \equiv \varrho(c^2+\Pi)$; in this way, Π_b and $\delta\Pi$ are absorbed to μ and δ_μ. In deriving this equation, as P_i cancels we do not need to impose the gauge condition and Eq. (6) is not used. Thus, Eq. (31) is naturally gauge invariant. This is a density perturbation equation valid to 1PN order; p, Π_{ij}, and Q_i are provided by specifying the nature of the fluid; see below for the axion case.

The Jeans scale dividing the gravity and pressure dominating scales can be derived by setting $p = \delta\Pi = \Pi_{ij} = Q_i = 0$. In Fourier space with $\Delta = -k^2$ and introducing the sound velocity as $\delta p/\varrho \equiv v_s^2\delta$, by setting the coefficients of δ terms equal to zero, assuming constant v_s, we can show

$$\frac{k_{\rm J}}{a} = \frac{\sqrt{4\pi G\varrho}}{v_s}\left(1-\frac{1}{\Omega}\frac{v_s^2}{c^2}\right), \tag{32}$$

where $\Omega \equiv 8\pi G\varrho/(3H^2)$. As we consider a general fluid, the axion can be regarded as a fluid. In order to derive the axion-Jeans scale to be derived later, we need to properly include the nonvanishing $\delta\Pi$ and $\dot{v}_s$, see below Eq. (41).

For a zero-pressure ideal fluid ($Q_i = 0 = \Pi_{ij}$), we have

$$\ddot{\delta}_\mu + 2\frac{\dot{a}}{a}\dot{\delta}_\mu - \frac{4\pi G\mu}{c^2}\delta_\mu + \frac{12\pi G\varrho a^2}{\Delta c^2}\left[\frac{\dot{a}}{a}\dot{\delta}+\left(2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)\delta\right] = 0. \tag{33}$$

In the absence of internal energy we replace $\delta_\mu \to \delta$ and $\mu/c^2 \to \varrho$. This was derived in Eq. (25) of [19]; comparison with relativistic linear perturbation was made in that paper. Equation (33) coincides with the 1PN limit of relativistic density perturbation equation in zero-shear gauge and uniform-expansion gauge both of which show proper Newtonian limit in the subhorizon limit for density and velocity perturbations and gravitational potential [20]. The PN correction terms become important near the horizon scale while the PN expansion is reliable in the subhorizon scale. In the comoving gauge and the synchronous gauge, Eq. (33) *without* the PN correction is exactly valid in all scales [21,22]. However, in these two gauge conditions we cannot properly identify the gravitational potential.

B. Axion hydrodynamics

To the background order, Eqs. (13) and (17) give

$$\dot{\varrho} + 3H\varrho = 0, \tag{34}$$

$$\dot{u} = -\frac{\hbar^2}{2m^2c^2}\left(\frac{\ddot{\sqrt{\varrho}}}{\sqrt{\varrho}}+3H\frac{\dot{\sqrt{\varrho}}}{\sqrt{\varrho}}\right) = \frac{3\hbar^2}{4m^2c^2}\left(\dot{H}+\frac{3}{2}H^2\right) = \frac{3\hbar^2}{8m^2}\Lambda. \tag{35}$$

Thus, $\varrho \propto a^{-3}$.

To the linear order perturbation, Eqs. (13)–(15) using Eq. (17) give

$$\dot{\delta}+\frac{1}{a}\nabla\cdot\mathbf{u}+\frac{1}{c^2}\left[3\dot{U}+\frac{1}{a}\nabla\cdot\mathbf{P}-\frac{\hbar^2\Delta}{4m^2a^2}\left(\dot{\delta}-2\frac{\dot{a}}{a}\delta\right)\right]=0, \tag{36}$$

$$\dot{\mathbf{u}}+\frac{\dot{a}}{a}\mathbf{u}-\frac{1}{a}\nabla U-\frac{\hbar^2\Delta}{4m^2a^3}\nabla\delta+\frac{1}{c^2}\frac{1}{a}\nabla\left(-2\Upsilon+\frac{\hbar^2}{4m^2}\ddot{\delta}\right)=0, \tag{37}$$

$$\frac{\Delta}{a^2}U+4\pi G\varrho\delta+\frac{1}{c^2}\left[2\frac{\Delta}{a^2}\Upsilon+3\ddot{U}+9\frac{\dot{a}}{a}\dot{U}+6\frac{\ddot{a}}{a}U+\frac{1}{a^2}(a\nabla\cdot\mathbf{P})^{\cdot}-4\pi G\varrho\frac{\hbar^2\Delta}{m^2a^2}\delta\right]=0. \tag{38}$$

Equation (16) is not needed. Without imposing temporal gauge condition we can derive

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}-4\pi G\varrho\delta+\frac{\hbar^2\Delta^2}{4m^2a^4}\delta+\frac{1}{c^2}\left\{\frac{12\pi G\varrho a^2}{\Delta}\left[\frac{\dot{a}}{a}\dot{\delta}+\left(2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)\delta\right]+\frac{\hbar^2\Delta}{2m^2a^2}\left(3\frac{\dot{a}}{a}\dot{\delta}+\frac{\hbar^2\Delta^2}{4m^2a^4}\delta\right)\right\}=0. \tag{39}$$

These equations can also be derived from Eqs. (28)–(31) by using the axion fluid quantities; to the linear order, ignoring the self-interaction terms, Eq. (B44) gives

$$\Pi_b=0,\qquad \frac{p_b}{\varrho_b}=-\frac{3\hbar^2}{4m^2c^2}\dot{H},$$
$$\frac{\delta p}{\varrho}=-\frac{\hbar^2\Delta}{4m^2a^2}\delta-\frac{\hbar^2}{4m^2c^2}(\ddot{\delta}+3H\dot{\delta}+3\dot{H}\delta),$$
$$\delta\Pi=-\frac{\hbar^2\Delta}{4m^2a^2}\delta,\qquad Q_i=0=\Pi_{ij}. \tag{40}$$

The background pressure p_b appearing with the c^{-2} factor does not have any role to the 1PN order. Compared with the zero-pressure ideal fluid in Eq. (33), the axion equation in Eq. (39) differs only in the quantum stress (with $\hbar^2$) terms appearing in both 0PN and 1PN orders.

From the coefficients of δ terms in Eq. (39), setting $\Delta\to -k^2$, we have the Jeans wave number

$$\frac{k_{\rm J}}{a}=2\sqrt{\frac{\sqrt{\pi G\varrho}m}{\hbar}}\left[1-\left(1-\frac{1}{2\Omega}\right)\frac{\sqrt{\pi G\varrho}\hbar}{mc^2}\right]. \tag{41}$$

Thus, the 1PN correction is of $\sqrt{G\varrho}/\omega_c\sim H/\omega_c=\hbar H/(mc^2)$ order. To 1PN order this differs from the fluid case in Eq. (32). As mentioned below Eq. (32), we can derive this result directly from Eq. (31) using Eq. (40) and

$$v_s^2=\frac{\hbar^2}{4m^2}\left(\frac{k^2}{a^2}+\frac{12\pi G\varrho}{c^2}\right). \tag{42}$$

The evolution of axion density perturbation in the non-relativistic limit was studied in [23].

C. Schrödinger formulation

From the Madelung transformation, we have

$$\varrho=m|\psi|^2,\qquad u=\frac{\hbar}{2im}\ln(\psi/\psi^*), \tag{43}$$

to the background, and

$$\delta=\frac{\delta\psi}{\psi}+\frac{\delta\psi^*}{\psi^*},\qquad \frac{2im}{\hbar}\delta u=\frac{\delta\psi}{\psi}-\frac{\delta\psi^*}{\psi^*}, \tag{44}$$

to perturbations. To the background order, Eq. (8) gives

$$\dot{\psi}+\frac{3}{2}\frac{\dot{a}}{a}\psi-\frac{\hbar}{2imc^2}\left(\ddot{\psi}+3\frac{\dot{a}}{a}\dot{\psi}\right)=0. \tag{45}$$

Using

$$\frac{\delta\psi}{\psi}=\frac{1}{2}\delta+\frac{im}{\hbar}\delta u, \tag{46}$$

the imaginary and real parts of Eq. (8) give

$$\dot{\delta}+\frac{\Delta}{a^2}\delta u+\frac{1}{c^2}\left[3\dot{U}+\frac{1}{a}P^i{}_{,i}-\frac{\hbar^2\Delta}{4m^2a^2}\left(\dot{\delta}-2\frac{\dot{a}}{a}\delta\right)\right]=0, \tag{47}$$

$$\delta\dot{u}-U-\frac{\hbar^2\Delta}{4m^2a^2}\delta+\frac{1}{c^2}\left(\frac{\hbar^2}{4m^2}\ddot{\delta}-2\Upsilon\right)=0. \tag{48}$$

Using $\mathbf{u}\equiv\frac{1}{a}\nabla u$, we have Eqs. (36) and (37), and Eq. (9) gives Eq. (38).

IV. STATIC LIMIT

We consider the static limit with $\mathbf{v}=\dot{\varrho}=0$, etc., in Minkowski background, thus $a\equiv 1$, $\Lambda=\varrho_b=0$, etc. We ignore the self-interaction.

A. Hydrodynamics

For the fluid, without imposing the gauge condition, Eqs. (4) and (5) give

$$\nabla\left(U+\frac{1}{c^2}2\Upsilon\right)=\frac{1}{\varrho(1+\frac{1}{c^2}\Pi)+\frac{p}{c^2}}\left[\nabla p+\left(1-\frac{1}{c^2}2U\right)\Pi^j_{i,j}\right], \tag{49}$$

$$\Delta\left(U+\frac{1}{c^2}2\Upsilon\right)=-4\pi G\left[\varrho\left(1+\frac{1}{c^2}\Pi+\frac{1}{c^2}2U\right)+3\frac{p}{c^2}\right]. \tag{50}$$

Combining the above equations, we have

$$\nabla\cdot\left(\frac{\nabla p+(1-\frac{1}{c^2}2U)\Pi^j_{i,j}}{\mu+p}\right)=-\frac{4\pi G}{c^4}\frac{\mu+3p}{1-\frac{1}{c^2}2U}, \tag{51}$$

which is consistent with the Oppenheimer-Volkoff equation in the spherically symmetric case to the 1PN order [24]. As P_i disappears in the above equations, all variables are naturally gauge invariant, and we do not need Eqs. (3) and (6) which give $(\varrho P^i)_{,i}=0$ and $P^k{}_{,ki}=\Delta P_i$ to 1PN order.

B. Axion hydrodynamics

For an axion, combining Eqs. (14) and (15), we have

$$\frac{\hbar^2\Delta}{2m^2}\left[\left(1+\frac{3\hbar^2}{2m^2c^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right)\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right]=4\pi G\varrho\left(1-\frac{3\hbar^2}{m^2c^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right). \tag{52}$$

This is valid independently of the gauge condition, and also follows from Eq. (51) using the fluid quantities in Eq. (B44).

By setting $\Delta\sqrt{\varrho}/\sqrt{\varrho}\to -k^2/2$ (1/2-factor to match with the Jeans scale) we have the equilibrium wave number

$$k_{\rm EQ}=2\sqrt{\frac{\sqrt{\pi G\varrho}m}{\hbar}}\left(1+\frac{9}{4}\frac{\sqrt{\pi G\varrho}\hbar}{mc^2}\right). \tag{53}$$

This can be compared with Jeans wave number in Eq. (41).

C. Schrödinger formulation

For an axion, combining the real part of Eq. (8) and Eq. (9), we have

$$\frac{\hbar^2\Delta}{2m^2}\left[\left(1+\frac{3\hbar^2}{2m^2c^2}\frac{\Delta\psi}{\psi}\right)\frac{\Delta\psi}{\psi}\right]=4\pi G\varrho\left(1-\frac{3\hbar^2}{m^2c^2}\frac{\Delta\psi}{\psi}\right), \tag{54}$$

which is valid independently of the gauge condition. The imaginary part gives $(P^i\psi^2)_{,i}=0$. For $u_{,i}=0$, we have $\Delta\psi/\psi=\Delta\sqrt{\varrho}/\sqrt{\varrho}$, and Eq. (54) leads to Eq. (52).

V. DISCUSSION

The PN approximation, being weakly relativistic but fully nonlinear, provides a complementary method to the relativistic perturbation theory which is fully relativistic but weakly nonlinear. Here we presented complete sets of 1PN approximation equations for a general imperfect fluid and an axion in the cosmological context. In the axion case we present the Schrödinger and Madelung hydrodynamic formulations where PN expansions are possible. All PN formulations are derived without fixing the temporal gauge condition.

The complete sets of equations for the three formulations are summarized in Sec. II. Detailed derivations are presented in two appendixes; these include the covariant formulations and the 1PN approximations for the axion. As applications we studied the gravitational instability and a static limit of the 1PN formulations. We analyzed these two cases without imposing the gauge condition, thus results are naturally gauge invariant, see Secs. III and IV.

In [8] we derived a relativistic axion density perturbation equation valid to fully nonlinear order by using the fully nonlinear and exact perturbation formulation made for a fluid [25]. We took the axion-comoving gauge setting time average of the longitudinal part of T^0_i equal to zero; we note that although we have not imposed the temporal gauge condition in our study of the PN order gravitational instability in Sec. III, the PN approximation does not allow the comoving gauge condition which implies vanishing perturbed lapse function, α (the Newtonian gravitational potential) in Eq. (B1), for a zero-pressure medium. In that study, by *assuming* $H/\omega_c\ll 1$ we arrived at the same equation known in a nonrelativistic limit except for relativistic contributions from the metric. By strictly ignoring the H/ω_c higher order term, which is $\hbar H/(mc^2)$ thus 1PN order, [8] has derived the nonrelativistic limit of the axion part.

In this work we presented a consistent 1PN extension for both gravity and axion parts. The nonrelativistic (0PN) and 1PN approximations of hydrodynamics as limits of the relativistic fully nonlinear perturbation formulation were presented in [26]. Thus, we derived 1PN approximation as the leading relativistic correction to the classical axion field. A complementary approach using an effective field theory for the nonrelativistic regime of scalar field models was studied in [27].

Here we treated the axion as a classical scalar field. Regarding the axion as a quantum scalar field, however, in [6,28] it was suggested that the axion fluid thermalizes by gravitational self-interactions and forms a Bose-Einstein condensate with cosmologically long-range correlation and galactic scale observational consequences. There are some conflicting ideas in the literature concerning the issue [29].

The PN approximation can be applied to situations where all relativistic effects are small but not negligible. The 1PN equations are fully nonlinear, and the equations are designed so that the relativistic effects appear as the PN correction terms in the more familiar Newtonian hydrodynamic equations. Thus, the PN formulation is easier for numerical simulations compared with the full-blown numerical relativity. By setting $a\equiv 1$, ignoring the background fluid quantities and Λ, the formulations are valid in the Minkowski background. The 1PN equations are generally valid and can be applied to any astrophysical system where a single component fluid or axion is dominating. Extension to multicomponent fluid in combination with axion is trivial; see [30] for multicomponent fluids and scalar fields.

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APPENDIX A: COVARIANT FORMULATION

We present covariant equations of the axion under the Klein and Madelung transformations. Although both transformations are applicable in the nonrelativistic limit, here we apply these in the relativistic and covariant level. The covariant $(1+3)$ equations for a general fluid can be found in [31]. In the case of an axion, what we need are fluid quantities of the axion and the equation of motion replacing (or complementing) the energy and momentum conservation equations. We will present these for the Schrödinger and Madelung formulations of the axion.

1. Scalar field

The fluid quantities are introduced based on a timelike four-vector u_a, normalized with $u^a u_a \equiv -1$, as [31]

$$T_{ab} = \mu u_a u_b + p(g_{ab} + u_a u_b) + q_a u_b + q_b u_a + \pi_{ab}, \tag{A1}$$

where μ, p, q_a. and π_{ab} are the energy density, pressure, energy flux, and anisotropic stress, respectively, with $q_a u^a \equiv 0 \equiv \pi_{ab} u^b$, $\pi_{ab} = \pi_{ba}$, and $\pi^a_a \equiv 0$. Thus, we have

$$\mu = T_{ab} u^a u^b, \qquad p = \frac{1}{3} T_{ab} h^{ab}, \qquad q_a = -T_{cd} u^c h^d_a,$$
$$\pi_{ab} = T_{cd} h^c_a h^d_b - p h_{ab}, \tag{A2}$$

where $h_{ab} \equiv g_{ab} + u_a u_b$ is the spatial-projection tensor. The fluid quantities have 13 independent components (μ, p, three u_a, three q_a, and five π_{ab}), whereas T_{ab} needs only 10 independent components. Thus, we can freely impose three frame conditions without any physical constraint. Often used ones are the normal-frame setting $u_i \equiv 0$, thus $u_a = n_a$, and the energy-frame setting $q_i \equiv 0$, thus $q_a = 0$.

The energy and the momentum conservation equations follow from $u_a T^{ab}{}_{;b} = 0$ and $h^c_a T^{ab}{}_{;b} = 0$, respectively,

$$\tilde{\dot{\mu}} + (\mu + p)\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + q^a a_a = 0, \tag{A3}$$

$$(\mu + p) a_a + h^b_a (p_{,b} + \pi^c_{b;c} + \tilde{\dot{q}}_b) + \left(\omega_{ab} + \sigma_{ab} + \frac{4}{3}\theta h_{ab}\right) q^b = 0, \tag{A4}$$

where $\tilde{\dot{\mu}} \equiv \mu_{,c} u^c$; the expansion scalar ($\theta$), the acceleration vector (a_a), the rotation tensor (ω_{ab}), and the shear tensor (σ_{ab}) are introduced as

$$\theta \equiv u^a{}_{;a}, \quad a_a \equiv \tilde{\dot{u}}_a \equiv u_{a;b} u^b, \quad \sigma_{ab} \equiv \theta_{ab} - \frac{1}{3}\theta h_{ab},$$
$$h^c_a h^d_b u_{c;d} = h^c_{[a} h^d_{b]} u_{c;d} + h^c_{(a} h^d_{b)} u_{c;d} \equiv \omega_{ab} + \theta_{ab}, \tag{A5}$$

with $A_{[ab]} \equiv \frac{1}{2}(A_{ab} - A_{ba})$ and $A_{(ab)} \equiv \frac{1}{2}(A_{ab} + A_{ba})$.

We consider a minimally coupled scalar field in Einstein's gravity. We choose our convention in the Lagrangian density as

$$\mathcal{L} = \sqrt{-g}\left[\frac{c^4}{16\pi G}(R - 2\Lambda) - \frac{1}{2}\phi^{;c}\phi_{,c} - V(\phi) + L_{\rm m}\right], \tag{A6}$$

where $L_{\rm m}$ is the matter part Lagrangian and Λ is the cosmological constant. For the scalar field, the equation of motion and the energy-momentum tensor are

$$\Box\phi = V_{,\phi}, \tag{A7}$$

$$T_{ab} = \phi_{,a}\phi_{,b} - \left(\frac{1}{2}\phi^{;c}\phi_{,c} + V\right) g_{ab}. \tag{A8}$$

The fluid quantities in Eq. (A2) give

$$\mu = \frac{1}{2}\tilde{\dot{\phi}}^2 + V + \frac{1}{2} h^{ab}\phi_{,a}\phi_{,b},$$
$$p = \frac{1}{2}\tilde{\dot{\phi}}^2 - V - \frac{1}{6} h^{ab}\phi_{,a}\phi_{,b}, \qquad q_a = -\tilde{\dot{\phi}} h^b_a \phi_{,b},$$
$$\pi_{ab} = h^c_a \phi_{,c} h^d_b \phi_{,d} - \frac{1}{3} h_{ab} h^{cd}\phi_{,c}\phi_{,d}. \tag{A9}$$

The equation of motion in Eq. (A7) gives

$$\tilde{\ddot{\phi}} + \theta\tilde{\dot{\phi}} + V_{,\phi} - h^b_a (h^{ac}\phi_{,c})_{;b} - h^b_a \phi_{,b} a^a = 0. \tag{A10}$$

By taking the energy-frame condition, $q_a \equiv 0$, we have $h^b_a \phi_{,b} = 0$, thus $u_a = -\phi_{,a}/\tilde{\dot{\phi}}$, and the fluid quantities and the equation of motion are simplified as

$$\mu = \frac{1}{2}\tilde{\dot{\phi}}^2 + V, \qquad p = \frac{1}{2}\tilde{\dot{\phi}}^2 - V, \qquad \pi_{ab} = 0, \tag{A11}$$

$$\tilde{\ddot{\phi}} + \theta\tilde{\dot{\phi}} + V_{,\phi} = 0. \tag{A12}$$

Using the fluid quantities in Eq. (A11), we can show that Eq. (A3) gives Eq. (A12) and Eq. (A4) is naturally valid.

The energy-frame condition $h_a^b\phi_{,b} = 0$ imposed in the field, however, is *not* suitable for the axion in a coherent oscillation stage. In the axion case, as we take time-average, $h_i^b\phi_{,b} = 0$ is not necessarily the same as $\tilde{\dot{\phi}} h_i^b\phi_{,b} = 0$; for example, $\pi_{ab} \neq 0$ in the axion case [8]. The difference did not appear in the linear order perturbation studied in [7,32]. In general, we can use Eqs. (A7) and (A9), instead.

2. Relativistic Schrödinger formulation

The Klein transformation is in Eq. (7). We consider a scalar field potential [33]

$$V = \sum_{n=1} \frac{\lambda_{2n}}{2n}\phi^{2n} = \sum_{n=1} \frac{\lambda_{2n}}{2n}\frac{(2n)!}{(n!)^2}\left(\frac{\hbar^2|\psi|^2}{2m}\right)^n, \quad \text{(A13)}$$

where in the second step we used the Klein transformation and *ignored* (by time-averaging) oscillating terms. In the following we consider up to $n = 2$, thus

$$V = \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2 + \frac{1}{3}\frac{m^2}{\hbar^4}g\phi^4 = \frac{1}{2}mc^2|\psi|^2 + \frac{1}{2}g|\psi|^4, \quad \text{(A14)}$$

where we set $\lambda_2 \equiv \frac{m^2c^2}{\hbar^2}$, $\lambda_4 \equiv \frac{4m^2}{3\hbar^4}g$, and $g \equiv \frac{4\pi\ell_s\hbar^2}{m}$.

The Klein-Gordon equation and the energy-momentum tensor become [34]

$$0 = \Box\phi - V_{,\phi} = \frac{\hbar}{\sqrt{2m}}\left\{e^{-imc^2t/\hbar}\left[\Box\psi - \frac{2imc}{\hbar}g^{0c}\psi_{,c} + \frac{imc}{\hbar}g^{ab}\Gamma^0_{ab}\psi - \frac{m^2c^2}{\hbar^2}(g^{00}+1)\psi - \frac{2m}{\hbar^2}g|\psi|^2\psi\right] + \text{c.c.}\right\}, \quad \text{(A15)}$$

$$T_{ab} = mc^2|\psi|^2\delta^0_a\delta^0_b + ic\hbar(\psi_{,(a}\delta^0_{b)}\psi^* - \psi^*_{,(a}\delta^0_{b)}\psi) + \frac{\hbar^2}{m}\psi_{,(a}\psi^*_{,b)}$$
$$- \frac{1}{2}g_{ab}\left[(1+g^{00})mc^2|\psi|^2 + ic\hbar(\psi^{;0}\psi^* - \psi^{*;0}\psi) + \frac{\hbar^2}{m}\psi^{;c}\psi^*_{,c} + g|\psi|^4\right], \quad \text{(A16)}$$

where we *ignored* oscillating terms in T_{ab}, and used

$$V_{,\phi} = \frac{\partial\psi}{\partial\phi}V_{,\psi} + \frac{\partial\psi^*}{\partial\phi}V_{,\psi^*}. \quad \text{(A17)}$$

These can be derived in action formulation. The field part of Lagrangian is

$$\mathcal{L} = -\sqrt{-g}\left[\frac{1}{2}\phi^{;c}\phi_{,c} + V(\phi)\right] = -\sqrt{-g}\left[\frac{1}{2}mc^2g^{00}|\psi|^2 + \frac{1}{2}ic\hbar(\psi^{;0}\psi^* - \psi^{*;0}\psi) + \frac{\hbar^2}{2m}\psi^{;c}\psi^*_{,c} + V(\psi,\psi^*)\right]. \quad \text{(A18)}$$

Variations with respect to ϕ, ψ, ψ^*, and g_{ab}, with $\delta\mathcal{L} = \frac{1}{2}\sqrt{-g}T^{ab}\delta g_{ab}$, lead to Eqs. (A7), (A8), (A15), and (A16).

Thus, the equation of motion becomes

$$\Box\psi - cg^{0c}\frac{2im}{\hbar}\psi_{,c} + cg^{ab}\Gamma^0_{ab}\frac{im}{\hbar}\psi - c^2(g^{00}+1)\frac{m^2}{\hbar^2}\psi - 8\pi\ell_s|\psi|^2\psi = 0. \quad \text{(A19)}$$

This is a Schrödinger equation in the relativistic form. More properly, it is the Klein-Gordon equation written in terms of ψ, and in the absence of the self-interaction term, it leads to the Schrödinger equation in the nonrelativistic limit with $c \to \infty$, see below.

Using Eqs. (A2) and (A16), the fluid quantities become

$$\mu = m|\psi|^2c^2\left[u^0u^0 + \frac{1}{2}(g^{00}+1)\right] + i\hbar c\left(u^0u^c + \frac{1}{2}g^{0c}\right)(\psi_{,c}\psi^* - \psi^*_{,c}\psi) + \frac{\hbar^2}{m}\left(|\psi_{,c}u^c|^2 + \frac{1}{2}\psi^{;c}\psi^*_{,c} + 2\pi\ell_s|\psi|^4\right),$$

$$p = \frac{1}{3}m|\psi|^2c^2\left[u^0u^0 - \frac{1}{2}(g^{00}+3)\right] + i\hbar\frac{1}{3}c\left(u^0u^c - \frac{1}{2}g^{0c}\right)(\psi_{,c}\psi^* - \psi^*_{,c}\psi) + \frac{1}{3}\frac{\hbar^2}{m}\left(|\psi_{,c}u^c|^2 - \frac{1}{2}\psi^{;c}\psi^*_{,c} - 6\pi\ell_s|\psi|^4\right),$$

$$q_a = -m|\psi|^2c^2u^0(u^0u_a + ct_{,a}) - i\hbar c\left[\frac{1}{2}(\psi_{,a}\psi^* - \psi^*_{,a}\psi)u^0 + (\psi_{,c}\psi^* - \psi^*_{,c}\psi)u^c\left(u^0u_a + \frac{1}{2}ct_{,a}\right)\right]$$
$$- \frac{\hbar^2}{m}\left[|\psi_{,c}u^c|^2u_a + \frac{1}{2}(\psi_{,a}\psi^*_{,c} + \psi^*_{,a}\psi_{,c})u^c\right],$$

$$\pi_{ab} = m|\psi|^2 c^2 \left[(u^0 u_a + ct_{,a})(u^0 u_b + ct_{,b}) - \frac{1}{3}(g_{ab} + u_a u_b)(g^{00} + u^0 u^0) \right]$$
$$+ i\hbar \left[c(u^0 u_{(a} + ct_{,(a)})(\psi_{,b)}\psi^* - \psi^*_{,b)}\psi) + c(u^0 u_{(a} + ct_{,(a)} u_{b)})(\psi_{,c}\psi^* - \psi^*_{,c}\psi) u^c \right.$$
$$\left. - \frac{1}{3}(g_{ab} + u_a u_b) c (g^{0c} + u^0 u^c)(\psi_{,c}\psi^* - \psi^*_{,c}\psi) \right]$$
$$+ \frac{\hbar^2}{m} \left[(\psi_{,(a} + \psi_{,c} u^c u_{(a})(\psi^*_{,b)} + \psi^*_{,d} u^d u_{b)}) - \frac{1}{3}(g_{ab} + u_a u_b)(\psi^{;c}\psi^*_{,c} + |\psi_{,c} u^c|^2) \right]. \quad \text{(A20)}$$

The energy-frame condition, $q_a \equiv 0$, gives

$$m|\psi|^2 c^2 u^0 (u^0 u_a + ct_{,a}) = -i\hbar c \left[\frac{1}{2}(\psi_{,a}\psi^* - \psi^*_{,a}\psi) u^0 + (\psi_{,c}\psi^* - \psi^*_{,c}\psi) u^c \left(u^0 u_a + \frac{1}{2} ct_{,a} \right) \right]$$
$$- \frac{\hbar^2}{m} \left[|\psi_{,c} u^c|^2 u_a + \frac{1}{2}(\psi_{,a}\psi^*_{,c} + \psi^*_{,a}\psi_{,c}) u^c \right]. \quad \text{(A21)}$$

Contracting with $ct^{;a} = g^{ab} ct_{,b} = g^{0a}$, we have

$$m|\psi|^2 c^2 u^0 (u^0 u^0 + g^{00}) = -i\hbar c \left[\frac{1}{2}(g^{0c} u^0 + g^{00} u^c) + u^0 u^0 u^c \right] (\psi_{,c}\psi^* - \psi^*_{,c}\psi)$$
$$- \frac{\hbar^2}{m} \left[|\psi_{,c} u^c|^2 u^0 + \frac{1}{2} g^{0c} (\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d}) u^d \right]. \quad \text{(A22)}$$

We have $ct_{,a} = \delta^0_a$, thus $u^0 u_a + ct_{,a} = h^0_a$ and $u^0 u^0 + g^{00} = h^{00}$ are components of the spatial projection tensor. The Schrödinger equation (A19) gives

$$m|\psi|^2 c^2 (g^{00} + 1) = -i\hbar c g^{0c} (\psi_{,c}\psi^* - \psi^*_{,c}\psi) + \frac{\hbar^2}{2m} [(\Box\psi)\psi^* + (\Box\psi^*)\psi - 16\pi\ell_s |\psi|^4]. \quad \text{(A23)}$$

Using these the fluid quantities become

$$\mu = m|\psi|^2 c^2 + i\hbar \frac{1}{2}\frac{c}{u^0}(g^{0c} u^0 - g^{00} u^c)(\psi_{,c}\psi^* - \psi^*_{,c}\psi)$$
$$+ \frac{\hbar^2}{2m} \left\{ \psi^{;c}\psi^*_{,c} - g^{0c}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d}) \frac{u^d}{u^0} - \frac{1}{2}[(\Box\psi)\psi^* + (\Box\psi^*)\psi] + 12\pi\ell_s |\psi|^4 \right\},$$
$$p = i\hbar \frac{1}{6}\frac{c}{u^0}(g^{0c} u^0 - g^{00} u^c)(\psi_{,c}\psi^* - \psi^*_{,c}\psi)$$
$$- \frac{\hbar^2}{6m} \left\{ \psi^{;c}\psi^*_{,c} + g^{0c}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d}) \frac{u^d}{u^0} + \frac{3}{2}[(\Box\psi)\psi^* + (\Box\psi^*)\psi] - 12\pi\ell_s |\psi|^4 \right\}. \quad \text{(A24)}$$

As we have

$$g^{0c} u^0 - g^{00} u^c = (g^{0c} + u^0 u^c) u^0 - (g^{00} + u^0 u^0) u^c, \qquad \text{etc.}, \quad \text{(A25)}$$

using Equations (A21) and (A22), we finally have

$$\mu = m|\psi|^2 c^2 + \frac{\hbar^2}{4m}\left\{2\psi^{;c}\psi^*_{,c} - 2g^{0c}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})\frac{u^d}{u^0} - [(\Box\psi)\psi^* + (\Box\psi^*)\psi] + 24\pi\ell_s|\psi|^4\right\}$$
$$+ \frac{\hbar^2}{4m}\frac{1}{|\psi|^2}\left\{(\psi^{;c}\psi^* - \psi^{*;c}\psi)(\psi_{,c}\psi^* - \psi^*_{,c}\psi) - g^{00}\left[(\psi_{,c}\psi^* - \psi^*_{,c}\psi)\frac{u^c}{u^0}\right]^2\right\}$$
$$+ \frac{i\hbar^3}{4m^2}\frac{1}{c|\psi|^2}\left[-(\psi^{;c}\psi^* - \psi^{*;c}\psi)(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})\frac{u^d}{u^0} + g^{0c}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})\frac{u^d}{u^0}(\psi_{,e}\psi^* - \psi^*_{,e}\psi)\frac{u^e}{u^0}\right],$$
$$p = -\frac{\hbar^2}{6m}\left\{\psi^{;c}\psi^*_{,c} + g^{0c}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})\frac{u^d}{u^0} + \frac{3}{2}[(\Box\psi)\psi^* + (\Box\psi^*)\psi] - 12\pi\ell_s|\psi|^4\right\}$$
$$+ \frac{\hbar^2}{12m}\frac{1}{|\psi|^2}\left\{(\psi^{;c}\psi^* - \psi^{*;c}\psi)(\psi_{,c}\psi^* - \psi^*_{,c}\psi) - g^{00}\left[(\psi_{,c}\psi^* - \psi^*_{,c}\psi)\frac{u^c}{u^0}\right]^2\right\}$$
$$+ \frac{i\hbar^3}{12m^2}\frac{1}{c|\psi|^2}\left[-(\psi^{;c}\psi^* - \psi^{*;c}\psi)(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})\frac{u^d}{u^0} + g^{0c}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})\frac{u^d}{u^0}(\psi_{,e}\psi^* - \psi^*_{,e}\psi)\frac{u^e}{u^0}\right],$$
$$\pi_{ab} = \frac{\hbar^2}{m}\left[(\psi_{,(a} + \psi_{,c}u^c u_{(a})(\psi^*_{,b)} + \psi^*_{,c}u^c u_{b)}) - \frac{1}{3}(g_{ab} + u_a u_b)(\psi^{;c}\psi^*_{,c} + |\psi_{,c}u^c|^2)\right]$$
$$+ \frac{\hbar^2}{m}\frac{1}{|\psi|^2}\left\{\frac{1}{4}(\psi_{,a}\psi^* - \psi^*_{,a}\psi)(\psi_{,b}\psi^* - \psi^*_{,b}\psi) + \frac{1}{2}(\psi_{,(a}\psi^* - \psi^*_{,(a}\psi)u_{b)}(\psi_{,c}\psi^* - \psi^*_{,c}\psi)u^c\right.$$
$$- \frac{1}{2}\left(u^0 u_{(a}\delta^0_{b)} + \frac{1}{2}\delta^0_a\delta^0_b\right)\left[(\psi_{,c}\psi^* - \psi^*_{,c}\psi)\frac{u^c}{u^0}\right]^2$$
$$\left.- \frac{1}{3}(g_{ab} + u_a u_b)\frac{1}{4}\left[(\psi^{;c}\psi^* - \psi^{*;c}\psi)(\psi_{,c}\psi^* - \psi^*_{,c}\psi) - g^{00}\left[(\psi_{,c}\psi^* - \psi^*_{,c}\psi)\frac{u^c}{u^0}\right]^2\right]\right\}$$
$$+ \frac{i\hbar^3}{m^2}\frac{1}{c(u^0)^2|\psi|^2}\left\{(u^0 u_{(a} + \delta^0_{(a})u_{b)}|\psi_{,c}u^c|^2 + \frac{1}{2}(u^0 u_{(a} + \delta^0_{(a})(\psi_{,b)}\psi^*_{,c} + \psi^*_{,b)}\psi_{,c})u^c\right.$$
$$\left.- \frac{1}{3}(g_{ab} + u_a u_b)\left[\frac{1}{2}g^{0c}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})u^d + |\psi_{,c}u^c|^2 u^0\right]\right\}(\psi_{,e}\psi^* - \psi^*_{,e}\psi)u^e$$
$$+ \frac{\hbar^4}{m^3}\frac{1}{c^2(u^0)^2|\psi|^2}\left\{\left[|\psi_{,c}u^c|^2 u_a + \frac{1}{2}(\psi_{,a}\psi^*_{,c} + \psi^*_{,a}\psi_{,c})u^c\right]\left[|\psi_{,c}u^c|^2 u_b + \frac{1}{2}(\psi_{,b}\psi^*_{,c} + \psi^*_{,b}\psi_{,c})u^c\right]\right.$$
$$\left.- \frac{1}{3}(g_{ab} + u_a u_b)\left[|\psi_{,e}u^e|^4 + \frac{1}{4}(\psi_{,c}\psi^*_{,d} + \psi^*_{,c}\psi_{,d})u^c(\psi^{;d}\psi^*_{,e} + \psi^{*;d}\psi_{,e})u^e\right]\right\}. \tag{A26}$$

For $\frac{\hbar^2}{m^2c^2} \to 0$, we have

$$\mu = m|\psi|^2 c^2, \qquad p = 0 = \pi_{ab}, \tag{A27}$$

thus the axion behaves as a zero-pressure fluid. Only for $\frac{\hbar^2}{m^2c^2} \to 0$, $m|\psi|^2$ can be properly identified as the fluid density $\varrho(\equiv\mu/c^2)$.

3. Relativistic Madelung formulation

The Madelung transformation is in Eq. (12). Applying the Madelung transformation to Eq. (A19), we have

$$\frac{\Box\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho - \frac{m^2}{\hbar^2}(u^{;c}u_{,c} + c^2 + c^2g^{00} - 2cg^{0c}u_{,c}) + \frac{im}{\hbar}\frac{1}{\varrho}[(\varrho u^{;c})_{;c} - cg^{0c}\varrho_{,c} + cg^{ab}\Gamma^0_{ab}\varrho] = 0. \tag{A28}$$

The imaginary and real parts, respectively, give

$$(\varrho u^{;c})_{;c} = c\varrho^{;0} - cg^{ab}\Gamma^0_{ab}\varrho, \tag{A29}$$

$$u^{;c}u_{,c} + c^2 - \frac{\hbar^2}{m^2}\left(\frac{\Box\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right) = -c^2 g^{00} + 2cu^{;0}. \tag{A30}$$

Using the Madelung transformation in Eq. (12) on Eq. (A20), we have

$$\begin{aligned}
\mu &= \varrho(cu^0 - u_{,c}u^c)^2 + \frac{\hbar^2}{m^2}\left[\frac{1}{2}\sqrt{\varrho}^{;c}\sqrt{\varrho}_{,c} + (\sqrt{\varrho}_{,c}u^c)^2 + \frac{1}{2}\sqrt{\varrho}\Box\sqrt{\varrho} - \frac{2\pi\ell_s}{m}\varrho^2\right], \\
p &= \frac{1}{3}\varrho\left[(cu^0 - u_{,c}u^c)^2 - c^2\right] + \frac{\hbar^2}{3m^2}\left[-\frac{1}{2}\sqrt{\varrho}^{;c}\sqrt{\varrho}_{,c} + (\sqrt{\varrho}_{,c}u^c)^2 - \frac{1}{2}\sqrt{\varrho}\Box\sqrt{\varrho} - \frac{2\pi\ell_s}{m}\varrho^2\right], \\
q_a &= \varrho(cu^0 - u_{,c}u^c)\left[(u - c^2 t)_{,a} - (cu^0 - u_{,c}u^c)u_a\right] - \frac{\hbar^2}{m^2}(\sqrt{\varrho}_{,c}u^c)[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,d}u^d)u_a], \\
\pi_{ab} &= \varrho\Bigg\{\left[(u - c^2 t)_{,a} - (cu^0 - u_{,c}u^c)u_a\right]\left[(u - c^2 t)_{,b} - (cu^0 - u_{,c}u^c)u_b\right] \\
&\quad - \frac{1}{3}(g_{ab} + u_a u_b)\left[u^{;c}u_{,c} - 2cg^{0c}u_{,c} + c^2 g^{00} + (cu^0 - u_{,c}u^c)^2\right]\Bigg\} \\
&\quad + \frac{\hbar^2}{m^2}\left\{\left[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,c}u^c)u_a\right][\sqrt{\varrho}_{,b} + (\sqrt{\varrho}_{,d}u^d)u_b\right] - \frac{1}{3}(g_{ab} + u_a u_b)\left[\sqrt{\varrho}^{;c}\sqrt{\varrho}_{,c} + (\sqrt{\varrho}_{,c}u^c)^2\right]\right\},
\end{aligned} \tag{A31}$$

where we used Eq. (A30).

The energy-frame condition, $q_a \equiv 0$, gives

$$u_{,a} = c^2 t_{,a} + (cu^0 - u_{,c}u^c)u_a + \frac{\hbar^2}{m^2}\frac{\sqrt{\varrho}_{,c}u^c}{\varrho(cu^0 - u_{,c}u^c)}\left[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,d}u^d)u_a\right]. \tag{A32}$$

Using this, Eq. (A30) becomes

$$(cu^0 - u_{,c}u^c)^2 = c^2 - \frac{\hbar^2}{m^2}\left(\frac{\Box\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right) + \frac{\hbar^4}{m^4}\frac{(\sqrt{\varrho}_{,c}u^c)^2}{\varrho^2(cu^0 - u_{,c}u^c)^2}g^{ab}\left[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,d}u^d)u_a\right]\left[\sqrt{\varrho}_{,b} + (\sqrt{\varrho}_{,d}u^d)u_b\right]. \tag{A33}$$

Thus, Eq. (A31) gives

$$\begin{aligned}
\mu &= \varrho c^2 + \frac{\hbar^2}{m^2}\left[\frac{1}{2}\sqrt{\varrho}^{;c}\sqrt{\varrho}_{,c} + (\sqrt{\varrho}_{,c}u^c)^2 - \frac{1}{2}\sqrt{\varrho}\Box\sqrt{\varrho} + \frac{6\pi\ell_s}{m}\varrho^2\right] \\
&\quad + \frac{\hbar^4}{m^4}\frac{(\sqrt{\varrho}_{,c}u^c)^2}{\varrho(cu^0 - u_{,c}u^c)^2}g^{ab}\left[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,d}u^d)u_a\right]\left[\sqrt{\varrho}_{,b} + (\sqrt{\varrho}_{,d}u^d)u_b\right], \\
p &= \frac{\hbar^2}{3m^2}\left[-\frac{1}{2}\sqrt{\varrho}^{;c}\sqrt{\varrho}_{,c} + (\sqrt{\varrho}_{,c}u^c)^2 - \frac{3}{2}\sqrt{\varrho}\Box\sqrt{\varrho} + \frac{6\pi\ell_s}{m}\varrho^2\right] \\
&\quad + \frac{\hbar^4}{3m^4}\frac{(\sqrt{\varrho}_{,c}u^c)^2}{\varrho(cu^0 - u_{,c}u^c)^2}g^{ab}\left[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,d}u^d)u_a\right]\left[\sqrt{\varrho}_{,b} + (\sqrt{\varrho}_{,d}u^d)u_b\right], \\
\pi_{ab} &= \frac{\hbar^2}{m^2}\left\{\left[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,c}u^c)u_a\right]\left[\sqrt{\varrho}_{,b} + (\sqrt{\varrho}_{,d}u^d)u_b\right] - \frac{1}{3}(g_{ab} + u_a u_b)\left[\sqrt{\varrho}^{;c}\sqrt{\varrho}_{,c} + (\sqrt{\varrho}_{,c}u^c)^2\right]\right\} \\
&\quad + \frac{\hbar^4}{m^4}\frac{(\sqrt{\varrho}_{,c}u^c)^2}{\varrho(cu^0 - u_{,c}u^c)^2}\Bigg\{\left[\sqrt{\varrho}_{,a} + (\sqrt{\varrho}_{,d}u^d)u_a\right]\left[\sqrt{\varrho}_{,b} + (\sqrt{\varrho}_{,d}u^d)u_b\right] \\
&\quad - \frac{1}{3}(g_{ab} + u_a u_b)g^{cd}\left[\sqrt{\varrho}_{,c} + (\sqrt{\varrho}_{,e}u^e)u_c\right]\left[\sqrt{\varrho}_{,d} + (\sqrt{\varrho}_{,e}u^e)u_d\right]\Bigg\}.
\end{aligned} \tag{A34}$$

For $\frac{\hbar^2}{m^2c^2} \to 0$, we have

$$\mu = \varrho c^2, \qquad p = 0 = \pi_{ab}, \tag{A35}$$

thus the axion behaves as a zero-pressure fluid; only for $\frac{\hbar^2}{m^2c^2} \to 0$, ϱ can be properly identified as the fluid density.

APPENDIX B: COSMOLOGICAL 1PN APPROXIMATION

1. Curvature and fluid quantities

Our metric convention is

$$g_{00} = -(1+2\alpha), \qquad g_{0i} = -\chi_i, \qquad g_{ij} = a^2(1+2\varphi)\delta_{ij}, \tag{B1}$$

with

$$\alpha \equiv \frac{\Phi}{c^2}, \qquad \varphi \equiv -\frac{\Psi}{c^2}, \qquad \chi_i \equiv a\frac{P_i}{c^3}. \tag{B2}$$

We consider the flat cosmological background, and index $0 = ct$. The spatial indices of χ_i and P_i are raised and lowered using δ_{ij} and its inverse. In order to properly include the 1PN expansion, we have to consider c^{-4} order in g_{00}, see Eq. (B19); thus, Φ includes c^{-2} order, and we expand the inverse metric g^{00} to Φ^2 order. Here are the metric tensor, connection, and curvatures to the 1PN order in PN expansion

$$\begin{aligned} g_{00} &= -\left(1+2\frac{\Phi}{c^2}\right), \qquad g_{0i} = -a\frac{P_i}{c^3}, \qquad g_{ij} = a^2\left(1-2\frac{\Psi}{c^2}\right)\delta_{ij}, \\ g^{00} &= -\left(1-2\frac{\Phi}{c^2}+4\frac{\Phi^2}{c^4}\right), \qquad g^{0i} = -\frac{1}{a}\frac{P^i}{c^3}, \qquad g^{ij} = \frac{1}{a^2}\left(1+2\frac{\Psi}{c^2}\right)\delta^{ij}, \end{aligned} \tag{B3}$$

$$\begin{aligned} \Gamma^0_{00} &= \frac{\dot\Phi}{c^3} - \frac{1}{c^5}\left(2\Phi\dot\Phi + \frac{1}{a}P^i\Phi_{,i}\right), \qquad \Gamma^0_{0i} = \frac{\Phi_{,i}}{c^2} - \frac{1}{c^4}(2\Phi\Phi_{,i} + aHP_i), \\ \Gamma^0_{ij} &= \frac{1}{c}a^2H\delta_{ij} - \frac{1}{c^3}a^2\left[\dot\Psi + 2H(\Phi+\Psi)\right]\delta_{ij} + \frac{1}{c^3}aP_{(i,j)}, \qquad \Gamma^i_{00} = \frac{1}{c^2}\frac{1}{a^2}\Phi^{,i} + \frac{1}{c^4}\frac{1}{a^2}\left[2\Psi\Phi^{,i} - (aP^i)^{\cdot}\right], \\ \Gamma^i_{0j} &= \frac{1}{c}H\delta^i_j - \frac{\dot\Psi}{c^3}\delta^i_j + \frac{1}{c^3}\frac{1}{2a}(P_j{}^{,i} - P^i{}_{,j}), \qquad \Gamma^i_{jk} = -\frac{1}{c^2}(\Psi_{,k}\delta^i_j + \Psi_{,j}\delta^i_k - \Psi^{,i}\delta_{jk}). \end{aligned} \tag{B4}$$

The Riemann curvature is

$$\begin{aligned} R^0{}_{00i} &= -\frac{1}{c^5}\left(\ddot a P_i - \frac{1}{a}\Phi_{,ij}P^j\right), \qquad R^0{}_{0ij} = L^{-2}\mathcal{O}(c^{-6}), \\ R^0{}_{i0j} &= \frac{1}{c^2}(a\ddot a\delta_{ij} - \Phi_{,ij}) + \frac{1}{c^4}\left\{-a^2\left[\dot\Psi + 2H(\Phi+\Psi)\right]^{\cdot}\delta_{ij} - a^2H\left[-\dot\Phi + 2H(\Phi+\Psi)\right]\delta_{ij}\right. \\ &\quad \left. + \Phi_{,i}\Phi_{,j} + 2\Phi\Phi_{,ij} - \Psi_{,i}\Phi_{,j} - \Psi_{,j}\Phi_{,i} + \Psi^{,k}\Phi_{,k}\delta_{ij} + (aP_{(i,j)})^{\cdot}\right\}, \\ R^0{}_{ijk} &= -\frac{1}{c^3}\left[2a^2(\dot\Psi + H\Phi)_{,[j}\delta_{k]i} + aP_{[j,k]i}\right], \\ R^i{}_{00j} &= \frac{1}{c^2}\left(\frac{\ddot a}{a}\delta^i_j - \frac{1}{a^2}\Phi^{,i}{}_j\right) + \frac{1}{c^4}\left\{-\left[\ddot\Psi + H(\dot\Phi + 2\dot\Psi)\right]\delta^i_j\right. \\ &\quad \left. + \frac{1}{a^2}(-2\Psi\Phi^{,i}{}_j\Phi^{,i}\Phi_{,j} - \Psi^{,i}\Phi_{,j} - \Psi_{,j}\Phi^{,i} + \Psi^{,k}\Phi_{,k}\delta^i_j) + \frac{1}{2a^2}\left[a(P^i{}_{,j} + P_j{}^{,i})\right]^{\cdot}\right\}, \end{aligned}$$

$$R^i{}_{0jk} = -\frac{1}{c^3}\left[2(\dot{\Psi}+H\Phi)_{,[j}\delta^i_{k]} + \frac{1}{a}P_{[j,k]}{}^{i}\right],$$
$$R^i{}_{j0k} = \frac{1}{c^3}\left[(\dot{\Psi}+H\Phi)^{,i}\delta_{jk} - (\dot{\Psi}+H\Phi)_{,j}\delta^i_k + \frac{1}{2a}(P^i{}_{,j} - P_j{}^{,i})_{,k}\right],$$
$$R^i{}_{jk\ell} = \frac{2}{c^2}\left(a^2H^2\delta^i_{[k}\delta_{\ell]j} + \Psi^{,i}{}_{[k}\delta_{\ell]j} - \Psi_{,j[k}\delta^i_{\ell]}\right). \quad \text{(B5)}$$

The Ricci and the scalar curvature are

$$R^0_0 = \frac{1}{c^2}\left(3\frac{\ddot{a}}{a} - \frac{\Delta}{a^2}\Phi\right) - \frac{1}{c^4}\left\{3\left[\ddot{\Psi} + H(\dot{\Phi}+2\dot{\Psi}) + 2\frac{\ddot{a}}{a}\Phi\right] - \frac{1}{a^2}\left[2(\Phi-\Psi)\Delta\Phi + (\Phi+\Psi)^{,i}\Phi_{,i}\right] - \frac{1}{a^2}(aP^i{}_{,i})^{\cdot}\right\},$$
$$R^0_i = \frac{1}{c^3}\left[-2(\dot{\Psi}+H\Phi)_{,i} + \frac{1}{2a}(P^k{}_{,ki} - \Delta P_i)\right], \qquad R^i_j = \frac{1}{c^2}\left[\left(\frac{\ddot{a}}{a}+2H^2\right)\delta^i_j + \frac{\Delta}{a^2}\Psi\delta^i_j + \frac{1}{a^2}(\Psi-\Phi)^{,i}{}_j\right],$$
$$R = \frac{1}{c^2}\left[6\left(\frac{\ddot{a}}{a}+H^2\right) + 2\frac{\Delta}{a^2}(2\Psi-\Phi)\right]. \quad \text{(B6)}$$

Using Chandrasekhar's 1PN notation in Eq. (B19) we recover Eqs. (1)–(8) in [3].

The normalized fluid four-vector is introduced as

$$u_i \equiv a\gamma\frac{v_i}{c} = a\left(1+\frac{v^2}{2c^2}\right)\frac{v_i}{c}, \qquad u_0 = -1 - \frac{1}{c^2}\left(\frac{1}{2}v^2+\Phi\right) - \frac{1}{c^4}\left[\left(\frac{3}{8}v^2+\frac{1}{2}\Phi+\Psi\right)v^2 - \frac{1}{2}\Phi^2 + P^iv_i\right],$$
$$u^i = \frac{1}{ac}\left[\left(1+\frac{v^2}{2c^2}+2\frac{\Psi}{c^2}\right)v^i + \frac{P^i}{c^2}\right], \qquad u^0 = 1 + \frac{1}{c^2}\left(\frac{1}{2}v^2-\Phi\right) + \frac{1}{c^4}\left[\left(\frac{3}{8}v^2-\frac{1}{2}\Phi+\Psi\right)v^2 + \frac{3}{2}\Phi^2\right], \quad \text{(B7)}$$

where $v^2 \equiv v^iv_i$.

We introduce

$$\mu \equiv \varrho(c^2+\Pi), \qquad q_i \equiv \frac{a}{c}Q_i, \qquad \pi_{ij} \equiv a^2\Pi_{ij}, \quad \text{(B8)}$$

where ϱ and $\varrho\Pi$ are mass density and internal energy density, respectively. The indices of Q_i and Π_{ij} are raised and lowered using δ_{ij} and its inverse; we have

$$q_i \equiv \frac{a}{c}Q_i, \qquad q_0 = -\frac{1}{c^2}Q_iv^i, \qquad q^i = \frac{1}{ac}Q^i, \qquad q^0 = \frac{1}{c^2}Q_iv^i,$$
$$\pi_{ij} \equiv a^2\Pi_{ij}, \qquad \pi_{0i} = -\frac{a}{c}\Pi_{ij}v^j, \qquad \pi_{00} = \frac{1}{c^2}\Pi_{ij}v^iv^j,$$
$$\pi^i_j = \left(1+2\frac{\Psi}{c^2}\right)\Pi^i_j, \qquad \pi^i_0 = -\frac{1}{ca}\Pi^i_jv^j, \qquad \pi^0_i = \frac{a}{c}\Pi_{ij}v^j, \qquad \pi^0_0 = -\frac{1}{c^2}\Pi_{ij}v^iv^j, \quad \text{(B9)}$$

thus, $\pi^c_c = 0$ implies

$$\Pi^i_i = \frac{1}{c^2}\Pi_{ij}v^iv^j. \quad \text{(B10)}$$

The energy-momentum tensor gives

$$T^0_0 = -\varrho c^2 - \varrho(\Pi + v^2) - \frac{1}{c^2}[\varrho(\Pi + v^2 + 2\Psi)v^2 + \varrho P^i v_i + pv^2 + 2Q_i v^i + \Pi_{ij}v^i v^j],$$
$$T^0_i = \varrho c a v_i + \frac{a}{c}[\varrho v_i(\Pi + v^2 - \Phi) + pv_i + Q_i + \Pi_{ij}v^j],$$
$$T^i_j = \varrho v^i v_j + p\delta^i_j + \Pi^i_j + \frac{1}{c^2}[\varrho v^i v_j(\Pi + v^2 + 2\Psi) + \varrho P^i v_j + pv^i v_j + Q^i v_j + Q_j v^i + 2\Psi\Pi^i_j], \tag{B11}$$

thus $T = -\varrho c^2 - \varrho\Pi + 3p$.

2. 1PN hydrodynamic formulation

We consider Einstein's equation in a form

$$R^a_b = \frac{8\pi G}{c^4}\left(T^a_b - \frac{1}{2}T\delta^a_b\right) + \Lambda\delta^a_b. \tag{B12}$$

The R^0_0, R^0_i, and R^i_j components, respectively, give

$$\frac{1}{c^2}\left(\frac{\Delta}{a^2}\Phi - 4\pi G\varrho - 3\frac{\ddot{a}}{a} + \Lambda c^2\right) + \frac{1}{c^4}\left\{3\left[\ddot{\Psi} + H(\dot{\Phi} + 2\dot{\Psi}) + 2\frac{\ddot{a}}{a}\Phi\right] - \frac{1}{a^2}[2(\Phi - \Psi)\Delta\Phi + (\Phi + \Psi)^{,i}\Phi_{,i}]\right.$$
$$\left. - \frac{1}{a^2}(aP^i{}_{,i})^{\cdot} - 8\pi G\left[\varrho\left(\frac{1}{2}\Pi + v^2\right) + \frac{3}{2}p\right]\right\} = 0, \tag{B13}$$

$$\frac{1}{c^3}\left[-(\dot{\Psi} + H\Phi)_{,i} + \frac{1}{4a}(P^k{}_{,ki} - \Delta P_i) - 4\pi G\varrho a v_i\right] = 0, \tag{B14}$$

$$\frac{1}{c^2}\left[\left(\frac{\ddot{a}}{a} + 2H^2\right)\delta^i_j + \frac{\Delta}{a^2}\Psi\delta^i_j + \frac{1}{a^2}(\Psi - \Phi)^{,i}{}_j\right] = \frac{8\pi G}{c^4}\left[\varrho v^i v_j + \Pi^i_j + \frac{1}{2}(\varrho c^2 + \varrho\Pi - p)\delta^i_j\right] + \Lambda\delta^i_j, \tag{B15}$$

where the left-hand sides are derived up to 1PN order; we kept c^{-4} order in Eq. (B15) only to have a proper cosmological background equation. In the cosmological background, to the background order, from Eqs. (B13) and (B15), we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\varrho_b\left(1 + \frac{\Pi_b}{c^2}\right) + 3\frac{p_b}{c^2}\right] + \frac{\Lambda c^2}{3}, \qquad \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho_b\left(1 + \frac{\Pi_b}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{B16}$$

Subtracting the background equation, Eq. (B15) becomes

$$\frac{\Delta}{a^2}\Psi\delta^i_j + \frac{1}{a^2}(\Psi - \Phi)^{,i}{}_j = 4\pi G\delta\varrho. \tag{B17}$$

Thus, trace and trace-free parts, respectively, give

$$\frac{\Delta}{a^2}\Psi = 4\pi G\delta\varrho, \qquad \Psi = \Phi. \tag{B18}$$

As mentioned, Φ still includes 1PN order contribution besides the 0PN (Newtonian) one. Compared with Chandrasekhar's notation

$$\Phi = -U - \frac{1}{c^2}(2\Upsilon - U^2), \qquad \Psi = -V, \qquad v_i = \bar{v}_i - \frac{1}{c^2}[(\Phi + 2\Psi)\bar{v}_i + P_i], \tag{B19}$$

where

$$\frac{1}{ac}\bar{v}^i \equiv \frac{dx^i}{dx^0} = \frac{u^i}{u^0}, \qquad u_i \equiv \frac{a}{c}\gamma v_i. \tag{B20}$$

Using U and V, we have $V = U$ to 0PN order; $\bar{v}_i$ is used in [1,3]. Subtracting the background equation, Eqs. (B13) and (B14) give

$$\frac{\Delta}{a^2}U + 4\pi G(\varrho - \varrho_b) + \frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Upsilon + 3\left(\ddot{U} + 3H\dot{U} + 2\frac{\ddot{a}}{a}U\right) - \frac{2}{a^2}U\Delta U + \frac{1}{a^2}(aP^i{}_{,i})^\cdot + 8\pi G\left[\frac{1}{2}(\varrho\Pi - \varrho_b\Pi_b) + \varrho v^2 + \frac{3}{2}(p - p_b)\right]\right\} = 0, \tag{B21}$$

$$(\dot{U} + HU)_{,i} + \frac{1}{4a}(P^k{}_{,ki} - \Delta P_i) = 4\pi G\varrho a v_i. \tag{B22}$$

The energy and momentum conservation equations are

$$-\frac{1}{c}T^b_{0;b} = \dot{\varrho} + 3H\varrho + \frac{1}{a}(\varrho v^i)_{,i} + \frac{1}{c^2}\left\{[\varrho(\Pi + v^2)]^\cdot + 3H(\varrho\Pi + p) + \frac{1}{a}[\varrho v^i(\Pi + v^2 + \Phi + 2\Psi) + \varrho P^i + pv^i + Q^i + \Pi^i_j v^j]_{,i} + \varrho\left[-3\dot{\Psi} + \frac{1}{a}(\Phi - 3\Psi)_{,i}v^i + 4Hv^2\right]\right\}, \tag{B23}$$

$$\frac{1}{a}T^b_{i;b} = \frac{1}{a^4}(a^4\varrho v_i)^\cdot + \frac{1}{a}(\varrho v^j v_i + p\delta^j_i + \Pi^j_i)_{,j} + \frac{1}{a}\varrho\Phi_{,i} + \frac{1}{c^2}\left\{\frac{1}{a^4}\{a^4[\varrho v_i(\Pi + v^2 - \Phi) + pv_i + Q_i + \Pi_{ij}v^j]\}^\cdot + \frac{1}{a}[\varrho v^j v_i(\Pi + v^2 + 2\Psi) + \varrho P^j v_i + pv^j v_i + Q^j v_i + Q_i v^j + 2\Psi\Pi^j_i]_{,j} + (\dot{\Phi} - 3\dot{\Psi})\varrho v_i + \frac{1}{a}(\Phi - 3\Psi)_{,j}(\varrho v^j v_i + \Pi^j_i) + \frac{1}{a}\Phi_{,i}[\varrho(\Pi + v^2) + p - 2\Phi] + \frac{1}{a}\Psi_{,i}\varrho v^2 + \varrho v^j P_{j,i}\right\}. \tag{B24}$$

To the background order, from Eq. (B23), we have

$$\left[\varrho_b\left(1 + \frac{\Pi_b}{c^2}\right)\right]^\cdot + 3H\left[\varrho_b\left(1 + \frac{\Pi_b}{c^2}\right) + \frac{p_b}{c^2}\right] = 0. \tag{B25}$$

Subtracting the background equation, and using U and Υ we have

$$(\varrho - \varrho_b)^\cdot + 3H(\varrho - \varrho_b) + \frac{1}{a}(\varrho v^i)_{,i} + \frac{1}{c^2}\left\{(\varrho\Pi - \varrho_b\Pi_b + \varrho v^2)^\cdot + 3H(\varrho\Pi - \varrho_b\Pi_b + p - p_b) + \frac{1}{a}[\varrho v^i(\Pi + v^2 - 3U) + \varrho P^i + pv^i + Q^i + \Pi^i_j v^j]_{,i} + \varrho\left(3\dot{U} + \frac{2}{a}U_{,i}v^i + 4Hv^2\right)\right\} = 0, \tag{B26}$$

$$\frac{1}{a^4}(a^4\varrho v_i)^\cdot + \frac{1}{a}(\varrho v^j v_i + p\delta^j_i + \Pi^j_i)_{,j} - \frac{1}{a}\varrho U_{,i} + \frac{1}{c^2}\left\{\frac{1}{a^4}\{a^4[\varrho v_i(\Pi + v^2 + U) + pv_i + Q_i + \Pi_{ij}v^j]\}^\cdot + \frac{1}{a}\left[\varrho v^j v_i(\Pi + v^2 - 2U) + \varrho P^j v_i + pv^j v_i + Q^j v_i + Q_i v^j - 2U\Pi^j_i\right]_{,j} + 2\dot{U}\varrho v_i + \frac{2}{a}U_{,j}(\varrho v^j v_i + \Pi^j_i) - \frac{2}{a}\varrho\Upsilon_{,i} - \frac{1}{a}U_{,i}[\varrho(\Pi + 2v^2) + p] + \varrho v^j P_{j,i}\right\} = 0. \tag{B27}$$

For a general fluid to 1PN order, the energy and momentum conservation equations are in Eqs. (B26) and (B27), and Einstein's equation provides Eqs. (B21) and (B22). These provide a complete set of equations valid to 1PN order without imposing the temporal gauge condition; see below Eq. (2) for gauge conditions. The background evolution is described by Eqs. (B16) and (B25).

3. 1PN Schrödinger formulation

To 1PN order, Eq. (A19) gives

$$\left(\frac{\Delta}{a^2}-8\pi\ell_s|\psi|^2\right)\psi+\frac{2im}{\hbar}\left(\dot{\psi}+\frac{3}{2}H\psi\right)-\frac{2m^2}{\hbar^2}\Phi\psi+\frac{1}{c^2}\left[-\ddot{\psi}-3H\dot{\psi}+2\Psi\frac{\Delta}{a^2}\psi+\frac{1}{a^2}(\Phi-\Psi)^{,i}\psi_{,i}\right.$$
$$\left.+\frac{2im}{\hbar}\left(-2\Phi\dot{\psi}+\frac{1}{a}P^i\psi_{,i}\right)+\frac{im}{\hbar}\left(\frac{1}{a}P^i_{,i}-\dot{\Phi}-3\dot{\Psi}-6H\Phi\right)\psi+\frac{4m^2}{\hbar^2}\Phi^2\psi\right]=0. \tag{B28}$$

Using the PN notation in Eq. (B19), we have

$$\left(\frac{\Delta}{a^2}-8\pi\ell_s|\psi|^2\right)\psi+\frac{2im}{\hbar}\left(\dot{\psi}+\frac{3}{2}H\psi\right)+\frac{2m^2}{\hbar^2}U\psi+\frac{1}{c^2}\left[-\ddot{\psi}-3H\dot{\psi}-2U\frac{\Delta}{a^2}\psi\right.$$
$$\left.+\frac{2im}{\hbar}\left(2U\dot{\psi}+\frac{1}{a}P^i\psi_{,i}\right)+\frac{im}{\hbar}\left(\frac{1}{a}P^i_{,i}+4\dot{U}+6HU\right)\psi+\frac{2m^2}{\hbar^2}(2\Upsilon+U^2)\psi\right]=0. \tag{B29}$$

Using the four-vector in Eq. (B7), Eq. (A26) gives the energy-frame condition, $q_i\equiv 0$,

$$m|\psi|^2av_i=-i\hbar\frac{1}{2}(\psi_{,i}\psi^*-\psi^*_{,i}\psi)+\frac{1}{c^2}\left\{m|\psi|^2(v^2+\Phi)av_i-i\hbar(\dot{\psi}\psi^*-\dot{\psi}^*\psi)av_i\right.$$
$$\left.-\frac{\hbar^2}{2m}\left[\psi_{,i}\dot{\psi}^*+\psi^*_{,i}\dot{\psi}+\frac{1}{a}(\psi_{,i}\psi^*_{,j}+\psi^*_{,i}\psi_{,j})v^j\right]\right\}, \tag{B30}$$

and the fluid quantities

$$\mu=m|\psi|^2c^2+\frac{\hbar^2}{2m}\left\{\frac{1}{a^2}\psi^{,i}\psi^*_{,i}+\frac{1}{2|\psi|^2a^2}(\psi^{,i}\psi^*-\psi^{*,i}\psi)(\psi_{,i}\psi^*-\psi^*_{,i}\psi)-\frac{1}{2a^2}[(\Delta\psi)\psi^*+(\Delta\psi^*)\psi]+12\pi\ell_s|\psi|^4\right\},$$
$$p=\frac{\hbar^2}{2m}\left\{-\frac{1}{3a^2}\psi^{,i}\psi^*_{,i}+\frac{1}{6|\psi|^2a^2}(\psi^{,i}\psi^*-\psi^{*,i}\psi)(\psi_{,i}\psi^*-\psi^*_{,i}\psi)-\frac{1}{2a^2}[(\Delta\psi)\psi^*+(\Delta\psi^*)\psi]+4\pi\ell_s|\psi|^4\right\},$$
$$+\frac{\hbar^2}{12mc^2}\left\{6|\dot{\psi}|^2-\frac{4}{a^2}\psi^{,i}\psi^*_{,i}\Psi+\frac{2}{a}(\dot{\psi}\psi^*_{,i}+\dot{\psi}^*\psi_{,i})v^i\right.$$
$$\left.+3\left[\ddot{\psi}+3H\dot{\psi}-2\Psi\frac{\Delta}{a^2}\psi-\frac{1}{a^2}(\Phi-\Psi)^{,i}\psi_{,i}\right]\psi^*+3\left[\ddot{\psi}^*+3H\dot{\psi}^*-2\Psi\frac{\Delta}{a^2}\psi^*-\frac{1}{a^2}(\Phi-\Psi)^{,i}\psi^*_{,i}\right]\psi\right\}$$
$$+\frac{\hbar^2}{6mc^2|\psi|^2}\frac{1}{a^2}\left\{a(\dot{\psi}\psi^*-\dot{\psi}^*\psi)(\psi_{,i}\psi^*-\psi^*_{,i}\psi)v^i+\frac{1}{2}[(\psi_{,i}\psi^*-\psi^*_{,i}\psi)v^i]^2+\Psi(\psi^{,i}\psi^*-\psi^{*,i}\psi)(\psi_{,i}\psi^*-\psi^*_{,i}\psi)\right\}$$
$$-\frac{i\hbar^3}{12m^2c^2|\psi|^2}\frac{1}{a^2}(\psi^{,i}\psi^*-\psi^{*,i}\psi)\left[(\psi_{,i}\dot{\psi}^*+\psi^*_{,i}\dot{\psi})+\frac{1}{a}(\psi_{,i}\psi^*_{,j}+\psi^*_{,i}\psi_{,j})v^j\right],\qquad Q_i\equiv 0,$$
$$\Pi_{ij}=\frac{\hbar^2}{ma^2}\left\{\psi_{,(i}\psi^*_{,j)}+\frac{1}{4|\psi|^2}(\psi_{,i}\psi^*-\psi^*_{,i}\psi)(\psi_{,j}\psi^*-\psi^*_{,j}\psi)-\frac{1}{3}\delta_{ij}\left[\psi^{,k}\psi^*_{,k}+\frac{1}{4|\psi|^2}(\psi^{,k}\psi^*-\psi^{*,k}\psi)(\psi_{,k}\psi^*-\psi^*_{,k}\psi)\right]\right\}$$
$$+\frac{\hbar^2}{mc^2}\left\{\frac{1}{a}\left(\dot{\psi}+\frac{1}{a}\psi_{,k}v^k\right)v_{(i}\psi^*_{,j)}+\frac{1}{a}\left(\dot{\psi}^*+\frac{1}{a}\psi^*_{,k}v^k\right)v_{(i}\psi_{,j)}-\frac{1}{3a^2}\psi^{,k}\psi^*_{,k}v_iv_j\right.$$
$$\left.-\frac{1}{3}\delta_{ij}\left[\frac{1}{a}(\dot{\psi}\psi^*_{,k}+\dot{\psi}^*\psi_{,k})v^k+\frac{1}{a^2}|\psi_{,k}v^k|^2\right]\right\}+\frac{\hbar^2}{mc^2|\psi|^2}\left\{\frac{1}{2a}(\psi_{,(i}\psi^*-\psi^*_{,(i}\psi)v_{j)}\left[\dot{\psi}\psi^*-\dot{\psi}^*\psi+\frac{1}{a}(\psi_{,k}\psi^*-\psi^*_{,k}\psi)v^k\right]\right.$$
$$\left.-\frac{1}{12a^2}(\psi^{,k}\psi^*-\psi^{*,k}\psi)(\psi_{,k}\psi^*-\psi^*_{,k}\psi)v_iv_j-\frac{1}{12}\delta_{ij}\left[\frac{2}{a}(\dot{\psi}\psi^*-\dot{\psi}^*\psi)(\psi_{,k}\psi^*-\psi^*_{,k}\psi)v^k+\frac{1}{a^2}[(\psi_{,k}\psi^*-\psi^*_{,k}\psi)v^k]^2\right]\right\}. \tag{B31}$$

Using the axion fluid quantities, Einstein's equation in (B21) and (B22) give

$$\frac{\Delta}{a^2}U + 4\pi Gm(|\psi|^2 - |\psi_b|^2) + \frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Upsilon + 3\left(\ddot{U} + 3H\dot{U} + 2\frac{\ddot{a}}{a}U\right) - \frac{2}{a^2}U\Delta U + \frac{1}{a^2}(aP^i{}_{,i})^{\cdot} + 8\pi G\frac{\hbar^2}{m}\left[-\frac{1}{2a^2}(\psi\Delta\psi^* + \psi^*\Delta\psi) + 6\pi\ell_s(|\psi|^4 - |\psi_b|^4)\right]\right\} = 0, \tag{B32}$$

$$(\dot{U} + HU)_{,i} + \frac{1}{4a}(P^k{}_{,ki} - \Delta P_i) = 2\pi Gi\hbar(\psi\psi^*_{,i} - \psi^*\psi_{,i}). \tag{B33}$$

Equations (B29), (B32), and (B33) provide a complete set of axion fluid to 1PN order without imposing the temporal gauge condition; see below Eq. (2) for gauge conditions. The background evolution is described by Eqs. (B16) and (B25), with

$$\varrho_b = m|\psi_b|^2, \qquad \varrho_b\Pi_b = 3p_b = \frac{6\pi\ell_s\hbar^2}{m}|\psi_b|^4. \tag{B34}$$

4. 1PN Madelung formulation

Under the Madelung transformation the imaginary and real parts of Eq. (B28), or directly from Eqs. (A29) and (A30), give

$$\dot{\varrho} + 3H\varrho + \frac{1}{a^2}(\varrho u^{,i})_{,i} + \frac{1}{c^2}\left[-(\varrho\dot{u})^{\cdot} - 3H\varrho\dot{u} + 2(\Phi+\Psi)\frac{1}{a^2}(\varrho u^{,i})_{,i} + \frac{1}{a}(\varrho P^i)_{,i} - \varrho(\dot{\Phi} + 3\dot{\Psi}) + \frac{1}{a^2}\varrho(\Phi-\Psi)^{,i}u_{,i}\right] = 0, \tag{B35}$$

$$\dot{u} + \frac{1}{2a^2}u^{,i}u_{,i} + \Phi - \frac{\hbar^2}{2m^2}\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right) + \frac{1}{c^2}\left\{-\frac{1}{2}\dot{u}^2 + (\Phi+\Psi)\frac{1}{a^2}u^{,i}u_{,i} + \frac{1}{a}P^iu_{,i} + \frac{\hbar^2}{2m^2}\left[\frac{\ddot{\sqrt{\varrho}}}{\sqrt{\varrho}} + 3H\frac{\dot{\sqrt{\varrho}}}{\sqrt{\varrho}} - 2(\Phi+\Psi)\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} + \Phi\frac{16\pi\ell_s}{m}\varrho - \frac{1}{a^2}(\Phi-\Psi)^{,i}\frac{\sqrt{\varrho}_{,i}}{\sqrt{\varrho}}\right]\right\} = 0. \tag{B36}$$

By identifying $\mathbf{u} \equiv \frac{1}{a}\nabla u$, we have

$$\dot{\varrho} + 3H\varrho + \frac{1}{a}\nabla\cdot(\varrho\mathbf{u}) + \frac{1}{c^2}\left[-(\varrho\dot{u})^{\cdot} - 3H\varrho\dot{u} - \varrho(\dot{\Phi} + 3\dot{\Psi}) + 2(\Phi+\Psi)\frac{1}{a}\nabla\cdot(\varrho\mathbf{u}) + \frac{1}{a}\varrho\mathbf{u}\cdot\nabla(\Phi-\Psi) + \frac{1}{a}\nabla\cdot(\varrho\mathbf{P})\right] = 0, \tag{B37}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{a}\nabla\Phi - \frac{\hbar^2}{2m^2}\frac{1}{a}\nabla\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right) + \frac{1}{c^2}\frac{1}{a}\nabla\left\{-\frac{1}{2}\dot{u}^2 + (\Phi+\Psi)\mathbf{u}^2 + \mathbf{u}\cdot\mathbf{P} + \frac{\hbar^2}{2m^2}\left[\frac{\ddot{\sqrt{\varrho}}}{\sqrt{\varrho}} + 3H\frac{\dot{\sqrt{\varrho}}}{\sqrt{\varrho}} - 2(\Phi+\Psi)\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} + \Phi\frac{16\pi\ell_s}{m}\varrho - \frac{1}{a^2}\frac{1}{\sqrt{\varrho}}(\nabla\sqrt{\varrho})\cdot\nabla(\Phi-\Psi)\right]\right\} = 0, \tag{B38}$$

and, for remaining $\dot{u}$, we can use

$$\dot{u} = -\Phi - \frac{1}{2}\mathbf{u}^2 + \frac{\hbar^2}{2m^2}\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right). \tag{B39}$$

Using Chandrasekhar's 1PN notation in Eq. (B19) we have

$$\dot{\varrho} + 3H\varrho + \frac{1}{a}\nabla\cdot(\varrho\mathbf{u}) + \frac{1}{c^2}\left[-(\varrho\dot{u})^{\cdot} - 3H\varrho\dot{u} - 4U\frac{1}{a}\nabla\cdot(\varrho\mathbf{u}) + 4\varrho\dot{U} + \frac{1}{a}\nabla\cdot(\varrho\mathbf{P})\right] = 0, \tag{B40}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{a}\nabla U - \frac{\hbar^2}{2m^2}\frac{1}{a}\nabla\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right)$$
$$+ \frac{1}{c^2}\frac{1}{a}\nabla\left[-\frac{1}{2}\dot{u}^2 - 2\Upsilon + U^2 - 2U\mathbf{u}^2 + \mathbf{P}\cdot\mathbf{u} + \frac{\hbar^2}{2m^2}\left(\frac{\ddot{\sqrt{\varrho}}}{\sqrt{\varrho}} + 3H\frac{\dot{\sqrt{\varrho}}}{\sqrt{\varrho}} + 4U\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - U\frac{16\pi\ell_s}{m}\varrho\right)\right] = 0, \tag{B41}$$

$$\dot{u} = U - \frac{1}{2}\mathbf{u}^2 + \frac{\hbar^2}{2m^2}\left(\frac{1}{a^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} - \frac{8\pi\ell_s}{m}\varrho\right). \tag{B42}$$

To 1PN order, using the four-vector in Eq. (B7), Eq. (A34) gives the fluid quantities. The energy-frame condition, $q_a \equiv 0$, in Eq. (A32) gives

$$u_{,i} = av_i\left[1 - \frac{1}{c^2}(\dot{u} + \Phi)\right] + \frac{\hbar^2}{m^2c^2}\left(\frac{\dot{\sqrt{\varrho}}}{\sqrt{\varrho}} + \frac{1}{a}\frac{\sqrt{\varrho}_{,k}}{\sqrt{\varrho}}v^k\right)\frac{\sqrt{\varrho}_{,i}}{\sqrt{\varrho}}. \tag{B43}$$

Thus, to 0PN order $u_{,i} = av_i$. Using the notation in Eq. (B8), the fluid quantities follow from Eq. (A34). We can identify ϱ as the mass density, and have

$$\varrho\Pi = \frac{\hbar^2}{2m^2}\left(\frac{1}{a^2}\sqrt{\varrho}^{,i}\sqrt{\varrho}_{,i} - \frac{1}{a^2}\sqrt{\varrho}\Delta\sqrt{\varrho} + \frac{12\pi\ell_s}{m}\varrho^2\right),$$
$$p = \frac{\hbar^2}{6m^2}\left\{-\frac{1}{a^2}\sqrt{\varrho}^{,i}\sqrt{\varrho}_{,i} - \frac{3}{a^2}\sqrt{\varrho}\Delta\sqrt{\varrho} + \frac{12\pi\ell_s}{m}\varrho^2 + \frac{1}{c^2}\left[3\sqrt{\varrho}(\ddot{\sqrt{\varrho}} + 3H\dot{\sqrt{\varrho}}) + 3(\dot{\sqrt{\varrho}})^2\right.\right.$$
$$\left.\left. + 4\dot{\sqrt{\varrho}}\frac{1}{a}\sqrt{\varrho}_{,i}v^i + \frac{2}{a^2}(\sqrt{\varrho}_{,i}v^i)^2 - \frac{2}{a^2}\Psi(\sqrt{\varrho}^{,i}\sqrt{\varrho}_{,i} + 3\sqrt{\varrho}\Delta\sqrt{\varrho}) - \frac{3}{a^2}(\Phi-\Psi)^{,i}\sqrt{\varrho}\sqrt{\varrho}_{,i}\right]\right\}, \qquad Q_i \equiv 0,$$
$$\Pi_{ij} = \frac{\hbar^2}{m^2a^2}\left\{\sqrt{\varrho}_{,i}\sqrt{\varrho}_{,j} - \frac{1}{3}\sqrt{\varrho}^{,k}\sqrt{\varrho}_{,k}\delta_{ij} + \frac{1}{c^2}\left[a\left(\dot{\sqrt{\varrho}} + \frac{1}{a}\sqrt{\varrho}_{,k}v^k\right)\left(\sqrt{\varrho}_{,i}v_j + \sqrt{\varrho}_{,j}v_i - \frac{2}{3}\delta_{ij}\sqrt{\varrho}_{,k}v^k\right)\right.\right.$$
$$\left.\left. + \frac{1}{3}\delta_{ij}(\sqrt{\varrho}_{,k}v^k)^2 - \frac{1}{3}\sqrt{\varrho}^{,k}\sqrt{\varrho}_{,k}v_iv_j\right]\right\}. \tag{B44}$$

This also follows from Eq. (B31). Notice that p and Π_{ij} are derived up to 2PN order as we need that order to derive the axion conservation equations in (B40) and (B41) from the fluid conservation equations in (B26) and (B27); we also need to use Eqs. (B42) and (B43).

Using the axion fluid quantities, Einstein's equations in (B21) and (B22) give

$$\frac{\Delta}{a^2}U + 4\pi G(\varrho - \varrho_b) + \frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Upsilon + 3\left(\ddot{U} + 3H\dot{U} + 2\frac{\ddot{a}}{a}U\right) - \frac{2}{a^2}U\Delta U + \frac{1}{a^2}(aP^i{}_{,i})^{\cdot}\right.$$
$$\left. + 8\pi G\varrho u_i^2 + 8\pi G\frac{\hbar^2}{m^2}\left[-\frac{1}{a^2}\sqrt{\varrho}\Delta\sqrt{\varrho} + \frac{6\pi\ell_s}{m}(\varrho^2 - \varrho_b^2)\right]\right\} = 0, \tag{B45}$$

$$(\dot{U} + HU)_{,i} + \frac{1}{4a}(P^k{}_{,ki} - \Delta P_i) = 4\pi G\varrho a u_i. \tag{B46}$$

These also follow from Eqs. (B32) and (B33) using the Madelung transformation.

Equations (B40), (B41), (B45), and (B46) provide a complete set of axions in Schrödinger formulation to 1PN order without fixing the temporal gauge condition; see below Eq. (2) for gauge conditions. The background evolution is described by Eqs. (B16) and (B25), with

$$\varrho_b\Pi_b = 3p_b = \frac{6\pi\ell_s\hbar^2}{m^3}\varrho_b^2. \tag{B47}$$

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