Linear response hydrodynamics of a relativistic dissipative fluid with spin

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We formulate a Lagrangian hydrodynamics including shear and bulk viscosity in the presence of spin density, and investigate it using the linear response functional formalism. The result is a careful accounting of all sound and vortex interactions close to local equilibrium. In particular, we demonstrate that the mixing of sound waves and vortices via polarization, first observed in the ideal fluid limit, extends to the shear mode once dissipative effects are included. This provides a realization within Lagrangian hydrodynamics of the symmetric shear polarization contribution recently advocated from transport and Zubarev approaches as well as phenomenological considerations. Once causal relaxational dynamics is included, this effect, seemingly puzzling because it results in a nondissipative coupling between a transient mode to an equilibrium quantity, can be understood as a competition between the Israel-Stewart and the polarization relaxation timescale, and a breakdown of local Markovianity. We close by discussing phenomenological implications of these results.

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I. INTRODUCTION

Relativistic spin hydrodynamics has been part of a vigorous theoretical investigation, triggered by the experimental discovery of a vorticity-correlated hyperon spin polarization and vector resonance spin alignment in heavy ion collisions [1]. Recently, several versions of hydrodynamics with spin have been proposed [2–17] but basic conceptual questions, such as the role of spin-vorticity coupling, pseudogauge dependence, entropy production and the definition of equilibrium, remain unanswered.

One approach that has the advantage of an immediate connection with both microscopic statistical mechanics and field theory is Lagrangian hydrodynamics [6] analyzed via linear response techniques. In this formalism, one abandons the definition of hydrodynamics as dictated by *conservation laws*, but instead develops it from a definition of *a free energy* to be locally minimized. The advantages, as written earlier, are a direct connection with microscopic entropy and, since conservation laws do not dictate the dynamics, a bypassing of the pseudogauge issue. The disadvantage is that away from the ideal limit free energy is not maximized *exactly* so a Schwinger-Keldysh formalism, and a precise fluctuation-dissipation relation, are needed to hold for the dynamics to be well defined [18].

The Lagrangian approach was previously used to understand the properties of a nondissipative fluid [19–22], then one with bulk [23] and shear [24] viscosity as well as Israel-Stewart (*IS*) corrections. It was used in [2,3] to understand the transport properties of a nearly ideal fluid with spin and to show that this limit clashed with causality [4]. Finally, [25] a rigorous connection was made between spin hydrodynamics and the linear response theory of [26,27], including a Higgs-like relation between the polarization "condensate" and space-time symmetries and a fluctuation dissipation relation between polarization susceptibility and spin relaxation time, whose long-time tail parallels the tail of hydrodynamic fluctuations [28].

In this work, we would like to extend our earlier analysis [25] to include dissipative effects as well as spin effects. This is because, while spin is a necessary contributor to dissipation [4], it is not its only source, since of course momentum diffusion by microscopic collisions is still present. As recently seen in [16], the interplay of the "pure dissipation" scale with the "polarization scale" can be quite nontrivial. Moreover, the broken time-reversal symmetry induced by dissipation, and the broken space-time symmetry induced by polarization, can combine in relatively nontrivial ways. In this work we will use the linear response analysis developed in [25] to elucidate all these issues in detail, clarify how polarization and dissipation combine in different regimes, and explain the role of interactions between "macroscopic" sound and diffusive modes and microscopic fluctuations as well as of the effective symmetries. In Sec. II, we review the QFT implementation via the Schwinger-Keldysh formalism and the construction of both dissipative and spin hydrodynamics as effective field theories. In Sec. III, we will put together the dissipative and polarization terms to study the interaction between them. This analysis is developed via Feynman diagrams in Sec. IV by deriving the EoS and transport corrections due to polarization and elucidating the regimes in the different length scale hierarchies. In Sec. V we provide an overview of the Markovian limit and discuss the inclusion of the memory effect to describe second-order polarized hydrodynamics. Finally, in Sec. VI we present the general remarks and summarize our findings. The detailed form of the Kubo formulas and correlations is given in the Appendix. Finally, we address the memory effect by means of hierarchy of the relaxation timescale.

Conventions: $\hbar = 1$. $c = 1$ metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. $\mu, \nu, \ldots$ all space-time coordinates, $I, J, K, \ldots$ spatial coordinates in the Lagrangian frame.

II. SCHWINGER-KELDYSH *CTP* FORMALISM

We have known hydrodynamics as an effective description of the infrared regime where the physical properties manifest from underlying microscopic interactions. We address hydrodynamics by using a bottom-up effective field theory language [19–21]. In this section, we introduce the closed time path formalism as a good qualitative description of what is going on in a dissipative polarized fluid in rotation.

A. *CTP* Wilsonian effective action

The Schwinger-Keldysh formalism has been developed to exploit the fluctuation-dissipation theorem, which calculates deviations from equilibrium. These deviations are modeled as perturbations on a heath bath. The key object to perform such analysis is the generating functional

$$Z[J] = \int \mathcal{D}\varphi\rho[\varphi(t,\mathbf{x})] \exp\left\{ iS[\varphi] + i\int_{\mathcal{C}} d^4x J\cdot\varphi \right\}, \quad (2.1)$$

where φ encodes $\{\phi, \Psi\}$ the infrared and ultraviolet degrees of freedom, respectively, J is the usual classical external source, and the ρ is the density matrix. The integration contour $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ represents the branch $\mathcal{C}_1$ running "toward" positive time axis and $\mathcal{C}_2$ running "backward" negative time axis. In order to build the n–point correlation function, we use $Z[J]$

$$G(x_1,\ldots,x_m) = i(-1)^m \frac{\delta^m Z[J]}{\delta J(x_1)\ldots\delta J(x_m)} = \langle \mathcal{T}\{\phi(x_1)\ldots\phi(x_m)\}\rangle, \quad (2.2)$$

where $\mathcal{T}$ is the time-ordered-product and $\langle\ldots\rangle$ the thermal expectation value.[1] Since the equilibrium is built around stationarity deviations, implemented via the *KMS* condition. It must necessarily come in two types, toward and away from equilibrium. Microscopically, the equilibrium condition demands that transitions to and from equilibrium must be equivalent, but in the long run, as equilibrium maximizes the number of microstates (whose logarithm is the entropy), fluctuations toward equilibrium, will be more frequent. With the purpose to approach any real process in the effective theory, these requirements are investigated by doubling the degrees of freedom

$$\varphi^+ = \frac{\varphi_1+\varphi_2}{2}, \qquad \varphi^- = \varphi_1 - \varphi_2,$$
$$J^+ = \frac{J_1+J_2}{2}, \qquad J^- = J_1 - J_2, \quad (2.3)$$

as well as the Hilbert space. Substituting these coordinates in (2.1), the generator functional in the Heisenberg representation becomes

$$Z[\varphi_\pm, J_\pm] = \mathrm{Tr}[U(+\infty,-\infty,\varphi_+,J_+)\rho_o[\varphi_\pm(t,\mathbf{x})] \times U^\dagger(+\infty,-\infty,\varphi_-,J_-)], \quad (2.4)$$

being ρ_o the initial density matrix. This transformation explicitly makes the connection between the "exact" quantum interaction with a thermal bath. As the unitary matrix is $U = \mathcal{T}\{\exp i\int d^4x[\mathcal{E}(x) - J^+(x)\varphi^+(x)]\}$, the generating function can be decomposed as

$$Z[\phi_\pm,\Psi_\pm] = \int \mathcal{D}\phi_\pm\mathcal{D}\Psi_\pm\rho[\phi_\pm,\Psi_\pm]\exp[iS[\phi_\pm] + iS[\Psi_\pm] + iS[\phi_\pm,\Psi_\pm]], \quad (2.5)$$

where $S[\phi_\pm]$ is the conservative stress-tensor, bounded from below, and $S[\Psi_\pm]$ is the action of the microscopic variable. The fact that there are two directions, toward and away from equilibrium, means that, from a microscopic viewpoint, the coarse-grained procedure produces fluctuations in $S[\phi_\pm,\Psi_\pm]$. Note that all nontrivial couplings between $\phi - \Psi$ will be necessary according to the dissipative-order needed to exploit the fluid behavior. Physically, even though we cannot keep track of inaccessible degrees of freedom, their feedback interferes with the dynamics of accessible degrees of freedom. After integrating out of the Ψ-sector in (2.5), we obtain

[1]The coordinate arguments is in 4-dimensional Minkowski space.

$$Z[\phi_+,\phi_-] = \text{Const} \times \int \mathcal{D}\phi_\pm \rho[\phi_\pm] \exp[\underbrace{iS[\phi_\pm] + i\lambda \int d^4x \phi^a \hat{G}_{ab} \phi^b}_{S_{CTP}}]. \tag{2.6}$$

We omit the derivatives upon ϕ to keep the notation light. For the sake of simplicity, we are interested in linearized perturbation to account for the time evolution of fluctuations [see Eq. (2.8)]. Summarizing the expression above for the effective action, we have

$$S_{CTP}[\phi^\pm, J^\pm] = \int_{t_i}^{t_f} d^4x \{\mathcal{L}_o[\phi^+, J^+] - \mathcal{L}_o^*[\phi^-, J^-] + \mathcal{L}_s[\phi_\pm, J_\pm]\}, \tag{2.7}$$

where t_i and t_f are the initial and final time, respectively. It is useful, in this language, to separate the contributions coming from different energy scales. The two copies of the nondissipative Lagrangian $\mathcal{L}_o$ and $\mathcal{L}_o^*$ manifest the conservation of variables related to macrodimensions. The inclusion of $\mathcal{L}_s$ represents the system-environment correlation, keeping the infrared dynamics almost closed. In addition, it enlarges the coupling space with thermal noise and thermodynamic forces that break symmetries. In addition, we demand $\frac{\delta^2 \mathcal{L}_s}{\delta\phi^+ \delta\phi^-} \neq 0$ since the collision of microscopic degrees of freedom and interactions with the external fields are regulated by the second derivative of action. The action (2.7) is subject to the "fluctuation-dissipation" restrictions $S_{CTP}[\phi^+,\phi^-] = -S^*_{CTP}[\phi^-,\phi^+]$ and the future-pointing constraint $\text{Im}S_{\text{eff}}[\phi^+,\phi^-] > 0$. One can develop a satisfactory definition of the Green's function in (2.2), 2×2 matrix in CTP internal space $\{+,-\}$

$$\hat{G}_{ab} = \begin{pmatrix} G_{++} & G_{-+} \\ G_{+-} & G_{--} \end{pmatrix},$$
$$G_{mn}(x,x') = \left[\frac{\delta}{i\delta J_m(x)}\right]\left[\frac{\delta}{i\delta J_n(x')}\right] e^{iW[J_+,J_-]}\Big|_{J_+=J_-=0}, \tag{2.8}$$

with the generating functional $iW(J(x)) \equiv \ln Z(J(x))$. The Green's functions satisfy the algebraic identity $G_{++} + G_{--} = G_{+-} + G_{-+}$. Following the restrictions $\delta J_+ = \delta J_- = 0$, we define the classical fields as $\phi_c(x) = \frac{\delta W[\hat{\phi}_\pm,\Omega_\pm]}{\delta J(x)}$. For arbitrary constant $a \in \mathbb{R}$, we have $W[\hat{\phi}_\pm, J_\pm] = W[\hat{\phi}_\pm + a, J_\pm]$ with reflective condition: $W[\phi_+, J_+; \phi_-, J_-] = W[\phi_-, J_-; \phi_+, J_+]$, and normalization condition: $W[\hat{\phi}, J; \hat{\phi}, J] = 0$. It is appropriate to define the generating functional of amputated $1PI$ by the functional Legendre transformation

$$\Gamma[\phi_+,\phi_-] = W[\hat{\phi}_\pm, J_\pm] - c^{ab}\int_{\mathcal{V}} d^4x J_a(x)\phi_b(x). \tag{2.9}$$

It is clear that for a "classical" system one can use the method of steepest descent to obtain an estimate for Γ. In this limit, one has

$$\frac{\delta\Gamma}{\delta\phi_+(x)}\Big|_{\phi_+=\phi_-=\langle\phi\rangle} = 0, \tag{2.10}$$

As the boundary condition $\phi^+(t_f) = \phi^-(t_f)$ introduces the reminiscent long fluctuation, we can still interpret the effective Lagrangian as a gradient expansion of infrared-variables. In analogy with statistical mechanics, where Γ would represent free energy. When (2.10) has ill-defined or multiple minima, it indicates a phase structure with the Landau theory of phase transitions. The CTP formalism then allows calculating dynamical behavior (transport coefficients) around the phase transition [29], via the expansion around equilibrium outlined below, of course, the semiclassical limit neglects fluctuations. If the method of steepest descent is a good approximation, these can be modeled by an expansion of (2.10) around the minimum provided by the effective degrees of freedom. One can get the leading order propagator term

$$\Gamma[\phi_+,\phi_-] = S_{CTP}[\phi_+,\phi_-] - \frac{i\hbar}{2}\ln\ \det\left(\frac{\delta^2 S_{CTP}}{\delta\phi_a(x)\delta\phi_b(x')}\right)^{-1} \times [\phi_+,\phi_-] + \ldots. \tag{2.11}$$

As we will show later, using the generating function technique, we get a self-consistent description of fluid and a general perspective of what symmetries are involved in dissipative phenomena.

B. Fluid dynamics as effective field theory

We devote this section to provide a big picture behind the generation of dissipative polarizable fluid under rotation by effective field theory. As we have known, hydro is a nonlinear effective theory of long-wavelength that encodes collective motion of microparticles. In the context of Lagrangian hydrodynamics (actually any kind of continuum matter), the "field" is nothing else than the position of a fluid cell, $\phi_{i=1,2,3}$. What distinguishes the ideal spinless fluid limit [19] are the reparametrization symmetries of these coordinates. As we will show later, using the generating function technique, we get a self-consistent description of fluid and general perspective of what symmetries are involved in dissipative phenomena.[2]

$$\phi^{iI} \to \phi^{iI} + \alpha^i, \quad \text{with} \quad \alpha^i = \text{const}, \tag{2.12a}$$

$$\phi^{iI} \to R^i_j \phi^{jJ}, \quad \text{with} \quad R^i_j \in SO(3), \tag{2.12b}$$

$$\phi^{iI} \to \xi^{iI}(\phi), \quad \text{with} \quad \det[\partial \xi^{iI}/\partial \phi^{jJ}] = 1, \tag{2.12c}$$

The symmetry (2.12c) forces the Lagrangian of ideal fluid to be of the form

$$\mathcal{L}_{\text{free}} = f(b), \qquad b = \sqrt{\det[B_{ij}]}, \qquad B_{ij} = \partial_\mu \phi_i \partial^\mu \phi_j, \tag{2.13}$$

being b the entropy density. This variational principle deduces Euler's equations, but the converse is not generally true. For further convenience, we shall replace the old-hydrodynamics variables with more suitable ones. In this case, we identify the hydrodynamic entropy vector as a gradient expansion

$$K^{\mu I} \equiv P^{\mu\gamma I}_k \partial_\gamma \phi^k, \qquad P^{\mu\gamma I}_k = \frac{1}{6} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{ijk} \partial_\alpha \phi^{iI} \partial_\beta \phi^{jJ}, \tag{2.14}$$

where $P^{\mu\gamma I}_k$ is a projector and $(I, J) = \{(0,0), (3,3)\}$. The conservation law of entropy [21] is

$$\partial_\mu K^{\mu I} = \partial_\mu (b u^{\mu I}) = 0, \tag{2.15}$$

where the ϕ^{iI} is invariant along its comoving frame $u^\mu \partial_\mu \phi^{iI} = 0$. The velocity vector norm is $u^\mu u_\mu = 1 \to b^2 = K^\mu K_\mu$. Note that the comoving projector is perpendicular to the flow direction

$$\Delta^{\mu\nu} = B^{-1}_{ij} \partial^\mu \phi^i \partial^\nu \phi^j = \eta^{\mu\nu} - u^\mu u^\nu. \tag{2.16}$$

In this language, Kelvin's theorem for circulation conservation in ideal hydrodynamics can be written as an application of Noether's theorem for such a diffeomorophism current going around a closed path [19] which spin breaks, as it has an internal direction, and dissipative effects break as well, in line with [22].

To continue our analytical approach, we shall show how spin variables can be incorporated into the Lagrangian formalism before introducing spin Lagrangian. First, one argues that the energy density, pressure of the medium, and vorticity are not enough thermodynamical quantities to settle out a polarized system [3,4]. Second, this new degree of freedom should act as a source breaking Kelvin's theorem. Following these lines, the last dynamical variable to define polarizable fluid should have a long time behavior matching

$$y_{\mu\nu}|_{\text{Infrared}} \sim u_\alpha \partial^\alpha \sum_i \hat{T}^i \theta_i(\phi) \equiv \chi(b, \omega^{\mu\nu}\omega_{\mu\nu})\omega^{\mu\nu}, \tag{2.17}$$

where $\hat{T}^i$ and θ_i are the generators and local phase,[3] respectively, and the vortical susceptibility $\chi(b, \omega^2)$ represents how inaccessible degrees of freedom coupled with macroscopic ones. Because we cannot neglect feedback of microscopic variables, the fluid turns out nonunitary by assumption. The relation between $y_{\mu\nu}$ and spin is the same as the relation between chemical potential and field phase in [2]. With the help of the energy positivity $y^2 \equiv y^{\mu\nu} y_{\mu\nu} > 0$ and $y^{\mu\nu} u_\mu = 0$, we restrict the polarization form. Note that the nonexistence of Goldstone modes requires polarization parallel to vorticity in thermodynamical equilibrium [2].

Even though the formulation of hydrodynamics system via conservation of stress tensor leads to remarkable results near-equilibrium, the gradient expansion partially presents the fluid behavior since the physics of fluctuation remains out of the partial differential equations [26]. Moving on to a more realistic description, we solve this problem by using the generating functional.

$$Z[b, y^2] = \int \mathcal{D}\phi_i \mathcal{D}y^{\mu\nu} \rho(\phi, y) e^{iS}, \tag{2.18}$$

We begin with a complete description of the dissipative polarizable fluid in accord with symmetries.

where the action S describes the physical model proposed in our work

$$S = S_{\text{free}} + S_{\text{shear}} + S_{\text{bulk}} + S_{\text{pol}}(Y_{\mu\nu}), \tag{2.19}$$

$$S_{\text{shear}} = \int d^4x z_{IJK}(b^2) b^2 B^{-1}_{ij} \partial^\mu \phi^{iI} \partial^\nu \phi^{jJ} \partial_\mu K^K_\nu, \tag{2.20}$$

$$S_{\text{bulk}} = \int d^4x h_{IJK}(b^2) K^{\mu I} K^{\nu J} \partial_\mu K^K_\nu, \tag{2.21}$$

$$S_{\text{free}} = \int d^4x F(b(1 - cy^2)). \tag{2.22}$$

Note that as shown in [2,4] the equilibrium component of polarization can be absorbed into S_{free} and in non-equilibrium polarization necessarily acquires relaxational Israel-Stewart like degrees of freedom, denoted by $Y_{\mu\nu}$. The relaxational parts of the Lagrangian, S_{pol} and S_{IS} will be

[2] CTP internal space are now the capital Latin indices $\{-,+\}$, while the small ones, spatial directions, run 1 to 3.

[3] These solutions are no longer stationary as in the ideal Euler's equations. After a sufficient length of time, if the source is absent, the gradient will vanish, and thus the fluid establish the homogeneous configuration.

TABLE I. Symmetries of various terms beyond local equilibrium hydrodynamics.

	Parity	Time	Charge	$SO(3)$	$S\mathrm{Diff}(\mathbb{R}^{1,3})$
Perfect fluid	Even	Even	Even	Unbroken	Unbroken
Bulk viscosity	Even	Even	Even	Unbroken	Unbroken
Shear viscosity	Even	Even	Even	Unbroken	Broken
Polarization	Odd	Odd	Even	Broken	Unbroken

discussed later in Sec. IV C where $S_{\mathrm{shear,bulk}}$ will also be modified.

Remembering that the coefficients $\{z_{IJK}, h_{IJK}\}$ becomes physical ones $\{\bar{z}_{IJK}, \bar{h}_{IJK}\}$ if we remove the CTP degeneracy $\phi^+ = \phi^-$ in the equations of motion [2]. Following the guidelines of the variational principle, we assume all relations are valid locally. The dissipative construction of the shear viscosity η involves the linear introduction of inverse matrix B^{-1}_{IJ}. This transport coefficient breaks the volume-preserving diffeomorphism group in Table I (a uniform ball-shaped volume element has different dissipative forces than a stick-shaped inhomogeneous element of the same volume). It causes instability by turning the action unbounded from below. For bulk viscosity ξ, it is enough to double K_μ in (2.21) since this dissipative mechanism still preserves the homogeneity, isotropy, and parity symmetry. Finally, in the local polarization phenomena, we slightly perturb the Lagrangian by introducing a term breaking the symmetry in (2.22). This ingredient, forcing each cell fluid from $SO(3)$ to $SO(2)$ group, arises a well-defined polarization with intrinsic anisotropic degrees of freedom. Furthermore, the physical origin of this "new" induced polarization comes from linking vortex to spin (2.17), which introduces a degeneracy that changes the structure stability. We will postpone the stability question for Sec. IV B when including relaxation terms.

III. LINEAR RESPONSE THEORY FOR DISSIPATIVE SPIN HYDRODYNAMICS

A. Fluctuation-dissipation theorem

The motivations for including Navier-Stokes viscosity in polarizable hydrodynamics models are based on theoretical and experimental perspectives. We can argue to provide a more realistic role for predicting the spatial distribution of spin [30], to correct the analytical predictions of polarization momentum dependence [31], to claim dissipative effects of $RHIC$ fluid [32], and for outstanding theoretical properties. We proceed with the approach introduced in Sec. II A, which includes angular momentum as per the prescription of [17], to the density matrix

$$\rho = \mathcal{Z}^{-1} \exp\left[-\beta\left(\mathcal{E} - v\cdot p - \frac{1}{2}\varpi_{\lambda\nu}\cdot J^{\lambda\nu}\right)\right], \qquad (3.1)$$

where the partition function $\mathcal{Z}$ with $\mathrm{Tr}[\rho] = 1$ is the normalizing constant, v is spatial velocity, p is momentum density, $\mathcal{E}$ is Hamiltonian, $\beta = 1/T$ is inverse of temperature, $\varpi_{\lambda\nu}$ is vorticity, and $J^{\lambda\nu}$ is finite total angular momentum. By disturbing the energy density, we have

$$\delta\mathcal{E}(t) = \int d^3x\left[\frac{\delta T}{T}(t,\mathbf{x})\epsilon(t,\mathbf{x}) + p^i(t,\mathbf{x})v_i(t,\mathbf{x}) + Y^{\mu\nu}(t,\mathbf{x})\omega_{\mu\nu}(t,\mathbf{x})\right], \qquad (3.2)$$

where ϵ is the energy density. We discard terms up to second order because the thermodynamics forces are small within the limit of the near-equilibrium state. The ρ evolves in according with

$$\rho(t) = U(t)\rho_0(t)U^{-1}(t), \qquad (3.3)$$

where $U(t)$ is a unitary matrix. After taking an average over the equilibrium ensemble, any operator $\mathcal{A}$ becomes

$$\langle\mathcal{A}\rangle = \mathrm{Tr}[\rho\mathcal{A}] = \langle\rho_{\mathrm{eq}}U^\dagger(t,t_0)\mathcal{A}U(t,t_0)\rangle. \qquad (3.4)$$

Using the interaction picture $U(t) = \mathcal{T}\{\exp(-i\int_0^t dt'\,\mathcal{E}(t'))\}$, we can obtain the perturbative information of how deviations out of equilibrium disappear. Following the linear response approach, the expectation value is calculated via

$$\begin{aligned}\delta\langle\mathcal{A}(t,\mathbf{x})\rangle &= \langle\mathcal{A}(t,\mathbf{x})\rangle - \langle\mathcal{A}(t,\mathbf{x})\rangle_{\mathrm{eq}}\\ &= i\int d^3\mathbf{x}'\int_{-\infty}^{t} dt' e^{\alpha t}\Theta(-t')\\ &\quad\times\langle[\mathcal{A}(t,\mathbf{x}),\delta\mathcal{E}(t',\mathbf{x}')]\rangle_{\mathrm{eq}},\end{aligned} \qquad (3.5)$$

where $\langle\mathcal{A}\rangle_{\mathrm{eq}} = \mathrm{Tr}[\rho_0\mathcal{A}]$ and α encodes the adiabatic switching operation for the external sources. The fluctuation-dissipation theorem establishes the response to small perturbation as the correlation between the perturbed observable and another conjugated one concerning to *energy* in (3.2). The retarded and advanced Green's functions are defined via the Heaviside function Θ by $G_{R/A}(t-t',\mathbf{x}-\mathbf{x}') = -i\Theta(\pm(t-t'))\langle[\mathcal{A}(t,\mathbf{x}),\mathcal{A}(t',\mathbf{x}')]\rangle$. The foregoing discussion allows us to elucidate new physical phenomena from the interaction of polarization, shear, and bulk viscosity. Following the macroscopic perspective, let us begin with the average of polarization

$$\begin{aligned}&\langle Y^{\mu\nu}(t,\mathbf{x})\rangle_\omega - \langle Y^{\mu\nu}(t,\mathbf{x})\rangle_{\mathrm{eq},\omega=0}\\ &\approx +i\int dt'\int_{\mathcal{V}} d^3x'\langle[Y^{\mu\nu}(t,\mathbf{x}),Y^{\alpha\beta}(t',\mathbf{x}')]\rangle_{\mathrm{eq}}\omega_{\alpha\beta}\\ &\quad + i\int dt'\int_{\mathcal{V}} d^3x\langle[Y^{\mu\nu}(t,\mathbf{x}),p^i_T(t',\mathbf{x}')]\rangle_{\mathrm{eq}}v^i_T\\ &\quad + i\int dt'\int_{\mathcal{V}} d^3x\langle[Y^{\mu\nu}(t,\mathbf{x}),p^i_L(t',\mathbf{x}')]\rangle_{\mathrm{eq}}v^i_L,\end{aligned} \qquad (3.6)$$

where the last two terms represent the coupling of polarization with transverse p_T^i and longitudinal p_L^i momentum, respectively. The p_T^i is the shear-induced by polarization [13], while the p_L^i is the anisotropic expansion of the fluid. We can gather varieties of dissipative phenomena by using the statistic correlation function to evaluate (3.6). Nonetheless, these three natural fluctuations will not have the same weight in all physical problems as a result of considering both initial and boundary conditions, as well as the free energy scale. Moreover, the small viscosity and weak coupling of linearized theory restrain the interaction among the fluctuations in (3.6), only proposed for nonlinear circumstances.

As we restrict ourselves to the first-order processes near-thermal equilibrium, the cumulative effects, due to different external sources, also run in a reverse trajectory. Hence, we derive the average of transverse momentum density as

$$\langle p_T^i(t,\mathbf{x})\rangle_\omega - \langle p_T^i(t,\mathbf{x})\rangle_{\mathrm{eq},\omega=0} \approx i\int dt' \int_{\mathcal{V}} d^3x \langle [p_T^i(t,\mathbf{x}), p_T^j(t',\mathbf{x}')]\rangle_{\mathrm{eq}} v_T^j + i\int dt' \int_{\mathcal{V}} d^3x' \langle [p_T^i(t,\mathbf{x}), Y^{\alpha\beta}(t',\mathbf{x}')]\rangle_{\mathrm{eq}} \omega_{\alpha\beta}. \tag{3.7}$$

The last term exhibits a vorticity resistance, called the rotational shear viscosity. It is a mechanism in which the spin-vortex coupling losses angular momentum due to the friction of rotational fluid. We shall recall that viscosity forces tend to quench the velocity gradient from fluid layers. This friction effect, a mechanism of momentum transfer between adjacent fluid layers, produces distinct velocity distribution curves, as shown qualitatively in Fig. 1. For rotational flows, it expresses the velocity increases from outer annuli layers to inner ones, so the dissipation rate depends on the height disc. We then assume that shear viscosity plays a role in the spatial structure of the spin by redistributing its location in the fluid. In practice, the vortical fluid deforms the spin current because spins lose angular momentum and move inward, whereas the shear viscosity transfer angular momentum to outer layers.

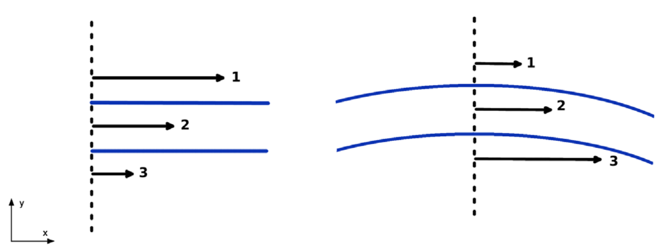


FIG. 1. A schematic view describing the velocity profile shape by arrows for parallel (left) and rotating (right) viscous fluid. The former follows $\sigma_{xy} \sim \eta \frac{\partial v_x}{\partial y}$, with y perpendicular to the interface, while the latter adopts the azimuthal ψ direction $\sigma_{xy} \sim \sigma_{r\psi} \sim (\frac{\partial v_\psi}{\partial r} - \frac{v_\psi}{r}) = \eta r \frac{\partial \Omega}{\partial r}$, with r the radius.

As spins lay out a vortex structure, the vortices interfere with the propagation and scattering of sound waves. This phenomenon was already studied in the literature [33]. Last but not least, the average of longitudinal momentum density is

$$\langle p_L^i(t,\mathbf{x})\rangle_\omega - \langle p_L^i(t,\mathbf{x})\rangle_{\mathrm{eq},\omega=0} \approx i\int dt' \int_{\mathcal{V}} d^3x \langle [p_L^i(t,\mathbf{x}), p_L^j(t',\mathbf{x}')]\rangle_{\mathrm{eq}} v_L^j + i\int dt' \int_{\mathcal{V}} d^3x' \langle [p_L^i(t,\mathbf{x}), Y^{\alpha\beta}(t',\mathbf{x}')]\rangle_{\mathrm{eq}} \omega_{\alpha\beta}, \tag{3.8}$$

where the first correlation function measures the deviation from equilibrium pressure, while the last one determines the antisymmetric pressure from the expansion of fluid with spin.

In principle, to achieve a complete description of spin-shear interplay, we shall establish further assumptions, more precisely at first-order, in accordance with Markovian diffusion. For that reason, the average of any operator at the equilibrium stage must satisfy defined thermodynamic relations in (3.6) at $t<0$ as $\partial \ln \mathcal{Z}/\partial \vec{\omega} = \langle \vec{J}\cdot\hat{\omega}\rangle/T$ and $\partial \ln \mathcal{Z}/\partial v^j = \langle p\cdot\hat{v}\rangle \hat{v}^j/T$. Hence these thermodynamic derivatives in the long wavelength limit are $\lim_{\mathbf{k}\to 0}\chi = \partial Y^{\mu\nu}/\partial\omega^{\mu\nu}$ and $\lim_{\mathbf{k}\to 0} w_o = \partial p^i/\partial v^i$, where w_o is enthalpy, with the initial conditions expressed appropriately as

$$p_{T,L}^i(0,\mathbf{x}) = w_o u_{T,L}^i, \qquad \partial_t \langle p_{T,L}^i(0,\mathbf{x})\rangle_{|t=0} = 0,$$
$$\langle Y^{\mu\nu}(0,\mathbf{x})\rangle = \chi\omega^{\mu\nu}, \qquad \partial_t \langle Y^{\mu\nu}(0,\mathbf{x})\rangle_{|t=0} = 0. \tag{3.9}$$

Beyond that, we shall discuss the general concept of dissipative current around local equilibrium of (3.5)

$$\delta\langle\mathcal{A}(x)\rangle = \int d^4x' G_{\alpha\beta}(x-x')F_\beta(x') + \mathcal{O}(F^2), \tag{3.10}$$

where the Green's function $G_{\alpha\beta}(x-x')$ depends on local currents $\delta\langle\mathcal{A}(x)\rangle$ and external thermodynamical sources $F_\beta(x)$. Note the currents corresponding to $\{Y^{\mu\nu}, p_T^i, p_L^i\}$ have the respective sources $\{\omega^{\mu\nu}, v_T^i, v_L^i\}$. The equation above helps expressing the linear relations of dissipative currents in terms of thermodynamics forces asymptotically close to equilibrium. Nonetheless, these same linear equations rule, in the statistical equilibrium, the decaying of the steady-state fluctuations. Thus we cannot predict if a fluctuation or a small external thermodynamical source creates a perturbation. This observation elucidates that different classes of independent elementary mechanisms involved in the irreversible integral (3.10) yield the same expected result due to microscopic reversibility. We then categorize this property according to the famous Onsager reciprocity $G_{\alpha\beta} = G_{\beta\alpha}$, associated with detailed balance [26].

B. Variational approach

In this subsection, we present an alternative way to derive linear relations between dissipative and spin terms near equilibrium. The motivation is that previous method can lack a clear physical meaning for generic two-point correlation functions. Even though thermodynamical identity and conservation laws are the tools for writing fluctuation-dissipation theorem, previous studies already anticipated the observable as spin requires knowledge that goes beyond the simple expansion in gradients of conserved quantities [13]. For instance, we cannot determine the correlation between acceleration of spin and energy density. To bypass this difficulty, we suggest another scheme in which metric fluctuations rather than external sources originate the small deviation from equilibrium. We introduce in (3.5) the stress tensor operator

$$\delta\langle\mathcal{T}^{\mu\nu}(t,k_i)\rangle=\int_{-\infty}^{\infty}dt'\Theta(-t')e^{\alpha t'}G^{\mu\nu}_{\lambda\delta}(t-t',k_i)\mathcal{S}^{\lambda\delta}(t',k_i), \tag{3.11}$$

where the new source $\mathcal{S}^{\lambda\delta}$ encodes not only metric fluctuation $h^{\mu\nu}$ but also gauge field ω^{μ}. We can replace the energy-momentum tensor for the conserved current $\mathcal{J}^{\mu\nu}$. We explicitly rewrite (3.11) by using the variational principle[4] [34]

$$G^R_{T^{\sigma\tau}T^{\mu\nu}}=-2\frac{\delta\mathcal{T}^{\sigma\tau}}{\delta h_{\mu\nu}}\bigg|_{h^{\alpha\beta}=\omega^{\alpha}=0},\qquad G^R_{T^{\mu\nu}J^{\sigma}}=-\frac{\delta\mathcal{T}^{\mu\nu}}{\delta\omega_{\sigma}}\bigg|_{h^{\alpha\beta}\equiv\omega^{\alpha}=0},$$
$$G^R_{J^{\sigma}T^{\mu\nu}}=-2\frac{\delta\mathcal{J}^{\sigma}}{\delta h_{\mu\nu}}\bigg|_{h^{\alpha\beta}=\omega^{\alpha}=0},\qquad G^R_{J^{\mu}J^{\nu}}=-\frac{\delta\mathcal{J}^{\mu}}{\delta\omega_{\nu}}\bigg|_{h^{\alpha\beta}=\omega^{\alpha}=0}. \tag{3.12}$$

By inspection, these equations give in the first order the equivalent of the propagator in (2.11). By conservation of energy-momentum, one finds (only the equilibrium d.o.f.s are kept for now)

$$\omega(G_{T^{x0},T^{x0}}(\omega,k_x)+\underbrace{F_y y-f}_{\epsilon})=k_xG_{T^{x0},T^{xy}}(\omega,k_x), \tag{3.13}$$

$$\omega(G_{T^{x0},T^{xy}}(\omega,k_x))=k_x(G_{T^{xy},T^{xy}}(\omega,k_x)+\underbrace{f-f_bb}_{P}), \tag{3.14}$$

where P is the fluid pressure. Note that the contact terms appear naturally because of the reminiscent "functions" $\delta(\omega)\delta^3(\mathbf{k})$. Let us then sum the two equations above

$$\omega^2G_{T^{x0},T^{x0}}(\omega,k_x)-k_x^2G_{T^{xy},T^{xy}}(\omega,k_x)=0. \tag{3.15}$$

Here, by suppressing the contact terms, we satisfy the conservation law. The correlation function decays to asymptotic configuration in long-wavelength limit where the well-known thermodynamic quantity emerges

$$\lim_{k_x\to0}G_{T^{0x},T^{0x}}(0,\mathbf{k})=\int d^3\mathbf{x}\int_0^{\beta}dt\langle T^{0x}(t,\mathbf{x})T^{0x}(0)\rangle$$
$$=\beta(\langle\mathcal{E}T^{0x}\rangle-\langle\mathcal{E}\rangle\langle T^{0x}\rangle). \tag{3.16}$$

The enthalpy is explicitly written analogously to [20,21]

$$\lim_{\omega\to0}\lim_{k\to0}\frac{1}{\omega}\mathrm{Re}G^R_{T^{0x},T^{0x}}(\omega,\mathbf{k})=\epsilon+P=F_yy-f_bb. \tag{3.17}$$

The explicit relation between the retarded Green function and the transport coefficient originated from Kubo formulas are

$$\frac{1}{12}\lim_{\omega\to0}\lim_{k\to0}\partial^3_{\omega}\mathrm{Re}G^R_{T^{0x},T^{0x}}(\omega,\mathbf{k})=\chi^2, \tag{3.18}$$

$$\lim_{\omega\to0}\frac{1}{\omega}\lim_{k\to0}\mathrm{Im}G^R_{T^{0x},T^{0x}}(\omega,\mathbf{k})$$
$$=f_b+b_o^3(2\bar{z}_{003}-3\bar{z}_{033}+4\bar{z}_{303}+3\bar{z}_{333}), \tag{3.19}$$

$$\lim_{\omega\to0}\lim_{k\to0}\frac{1}{k}\partial_{\omega}G^R_{J^x,\omega^{xy}}=b_o^3(\bar{h}_{003,03}+\bar{h}_{303,03}+\bar{h}_{333,03}+\bar{h}_{033,03}+2\bar{h}_{300,03})-4b_o^5(\bar{h}_{333,00}+\bar{h}_{303,00})$$
$$+4b_o^5(\bar{h}_{003,33}+\bar{h}_{303,33}+\bar{h}_{333,33}+\bar{h}_{033,33}+\bar{h}_{330,33}+\bar{h}_{300,33}). \tag{3.20}$$

The Eqs. (3.18)–(3.20) are valid within a limited range (near-equilibrium) and fail to include nonperturbative analysis. We have then identified the relevant couplings from polarizable dissipative fluid in terms of the effective action expansion (2.19).

C. Interactions of hydrodynamic modes

In the previous section, the results obtained are governed by a linearized hydrodynamic regime in which we characterize slow internal fluctuations by an averaging over the equilibrium ensemble. To go beyond linear response while remaining in the perturbative domain, we analyze the correlation function behavior in more complex circumstances. Following this thought, the structures closely

[4]Note that the ϕ_Is and $h_{\mu\nu}$ are equivalent ways of encoding metric perturbations, so the canonical energy-momentum tensor described in the Appendix A and the metric one are equivalent.

connected with thermal fluctuations are a dominant factor in studying nonhydrodynamics modes. Hence, one can investigate the collective excitations appearing in a limited region $(\omega, \mathbf{k})$, sensitive to microscopic dynamics. Furthermore, we also consider the hydrodynamics modes subjected to nonanalytical conditions whose correlation functions decay at long-time power law tails.

Our main point is to understand how vorticity effects and spin thermal fluctuations interplay in a dissipative polarizable fluid. We continue to study this system from a perturbative perspective by following the approach [28], adapted to the Lagrangian picture. Examining the stress tensor in the co-moving frame, up to the second-power of the hydrodynamical variable, we have

$$\begin{aligned} T^{pq} \sim & \left(c_s^2[\partial\pi^2] + \frac{1}{2}(1+c_s^2)(\dot{\pi}[\partial\pi] + \dot{\pi}^2) + \frac{1}{6}(3c_s^2 + f_3)[\partial\pi]^2 \right)\delta^{pq} - 2(z_{00K} + h_{00K,00})\delta_k^p[\partial\dot{\pi}]\partial^q\pi^{kK} \\ & + z_{IJ0}\left(\pi^{mI}\pi^{nJ}S_{mn}^{pq} + \left(\dot{\pi}^2 + \frac{1}{2}[\partial\pi]^2 - [\partial\pi\cdot\partial\pi] \right)\delta^{pq}\delta_J^I + (\dot{\pi}\cdot\partial\pi^{pI} - [\partial\pi]\dot{\pi}^{pI})\delta^{qJ} \right) + 2F_b\chi^2 \\ & \times (g_{i[\rho}\partial_{0]}\dot{\pi}^i\ddot{\pi}^\rho\delta^{pq} + \dot{\pi}^i H_{ij}^{pq}\dot{\pi}^j) + 2F_b\chi y_\rho^p\delta^{0q} + \frac{c_P^2}{3}F_b y_\sigma^p\epsilon^{q\rho\alpha}\epsilon_{0\rho i}\partial^\sigma\partial_\alpha\pi^i + \ldots, \end{aligned} \tag{3.21}$$

where the dots contain third or higher-order terms of π. The f_3 corresponds to f''', c_s is the sound speed, c_P is the longitudinal perturbation emitted by vortex-spin source, and the projectors read $S_{mn}^{ab} = \frac{1}{2}(\delta_m^a\delta_n^b + \delta_n^a\delta_m^b - \frac{2}{3}\delta^{ab}\delta_{mn})$ and $H_{kl}^{ij} = (\delta_k^i\delta_l^j - \delta_l^i\delta_k^j)$. The spin current is

$$\begin{aligned} J^p \sim & \left(\dot{\pi}^2\delta_0^p - \frac{1}{2}(c_s^2 - f_3)\partial^p\pi\cdot\dot{\pi} \right) + \frac{1}{6}(h_{0JK,00} + z_{0JK})\dot{\pi}^J\cdot(\dot{\pi}^K\delta_0^p + \partial^p\pi^K) + \frac{1}{6}z_{IJ0}\epsilon^{pab}\epsilon_{0mn}\partial_a\pi^{mI} \\ & \times \partial_b\pi^{nJ} + 2z_{IJK}(\dot{\pi}^I\cdot\dot{\pi}^J\delta^{pK} + \dot{\pi}^I\cdot\partial\pi^{pJ}\delta_0^K) + z_{000}(\dot{\pi}^2 + [\partial\pi]^2 - \partial^p[\partial\pi^2]) + w_o c_p^2 F_b\chi\partial^p\partial_j\pi^j \\ & - 2w_o F_b(\chi^2\partial^p\dot{\pi}^I\cdot\partial\dot{\pi}^I - \chi\partial^p\chi[\partial\dot{\pi}\cdot\partial\dot{\pi}]) + \ldots. \end{aligned} \tag{3.22}$$

This current, a conserved dynamical variable, presents the irreversible flux as a decomposition of polarization J_P and hydro J currents. The physical meaning of T^{pq} and J^p is to explore the influence of collective excitations on transport coefficient values. These short-lived modes involve nonequilibrium states, which correct the bare Green's function in Eqs. (3.18)–(3.20).

To evaluate the quadratic-order correlation (3.21), we assume Gaussian fluctuations (Markovian dynamics). It is because the fluctuation-dissipation theorem lies in regions where the stable steady states have a well-distinct small and large-scale time [26].

$$\begin{aligned} G^{(2)}_{T^{ij}T^{kl}}(t,\mathbf{x}) &= \frac{T}{w_0^2}\langle \pi_i(t,\mathbf{x})\pi_j(t,\mathbf{x})\pi_k(0)\pi_l(0)\rangle_{\text{eq}}, \\ &= \frac{T}{w_0^2}\langle \pi_i(t,\mathbf{x})\pi_k(0)\rangle_{\text{eq}}\langle \pi_j(t,\mathbf{x})\pi_l(0)\rangle_{\text{eq}}, \\ &= \frac{2T}{w_o^2}\int\frac{d\omega'}{2\pi}\int\frac{d^3k'}{(2\pi)^3}G^{(0)}_{T^{0i}T^{0j}}(\omega',\mathbf{k}')G^{(0)}_{T^{0k}T^{0l}}(\omega-\omega',\mathbf{k}-\mathbf{k}'), \\ &= (S^{ij}_{mn}S^{kl}_{pq} + H^{ij}_{mn}H^{kl}_{pq})\frac{2T}{w_o^2}\int\frac{d\omega'}{2\pi}\int\frac{d^3k'}{(2\pi)^3}G^{(0)}_{T^{0i}T^{0j}}(\omega',\mathbf{k}')G^{(0)}_{T^{0k}T^{0l}}(\omega-\omega',\mathbf{k}-\mathbf{k}'), \end{aligned} \tag{3.23}$$

factorizing the Green function as the product of two zero-order ones. In particular, we evaluate (3.23) in the long-wavelength limit $\chi^2\mathbf{k}^2 \ll \gamma(z,h)\mathbf{k} \ll c_s, c_P$ by

$$\begin{aligned} G^{(0)}_{T^{0i}T^{0j}}(\omega,\mathbf{k}) = \frac{w_o T}{2}\Bigg[& \left(\delta^{ij} - \frac{k^ik^j}{\mathbf{k}^2}\right)\frac{\mathbf{k}^2 - \gamma_\eta\omega^2\mathbf{k} - 2\chi^2(\omega^4 - \mathbf{k}^2\omega^2)}{f_b\omega^2 - i\gamma(z,h)\omega\mathbf{k}^2 - 2\chi^2\omega^4} \\ & + \left(\frac{k^ik^j}{\mathbf{k}^2}\right)\frac{\omega^2 + \gamma_\eta\omega\mathbf{k}^2 - \chi^2(3\omega^4 + 2\omega^2\mathbf{k}^2)}{f_b(\omega^2 - c_s^2\mathbf{k}^2) - i\gamma_\eta(2\omega^2\mathbf{k} + 3\omega\mathbf{k}^2) - i\gamma(z,h)\omega\mathbf{k}^2 - 2\chi^2(\omega^4 - \mathbf{k}^2\omega^2)}\Bigg], \end{aligned} \tag{3.24}$$

where $\gamma(z,h) = \gamma_\eta(z) + \gamma_\xi(h)$ is the function composed by elementary components of the shear $\{z_{IJK}\}$ and bulk $\{h_{IJK,LM}\}$ viscosity, respectively. Making the following association with (3.23), we obtain

$$\mathcal{V}\langle\{T^{ij}(t,\mathbf{x}),T^{kl}(0)\}\rangle = \frac{T}{w_o^2}(S^{ij}_{mn}S^{kl}_{pq}+H^{ij}_{mn}H^{kl}_{pq})G^{(2)}_{T_{mn}T_{pq}}(t,\mathbf{x}), \tag{3.25}$$

where $\mathcal{V}$ is the spatial volume. For $t>0$, to characterize the hydrodynamic regime, we take the limit at zero momentum and low frequency

$$\mathcal{V}\langle\{T^{ij}(t,\mathbf{x}),T^{kl}(0)\}\rangle \sim \mathcal{O}(\Lambda) + \frac{H^{ij}_{kl}}{w_o}\frac{T^2}{6\pi}\left(\frac{\chi^{-5/2}}{t^{3/2}} + \left[3+\left(\frac{1}{6}\right)^{1/2}\right]\frac{\chi^{-3/2}}{2t^{1/2}}\right) - \frac{S^{ij}_{kl}}{60\pi}\left[7+\left(\frac{3}{2}\right)^{3/2}\right] \times \frac{1}{(\gamma_\eta(z)t)^{3/2}} + (\text{exponential decay}), \tag{3.26}$$

where $\mathcal{O}(\Lambda)$ represents the correction for transport coefficient from shear and sound collective modes. Before starting this discussion, let us analyze the symmetry properties of the excitations. The broken rotation group in Table I determines the spin alignment (orientation) as a new hydrodynamic degree of freedom responsible for describing the equilibrium thermodynamic state. This physical quantity acts as a nonconservative source generating a nonvanishing current from the conversion of angular into spin momentum. We found this current-current correlation, more precisely the autocorrelation of spin velocity $\langle v_s(t), v_s(0)\rangle$, decays by a time power-law $\sim t^{-3/2}$. Supposing at $t=0$, the average velocity of a spin particle $\langle v_s\rangle^2$ decreases slowly through an elliptical shell region in the neighborhood of a rotating fluid. Since the randomization motion governs this whole process, the mechanism of vorticity diffusion is responsible for spreading out the average spin velocity inside this shell column. As the neighborhood rearranges at a time τ longer than the spin takes to approach a "local equilibrium," we expect therefore the decay of $\langle v_s\rangle^2$ be $t\sim A/(\theta\times r)^2\tau$, with the azimuthal angle θ, and the transverse area of elliptical shell column A. Recalling this diffusion process, shear momentum induced from vortical susceptibility, depends on the shell surface radius r Fig. 1.

The contribution of each separable sum in (3.26) represents a conserved quantity of collective modes. Indeed, these modes emerge from length scales ∼ microscopic degrees of freedoms, not-generated by old-fashioned hydrodynamic [35]. This relevant aspect can be understood as a breaking down of gradient expansion signature for out-of-equilibrium systems. In this case, the fluctuation (3.26), at the initial condition, is sensible to the presence of nonanalytical and exponential terms.

IV. FEYNMAN DIAGRAMS FOR POLARIZABLE FLUID WITH DISSIPATION

This section uses Feynman diagram techniques to formulate a guide to how the polarization current modifies with the inclusion of well-know dissipative effects. We apply the ideas developed in [24,25] to explore the behavior of a dissipative polarizable fluid. Even though our effective action (2.19) in $(3+1)$ dimensions is nonrenormalizable because of the coupling χ with negative mass dimension, it should not be interpreted as ultraviolet completion of a spinless fluid. Our independent d.o.f.s are summarized in the Table II below.

A. The linearized effective action

Let us incorporate perturbative effects into the generating functional

$$Z[b,y^2] = \int \mathcal{D}\phi^i\mathcal{D}y^{\mu\nu}\rho(\phi^i,y)e^{i\int d^4x(\mathcal{L}_{\text{shear}}+\mathcal{L}_{\text{pol}})}, \tag{4.1}$$

by expanding at the long-wavelength and low frequencies, the Lagrangian (2.19) up to fourth-order. We can easily separate the perturbative information from the free hydromodes of the 2-point correlator function $\langle\delta K^0,\delta K^0\rangle$ at $\mathcal{O}(w_o)$ order

$$\mathcal{L}_{\text{free}} = w_o f_b\left(\frac{1}{2}\dot{\pi}^2 - \frac{1}{2}c_s^2[\partial\pi]^2 + \frac{1}{2}[\partial\pi^T\partial\pi]\right) - z_{0JK}(\dot{\pi}^J\cdot\partial[\partial\pi^K] + \partial^2\pi^J\cdot\partial\pi^K + [\partial\pi^J\cdot\partial\dot{\pi}^K]) + 2z'_{0JK}[\partial\dot{\pi}^J][\partial\pi^K]) + F_b\left(\frac{1}{2}\dot{\pi}^2 - \frac{1}{2}c_p^2[\partial\pi]^2 - \chi^2(\partial_\mu\dot{\pi}\cdot\partial^\mu\dot{\pi} + [\partial\dot{\pi}\cdot\partial\dot{\pi}])\right), \tag{4.2}$$

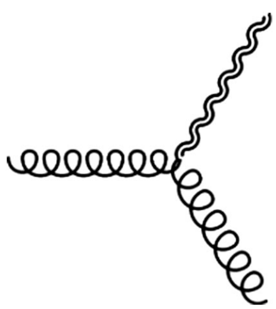
from the interacting

$$\begin{aligned}\mathcal{L}_{\rm int} \supset\, & w_o^2\left[\frac{c_s^2}{2}[\partial\pi][\partial\pi^2] + \frac{1}{2}(1+c_s^2)[\partial\pi]\dot{\pi}^2 - \frac{1}{6}(3c_s^2+f_3)[\partial\pi]^3 - \dot{\pi}\cdot\partial\pi\cdot\dot{\pi} - c_s^2[\partial\pi]\det\partial\pi\right]\\ & + w_o z_{IJK}[\pi^I\cdot\partial\pi^J\cdot\partial\pi^K + 3[\partial\pi^I]\pi^J\cdot\partial\pi^K - \pi^I\cdot\pi^J[\partial\dot{\pi}^K] - (\partial\pi^I)\cdot(\pi^J\cdot\partial\pi^K)]\\ & + w_o\chi^2[[\partial\pi]([\partial\dot{\pi}\cdot\partial\dot{\pi}] + (\partial_\mu\dot{\pi})\cdot(\partial^\mu\dot{\pi}))](1+c_s^2) + z_{IJK}[|\partial\pi^T\cdot\dot{\pi}^I|^2\delta_K^J + [\partial\pi]^2\dot{\pi}^I\cdot\dot{\pi}^J\delta_0^K\\ & + [\partial\pi]\dot{\pi}^I\cdot\partial\dot{\pi}^J\cdot\dot{\pi}^K + \ldots] + z_{III}\chi^2[[\partial\dot{\pi}][\partial\dot{\pi}\cdot\partial\dot{\pi}] + \partial^2[\partial^2\pi]\ddot{\pi}\cdot\pi + 2[\partial^2\pi]^2[\partial\dot{\pi}] + \ldots],\end{aligned} \tag{4.3}$$

and self-interacting ones

$$\begin{aligned}\mathcal{L}_{\rm self-int} \supset\, & w_o^2\left[\frac{1}{2}[\partial\pi]\dot{\pi}^2 + \frac{c_s^2}{2}[\partial\pi]^3 + \frac{1}{2}[\partial\pi]^2[\partial\pi^2] + \frac{(1+c_s^2)}{2}[\partial\pi]^2\dot{\pi}^2 - \frac{1}{6}(3c_s^2+f_3)[\partial\pi]^4 - [\partial\pi]\right.\\ & \left.\times\,\dot{\pi}\cdot\partial\pi\cdot\dot{\pi} - c_s^2[\partial\pi]\det\partial\pi] + z^2[(\partial[\partial\pi])\cdot(\partial[\partial\pi])[\partial\pi] + 2\dot{\pi}\cdot\partial\dot{\pi}[\partial\dot{\pi}] + 2[\partial\pi][\partial\dot{\pi}]^2 + \ldots\right]\\ & + 2\chi_b\chi[[\partial\pi][\partial\dot{\pi}\cdot\partial\dot{\pi}] + [\partial\pi](\partial_\mu\dot{\pi})\cdot(\partial^\mu\dot{\pi})] + \chi\partial_{\omega^2}\chi[(\partial_\mu\dot{\pi}^2)\cdot(\partial_\mu\dot{\pi}^2) + 2(\partial_\mu\dot{\pi})(\partial_\mu\dot{\pi})\\ & \times[\partial\dot{\pi}\cdot\partial\dot{\pi}] + [\partial\dot{\pi}\cdot\partial\dot{\pi}]^2 + \ldots].\end{aligned} \tag{4.4}$$

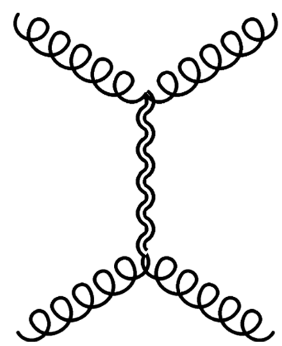
To derive the Feynman diagrams, we need to present $W(J)$ (4.1) in a manageable form. The main Feynman propagators are in the Table II. For the general scattering process, all amplitude are calculated in a long-wavelength approximation. The Feynman rules: all "in-" and "out-" states are *on-shell*, it means, they satisfy the Euler hydrodynamics equation, internal line $1/w_o$, external line $1/\sqrt{w_o}$.[5] Consider the amplitude decay mechanism of the diagram below



where a transverse polarization decay into a transverse polarization and excitation. The amplitude of this tree-level diagram is

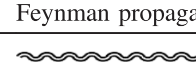
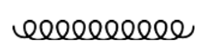
$$\begin{aligned} i\mathcal{M}_{\mathcal{T}\to\mathcal{T}\mathcal{T}} = \frac{(\hat{\epsilon}\cdot\hat{k})\mathbf{k}^2}{\sqrt{w_o}}\{ & z_{I00}[2\sin(3\theta/2)]\\ & + z_{III,00}[\sin\theta(c_s^2 - 2\cos\theta)] - 4c_s\sin(\theta/2)\\ & + z_{0I0}\chi^2[\omega\mathbf{k}\sin\theta + \mathbf{k}^2\sin(2\theta)]\},\end{aligned} \tag{4.5}$$

where θ is the angle between the ingoing and outgoing transverse mode. The kinematical restriction of energy and angular momentum conservation, as well as the breaking symmetry, impose constraints on the amplitude. Before going further, we shall note that the energy required for the vortex to act as the emission of transverse excitations has to be greater than already found in [25]. If these specific conditions are satisfied, we can draw a class of relevant Feynman diagrams which manifest the coupling of η with χ^2. Now we restrict ourselves to three-level scattering problems $\mathcal{T}\mathcal{T}\to\mathcal{T}\mathcal{T}$ in which the polarized dissipative problems take new phenomena.



where the scattering amplitude for the diagram above is

TABLE II. We list each Feynman line of polarizable fluid with dissipation.

	Feynman propagator	
Transverse excitation		$\frac{i}{\omega^2 - c_s^2\mathbf{k}^2 + i\gamma_\eta(z)\omega\mathbf{k}^2}$
Longitudinal excitation		$\frac{i}{\omega + i\gamma_\eta(z)\mathbf{k}^2}$
Transverse Polarization		$\frac{i(\omega^2-\mathbf{k}^2)}{(\omega^2-\mathbf{k}^2)-\chi^{-2}}$
Longitudinal Polarization		$\frac{i}{\omega^2 - c_p^2\mathbf{k}^2 - \chi^{-2}}$

[5] A pedagogical review could be found in [19].

$$i\mathcal{M}_{\mathcal{TT}\to\mathcal{TT}} = \frac{1}{w_o}\Big\{ z_{III}\chi^2 c_p^2[(\omega^2\mathbf{k}^4 - (\omega\mathbf{k})^3(3-\cos\theta)) + \omega^4\mathbf{k}^2(\hat{k}_1\cdot\hat{k}_2)] + z_{III}[\omega^3\mathbf{k} + (\omega\mathbf{k})^2\cos(\theta/2)] + \omega^4(2\mathbf{k}^2\partial_b\chi + \omega^3\partial_{\omega^2}\chi)[(\hat{k}_1\cdot\hat{k}_2)+\cos\theta]^2 + \frac{1}{2}\mathbf{k}^4 c_s^2(\cos\theta - \cos(2\theta))\Big\} iD_{ij}^{(0)}(\omega,\mathbf{k}), \tag{4.6}$$

the terms proportional to z_{III} contribute to the sound generation effect and χ^2 provides relevant information to the transverse generation by vortices source. These phenomena are qualitatively different from those studied in [25] because both the sound mode and dissipative mode are included. Unavoidably, as we have seen above, this means that shear modes, not just propagating sound waves but the symmetric shear tensor fields sourcing heat via shear viscosity, couple to polarization via $z_{III}\chi^2$. It is not surprising that χ^2 turns the fluid dynamics into a non-renormalizable theory. In the next sections, we shall explore the more direct physical consequences of this new process.

B. Polarization mass correction

The case of virtual correction for ideal polarized fluid has been discussed in [25]. Our starting point is the free Lagrangian (4.2). from the longitudinal polarization and excitation terms one can obtain the propagators

$$iH_{ij}^{(0)}(\omega,\mathbf{k}) = \frac{1}{w_o}\frac{i}{\omega^2 - c_p^2\mathbf{k}^2 - \chi_0^{-2}},$$
$$iD_{ij}^{(0)}(\omega,\mathbf{k}) = \frac{1}{w_o}\frac{i}{\omega^2 - c_s^2\mathbf{k}^2 + i\gamma_\eta(z_0)\omega\mathbf{k}^2}, \tag{4.7}$$

where we denote the bare quantities with the subscript 0. The beauty of this propagator is the presence of shear viscosity and spin degrees of freedom. As we are primarily interested in the complete correction of the 2-point Green's function, we include the sum of all relevant one-particle-irreducible contribution

$$iH_{ij}(\omega,\mathbf{k}) = iH_{ij}^{(0)}(\omega,\mathbf{k}) + iH_{im}^{(0)}(\omega,\mathbf{k})i\tilde{\Sigma}_{mn}(\omega,\mathbf{k}) \times iH_{nj}^{(0)}(\omega,\mathbf{k}) + \ldots,$$
$$iD_{ij}(\omega,\mathbf{k}) = iD_{ij}^{(0)}(\omega,\mathbf{k}) + iD_{im}^{(0)}(\omega,\mathbf{k}) \times i\tilde{\Sigma}_{mn}(\omega,\mathbf{k})iD_{nj}^{(0)}(\omega,\mathbf{k}) + \ldots, \tag{4.8}$$

where the self-energy function $\tilde{\Sigma}_{mn}$ encodes all reliable aspects of dissipative and polarized effects in effective field theory. We begin with the trivial sum for the inverse propagator

$$H_{ij}^{-1}(\omega,\mathbf{k}) = \omega^2 - c_p^2\mathbf{k}^2 - \chi_0^{-2} + \tilde{\Sigma}_{\omega,\mathbf{k}},$$
$$D_{ij}^{-1}(\omega,\mathbf{k}) = \omega^2 - c_s^2\mathbf{k}^2 + i\gamma_\eta(z_0)\omega\mathbf{k}^2 + \tilde{\Sigma}_{\omega,\mathbf{k}}, \tag{4.9}$$

where $\tilde{\Sigma}_{\omega,\mathbf{k}}$ is the sum of all one-particle-irreducible Feynman diagrams. Physically, these diagram's role is analogous to the electroweak mixing of η and η' mesons: shear-polarization coupling can, beyond leading order, lead to physical states which are mixtures of states defined by their symmetry properties.

We therefore decompose this self-energy contribution as

$$\tilde{\Sigma}_{\omega,\mathbf{k}} = i(\Sigma_{\omega,\mathbf{k}}^P + \Sigma_{\omega,\mathbf{k}}^S)\mathbb{1}, \tag{4.10}$$

being $\mathbb{1}$ the unitary matrix. We label the polarized and shear contributions by the superscripts S and P, respectively. In the lowest order, the relevant contributions for the one-loop diagrams are[6]

$$i\Sigma_{\omega,\mathbf{k}}^P = -\chi^2\int\frac{d^4q}{(2\pi)^4}H_{im}^{(0)}(q)H_{mj}^{(0)}(k-q) - \chi z_{IJK} \times \int\frac{d^4q}{(2\pi)^4}H_{im}^{(0)}(q)D_{mj}^{(0)}(k-q) - z^2\int\frac{d^4q}{(2\pi)^4}D_{im}^{(0)}(q)D_{mj}^{(0)}(k-q). \tag{4.11}$$

Our effective field method requires a careful calculation to separate the infrared and ultraviolet contributions, and as a consequence, the $SO(3)$ symmetry is broken. The self-energy diagrams are

Each loop in Fig. 2 displays one general property of dissipative fluids with spin. The (a) loop, already investigated in [25], corresponds to the dissipative vortex mass. The (b) loop tells us the interaction between shear and polarization, responsible for driving the vortex toward the principal axes, loses angular momentum from vortex-spin coupling to fluid. The loop (c) is mediated by compressional modes due to χz^2 coupling. Let us now estimate the one-loop diagrams for the longitudinal excitation.

The (a) loop provides a dissipative mass to sound waves generated by the vortex-spin coupling. Such a process produces an infrared limit to compressional modes given by the vortex mass. In fact, such configuration arises only if the momentum of internal line is $k^2 > \chi^{-2}$. The (b) loop mediates the interaction between sound waves and vortex-coupling. Below the energy scale χ^{-2}, the vortices are massless low-energy degrees of freedom where sound waves scattering elastically with spins. Above this gap,

[6]The polarization Feynman propagator in (4.7) is free of ultraviolet divergences if we consider a relaxation time [4].

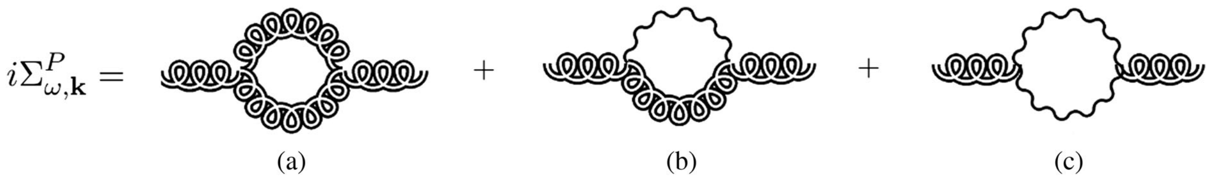


FIG. 2. The diagrams corresponds to the self-energy correction of the longitudinal polarization propagator.

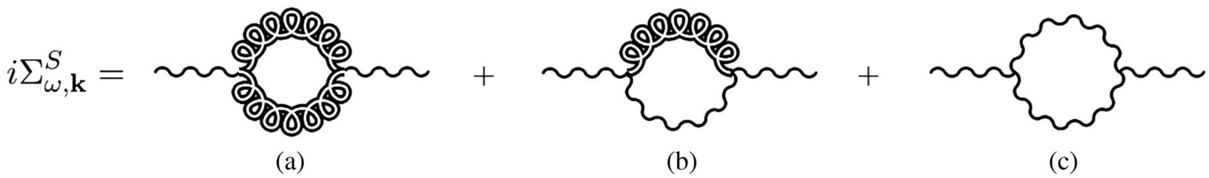


FIG. 3. The one-loop corrections to the longitudinal excitation.

we have access to inelastic scattering and absorption processes of sound waves in the vortex-spin. In the general case, we observe anisotropy due to the angle between spin and sound vector. We here restrict ourselves to sound waves propagating parallel to spin in Fig. 3, so that the orientation of sound wave velocity is not anisotropic. We can compute these implications in the dynamics of longitudinal compressional modes by insertion of (4.10) into $iD^{(0)}_{ij}$ [refer to Eq. (4.16)].

Our framework allows us to absorb the divergence in the parameters of effective field theory, a necessary procedure even though this theory is nonrenormalizeable [36]. The renormalization constants are expanding around tree-level solution

$$Z_i = 1 + \sum_{j=1}^{\infty} \frac{1}{\epsilon^j} Z_{ij}(z,\chi), \tag{4.12}$$

where the analytical functions Z_{ij} are independent of ϵ and exclusively dependent on hydrodynamic couplings

$$Z_{ij} = \begin{cases} 1+\delta Z_{ii}, & i=j \\ \delta Z_{ij}, & i \neq j \end{cases} \tag{4.13}$$

We split the bare fields and insert the renormalized constants to render ultraviolet finite states to Green's function (2.8).

$$\begin{pmatrix} \chi_0 \\ z^0_{IJK} \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{2}\delta Z_{\chi\chi} & \delta Z_{\chi z} \\ \delta Z_{z\chi} & 1+\frac{1}{2}\delta Z_{zz} \end{pmatrix} \begin{pmatrix} \chi \\ z_{IJK} \end{pmatrix}, \tag{4.14}$$

where the renormalized parameters $\{z_{IJK}, \chi\}$ are finite. The expansion of this matrix only correspond to the vertex corrections because the field π is renormalized independently $\pi_0 = Z^{1/2}_{\pi\pi}\pi = (1+\frac{1}{2}\delta Z_{\pi\pi})\pi$. By linking the bare parameters with the measured ones, we can renormalize the coupling and fields. Since this matrix is no longer diagonal, we cannot express the renormalized parameters as eigenvalues of the bare ones. As we expected from the assumption of microscopic reversibility (5.1), the dissipative currents of shear and polarization take place simultaneously on a fluid cell. Switching these probe forces produces the same effect (commutation), so the counterterms of nondiagonal elements produce an orthogonal matrix[7]

$$\delta Z_{\chi z} = \delta Z_{z\chi}. \tag{4.15}$$

With the previous notation, we can obtain the renormalized propagator (4.7). In the case of polarization, the on-shell renormalization condition fixes the dissipative mass, found in [25], by the restrictions $\Sigma^P_{\omega,\mathbf{k}}(k^2)|_{k^2=\chi^{-2}} = 0$ and $\frac{d}{dk^2}\Sigma^P_{\omega,\mathbf{k}}(k^2)|_{k^2=\chi^{-2}} = 0$. Up to first order, the sum of all $1PI$ diagrams is $\Sigma^P_{\omega,\mathbf{k}} = Z_{\pi\pi} + \frac{k^2}{2}\delta Z_{\pi\pi} - (\frac{1}{2}\delta Z_{\chi\chi} + \frac{1}{2}\delta Z_{\pi} + \delta Z_{\chi z})\chi^{-2}$. Next, we write the exact propagator of longitudinal polarization and excitation as

$$iH_{ij}(\omega,\mathbf{k}) = \frac{1}{w_o}\frac{iL_{ij}}{\omega^2 - c_p^2\mathbf{k}^2 - \chi^{-2}},$$
$$iD_{ij}(\omega,\mathbf{k}) = \frac{1}{w_o}\frac{iL_{ij}}{\omega^2 - c_s^2\mathbf{k}^2 + i\gamma_\eta(z)\omega\mathbf{k}^2}, \tag{4.16}$$

where the renormalized coupling constants are $\chi^2(\omega) = \chi_0^2 + \mathrm{Im}\Sigma^S_{\omega,\mathbf{k}} + \mathrm{Re}\Sigma^P_{\omega,\mathbf{k}}$ and $\gamma_\eta(\omega) = \gamma_\eta(z_0) + \mathrm{Im}\Sigma^P_{\omega,\mathbf{k}} + \mathrm{Re}\Sigma^S_{\omega,\mathbf{k}}$. According to the Lagrangian Eqs. (4.2)–(4.4), we evaluate the relativistic correction to shear and vortical susceptibility up to the first order. The physical measurable quantities are

[7]The detailed balance condition shows that transport coefficients are not dependent on the path history, but rather on their immediate predecessor states.

$$\gamma_\eta(\omega) = \gamma_\eta + \frac{23T}{30w_o\pi}\left[\frac{|\omega|^{1/2}}{8\gamma_\eta^{3/2}} + \left(\frac{\omega\gamma_\eta^{-1/2}}{2c_s^2} + \omega^{3/2}\gamma_\eta^{-5/2}\right)\ln\left(\frac{\Lambda^2}{\omega^2}\right) + \left(\frac{1}{16} + \frac{1}{7\gamma_\eta^2\omega^2}\right)\Theta(\omega^2)\right]$$
$$+ \frac{4T^2}{27\pi^2}\left[\frac{(\chi\gamma_\eta)^{1/2}}{w_o(\chi^2\omega^2+1)^{2/3}} + \frac{2\chi^{5/2}}{w_o\gamma_\eta^{1/2}}\left(\frac{|\omega|^{1/2}}{(2+\chi^2\omega^2)}\right) + \left(\frac{1}{3} + \frac{\omega^2\chi^2}{15}\right)\sqrt{1-\omega^2\chi^2}\left(\frac{\gamma_\eta}{|\omega|}\right)^{1/2}\Theta(\omega^2-\chi^{-2})\right], \quad (4.17)$$

$$\chi^2(\omega) = \chi^2 + \frac{T^2}{(4\pi)^2 w_o}\left(1 + \frac{2}{3}\chi^2 + \left(6 + 2\omega^2\chi^2 + \frac{\chi^4}{4}\right)\ln(\chi^2\Lambda^2)\right) + \frac{T^2\chi^{3/2}|\omega^{1/2}|}{3\pi^2 w_o}\left(4 - \frac{\chi^{-2}-\mu^2}{\omega^2}\right)$$
$$\times\sqrt{\left(1-\frac{\chi^{-2}-\mu^2}{\omega^2}\right) - 4\chi^2\mu^2} + \frac{7T^2}{25\pi^2 w_o}\left(1 + \frac{1}{6}\omega^2\chi^2\right)\ln\left(2 - \frac{\omega^2-\chi^{-2}}{\mu^2}\right) + \frac{4T^2}{27\pi^2}$$
$$\times\left[\frac{(\chi\gamma_\eta)^{1/2}}{w_o(\chi^2\omega^2+1)^{2/3}} + \frac{2\chi^{5/2}}{w_o\gamma_\eta^{1/2}}\left(\frac{|\omega|^{1/2}}{(2+\chi^2\omega^2)}\right) + \left(\frac{1}{3} + \frac{\omega^2\chi^2}{15}\right)\sqrt{1-\omega^2\chi^2}\left(\frac{\gamma_\eta}{|\omega|}\right)^{1/2}\Theta(\omega^2-\chi^{-2})\right]. \quad (4.18)$$

We omit the script of $\{\gamma_\eta^{\rm ren}, \chi^2_{\rm ren}\}$ to keep the notation light. This process reflects, in the Lagrangian picture, the phenomenon identified in [13,17]. The symmetric shear and polarization states have the same symmetry properties. Hence, it is natural to expect them to mix. [13,17] characterize the mixing process as nondissipative, and, indeed, the equations above make it clear that they are based on microscopic susceptibility and occur under conditions of detailed balance. Then the investigation of effective field theory creates a scenario in which the one-loop corrections in 2 and 3 are simultaneous.

However, one should note that the symmetric shear is *not* an equilibrium quantity and generally relaxes to zero as global equilibrium is reached in a fluid, as it carries no conserved quantum numbers. Polarization also relaxes, generally not to zero but to the *anti-* symmetric vorticity gradient carrying angular momentum [4]. The key here is the realization that under detailed balance the mixing happens instantaneously. This generally violates causality, as we showed before in [4]. The next section explores how extending the action to second-order clarifies the relationship between the nondissipative mixing of transient quantities, and also determines under what conditions this mixing really occurs.

C. Second-order fluid action

From the previous section, we introduce the first-order correction of hydrodynamics as an effective field theory. This dynamics still presents well-known tensions with relativistic causality [37]. The origin of these problems in Eqs. (2.20)–(2.22) is the entropy divergence at Lagrangian level, which leads to acausality and instability modes.[8] To remedy these failures, we require the inclusion of additional degrees of freedom breaking the symmetries associated with the equilibrium but having relaxational dynamics (although fluctuations mean these are not uniquely defined [18]) [25]. Microscopic interactions are responsible for the relaxation of degrees of freedom and the second law of thermodynamics means that in the absence of backreaction dynamics [6] should be relaxational concerning their source. This requirement, which turns out the Lagrangian of Eqs. (2.20)–(2.22) causal and stable, enlarges the parameter space of our theory via the introduction of new couplings. The new Lagrangian, written using the doubled variable prescription outlined in Sec. 2 is [23,24]

$$S = S_{\rm free} + S_{\rm IS-shear} + S_{\rm IS-bulk} + S_{\rm IS-pol}, \quad (4.19)$$

$$S_{\rm IS-shear} = \int d^4x\left(\frac{\tau_\eta}{2}(\pi_-^{\mu\nu}u_+^\alpha\partial_\alpha\pi^+_{\mu\nu} - \pi_+^{\mu\nu}u_-^\alpha\partial_\alpha\pi^-_{\mu\nu}) + \frac{\pi_\pm^{\mu\nu 2}}{2} + z_{IJK}(b^2)b^2B^{-1}_{ij}\partial^\mu\phi^{iI}\partial^\nu\phi^{jJ}\partial_\mu K_\nu^K\right), \quad (4.20)$$

$$S_{\rm IS-bulk} = \int d^4x\left(\frac{\tau_\xi}{2}(\Pi_- u_+^\alpha\partial_\alpha\Pi_+ - \Pi_+ u_-^\alpha\partial_\alpha\Pi_-) + \frac{\Pi_\pm^2}{2} + h_{IJK}(b^2)K^{\mu I}K^{\nu J}\partial_\mu K_\nu^K\right), \quad (4.21)$$

$$S_{\rm IS-pol} = \int d^4x\left(\frac{\tau_\chi}{2}(Y_-^{\mu\nu}u_+^\alpha\partial_\alpha Y^+_{\mu\nu} - Y_+^{\mu\nu}u_-^\alpha\partial_\alpha Y^-_{\mu\nu}) + \frac{Y_\pm^{\mu\nu 2}}{2} + F(b(1-cy^2))\right). \quad (4.22)$$

The on-shell equations of motion are

[8] We relate these effects to the inclusion of correlations between microstates having no equilibrium counterpart (2.5). It means the perturbative calculation at the local level of the Ψ–sector has no meaningful hydrodynamic.

$$\begin{Bmatrix} \tau_\eta \\ \tau_\xi \\ \tau_\chi \end{Bmatrix} u_\alpha \partial^\alpha \begin{Bmatrix} \pi^{\mu\nu} \\ \Pi \\ Y^{\mu\nu} \end{Bmatrix} + \begin{Bmatrix} \pi^{\mu\nu} \\ \Pi \\ Y^{\mu\nu} \end{Bmatrix} = \begin{Bmatrix} \Delta^{\mu\nu\alpha\beta}\partial_\alpha u_\beta \\ \Delta^{\alpha\beta}\partial_\alpha u_\beta \\ \chi\omega^{\mu\nu} \end{Bmatrix}, \tag{4.23}$$

where $\Delta^{\mu\nu\alpha\beta} = \frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta})$. The $\pi^{\mu\nu}$, Π, and $Y^{\mu\nu}$ are new dynamical variables, which relax to their first-order gradient expansion: shear $\pi^{\mu\nu}_{|\mathrm{shear}}$, bulk $\Pi_{|\mathrm{bulk}}$, and polarization $Y^{\mu\nu}_{|\mathrm{pol}}$ by following their respective characteristic timescale. The transversality condition $u_\mu\pi^{\mu\nu} = 0$ can be enforced as it was in [24], by writing $\pi_{\mu\nu} \equiv X_{IJ}\partial_\mu\phi^I\partial_\nu\phi^J$, and considering X_{IJ} as degrees of freedom. In this work we do not use this parametrization as it does not affect the results.

In absence of homogeneous part, the fluid dynamic force decays exponentially to zero on the timescale $\tau_{\eta,\xi,\chi}$ dictated by the underlying microscopic interactions. As per the Israel-Stewart prescription, the acausal modes into Eqs. (3.18)–(3.20) must be crucially cut off after replacing the transport coefficients by

$$\begin{pmatrix} \eta \\ \xi \\ \chi \end{pmatrix} \to \begin{pmatrix} \eta \\ \xi \\ \chi \end{pmatrix} \frac{1}{1 + i\omega\tau_{\eta,\xi,\chi}}. \tag{4.24}$$

The new Green-Kubo formulas are

$$\frac{1}{2}\lim_{\omega\to 0}\lim_{k\to 0}\partial^2_\omega \mathrm{Im} G^R_{J^z,\omega^{xy}}(\omega,\mathbf{k}) = \chi^2\tau_\chi, \tag{4.25}$$

$$\frac{1}{2}\lim_{\omega\to 0}\lim_{k\to 0}\partial^2_k \mathrm{Im} G^R_{J^z,T^{xy}}(\omega,\mathbf{k}) = \tau_\eta(\bar z_{333} + \bar z_{303} + \bar z_{033} + \bar z_{003}) + \tau_\xi((\bar h_{003,03} + \bar h_{303,03} + \bar h_{333,03} + \bar h_{033,03} - 2\bar h_{333,03} - 2\bar h_{303,03} + 2\bar h_{330,03} + 2\bar h_{300,03}) - 4b_o^5(\bar h_{333,00} + \bar h_{303,00}) + 4b_o^5(\bar h_{003,33} + \bar h_{303,33} + \bar h_{333,33} + \bar h_{033,33} + \bar h_{330,33} + \bar h_{300,33}) - b_o^3(\bar h_{003} + \bar h_{303} + 2\bar h_{333} + 2\bar h_{033} + \bar h_{333} + \bar h_{300})). \tag{4.26}$$

Our main goal is to restore the causal behavior of fluctuations encoded in kernel (3.10). In a "bottom-up" effective theory, τ_χ, τ_η, and τ_χ are not arbitrary but reflect how different fluctuations conspire. We can determine them from thermodynamical identities and conserved quantities. However, as shown in [20], there is no well-defined limit when $\eta, \chi, \tau_\eta, \tau_\chi$ go to zero without IR instabilities due to vorticity.

The new set of variables in (4.23) arises nonhydrodynamics modes obeying the causality and stability conditions. These collective modes keep track of microscopic processes, which help us extend our analysis outside the hydrodynamic regime. Using the same procedure in III C, we evaluate the tensor-tensor correlation function for $t > 0$

$$\begin{aligned}\langle\{T^{ij}(t,\mathbf{x}), T^{kl}(0)\}\rangle \sim \mathcal{O}(\Lambda) &+ \frac{H^{ij}_{kl}}{w_o}\frac{T^2}{6\pi}\left[\frac{\chi^{-5/2}}{((1+\tau_\chi^2/\chi^2)t)^{3/2}} + \left(3 + \left(\frac{1}{6}\right)^{1/2}\right)\frac{\chi^{-3/2}}{2((1+\tau_\chi^2/\chi^2)t)^{1/2}}\right] \\ &+ \left(7 + \left(\frac{3}{2}\right)^{3/2}\right)\frac{S^{ij}_{kl}}{60\pi}\frac{T^2}{(\gamma_\eta(z)t)^{3/2}} + \frac{H^{ij}_{kl}T^2}{((\gamma_\eta(z) + \frac{4}{3}\gamma_\xi(z))t)^{3/2}} + \frac{T^2}{2\pi w_o}\left[\frac{e^{-\gamma_\eta \mathbf{k}^2 t}}{32}\right. \\ &\left. + \frac{1}{3}k^i k^j e^{-\frac{1}{2}\mathbf{k}^2\gamma(z,h)t}\cos(|\mathbf{k}|c_s t)\right]\frac{\langle \bar{p^2}\rangle}{\mathcal{V}} + \frac{T^2}{2\pi}e^{-\chi^2\mathbf{k}^4 t/f_b b}\frac{\langle\overline{y_2}\rangle}{\mathcal{V}} + \ldots\end{aligned} \tag{4.27}$$

This generalized matrix corresponds to the transverse conserved quantity of the stress dynamics. Clearly, we label the first and last exponential contributions as the fast variable in which the former is connected with the relaxing mode of shear dissipation, and the latter is related to vortex diffusivity. The manifestation of the oscillatory term, the longitudinal momentum solution, is irrelevant for long-time tails. The higher-order terms for (4.27) will lead to subdominant effects in the long-time tails with time power-law of $t^{5/2}$ and $t^{7/2}$. One essential feature is that the polarization dynamics satisfy the bounded causality relation $\frac{dP}{ds} \le \frac{\chi^2}{\tau_\chi^2}$ [4]. The pole solution for small wavenumber is

$$\omega(\mathbf{k}) \simeq -\frac{i}{\tau_\eta} - \frac{i}{\tau_\chi} + i\gamma_\eta(z)\mathbf{k}^2 + i\frac{\chi^2}{f_b b}\mathbf{k}^4 + \ldots \tag{4.28}$$

This dispersion relation, partially in accord with [28], rules the system evolution at initial time when the gradient expansion scale is weaker than nonhydrodynamics modes. The lifetime of collective modes (4.28), characterized by the inverse of damping behavior, increases with $\mathbf{k}$ and their decaying evolve in according to $\mathrm{Re}[\omega(\mathbf{k}\to 0)] =$ finite damping terms. In particular, we can evaluate the exponential decay time of $\chi \sim e^{-\omega(\mathbf{k})t}$ by $\mathrm{Re}[\omega(\mathbf{k})] \propto \mathbf{k}^4$. We identify this high wave number dependence as the aspect denoting how fast spin modes increase in nonhydrodynamic regime. Note that (4.28) exhibits an infinite lifetime for long-wavelength limit and survive for $\mathbf{k}\to 0$. This last interpretation is, in fact, reinforced by conservation laws in hydrodynamics picture. It means the balance equation of transverse p_T and longitudinal p_L momentum decouples. So the relaxation process limits the transverse macroscopic current, purely diffusive. Indeed, a more careful analysis of the relaxation time opens the door for an interesting qualitative result since this calculation is affected by nonlocal time contribution (see discussion in Sec. V). Such statement is related to the critical wave number in which fluid shows the dynamics of transverse excitations. First of all, the transverse momentum (3.7) of polarizable dissipative fluid is nontrivial and can be divided into two parts: spin and hydro-particle. For considered task, one evaluate the corrections induced by thermal fluctuations from spin and shear viscosity in case of domain spectrum approaches interatomic distance (high frequency). We should recall the eigenvalues of stress tensor operator (4.27) express, on the microscopic scale, the propagation of both shear viscosity and transverse polarization from kinetic modes. At this point, we can determine the emergence of these collective modes based on the kinetic characteristic of Maxwell relaxation time [38] by following the procedure in [39]

$$\mathbf{k}_\eta \sim \left[\frac{\epsilon G_\eta}{\eta^2}\right]^{1/2}_{\mathbf{k}\to 0}, \qquad \mathbf{k}_\chi \sim \left[\frac{f_b b}{8G_\chi \chi^4}\right]^{1/4}_{\mathbf{k}\to 0}, \tag{4.29}$$

where $\tau_\eta = \eta/G_\eta$ and $\tau_\chi = \chi^2 G_\chi$. The modes of shear-stress $\mathbf{k}_\eta$ and transverse polarization $\mathbf{k}_\chi$ show wave-solutions propagation for $\mathbf{k} > \mathbf{k}_\eta$ and $\mathbf{k}_\chi$, while they present diffusive behavior for length $> L_\eta$ and L_χ, where $L \sim 2\pi/\mathbf{k}$ is the upper bound limit to observe excitations. If $L_\eta > L_\chi$, the shear waves have a dominant role in the collective modes before the manifestation of wavelike transverse polarization. On the other hand, if $L_\eta < L_\chi$, the wavelike transverse polarization manifests the collective modes before shear waves. The former corrects the $G^R_{T^{0i}T^{0j}}(\omega,\mathbf{k})$ by $\langle J^T_P(t,\mathbf{k}) J^T_P(t=0,\mathbf{k})\rangle$.[9] The later corrects positively the shear viscosity by increasing more the spin displacement toward the principal axes of dissipative rotational fluids, e.g., transferring (x)momentum in the ($-y$) direction Fig. 1. It turns out that after increasing $\mathbf{k}$ to the spectrum region $< L_\eta$ and $< L_\chi$, both of transverse modes are relevant for correcting the transport coefficients. In this region, they become frequency dependent $\{\gamma_\eta(\omega), \chi(\omega)\}$ and reduce to the "bare" quantities $\{\gamma_\eta, \chi\}$ in $\lim_{\omega\to 0}$. In our approach, we cannot determine by analytical methods the magnitude of $\mathbf{k}_\eta$ and $\mathbf{k}_\chi$, as it is done in [39]. In other words, we cannot distinguish which wave occurs first because it depends on how η and χ compete. It means the collective excitation of the shear and polarization are originating from different microscopic forces depending on the spatial scale. Many hydrodynamics treatments are build-up for low $(\omega, \mathbf{k})$. Hence, they cannot provide a satisfactory prediction for high frequency by a straightforward extrapolation of higher derivatives. That is why the results (4.29) are important to understand the relevant aspects of fluid dynamics. We can only make progress in investigating further questions by appealing to experimental data, which will be postponed for future work.

V. MEMORY EFFECT

In relativistic nuclear collisions, hydrodynamics models validate the interpretation of data in the QGP as a quasi-ideal fluid [32]. Strictly speaking, the hydrosignature reproduces a wealth information about the QGP because we assume local thermal equilibrium to collective flow. However, most proposal theoretical models run into conceptual issues[10] in predicting polarizable fluid because the presence of spin excitations in a perfect fluid creates inherently an out-of-equilibrium scenario because of causality bounds [4]. The complexity of such a system concerns the incorporation of micro- and macroscales. It implies that we can cover the same phenomena from "bottom" to "up" by different coarse-grained processes. The strategy to set the appropriate hierarchical levels regards how far fluctuations are from linearized hydrodynamics. We have shown in the previous sections that the relevant aspects for a more accurate description of fluid dynamics are related to the high energy spectrum region.

The aforementioned Secs. III and IV show that fluctuation-dissipation theorem becomes an indispensable tool for describing the rotation of heavy ion collisions. To do this, we assume reversibility temporal and time-symmetry as the majors rule at the microscopic scale. As we have seen, the cross-phenomena between η and χ (4.15) stands, within a certain approximation, due to the Markov characteristic of linear approximation. By comparing (3.6) and (3.7), whenever the transfer of angular momentum

[9] J^T_P the microscopic current of transverse polarization, encoded in (3.22).

[10] The theory behind spin dependent momentum distribution shows disagreement when contrasted with experiments [1,40].

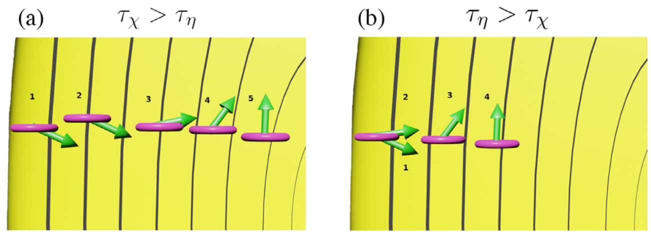


FIG. 4. This schematic picture shows how the inclusion of shear process redistributes the spin location in the polarizable fluid.

happens, the displacement of spin toward the rotational axes follows δt time later. On the other hand, the inverse process must occur at the same time. Hence, since the fluid does not depend on the previous history, the correlation function for p_T^i and $Y^{\mu\nu}$ in (3.10) assumes the form of

$$\langle p_T^i(t), Y^{\mu\nu}(t+\delta t, \Omega)\rangle_{\mathrm{eq}} = \langle Y^{\mu\nu}(t, -\Omega), p_T^i(t+\delta t)\rangle_{\mathrm{eq}}, \tag{5.1}$$

where Ω is an odd external vorticity field. The microscopic interactions yield the same correlation function for both processes and demand a special connection between η and χ. Thus this irreversible current demands that the scale time of spin evolution is comparable with the hydrovariables in the near-equilibrium state. In fact, the dynamics generator leads to the same physical state that the time-reversed symmetry generator. So far, the arguments raised by Markov approximation are well suited for a near-equilibrium system.

By looking at the conservation of stress tensor up to the first order $\partial_\mu\langle T^{\mu\nu}\rangle = \langle\partial_\mu T^{\mu\nu}\rangle = 0$, the Markovian approximation is another reason to adopt Noether's theorem to investigate polarizable fluids. Collective behavior can be generalized to a continuous medium, and the interaction between fluid variables and the background configuration can be neglected. In what follows, fluctuations are not observed in the l_{hydro} scale, and stochastic variables are present in linear equations as white noise. It is easy to see if we compare the relaxation time of hydrovariables-t and thermofluctuations-t_m, what one calls "system" and "bath," respectively. Thus, the environment restores its stationary solution for $t \gg t_m$ since the perturbation of hydro-variables does not survive for a long-time.

At a coarse-grained level, the fluid under certain physical circumstances can lead to a process where the memory effects influence the macroscopic dynamics [41]. In one of these, the relaxation timescale of hydrodynamical and nonhydrodynamical variables are comparable [42]. The need for a non-Markovian process may not seem obvious when theories beyond local equilibrium dismiss that the relaxation time of microscopic variables can reflect macroscopic dynamics. If the hydrodynamics fluctuations are within the second order dissipative equation, the fluid with spin faces a memory effect.

Since χ and η have different symmetries I, the rate of mechanical relaxation must provide two different scenarios in hydrodynamics.[11] To discuss both of them, let us first consider the spin out of equilibrium (not aligned with the external vortical fluid). For the first case $\tau_\chi > \tau_\eta$, the transfer of angular momentum from spin to fluid happens at τ_η time, and so the spin moves in the direction to inner layers: points 1 and 2 in Fig. 4. The spin alignment toward the external vortex direction only follows $\tau_\chi - \tau_\eta$ later: points 2–5 in Fig. 4. On the other hand, for the case $\tau_\eta > \tau_\chi$, the flow produces a scenario where the polarization alignment occurs before the spin moves toward the principal rotation axis. In this inverse process, whenever the spin begins to align at τ_χ time: points 1 and 2 in Fig. 4, its only shifts to the rotational axis $\tau_\eta - \tau_\chi$ later: points 2–4 Fig. 4. In both situations, the fluid reaches a thermodynamical equilibrium when the polarization current is parallel to the external vorticity field.

These characteristics appear when the time correlation between microstates is not neglected, and so $\partial_\mu\langle T^{\mu\nu}\rangle \neq \langle\partial_\mu T^{\mu\nu}\rangle$. Then the friction, including memory effect, is not instantaneous but depends on the previous steps. Consequently, the fluctuations turn the white noise degrees of freedom into colored ones. We can only restore the equality if freezing the environment for a short time in which nonhydrodynamics modes are relevant. Other scenarios are possible: $\tau_\eta \ll \tau_\chi$, $\tau_\eta \gg \tau_\chi$, or $\tau_\eta \sim \tau_\chi$. The classification of all these cases leads to a clear meaning of the interaction between spin and fluid particles.

The manifestation of non-Markov properties becomes evident if we examine the spectral properties of the physical variables. This framework should encode knowledge for out-equilibrium process since each relaxation time depends upon the nature of interaction: range-action, symmetry, potential, and microscopic forces. Taking the evolution time of the density matrix in the interaction picture by (3.3), we have

$$\rho_I(t) = e^{i\mathcal{E}_0 t}\rho_{\mathrm{eq}}(t)e^{-i\mathcal{E}_0 t}, \qquad \mathcal{E}_0 = \mathcal{E}_0(\phi) + \mathcal{E}_0(\Psi). \tag{5.2}$$

[11] We also expect the specification of symmetric and antisymmetric interaction between microscopic degrees of freedom yields different dynamics for the spin particle.

Showing that we can rewrite in a compact form

$$\dot{\rho}_I(t) = i[\rho_I(t), \mathcal{E}_I(t)], \tag{5.3}$$

where the interaction Hamiltonian coming from the interacting Lagrangian in (2.7) evolves as $\mathcal{E}_I(t) = e^{i\mathcal{E}_0 t}\mathcal{E}_I(0)e^{-i\mathcal{E}_0 t}$. The general solution of (5.2) is $\rho_I(t) = \rho(0) + i\int_0^t dt'[\rho_I(t'), \mathcal{E}_I(t)]$. An interactive recursive solution with (5.3) leads to

$$\dot{\rho}_I(t) = i[\rho_I(0), \mathcal{E}_I(t)] + \int_0^t dt'[[\mathcal{E}_I(t'), \rho_I(t')], \mathcal{E}_I(t)]. \tag{5.4}$$

The factorization $\rho_I = \rho_\phi \otimes \rho_\Psi$ and $\mathrm{Tr}\{\mathcal{E}_I(t)\rho_I(0)\} = 0$ are fundamental assumptions to implement a Markovian approximation. However, the second interactive term is a noncommutative dissipative process

$$\int_{t_0}^t d\tau_\chi \mathcal{E}_I(\tau_\chi) \int_{t_0}^{\tau_\chi} d\tau_\eta \mathcal{E}_I(\tau_\eta) \int_{t_0}^{\tau_\eta} dt' \mathcal{E}_I(t')$$
$$\neq \int_{t_0}^t d\tau_\eta \mathcal{E}_I(\tau_\eta) \int_{t_0}^{\tau_\eta} d\tau_\chi \mathcal{E}_I(\tau_\chi) \int_{t_0}^{\tau_\chi} dt' \mathcal{E}_I(t'), \tag{5.5}$$

where the left-hand side (lhs) and right-hand side (rhs) correspond to the left and right picture of 4, respectively. The presence of noncommutative diffusion dictates $\delta Z_{\chi z} \neq \delta Z_{z\chi}$ in effective field theory language. In particular, it shows that the memory effects influence loop corrections. For the rhs (5.5), it is seen that the (a) loop of 2 and 3 occurs at τ_χ time, and so the information of this early states is transferred to the other ones at $\tau_\eta - \tau_\chi$ time. On the other hand, for the lhs (5.5), the c) loop of 2 and 3 arises at τ_η time, while the other ones at $\tau_\chi - \tau_\eta$ time.

The aforementioned discussion opens a window of theoretical effort to deal with the discrepancy found in the QGP. We will see in the future publication of how the non-Markovian effects are appropriate adjectives.

VI. SUMMARY AND OUTLOOK

In this work we have examined Lagrangian hydrodynamics with both polarization and dissipative effects, from the perspective of the field theory. We have examined the interplay of viscosity, vortical susceptibility and sound wave backreaction on fluid dynamics. We hope that this is a step toward a full theory of hydrodynamics with spin.

Our main conclusion is that the shear forces and the polarization dynamics "do not commute," resulting in several different regimes determined by the respective relaxation timescales. While this paper is theoretical and specific to the Lagrangian picture and the linear response approach, it is directly related to different topics, both theoretical and phenomenological which have been discussed in the literature. The fact that viscous forces interact with polarization has been realized in the context of coarse-graining Zubarev hydrodynamics [43] and transport theory [13]. It has been advocated as a solution of the longitudinal polarization phenomenological puzzle [31,44,45]. However, since the symmetric shear is not an equilibrium nor a conserved quantity, a general effective theory of this dynamics was missing. This work has clarified the regime where such terms are significant in the scale expansion.

If [44,45] will become accepted as an explanation of the longitudinal spin puzzle, it would imply that the spin relaxation time is in fact not small with respect to the relaxation time of shear quantities, since,as we show, this is the regime where the shear forces drive polarization. It would likely mean that spin and vorticity are not in equilibrium throughout hydrodynamic evolution, and the Cooper-Frye type freeze out assuming that which has so far been used for phenomenology [46] needs correcting.

The "mass correction" derived in Sec. IV B is equally interesting phenomenologically. It would mean vorticity is not linearly proportional to angular momentum but acquires components dependent on characteristic vortex size ω weighted by microscopic parameters [χ^2 and $\gamma(z, h)$]. This is again of potential phenomenological interest in the transverse polarization (produced at the scale comparable to the system size) vs longitudinal polarization (produced on finer scales determined by anisotropic flow), as well as the impact parameter dependence of global polarization. One would need to input a frequency dependent polarization susceptibility correction into the freeze-out code (in practice, a correction to the Boltzmann polarization factor depending on the vortex size) to estimate such effects quantitatively. A conclusive deviation of the linear dependence of polarization with respect to impact parameter (not seen as yet [1]) could provide evidence for such "anomalous vorticity propagation."

On the theory side, while the approach presented here is based on the fluctuation-dissipation theorem, the backreaction on hydrodynamic evolution of fluctuations has not yet been explored. It is reasonable that, analogously for [34] spin fluctuations will affect evolution more in a regime where τ_Y is sub-dominant, while hydrodynamic fluctuations become important when shear relaxation time is small. A full understanding of fluctuations is necessary if spin is to drive a "ferromagnetic type" (or rather "ferrovortetic") phase transition, as originally discussed in [4]. The presence of a phase transition will add another scale, to be studied using the Landau theory of phase transitions [4,35] and either Maxwell construction or nucleation, depending on fluctuation probabilities. Large scale vortical structure, dissipation, fluctuation and phase structure would then each has a dominant regime.

In conclusion, we have developed an effective theory of dissipative hydrodynamics with spin, based on the

Lagrangian picture and linear response theory. We hope this is a step toward the still elusive goal to understand, both at a theoretical and phenomenological level, the effect that spin dynamics has on hydrodynamic evolution.

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APPENDIX: DERIVATION OF GREEN-KUBO RELATIONS

In this appendix, we follow [2] to briefly demonstrate the derivation of the Green's functions from the higher-order stress tensor. The main object to accomplish this task is the Lagrangian which describes our polarizable dissipative fluid system in (2.19). Thus, we can expand the stress tensor for higher orders defined as

$$T^{\mu\nu} = \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^I)} - \partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\phi^I)} + \partial_\beta\partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\partial_\gamma\phi^I)} - \ldots\right\}\partial^\nu\phi^I + \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\phi^I)} - \partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\partial_\gamma\phi^I)} + \ldots\right\}\partial_\beta\partial^\nu\phi^I + \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\partial_\gamma\phi^I)} - \ldots\right\}\partial_\beta\partial_\gamma\partial^\nu\phi^I + \cdots - \eta^{\mu\nu}\mathcal{L}. \tag{A1}$$

note that because ϕ_Is are physical coordinates (rather than internal space ones), this definition of $T_{\mu\nu}$ maintains its symmetry as long as polarization degrees of freedom are not involved.

It is convenient to introduce another physical object to the current vector

$$J^\mu = i\epsilon\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^I)} - \partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\phi^I)} + \ldots\right\}\phi^I + \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\beta\phi^I)} + \ldots\right\}\partial^\beta\phi^I + \ldots, \tag{A2}$$

upon substituting (2.19) into (A1), we obtain

$$\begin{aligned} T^{\mu\nu} = \Big\{ & w_o u_\gamma P^{\gamma\mu}(c_s^2 f_b + (\gamma_\eta - 2\gamma_\xi)b^3\Delta^{\alpha\beta}_{IJ}\partial_\alpha K^K_\beta + \gamma_\xi\epsilon^{\zeta\rho\beta}\epsilon_{0LM}\partial_\zeta\phi^I\partial_\rho\phi^J K^K_\beta - c_p^2 bF_y\omega^2\partial_b\chi^2 + F_b(1-cy^2)) \\ & + w_o\gamma_\eta(\Delta^{IJ}_{\alpha\zeta}K^K_\beta + K^I_\alpha\Delta^{JK}_{\beta\zeta})\partial^\zeta K^\alpha P^{\mu\beta} + F_y\gamma_\chi(2\omega^2\partial_{\omega^2}\chi + \chi)\left(\omega^2 u^\mu - y_{\beta\theta}(\dot{K}^\beta - u_\alpha\nabla^\beta K^\alpha)P^{\theta\mu} + \frac{1}{6}\epsilon^{\mu\theta\sigma\gamma}y^\rho_\theta\partial_\rho\partial_\sigma\phi\partial_\gamma\phi\right) \\ & - \partial_\beta(z_{IJK}b^2\Delta^{IJ}_{\alpha\sigma}P^{\alpha\{\mu}_K\delta^{\beta\}\sigma} - 2c_p^2F_y\gamma_\chi(1+2\omega\partial_{\omega^2}\chi)y_{\alpha\theta}P^{\theta\{\beta}\delta^{\mu\}\alpha})\Big\}\partial^\nu\phi \\ & + \{w_o\gamma_\eta\Delta^{IJ}_{\alpha\sigma}P^{\alpha\{\mu}_K\delta^{\beta\}\sigma} - 2c_p^2F_y(1+2\omega\partial_{\omega^2}\chi)\gamma_\chi y_{\alpha\theta}P^{\theta\{\beta}\delta^{\mu\}\alpha}\}\partial_\beta\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}, \end{aligned} \tag{A3}$$

where we introduce the transport coefficient and thermodynamic derivative

$$\begin{aligned} & c_s^2 = \frac{f_{bb}b}{f_b}, \qquad \gamma_\xi \equiv \frac{1}{w_o}\sum_{IJK}\left(h_{IJK} + \sum_{LM}h_{IJK,LM}\right), \qquad \chi \equiv \frac{\partial Y^{\mu\nu}}{\partial\omega^{\mu\nu}}, \qquad \gamma_\chi \equiv \frac{\chi^2}{w_o}, \\ & c_p^2 \equiv \frac{1}{2}\left(1 + \frac{F_{bb}}{F_b}\right), \qquad \gamma_\eta \equiv \frac{1}{w_o}\sum_{IJK}\left(z_{IJK} + \sum_{MN}z_{IJK,LM}\right), \qquad w_o = F_y y - f_b b. \end{aligned} \tag{A4}$$

The linearized "equation of motion" describes the macroscopic near-equilibrium system. Now, introducing the identity $u_\mu\Delta^{\mu\nu} = 0$ to fulfill the requirement of projection, we let the equation of motion $\partial_\mu\langle T^{\mu\nu}\rangle = 0$ onto parallel and perpendicular fluid velocity

$$u_\nu\partial_\mu\langle T^{\mu\nu}\rangle = 0, \tag{A5a}$$

$$\Delta^\alpha_\nu\partial_\mu\langle T^{\mu\nu}\rangle = 0. \tag{A5b}$$

By following the Taylor expansion around the static equilibrium, we shall express the main out-equilibrium field $\phi^{iI}(x) = x^{iI} + \pi^{iI} + \frac{1}{2!}\pi\cdot\partial\pi^{iI} + \frac{1}{3!}\pi\cdot\partial(\pi\cdot\partial\pi^{iI}) + \cdots$. To calculate the collective modes near-equilibrium limit, we linearize the equations above around $\phi^{iI} = x^{iI} + \pi^{iI}$ where x^{iI} is the hydrostatic background. The four-velocity is

$$u^\mu = u_0^\mu + \delta u^\mu, \tag{A6}$$

where $u_0 = 1 + \delta g_{00}/2$ and $u_\mu \delta u^\mu = 0$, with $\nabla^\mu = \Delta^{\mu\alpha}\partial_\alpha$. One checks by (2.15) that

$$u^\mu \simeq \delta_0^\mu \left(1 + \frac{1}{2}\dot{\pi}^2\right) + \delta_i^\mu(-\dot{\pi}^i + \dot{\pi}\cdot\partial\pi^i). \tag{A7}$$

Using the fact that

$$\omega_{\mu\nu} = 2\nabla_{[\mu} w u_{\nu]} = 2w(\nabla_{[\mu}u_{\nu]} - \dot{u}_{[\mu}u_{\nu]} + u_{[\mu}\nabla_{\nu]}\ln w) \underbrace{\simeq}_{\text{linear}} \nabla_\mu u_\nu - \nabla_\nu u_\mu,$$

we similarly derive the same expansion for

$$\omega^2 \simeq -(\partial_\mu \dot{\pi})\cdot(\partial^\mu \dot{\pi}) - [\partial\dot{\pi}\cdot\partial\dot{\pi}]. \tag{A8}$$

Following the linearization, the stress tensor and current for the first order are

$$T^{\mu\nu} \supset f_b b(\dot{\pi}\delta_0^\mu - \partial_l \dot{\pi}^j \delta_j^\mu - c_s^2[\partial\pi]\delta_l^\mu)\delta_0^\nu + z_{IJK}(3[\partial\dot{\pi}]\delta^{\mu\nu} - \delta_I^\mu \partial^\nu \dot{\pi}^I) + bh_{IJK,LM}2[\partial\dot{\pi}]\delta^{\mu\nu} - z_{IJK} \times (4\partial_l[\partial\pi]\delta_l^\mu\delta_0^\nu - \partial^\mu\partial^\nu\pi + \partial^\mu\dot{\pi}^J\delta_J^\nu - \partial_l\partial^m\pi\delta_l^\mu\delta_m^\nu + 2[\partial\dot{\pi}]\delta^{\mu\nu}) + \chi^2(\ddot{\pi}^l\delta_l^\mu + \partial^2\dot{\pi}^l\delta_l^\mu + \partial^\mu\ddot{\pi} + \partial^\mu[\partial\ddot{\pi}])\delta_0^\nu - \eta^{\mu\nu}[\partial\pi], \tag{A9}$$

The Green-Kubo formalism for variational principle points out $g^{\mu\nu}$ and $\omega^{\mu\nu}$ as the independent background. To avoid self-interaction and second order terms, we use $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ through the covariant derivative

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \frac{1}{2}\eta^{\nu\beta}(\partial_\mu h_{\beta\rho} + \partial_\rho h_{\beta\mu} - \partial_\beta h_{\mu\rho})u_0^\rho. \tag{A10}$$

Choosing the metric direction parallel to the external vortex field $\delta g^{\mu\nu} = \delta g^{\mu\nu}(t, x_3)$. We employ the definition in (3.12) to evaluate the retarded functions $\delta\phi^{iI}(\omega, \mathbf{k}) = \int d\omega d\mathbf{k}^3 e^{i\omega t - i\mathbf{k}x}\delta\phi^{iI}(t, x)$. The correlation functions for dissipative spin hydrodynamics are

$$G_{T^{xz},T^{xz}} = \frac{(\omega^2 + \omega\mathbf{k}) + \chi^2(\omega^5 - 2\omega^3\mathbf{k}^2) + iz_{IJK}\chi^2(2\omega^5 - \omega^4\mathbf{k} + \omega^2\mathbf{k}^3) + z_{IJK}^2(\omega\mathbf{k}^4 - 4\omega^2\mathbf{k}^3 - 3\omega^3\mathbf{k}^2) + iz_{IJK}\omega^3 - 4\chi^4(\omega^6 - \omega^4\mathbf{k}^2)}{\omega^2 + \mathbf{k}^2 - iz_{IJK}\omega\mathbf{k}^2 - 2ih_{IJK,LM}\omega\mathbf{k}^2 - 2\chi^2\omega^4}, \tag{A11}$$

$$G_{T^{xy},T^{xy}} = \frac{\omega\mathbf{k} - h_{IJK,LM}(\omega^4 - c_s^2\omega^2\mathbf{k}^2) + h_{IJK,LM}\chi^2(\omega^4\mathbf{k}^2 - \omega^3\mathbf{k}^3) + z_{IJK}\chi^2(\omega^6 - \omega^4\mathbf{k}^2) - 3z_{IJK}^2(\omega^4\mathbf{k} + \omega^3\mathbf{k}^2) - 2z_{IJK}'^2\omega^2\mathbf{k}^2 + \chi^4(2\omega^7 + 2\omega^6\mathbf{k} - \omega^4\mathbf{k}^3)}{\omega^2 - c_s^2\mathbf{k}^2 - iz_{IJK}(2\omega^2\mathbf{k} + 3\omega\mathbf{k}^2) - 2ih_{IJK,LM}\omega\mathbf{k}^2 - 2\chi^2(\omega^4 - \mathbf{k}^2\omega^2)}, \tag{A12}$$

$$G_{T^{0z},T^{0z}} = \frac{z_{IJK}(\omega^4 - 2\mathbf{k}^4) - 2\chi^2(\omega^4 + \omega^2\mathbf{k}^2) + z_{IJK}^2(\omega^4 - 4\omega^2\mathbf{k}^2 + \omega^3\mathbf{k} - 4\omega\mathbf{k}^3) + 8h_{IJK,LM}\times\chi^2\omega^2\mathbf{k}^2 + 2\chi^4(\omega^5\mathbf{k} - \omega^4\mathbf{k}^2 - \omega^2\mathbf{k}^4)}{\omega^2 - iz_{IJK}\omega\mathbf{k}^2 - 2ih_{IJK,LM}\omega\mathbf{k}^2 - 2\chi^2\omega^4}, \tag{A13}$$

$$G_{T^{0x},T^{0x}} = \frac{(4z_{IJK} + h_{IJK,LM})\omega^2\mathbf{k}^2 + \chi^2(2\omega^5 - \omega^3\mathbf{k}^2) + z_{IJK}\chi^2(2\omega^5\mathbf{k} + 5\omega^4\mathbf{k}^2 - 3\omega^2\mathbf{k}^4) + 2h_{IJK,LM}\chi^2(\omega^4\mathbf{k}^2 + \omega^2\mathbf{k}^4) - 4\chi^4(\omega^5\mathbf{k} - \omega^3\mathbf{k}^3)}{\omega^2 - iz_{IJK}\omega\mathbf{k}^2 - 2ih_{IJK,LM}\omega\mathbf{k}^2 - 2\chi^2\omega^4}, \tag{A14}$$

$$G_{T^{00},T^{00}} = \frac{(\omega\mathbf{k} + \mathbf{k}^2) - 2z_{IJK}(\omega^3\mathbf{k} + 2\omega^2\mathbf{k}^2 + 5\omega\mathbf{k}^3) + h_{IJK,LM}(4\omega^2\mathbf{k}^2 + \omega\mathbf{k}^3) + \chi^2(\omega^3\mathbf{k} + \omega^2\mathbf{k}^2) + z_{IJK}^2(5\omega\mathbf{k}^3 + \omega^2\mathbf{k}^2) + 2\chi^2(z_{IJK} + h_{IJK,LM})(\omega^4\mathbf{k}^2 - \omega^2\mathbf{k}^4 + \omega^3\mathbf{k}^3) - \chi^4(\omega^5\mathbf{k} + \omega^4\mathbf{k}^2 - \omega^3\mathbf{k}^3)}{\omega^2 - c_s^2\mathbf{k}^2 - iz_{IJK}(2\omega^2\mathbf{k} + 3\omega\mathbf{k}^2) - 2ih_{IJK,LM}\omega\mathbf{k}^2 - 2\chi^2(\omega^4 - \mathbf{k}^2\omega^2)}. \tag{A15}$$

It is straightforward the above calculation for conserved current

$$G_{J^z,\omega^{xy}} = \frac{4i\omega^2\mathbf{k} + 4\chi^2 z_{IJK}\omega^3\mathbf{k}}{(\omega^2 - c_s^2\mathbf{k}^2) - i\chi^2(\omega^4 - \omega^2\mathbf{k}^2)} - 2\chi^2\omega^2, \tag{A16}$$

$$G_{J^x,\omega^{xy}} = \frac{2z_{IJK}^2\omega^2\mathbf{k}^2}{(\omega^2 - c_s^2\mathbf{k}^2) - i\chi^2(\omega^4 - \omega^2\mathbf{k}^2)} - z_{IJK}\omega\mathbf{k}, \tag{A17}$$

$$G_{J^0,T^{00}} = \frac{\omega^2\mathbf{k} + 3z_{IJK}^2\omega\mathbf{k}^4 - \chi^4(\omega^5\mathbf{k} + \omega^3\mathbf{k}^3)}{(\omega^2 - c_s^2\mathbf{k}^2) + 8iz_{IJK}\omega\mathbf{k}^2 - \chi^2(\omega^4 - \omega^2\mathbf{k}^2)} + z_{IJK}(\mathbf{k}^2 - \omega k) - 2\chi^2(\omega^2 - \mathbf{k}^2), \tag{A18}$$

$$G_{J^z,T^{xy}} = \frac{z_{IJK}\omega\mathbf{k}^2 + 3z_{IJK}^2\omega\mathbf{k}^4 + z_{IJK}\chi^2(\omega^2\mathbf{k}^3) - \chi^4(\omega^4\mathbf{k}^2 + \omega^3\mathbf{k}^3)}{(\omega^2 - c_s^2\mathbf{k}^2) + 8iz_{IJK}\omega\mathbf{k}^2 - \chi^2(\omega^4 - \omega^2\mathbf{k}^2)} + iz_{IJK}(\mathbf{k}^2 - \omega\mathbf{k})\mathbf{k} + \chi^2\omega^2. \tag{A19}$$

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