Index theorem on magnetized blow-up manifold of $T^2/\mathbb{Z}_N$

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We investigate blow-up manifolds of $T^2/\mathbb{Z}_N (N = 2, 3, 4, 6)$ orbifolds with magnetic flux. First, we construct the blow-up manifolds and zero-mode wave functions on them more precisely. In particular, through an appropriate singular gauge transformation, winding numbers of wave functions on $T^2/\mathbb{Z}_N$ can be replaced with localized curvature and localized flux at orbifold fixed points. In addition, since the blow-up manifolds have no singularities, we apply the Atiyah-Singer index theorem to them; the chiral zero-mode number is given by the total magnetic flux. It can be also applied for $T^2/\mathbb{Z}_N$ orbifolds through the blow-up process, and then we find that it is consistent with the zero-mode counting formula in M. Sakamoto *et al.* [Zero-mode counting formula and zeros in orbifold compactifications, Phys. Rev. D **102**, 025008 (2020).]. Furthermore, the Atiyah-Singer index theorem shows that an additional degree of freedom of localized flux gives new chiral zero modes. We study their wave functions and then we find that they correspond to localized modes at the orbifold singular points. We also calculate their Yukawa couplings.

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I. INTRODUCTION

The Atiyah-Singer index theorem [1] states that the index of a Dirac operator $\not{D}$

$$\mathrm{Ind}(i\not{D}) \equiv n_+ - n_- \tag{1.1}$$

is a topological invariant. Here, $n_\pm$ are the numbers of $\pm$ chiral zero modes for the Dirac operator. The index theorem has been applied to many areas in physics, such as the chiral anomaly in gauge theory [2,3], the Witten index [4], and anomaly inflow [5,6].

In particular, we are interested in counting the number of chiral zero modes appearing in the four-dimensional (4D) effective field theories. The Standard Model has a lot of mysteries unanswered, including the generation problem of the quarks and leptons and the fermion mass hierarchy, and also to naturally explain their flavor structure. String theory and higher-dimensional theory are strong candidates beyond the Standard Model. Many proposals have been made to solve the generation problem, but known mechanisms to produce degenerate chiral zero modes are very limited. It is known to obtain the chiral spectra as magnetic-flux compactifications in type-I and II string theory [7–13] and heterotic string theory [14–17]. These models have provided semirealistic models of string phenomenology, e.g., three generation models [18,19], fermion mass hierarchy [20], and flavor structure [21–27].

The Atiyah-Singer (AS) index theorem for a two-dimensional (2D) compact manifold $\mathcal{M}^2$ with magnetic flux is known as [28,29]

$$n_+ - n_- = \frac{1}{2\pi}\int_{\mathcal{M}^2} F, \tag{1.2}$$

where F is a 2-form field strength of the flux. We should here stress that the index $n_+ - n_-$ is determined only by the flux but not the curvature of the 2D manifold $\mathcal{M}^2$. A simple application of the index theorem (1.2) is to take $\mathcal{M}^2$ to be a 2D torus T^2,

$$n_+ - n_- = \frac{1}{2\pi}\int_{T^2} F = M, \tag{1.3}$$

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where M is an integer and corresponds to a magnetic-flux quantization number.

The application of the index theorem to $T^2/\mathbb{Z}_N(N = 2,3,4,6)$ magnetized orbifolds will be phenomenologically and mathematically interesting because the index $n_+ - n_-$ gives the generation number in 4D effective field theories and it turns out to complicatedly depend on the flux quanta M, the $\mathbb{Z}_N$ eigenvalues under the $\mathbb{Z}_N$ transformation, the Scherk-Schwarz (SS) twist phases (α_1, α_2) and N (see the last columns in Tables I–V of the Appendix) [30–35]. In Ref. [30], a complete list of the index has been shown to satisfy the zero-mode counting formula[1]

$$n_+ - n_- = \frac{M}{N} - \frac{V_+}{N} + 1, \tag{1.4}$$

where V_+ is the sum of the winding numbers at the fixed points of $T^2/\mathbb{Z}_N$ orbifolds.

The first term on the right-hand side of Eq. (1.4) could be understood from Eq. (1.3) because the area of the $T^2/\mathbb{Z}_N$ orbifolds reduces to $1/N$ of that of the torus T^2. The origin of the second and third terms on the right-hand side of Eq. (1.4) is, however, unclear, and those terms seem not to be related to any flux on the $T^2/\mathbb{Z}_N$ orbifolds. In fact, Eq. (1.4) has not been derived as the AS index theorem in Ref. [30] and the relation (1.4) has been verified by computing the values of $n_+ - n_-$ and $M/N - V_+/N + 1$, separately and then by simply comparing them.

Our main purposes of this paper are to understand the formula (1.4) as the AS index theorem and clarify physical and geometrical meanings of the formula. There is, however, a problem. The $T^2/\mathbb{Z}_N$ orbifolds have singularities, and the AS index theorem cannot directly be applied to singular "manifolds". Our strategy to overcome the problem is to construct smooth blow-up manifolds of the $T^2/\mathbb{Z}_N$ orbifolds by removing cones around the singularities of the $T^2/\mathbb{Z}_N$ orbifolds and replacing them with parts of the 2D sphere S^2 [37,38]. Then, we can apply the AS index theorem directly to the blow-up manifolds. From the blow-up procedure, we can confirm the formula (1.4) as the AS index theorem and clarify the physical and geometrical meanings of the second and the third terms on the right-hand side of Eq. (1.4).

In addition, according to the AS index theorem, the additional degree of freedom of localized fluxes at the singular points of orbifolds [39–41][2] shows that there exist new chiral zero modes. We examine the profile of such chiral zero-mode wave functions in more detail. Then, we can find that they correspond to localized modes at the orbifold singular point of $T^2/\mathbb{Z}_N$ orbifolds. We also study their Yukawa couplings.

This paper is organized as follows: In Sec. II, we briefly review zero modes on the $T^2/\mathbb{Z}_N(N = 2,3,4,6)$ orbifolds with magnetic fluxes. In Sec. III, we construct the blow-up manifolds and compute zero-mode wave functions on them with magnetic fluxes in more detail, where $\mathbb{Z}_N$ invariant modes which have no winding numbers have been analyzed in Ref. [37]. By using the results, we derive the AS index theorem on the blow-up manifolds and reinterpret the zero-mode counting formula in Sec. IV. In particular, the AS index theorem shows that localized fluxes induce new zero modes. Then, we study wave functions of the new zero modes, which correspond to localized zero modes on $T^2/\mathbb{Z}_N$ orbifolds in Sec. V. We also discuss their Yukawa couplings in Sec. VI. We conclude this study in Sec. VII. In Appendix A, we show the detailed results of Sec. IV. In Appendix B, we show the detailed calculation of the normalization of bulk zero modes. In Appendix C, we show the detailed calculation of the normalization of localized zero modes.

II. MAGNETIZED $T^2/\mathbb{Z}_N$ ORBIFOLD

A. Magnetized T^2

We review the $U(1)$ gauge theory on a 2D torus with homogeneous magnetic flux [20]. First of all, let us consider the six-dimensional (6D) spacetime, which contains 4D Minkowski space-time $\mathcal{M}^4$ and an extra 2D torus T^2 with magnetic flux. The Lagrangian of a 6D Weyl fermion in magnetic flux background is given by

$$\mathcal{L}_{6\mathrm{D}} = i\bar{\Psi}\Gamma^M D_M \Psi, \qquad \Gamma_7\Psi = \Psi, \tag{2.1}$$

where $M(= 0,1,2,3,5,6)$ is the 6D spacetime index and $D_M = \partial_M - iqA_M$ is the covariant derivative. Γ^M is 6D gamma matrix and Γ_7 denotes the 6D chirality operator.

By the Kaluza-Klein mode expansion, the 6D Weyl fermion $\Psi(x,z)$ can be decomposed into 4D Weyl left/right-handed fermions $\psi^{(4)}_{\mathrm{L/R}}(x)$ as

$$\Psi(x,z) = \sum_{n,j} (\psi^{(4)n,j}_{\mathrm{R}}(x) \otimes \psi^{(2)n,j}_{+}(z) + \psi^{(4)n,j}_{\mathrm{L}}(x) \otimes \psi^{(2)n,j}_{-}(z)), \tag{2.2}$$

where x^μ $(\mu = 0,1,2,3)$ denotes the 4D Minkowski coordinate and z is the complex coordinate on T^2. The 2D Weyl fermions $\psi^{(2)n,j}_{\pm}(z)$ are expressed as the form

$$\psi^{(2)n,j}_{+}(z) = \begin{pmatrix} \psi^{n,j}_{+}(z) \\ 0 \end{pmatrix}, \quad \psi^{(2)n,j}_{-}(z) = \begin{pmatrix} 0 \\ \psi^{n,j}_{-}(z) \end{pmatrix}, \tag{2.3}$$

where n and j label the Landau level and the degeneracy of mode functions on each level, respectively.

[1] By use of the trace formula, Eq. (1.4) has been derived for $M = 0$ in Ref. [35] and for arbitrary M with $N = 2$ in Ref. [36].

[2] Even if localized fluxes vanish at the tree level, they may be induced by loop effects [39–42].

Here, T^2 is constructed by dividing a complex plane $\mathbb{C}$ into a 2D lattice Λ, i.e., $T^2 \simeq \mathbb{C}/\Lambda$. We define the complex coordinate of T^2, $z \equiv y_1 + \tau y_2$, such that the identifications are $z \sim z+1 \sim z+\tau$, where $y_i (0 \leq y_i < 1)$ with $i = 1, 2$ denotes the real coordinate along one lattice vector e_i and $\tau \equiv e_2/e_1 \in \mathbb{C}(\mathrm{Im}\tau > 0)$ denotes the complex structure modulus of T^2. The area of T^2 becomes $\mathrm{Im}\tau$. We note that the curvature of T^2 is zero.

The nonzero magnetic flux f on the torus can be obtained as $f = \int_{T^2} F$ with the field strength

$$F(z) = \frac{if}{2\mathrm{Im}\tau} dz \wedge d\bar{z}. \tag{2.4}$$

For $F = dA$, the (1-form) vector potential is

$$A(z;\zeta) = \frac{f}{2\mathrm{Im}\tau} \mathrm{Im}((\bar{z}+\bar{\zeta})dz) \equiv A_z(z;\zeta)dz + A_{\bar{z}}(z;\zeta)d\bar{z}, \tag{2.5}$$

and $A_z(z;\zeta)$, $A_{\bar{z}}(z;\zeta)$ are explicitly given by

$$A_z(z;\zeta) = -\frac{i}{2}\frac{\pi M}{\mathrm{Im}\tau}(\bar{z}+\bar{\zeta}), \qquad A_{\bar{z}}(z;\zeta) = \frac{i}{2}\frac{\pi M}{\mathrm{Im}\tau}(z+\zeta), \tag{2.6}$$

where $\zeta \equiv \zeta_1 + \tau\zeta_2$ $(\zeta_1, \zeta_2 \in \mathbb{R})$ denotes the Wilson line. Then, we obtain

$$\begin{aligned} A(z+1;\zeta) &= A(z;\zeta) + d\left(\frac{f}{2\mathrm{Im}\tau}\mathrm{Im}(z+\zeta)\right) \\ &\equiv A(z;\zeta) + d\Lambda_1(z+\zeta), \end{aligned} \tag{2.7}$$

$$\begin{aligned} A(z+\tau;\zeta) &= A(z;\zeta) + d\left(\frac{f}{2\mathrm{Im}\tau}\mathrm{Im}(\bar{\tau}(z+\zeta))\right) \\ &\equiv A(z;\zeta) + d\Lambda_2(z+\zeta), \end{aligned} \tag{2.8}$$

where $\Lambda_1(z+\zeta)$ and $\Lambda_2(z+\zeta)$ are gauge parameters. It follows from Eqs. (2.7) and (2.8) that the torus lattice shifts can be reinterpreted as gauge transformations.

The 2D Weyl fermions are required to satisfy the pseudoperiodic boundary conditions (BCs)

$$\begin{aligned} \psi_{\pm}^{n,j}(z+1;\zeta) &= U_1(z)\psi_{\pm}^{n,j}(z;\zeta), \\ \psi_{\pm}^{n,j}(z+\tau;\zeta) &= U_2(z)\psi_{\pm}^{n,j}(z;\zeta), \end{aligned} \tag{2.9}$$

with

$$U_i(z) = e^{i\Lambda_i(z+\zeta)} e^{2\pi i \alpha_i} \quad (i = 1, 2), \tag{2.10}$$

where α_i $(i = 1, 2)$ are called Scherk-Schwarz (SS) twist phases which are allowed to be any real numbers. The consistency condition with the contractible loop, $z \to z+1 \to z+1+\tau \to z+\tau \to z$, leads to the magnetic-flux quantization condition

$$\frac{f}{2\pi} \equiv M \in \mathbb{Z}. \tag{2.11}$$

The 2D Weyl fermions satisfy the equations

$$\begin{aligned} -2D_z\psi_{-}^{n,j}(z;\zeta) &= -2(\partial_z - iA_z(z;\zeta))\psi_{-}^{n,j}(z;\zeta) \\ &= m_n\psi_{+}^{n,j}(z;\zeta), \end{aligned} \tag{2.12}$$

$$2D_{\bar{z}}\psi_{+}^{n,j}(z;\zeta) = 2(\partial_{\bar{z}} - iA_{\bar{z}}(z;\zeta))\psi_{+}^{n,j}(z;\zeta) = m_n\psi_{-}^{n,j}(z;\zeta). \tag{2.13}$$

We focus on zero modes with $m_n = 0$. From Eqs. (2.12) and (2.13), zero modes satisfy

$$\begin{aligned} \left(\partial_z - \frac{\pi M}{2\mathrm{Im}\tau}(\bar{z}+\bar{\zeta})\right)\psi_{-}^{0,j}(z;\zeta) &= 0, \\ \left(\partial_{\bar{z}} + \frac{\pi M}{2\mathrm{Im}\tau}(z+\zeta)\right)\psi_{+}^{0,j}(z;\zeta) &= 0. \end{aligned} \tag{2.14}$$

In the case of $M > 0$, only $\psi_{+}^{0,j}$ has the normalizable solutions that satisfy the pseudoperiodic BCs (2.9) and they are given as

$$\begin{aligned} \psi_{T^2,+}^{0,(j+\alpha_1,\alpha_2),M}(z;\zeta) &= e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z+\zeta|^2} g^{(j+\alpha_1,\alpha_2),M}(z;\zeta) \\ &\qquad (j = 0, 1, \ldots, M-1), \end{aligned} \tag{2.15}$$

$$\begin{aligned} g^{(j+\alpha_1,\alpha_2),M}(z;\zeta) &= \mathcal{N}_{T^2}^{0,j} e^{\frac{\pi M}{2\mathrm{Im}\tau}(z+\zeta)^2} e^{2\pi i\frac{j+\alpha_1}{M}(\alpha_2 - M\zeta_1)} \vartheta \\ &\qquad \begin{bmatrix} \frac{j+\alpha_1}{M} \\ -\alpha_2 \end{bmatrix} (M(z+\zeta), M\tau), \end{aligned} \tag{2.16}$$

where, $j = 0, 1, \ldots, |M|-1$ stand for the degeneracy of zero-mode solutions. Here, $\mathcal{N}_{T^2}^{0,j}$ denotes a normalization constant determined by

$$\int dz d\bar{z} (\psi_{T^2,+}^{n,(j+\alpha_1,\alpha_\tau),M})^* \psi_{T^2,+}^{n,(k+\alpha_1,\alpha_\tau),M} = \delta_{j,k}, \tag{2.17}$$

and the Jacobi ϑ-function is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau) = \sum_{l=-\infty}^{\infty} e^{i\pi(a+l)^2\tau} e^{2\pi i(a+l)(z+b)}. \tag{2.18}$$

Note that the Wilson line $\zeta = \zeta_1 + \tau\zeta_2$ can be pushed on the SS phases as

$$\alpha_1 \to \alpha_1' = \alpha_1 + M\zeta_2, \qquad \alpha_2 \to \alpha_2' = \alpha_2 - M\zeta_1, \tag{2.19}$$

by the $U(1)$ local and gauge transformation,

$$\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2,+}(z;\zeta) \to V_\zeta^{-1}(z)\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2,+}(z;\zeta) = \psi^{n,(j+\alpha_1',\alpha_2'),M}_{T^2,+}(z;0), \tag{2.20}$$

$$A(z;\zeta) \to A(z;\zeta) + iV_\zeta^{-1}(z)dV_\zeta(z) = A(z;0), \tag{2.21}$$

with

$$V_\zeta^{-1}(z) \equiv e^{-\pi i M \frac{\mathrm{Im}(\bar{\zeta}z)}{\mathrm{Im}\tau} - \pi i M \zeta_1\zeta_2}, \tag{2.22}$$

as shown in Ref. [32]. Hence, hereafter, we set $\zeta = 0$. On the other hand, in the case of $M < 0$, only $\psi_{-,0,j}$ has the normalizable solutions, and they are given in a similar way.

The above results are consistent with the AS index theorem on the torus with magnetic flux, i.e.,

$$n_+ - n_- = \frac{1}{2\pi}\int_{T^2} F = M. \tag{2.23}$$

The index theorem (2.23) shows that the number of the independent chiral zero modes is decided by the magnetic flux quantization number M on the magnetized torus and further that the generation number of this model is given by M. We emphasize that the index $n_+ - n_-$ depends only on the flux.

B. Magnetized $T^2/\mathbb{Z}_N$

In this subsection, we review the $U(1)$ gauge theory on twisted orbifolds $T^2/\mathbb{Z}_N$ with magnetic flux [32,33]. It has been known that there are only four kinds of the $T^2/\mathbb{Z}_N$ orbifolds with $N = 2, 3, 4, 6$. The $T^2/\mathbb{Z}_N$ orbifolds are defined by the torus identification and the additional $\mathbb{Z}_N$ one

$$z \sim \rho z \qquad (\rho = e^{2\pi i/N}(N = 2,3,4,6)). \tag{2.24}$$

For $N = 2$, there is no restriction on τ except for $\mathrm{Im}\tau > 0$. On the other hand, for $N = 3, 4, 6$, τ should be fixed at $\tau = \rho$ due to the analysis of crystallography.

An important feature of the $T^2/\mathbb{Z}_N$ orbifolds is the existence of the fixed points z_I^{fp} defined by

$$z_I^{\mathrm{fp}} = \rho z_I^{\mathrm{fp}} + u + v\tau \quad \text{for } \exists u, v \in \mathbb{Z}. \tag{2.25}$$

The $\mathbb{Z}_N$ fixed points on the $T^2/\mathbb{Z}_N$ orbifolds are given by

$$z_I^{\mathrm{fp}} = \begin{cases} 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} & \text{on } T^2/\mathbb{Z}_2, \\ 0, \frac{2+\tau}{3}, \frac{1+2\tau}{3} & \text{on } T^2/\mathbb{Z}_3, \\ 0, \frac{1+\tau}{2} & \text{on } T^2/\mathbb{Z}_4, \\ 0 & \text{on } T^2/\mathbb{Z}_6, \end{cases} \tag{2.26}$$

and the respective values of (u, v) in Eq. (2.25) are

$$(u,v) = \begin{cases} (0,0), (1,0), (0,1), (1,1) & \text{on } T^2/\mathbb{Z}_2, \\ (0,0), (1,0), (1,1) & \text{on } T^2/\mathbb{Z}_3, \\ (0,0), (1,0) & \text{on } T^2/\mathbb{Z}_4, \\ (0,0) & \text{on } T^2/\mathbb{Z}_6. \end{cases} \tag{2.27}$$

Note that there are additional fixed points for $N = 4, 6$, since the $\mathbb{Z}_4(\mathbb{Z}_6)$ group includes $\mathbb{Z}_2$ ($\mathbb{Z}_2$ and $\mathbb{Z}_3$) as its subgroup. They are not invariant under the $\mathbb{Z}_4(\mathbb{Z}_6)$ transformation, but invariant under the $\mathbb{Z}_2$ ($\mathbb{Z}_2$ and $\mathbb{Z}_3$) transformation up to the torus shifts. The additional fixed points are found as

$$\mathbb{Z}_2 \text{ fixed points: } z_I^{\mathrm{fp}} = \frac{1}{2}, \frac{\tau}{2} \qquad \text{on } T^2/\mathbb{Z}_4, \tag{2.28}$$

$$\mathbb{Z}_3 \text{ fixed points: } z_I^{\mathrm{fp}} = \frac{1+\tau}{3}, \frac{2+2\tau}{3} \qquad \text{on } T^2/\mathbb{Z}_6, \tag{2.29}$$

$$\mathbb{Z}_2 \text{ fixed points: } z_I^{\mathrm{fp}} = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \qquad \text{on } T^2/\mathbb{Z}_6. \tag{2.30}$$

We should emphasize that the fixed points are singular points on the $T^2/\mathbb{Z}_N$ orbifolds.

For the orbifold identification, the Scherk-Schwarz phases (α_1, α_2) must be quantized such as

$$(\alpha_1,\alpha_2) = (0,0), (1/2,0), (0,1/2), (1/2,1/2) \quad \text{on } T^2/\mathbb{Z}_2, \tag{2.31}$$

$$\alpha = \alpha_1 = \alpha_2 = \begin{cases} 0, 1/3, 2/3 & (M = \text{even}) \\ 1/6, 3/6, 5/6 & (M = \text{odd}) \end{cases} \quad \text{on } T^2/\mathbb{Z}_3, \tag{2.32}$$

$$\alpha = \alpha_1 = \alpha_2 = 0, 1/25 \qquad \text{on } T^2/\mathbb{Z}_4, \tag{2.33}$$

$$\alpha = \alpha_1 = \alpha_2 = \begin{cases} 0 & (M = \text{even}) \\ 1/2 & (M = \text{odd}) \end{cases} \qquad \text{on } T^2/\mathbb{Z}_6. \tag{2.34}$$

Let us discuss $\mathbb{Z}_N$ eigenfunctions on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux. They should obey the boundary conditions (2.9) and the orbifold boundary conditions

$$\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(\rho z) = \rho^m \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z), \tag{2.35}$$

$$\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(\rho z) = \rho^{m+1} \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z), \tag{2.36}$$

where $\rho^m (m = 0, 1, \ldots, N-1)$ in Eq. (2.35) denotes the $\mathbb{Z}_N$ eigenvalue. If the $\mathbb{Z}_N$ eigenvalue of $\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z)$ is

ρ^m, then that of $\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z)$ has to be ρ^{m+1}. The difference in eigenvalues comes from a rotation matrix acting on 2D spinors, and it can also be understood from the relations (2.12) and (2.13). Then, the $\mathbb{Z}_N$ eigenfunctions can be constructed by the following linear combinations of the wave functions on the torus

$$\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z) = \mathcal{N}^{n,j}_{T^2/\mathbb{Z}_N,+}\sum_{k=0}^{N-1}\rho^{-km}\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2,+}(\rho^k z), \tag{2.37}$$

$$\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z) = \mathcal{N}^{n,j}_{T^2/\mathbb{Z}_N,-}\sum_{k=0}^{N-1}\rho^{-k(m+1)}\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2,-}(\rho^k z), \tag{2.38}$$

where $\mathcal{N}^{n,j}_{T^2/\mathbb{Z}_N,\pm}$ are normalization constants determined by

$$\int dz d\bar{z} (\psi^{n,(j+\alpha_1,\alpha_\tau),M}_{T^2/\mathbb{Z}_N^m,\pm})^* \psi^{n,(k+\alpha_1,\alpha_\tau),M}_{T^2/\mathbb{Z}_N^m,\pm} = \delta_{j,k}. \tag{2.39}$$

Especially, zero modes with the $\mathbb{Z}_N$ eigenvalue ρ^m are given by

$$\psi^{0,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z) = e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2} h^{(j+\alpha_1,\alpha_2),M}_{+,m}(z), \tag{2.40}$$

$$h^{(j+\alpha_1,\alpha_2),M}_{+,m}(z) = \mathcal{N}^{0,j}_{T^2/\mathbb{Z}_N,+}\sum_{k=0}^{N-1}\rho^{-km} g^{(j+\alpha_1,\alpha_2),M}(\rho^k z). \tag{2.41}$$

Here, $h^{(j+\alpha_1,\alpha_2),M}_{+,m}(z)$ denotes the holomorphic function of z.

Let us investigate the $\mathbb{Z}_N$ eigenfunctions $\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z)$ around the fixed points $z_I^{\mathrm{fp}} \equiv y_{1I}^{\mathrm{fp}} + \tau y_{2I}^{\mathrm{fp}}$ by modifying Eq. (2.35). Their property will become important later. First, we define the coordinate Z such that $Z = 0$ at the fixed point z_I^{fp}, i.e., $Z \equiv z - z_I^{\mathrm{fp}}$. Next, we rewrite z by Z as $z = (z - z_I^{\mathrm{fp}}) + z_I^{\mathrm{fp}} = Z + z_I^{\mathrm{fp}}$. This means that the second term, z_I^{fp}, can be regarded as the Wilson line $\zeta = z_I^{\mathrm{fp}}$ ($\zeta_1 = y_{1I}^{\mathrm{fp}}$, $\zeta_2 = y_{2I}^{\mathrm{fp}}$) from the viewpoint of the coordinate Z. (See the previous subsection.) Then, the Wilson line can be pushed on SS phases by the $U(1)$ local and gauge transformation,

$$\begin{aligned}\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z) &= \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(Z + z_I^{\mathrm{fp}}) \\ &= V_{z_I^{\mathrm{fp}}}(Z)\psi^{n,(j+\beta_1,\beta_2),M}_{T^2/\mathbb{Z}_N^m,+}(Z),\end{aligned} \tag{2.42}$$

where (β_1, β_2) are defined by

$$(\beta_1, \beta_2) \equiv (\alpha_1 + M y_{2I}^{\mathrm{fp}}, \alpha_2 - M y_{1I}^{\mathrm{fp}}) \pmod 1. \tag{2.43}$$

On the other hand, the left-hand side of Eq. (2.35) can be written by Z as

$$\begin{aligned}\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(\rho z) &= \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(\rho Z + \rho z_I^{\mathrm{fp}}) \\ &= \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(\rho Z + z_I^{\mathrm{fp}} - u - v\tau) \\ &= U_2^{-v}(\rho Z + z_I^{\mathrm{fp}} - u) U_1^{-u}(\rho Z + z_I^{\mathrm{fp}}) \\ &\quad \times V_{z_I^{\mathrm{fp}}}(\rho Z)\psi^{n,(j+\beta_1,\beta_2),M}_{T^2/\mathbb{Z}_N^m,+}(\rho Z),\end{aligned} \tag{2.44}$$

where we use Eq. (2.25). Thus, the mode functions $\psi^{(j+\beta_1,\beta_2)}_{T^2/\mathbb{Z}_N^m,+}(Z)$ transform under $\mathbb{Z}_N$ twist around z_I^{fp} as

$$\psi^{n,(j+\beta_1,\beta_2),M}_{T^2/\mathbb{Z}_N^m,+}(\rho Z) = \rho^{\chi_{+I}}\psi^{n,(j+\beta_1,\beta_2),M}_{T^2/\mathbb{Z}_N^m,+}(Z), \tag{2.45}$$

with

$$\chi_{+I} = N\left\{u\alpha_1 + v\alpha_2 + \frac{M}{2}(uv + u y_{2I}^{\mathrm{fp}} - v y_{1I}^{\mathrm{fp}})\right\} + m \pmod N, \tag{2.46}$$

where we use the result,

$$\begin{aligned}V^{-1}_{z_I^{\mathrm{fp}}}(\rho Z)V_{z_I^{\mathrm{fp}}}(Z) &= e^{-\pi i M\frac{\mathrm{Im}((\bar{z}_I^{\mathrm{fp}} - \bar{\rho}\bar{z}_I^{\mathrm{fp}})\rho Z)}{\mathrm{Im}\tau}} \\ &= U_1^{-u}(\rho Z)U_2^{-v}(\rho Z)e^{2\pi i(u\alpha_1 + v\alpha_2)}.\end{aligned} \tag{2.47}$$

Note that Eq. (2.45) with $z_I^{\mathrm{fp}} = 0$ corresponds to Eq. (2.35). Hence, we get the winding numbers χ_{+I} of the $\mathbb{Z}_N$ mode functions $\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z)$ around the fixed points z_I^{fp}.

We are interested in the numbers of chiral zero modes on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux. Although the AS index theorem on T^2 is known as (2.23), the AS index theorem cannot, however, be applied to orbifolds directly because they have singular points. On the other hand, in the previous paper [30], the following zero-mode counting formula on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux has been obtained:

$$n_+ - n_- = \frac{M}{N} - \frac{V_+}{N} + 1, \tag{2.48}$$

where V_+ is the sum of the winding numbers at the fixed points of the $T^2/\mathbb{Z}_N$ orbifolds. It should be emphasized that the equality between the left-hand side and the right-hand side of Eq. (2.48) has been verified in each case in Ref. [30], but the formula (2.48) has not been established as an index theorem. The first term on the right-hand side of Eq. (2.48) can be understood as the contribution of the flux and the factor $1/N$ comes from the fact that the area of the $T^2/\mathbb{Z}_N$ orbifold is $1/N$ of that of the torus T^2. On the other hand, physical roles of the second and the third terms of

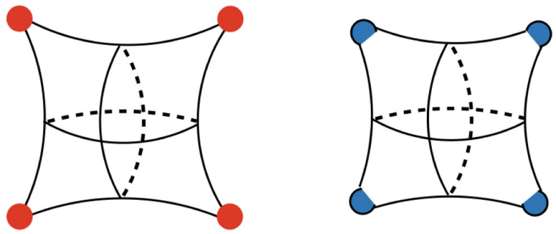

FIG. 1. The left figure shows $T^2/\mathbb{Z}_2$ orbifold and the red points represent the fixed points of $T^2/\mathbb{Z}_2$ orbifold. By cutting around the fixed points and embedding the part of S^2 as caps, we can construct the blow-up manifold as shown in the right figure.

Eq. (2.48) are unclear, because they are not related to any flux on the orbifolds. In particular, it is curious why the factor $+1$ is needed on the right-hand side of the formula (2.48).

In order to apply the AS index theorem to the orbifold models, we consider removing the singular points from the orbifolds. To this end, we replace the $T^2/\mathbb{Z}_N$ orbifolds with smooth manifolds without singularities by cutting out the singularities of the magnetized $T^2/\mathbb{Z}_N$ orbifolds and attaching smooth manifolds (parts of S^2) to them, as shown in Fig. 1. The smooth manifolds without singularities are called blow-up manifolds of the $T^2/\mathbb{Z}_N$ orbifolds. Then, we can apply the AS index theorem to the blow-up manifolds directly.

III. BLOW-UP MANIFOLD OF MAGNETIZED $T^2/\mathbb{Z}_N$ ORBIFOLD

In this section, we construct the blow-up manifolds of the magnetized $T^2/\mathbb{Z}_N$ orbifolds by replacing orbifold singularities with parts of S^2. Then, we compute wave functions on the blow-up manifolds by connecting those on the orbifolds with those on parts of S^2 smoothly without losing the orbifold information. In the blow-up process, in particular, we obtain two remarkable features; one is that winding numbers of wave functions on the $T^2/\mathbb{Z}_N$ orbifolds are related to localized flux and localized curvature on the blow-up manifolds, and the other is that not only curvature but also magnetic flux are not modified under the blow-up process.

A. Magnetized S^2

First of all, we review zero-mode functions on S^2 with magnetic flux [43]. Let z' be the complex coordinate on $S^2 \simeq \mathbb{CP}^1$ defined by projecting a point of S^2 into the complex plane passing through the center of S^2 from the north pole of S^2, as shown in Fig. 2. The radius of S^2 is taken to be R.

The magnetic flux on S^2 is quantized as

$$\frac{1}{2\pi}\int_{S^2} F' = M', \tag{3.1}$$

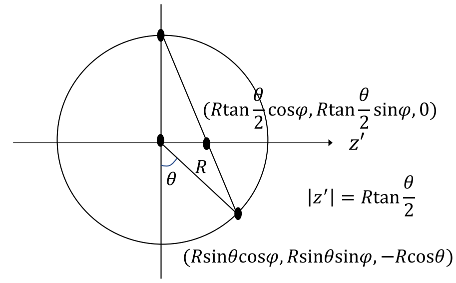

FIG. 2. The cross section of S^2 with the radius R is shown. We project a point of S^2 with the 3D coordinate, $(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, -R\cos\theta)$, from the north pole of S^2, into the point on the complex plane passing through the center of S^2 whose 3D coordinate is $(R\tan\frac{\theta}{2}\cos\varphi, R\tan\frac{\theta}{2}\sin\varphi, 0)$, where (R,θ,φ) are spherical coordinate parameters. We define the complex coordinate of the complex plane $\mathbb{CP}^1$, z', such that $z' = |z'|e^{i\varphi} = R\tan\frac{\theta}{2}e^{i\varphi}$ at the point with the 3D coordinate $(R\tan\frac{\theta}{2}\cos\varphi, R\tan\frac{\theta}{2}\sin\varphi, 0)$. Then, we denote the coordinate of a point on S^2 as the complex coordinate of the projected point on $\mathbb{CP}^1$, z'.

where M' is an integer. The field strength is

$$\frac{F'}{2\pi} = \frac{i}{2\pi}\frac{R^2M'}{(R^2+|z'|^2)^2}dz' \wedge d\bar{z}'. \tag{3.2}$$

The gauge potentials on S^2 are given by

$$A_{\bar{z}'} = \frac{i}{2}\frac{M'}{R^2+|z'|^2}z', \qquad A_{z'} = -\frac{i}{2}\frac{M'}{R^2+|z'|^2}\bar{z}'. \tag{3.3}$$

The mode functions on the magnetized S^2 obey the Dirac equations

$$\frac{R^2+|z'|^2}{R}i\left(\partial_{\bar{z}'} + i\frac{1}{2}\omega_{\bar{z}'} - iA_{\bar{z}'}\right)\psi^{n',M'}_{S^2,+}(z') = m_{n'}\psi^{n',M'}_{S^2,-}(z'), \tag{3.4}$$

$$\frac{R^2+|z'|^2}{R}i\left(\partial_{z'} - i\frac{1}{2}\omega_{z'} - iA_{z'}\right)\psi^{n',M'}_{S^2,-}(z') = m_{n'}\psi^{n',M'}_{S^2,+}(z'), \tag{3.5}$$

with

$$\omega_{\bar{z}'} = \frac{i}{2}\frac{2}{R^2+|z'|^2}z', \qquad \omega_{z'} = -\frac{i}{2}\frac{2}{R^2+|z'|^2}\bar{z}'. \tag{3.6}$$

Here, $\omega_{\bar{z}'}$ and $\omega_{z'}$ are the spin connections that come from the nonvanishing curvature on S^2,

$$\frac{1}{2\pi}\int_{S^2} R' = \chi(S^2) = 2. \tag{3.7}$$

Here, R' is the curvature on S^2 and χ is the Euler characteristic. Note that the spin connections (3.6) can be obtained by replacing the flux M' in the gauge potentials (3.3) by the Euler characteristic $\chi(S^2) = 2$.

The positive chirality zero mode solutions of Eq. (3.4) with $m_n = 0$ are given by

$$\psi^{0,M'}_{S^2,+}(z') = \frac{f^{M'}_{+}(z')}{(R^2 + |z'|^2)^{\frac{M'-1}{2}}}, \tag{3.8}$$

where $f^{M'}_{+}(z')$ is a holomorphic function of z'. These solutions are normalizable and well-defined on S^2 only if $M' > 0$ and $f^{M'}_{+}(z')$ is expressed as a $(M'-1)$th-order polynomial, which means that the number of the independent solutions is M'. On the other hand, normalizable and well-defined negative-chirality zero modes on S^2 are obtained in a similar way only if $M' < 0$, and an antiholomorphic function $f^{M'}_{-}(\bar{z}')$ is expressed as a $(|M'|-1)$th-order polynomial.

The above results are consistent with the AS index theorem on the magnetized S^2, i.e.,

$$n_+ - n_- = \frac{1}{2\pi}\int_{S^2} F' = M'. \tag{3.9}$$

The number of the chiral zero modes turns out to be given by the flux quantization number M', as it should be. It is important to emphasize that although the flux and the curvature exist in the magnetized S^2 model, only the flux contributes to the AS index theorem, as mentioned in the introduction.

B. Construction of blow-up manifold of $T^2/\mathbb{Z}_N$ orbifold

In this subsection, we review the construction of blow-up manifolds of $T^2/\mathbb{Z}_N$ orbifolds [37]. Since the $T^2/\mathbb{Z}_N$ orbifolds have singularities at the fixed points and around the fixed points become cones, we replace the cones with parts of S^2 to remove the singularities, as shown in Fig. 3. Figure 3 shows the case that the deficit angle around a fixed point is $2\pi(N-1)/N$, and we replace the cone whose slant height is r with $(N-1)/2N$ part of S^2 whose radius is $R = r/\sqrt{N^2-1}$. The left figure shows the development of the cone, and the right figure shows the cross section of the cone and S^2 with the radius $R = r/\sqrt{N^2-1}$. Here, the curvature around the singularity is $(N-1)/N$ which comes from the deficit angle. On the other hand, since the curvature of S^2 is $\chi(S^2) = 2$, the curvature of the embedded region is $(N-1)/N$. That is, this blow-up process does not change the topological invariant number. Similarly, we can apply this procedure for the other fixed points of the orbifolds.

We denote the coordinates of $T^2/\mathbb{Z}_N$ and S^2 as z and z', respectively. They are related at the connection points through the coordinate w, i.e., $z|_{z=re^{i\varphi/N}} \leftrightarrow w = \frac{N+1}{N} z'|_{z'=\frac{r}{N+1}e^{i\varphi}}$.

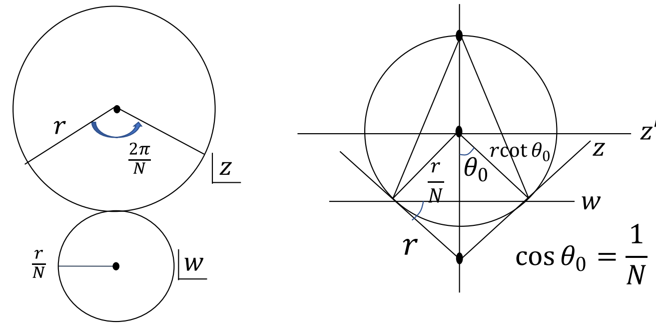

FIG. 3. The left figure shows the development of the cone around a fixed point of $T^2/\mathbb{Z}_N$ orbifold. The slant height and the radius of the base of the cone are r (called the blow-up radius) and r/N, respectively. The right figure shows the cross section of the cone and the S^2 with radius $R = r\cot\theta_0 = r/\sqrt{N^2-1}$. The blow-up manifold of $T^2/\mathbb{Z}_N$ is constructed by replacing the cone with $(N-1)/2N$-part of S^2, where $\sin(\theta_0/2) = (N-1)/2N$. Here, z and z' denote the coordinates of $T^2/\mathbb{Z}_N$ and S^2, respectively, and they are related at the connection points through the coordinate w, i.e., $z|_{z=re^{i\varphi/N}} \leftrightarrow w = \frac{N+1}{N} z'|_{z'=\frac{r}{N+1}e^{i\varphi}}$.

Next, we discuss zero-mode wave functions on magnetized blow-up manifolds, which can be obtained by smoothly connecting wave functions on the magnetized $T^2/\mathbb{Z}_N$ orbifold in Eq. (2.40) with those on the magnetized S^2 in Eq. (3.8) at the connection line. However, there is an obstacle. If zero modes on the $T^2/\mathbb{Z}_N$ orbifold have nonzero winding numbers, they cannot be connected to zero modes on S^2 because the boundary conditions for $\psi^{0,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(z)$ around the fixed point are different from those of $\psi^{0,M'}_{S^2,+}(z')$. $\psi^{0,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(z)$ obey the boundary condition (2.45). On the other hand, $\psi^{0,M'}_{S^2,+}(z')$ have no phase. In the next subsection, we resolve this obstacle by using singular gauge transformation.

C. Singular gauge transformation

In order to connect wave functions on $T^2/\mathbb{Z}_N$ to those on S^2, we remove nonzero winding numbers from wave functions on $T^2/\mathbb{Z}_N$ by the following singular gauge transformation[3] $\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(z) \to \tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(z)$ such that $\tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(z)$ has no winding number:

$$\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(\rho z) = \rho^m \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(z) \to \tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(\rho z) = \tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}^m_N,+}(z). \tag{3.10}$$

[3] See Refs. [39–41,44] and also Ref. [45] for magnetized S^2 with vortices.

Here, we have considered the case where the fixed point is $z_I^{\rm fp} = 0$. Note that the following analysis can be applied even for the other fixed points by the following replacement:

$$\begin{aligned} z &\to Z, \\ (\alpha_1, \alpha_2) &\to (\beta_1, \beta_2), \\ m &\to \chi_{+I}. \end{aligned} \tag{3.11}$$

The singular gauge transformation is defined by

$$A \to \tilde{A}(z) \equiv A(z) + \delta A(z), \tag{3.12}$$

$$\delta A(z) = iU_{\xi^F} dU_{\xi^F}^{-1} \simeq -i\frac{\xi^F}{2}\frac{1}{z}dz + i\frac{\xi^F}{2}\frac{1}{\bar{z}}d\bar{z}, \tag{3.13}$$

with

$$\begin{aligned} U_{\xi^F}(z) &= \left(\frac{\psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2/\mathbb{Z}_N^1,+}(z)}{(\psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2/\mathbb{Z}_N^1,+}(z))^*}\right)^{\frac{\xi^F}{2}} = \left(\frac{g_1(z)}{(g_1(z))^*}\right)^{\frac{\xi^F}{2}} \\ &\simeq \left(\frac{g_1^{(1)}(0)z}{(g_1^{(1)}(0)z)^*}\right)^{\frac{\xi^F}{2}}, \end{aligned} \tag{3.14}$$

where we use

$$\begin{aligned} \psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2/\mathbb{Z}_N^1,+}(z) &= \psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2,+}(z) = e^{-\frac{\pi}{2{\rm Im}\tau}|z|^2} g_1(z) \\ g_1(z) &\equiv g^{(\frac{1}{2},\frac{1}{2}),1} = e^{\frac{\pi}{2{\rm Im}\tau}z^2} e^{\frac{\pi i}{2}} \vartheta\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}(z,\tau), \end{aligned} \tag{3.15}$$

and $g_1^{(k)}(z) \equiv \frac{d^k g_1(z)}{dz^k}$. The rightmost-hand sides of Eqs. (3.13) and (3.14) are approximate expressions near $z = 0$. Under the singular gauge transformation, the field strength is modified as

$$\frac{F}{2\pi} \to \frac{\tilde{F}}{2\pi} \equiv \frac{F}{2\pi} + \frac{\delta F}{2\pi}, \tag{3.16}$$

$$\frac{\delta F}{2\pi} = i\xi^F \delta(z)\delta(\bar{z}) dz \wedge d\bar{z}. \tag{3.17}$$

Here, from Eq. (3.16), ξ^F/N can be regarded as a localized flux at the fixed point $z = 0$.

We further need to consider a singular gauge transformation for the spin connection in a way similar to the gauge potentials to remove winding numbers both of $\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z)$ and $\psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z)$. It is defined by

$$\omega \to \tilde{\omega} = \omega + \delta\omega = \delta\omega, \qquad (\omega = 0), \tag{3.18}$$

$$\delta\omega = iU_{\xi^R} dU_{\xi^R}^{-1} \overset{z\simeq 0}{\simeq} -i\frac{\xi^R}{2}\frac{1}{z}dz + i\frac{\xi^R}{2}\frac{1}{\bar{z}}d\bar{z}, \tag{3.19}$$

with

$$\begin{aligned} U_{\xi^R}(z) &= \left(\frac{\psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2/\mathbb{Z}_N^1,+}(z)}{(\psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2/\mathbb{Z}_N^1,+}(z))^*}\right)^{\frac{\xi^R}{2}} \\ &= \left(\frac{g_1(z)}{(g_1(z))^*}\right)^{\frac{\xi^R}{2}} \overset{z\simeq 0}{\simeq} \left(\frac{g_1^{(1)}(0)z}{(g_1^{(1)}(0)z)^*}\right)^{\frac{\xi^R}{2}}. \end{aligned} \tag{3.20}$$

Similarly, ξ^R/N can be regarded as a localized curvature at the fixed point, where it corresponds to $(N-1)/N$ in the case of the deficient angle $2\pi(N-1)/N$.

From Eqs. (3.14) and (3.20), the wave functions are transformed under the singular gauge transformation as

$$\begin{aligned} \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z) &\to \tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z) \\ &= U_{\xi^F}(z) U_{\xi^R}^{-1/2}(z) \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z), \end{aligned} \tag{3.21}$$

$$\begin{aligned} \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z) &\to \tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z) \\ &= U_{\xi^F}(z) U_{\xi^R}^{1/2}(z) \psi^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z). \end{aligned} \tag{3.22}$$

Then, Eqs. (2.35) and (2.36) are modified as

$$\tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(\rho z) = \rho^{\xi^F - \frac{\xi^R}{2} + m} \tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z), \tag{3.23}$$

$$\tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(\rho z) = \rho^{\xi^F + \frac{\xi^R}{2} + m + 1} \tilde{\psi}^{n,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,-}(z). \tag{3.24}$$

Note that the contributions of the localized curvature ξ^R act with opposite signs to the chirality positive and negative wave functions. In addition, Eq. (2.9) and equivalently Eq. (2.10) are also modified by replacing M and α_i with $M + \xi^F \mp \xi^R/2$ and $\alpha_i + \xi^F/2 \mp \xi^R/4$, respectively.

We arrive at the conditions to obtain wave functions with vanishing winding numbers as

$$\xi^F = \frac{N-1}{2} - m + \ell N \quad \text{for } {}^\forall \ell \in \mathbb{Z}, \tag{3.25}$$

where we used $\xi^R = N - 1$ for the $\mathbb{Z}_N$ fixed point. It is interesting to point out that a new degree of freedom ℓ appears. It comes from mod N property of Eqs. (3.23) and (3.24). In Sec. V, we discuss the physical meaning of the new degree of freedom of the localized flux in detail. Zero-mode wave functions in Eq. (2.41), in particular, can be expressed as

$$\tilde{\psi}^{0,(j+\alpha_1,\alpha_\tau),M}_{T^2/\mathbb{Z}_N^m,+}(z) = |g_1(z)|^{m-\ell N} e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2} \tilde{h}^{(j+\alpha_1,\alpha_\tau),M}_{+,m}(z)$$
$$\simeq |z|^{m-\ell N} e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2} |g_1^{(1)}(0)|^{m-\ell N} \tilde{h}^{(j+\alpha_1,\alpha_\tau),M}_{+,m}(z),$$
$$\tilde{h}^{(j+\alpha_1,\alpha_\tau),M}_{+,m}(z) = \mathcal{N}^{0,j}_{T^2/\mathbb{Z}_N,+} (g_1(z))^{-m+\ell N} \sum_{k=0}^{N-1} \rho^{-km} g^{(j+\alpha_1,\alpha_\tau),M}(\rho^k z)$$
$$\simeq \mathcal{N}^{0,j}_{T^2/\mathbb{Z}_N,+} N \frac{(g^{(j+\alpha_1,\alpha_\tau),M})^{(m)}(0)}{m!} (g_1^{(1)}(0))^{-m+\ell N} z^{\ell N}, \tag{3.26}$$

where we also show the approximation near $z = 0$ at the lowest order. Hereafter, we denote the coefficient shortly as

$$C^j \equiv \mathcal{N}^{0,j}_{T^2/\mathbb{Z}_N,+} \frac{(g^{(j+\alpha_1,\alpha_\tau),M})^{(m)}(0)}{m!} \left(\frac{g_1^{(1)}(0)}{|g_1^{(1)}(0)|}\right)^{-m+\ell_0 N}. \tag{3.27}$$

Similarly, the same argument can be applied for $z_I^{\mathrm{fp}} \neq 0$ by the replacement (3.11). Especially, we obtain the following relationship:

$$\xi_I^F = \frac{\xi_I^R}{2} - \chi_{+I} + \ell_I N \quad \text{for } {}^\forall \ell_I \in \mathbb{Z}. \tag{3.28}$$

It means that the winding number χ_{+I} can be rewritten in terms of the localized flux ξ_I^F and the localized curvature ξ_I^R. In other words, what we have done with the singular gauge transformations (3.21) and (3.22) is to replace the information of the winding number on the orbifolds with the localized flux and localized curvature at the fixed points of $T^2/\mathbb{Z}_N$. This operation is expected to connect wave functions on $T^2/\mathbb{Z}_N$ with those on S^2 without losing the orbifold information. In the next subsection, let us see zero-mode wave functions on the blow-up manifolds of magnetized $T^2/\mathbb{Z}_N$.

D. Wave functions on blow-up manifold of magnetized $T^2/\mathbb{Z}_N$

Now, we can explore zero-mode wave functions on the blow-up manifolds of magnetized $T^2/\mathbb{Z}_N$. Wave functions on the blow-up regions (parts of S^2 regions) are those on S^2 in Eq. (3.8), $\psi^{0,M'}_{S^2,+}(z')$, while wave functions on the bulk region (remaining region of $T^2/\mathbb{Z}_N$ by cutting out regions around fixed points) are those on $T^2/\mathbb{Z}_N$ in Eq. (3.26), $\tilde{\psi}^{0,(j+\alpha_1,\alpha_2),M}_{T^2/\mathbb{Z}_N^m,+}(z)$. Then, we should connect wave functions on bulk regions and those on the blow-up regions smoothly at the junction points. Note that the renewed point from Ref. [37] is using Eq. (3.26) instead of Eq. (2.41). Thus, we can treat wave functions with $\mathbb{Z}_N$ charge m more precisely.

The junction conditions are given by

$$\tilde{\psi}^{0,(j+\alpha_1,\alpha_\tau),M}_{T^2/\mathbb{Z}_N^m,+}(z)|_{z=re^{i\varphi/N}} = \psi^{0,M'}_{S^2,+}(z')|_{z'=\frac{r}{N+1}e^{i\varphi}},$$
$$\frac{1}{e^{-i\frac{\varphi}{N}}} \frac{d\tilde{\psi}^{0,(j+\alpha_1,\alpha_\tau),M}_{T^2/\mathbb{Z}_N^m,+}(z)}{dz}\bigg|_{z=re^{i\varphi/N}} = \frac{1}{\frac{N+1}{N}e^{-i\varphi}} \frac{d\psi^{0,M'}_{S^2,+}(z')}{dz'}\bigg|_{z'=\frac{r}{N+1}e^{i\varphi}}, \tag{3.29}$$

where the derivatives of their coordinates can be written as

$$e^{-i\frac{\varphi}{N}}dz = e^{-i\frac{\varphi}{N}} \frac{\partial z}{\partial |z|} d|z| + e^{-i\frac{\varphi}{N}} \frac{\partial z}{\partial (\frac{\varphi}{N})} d\left(\frac{\varphi}{N}\right) = d|z| + ird\left(\frac{\varphi}{N}\right),$$
$$\frac{N+1}{N} e^{-i\varphi} dz' = \frac{N+1}{N} e^{-i\varphi} \frac{\partial z'}{\partial |z'|} d|z'| + \frac{N+1}{N} e^{-i\varphi} \frac{\partial z'}{\partial \varphi} d\varphi = \frac{N+1}{N} d|z'| + i\frac{r}{N} d\varphi. \tag{3.30}$$

Indeed, we find that the following relations:

$$\frac{N+1}{N} d|z'| = \frac{N+1}{N} \frac{\partial |z'|}{\partial \theta} d\theta = \frac{N+1}{N} \frac{R}{2\cos^2\frac{\theta_0}{2}} d\theta = \frac{N+1}{N} \frac{R}{1+\cos\theta_0} d\theta = Rd\theta = d|z|,$$
$$rd\left(\frac{\varphi}{N}\right) = \frac{r}{N} d\varphi, \tag{3.31}$$

are satisfied at the connecting points, as seen in Fig. 3.

First, from nonholomorphic parts of wave functions in Eqs. (3.26) and (3.8), the junction conditions in Eq. (3.29) provide

$$\frac{\pi r^2}{N\mathrm{Im}\tau}M+\frac{N-1}{2N}-\frac{m}{N}+\ell=\frac{N-1}{2N}M'. \tag{3.32}$$

By using the relation (3.25), it can be rewritten as

$$\frac{\pi r^2}{N\mathrm{Im}\tau}M+\frac{\xi^F}{N}=\frac{N-1}{2N}M'. \tag{3.33}$$

The flux condition is generalized from that in Ref. [37]; the left-hand side shows the cut out flux from $T^2/\mathbb{Z}_N$ orbifold, which is the flux including the localized flux on the cone of $T^2/\mathbb{Z}_N$ orbifold, while the right-hand side shows the embedded flux, which is the flux on the part of S^2. Thus, it means that the magnetic flux is not modified under the blow-up process. This is important in deriving the AS index theorem, as we will see in the next section. In particular, in the orbifold limit $r\to 0$, Eq. (3.40) is expressed as

$$\frac{\xi^F}{N}=\frac{N-1}{2N}M'\bigg|_{r=0}, \tag{3.34}$$

which shows that the flux on the embedded area of S^2 [right-hand side of Eq. (3.34)] corresponds to the localized flux on the orbifold fixed point [left-hand side of Eq. (3.34)].

On the other hand, from holomorphic parts, the holomorphic function on the part of S^2 region $f_{S^2}^{M'}(z')$ can be determined as

$$f_+^{M'}(z')=C'^jz'^\ell,\quad C'^j=C^jNr^me^{-\frac{\pi M}{2\mathrm{Im}\tau}r^2}\left(\frac{r}{N+1}\right)^{M'-1-\ell}\left(\frac{N-1}{2N}\right)^{-\frac{M'-1}{2}}. \tag{3.35}$$

Note that the holomorphicity of bulk modes with positive flux M requires $\ell\geq 0$; otherwise, they will diverge at $z'=0$. The divergence induced by the negative localized flux ℓ would be removed by introducing vortices analyzed in Ref. [45], which is beyond the scope of this paper. In the following analysis, we focus on the $\ell\geq 0$ case.

Therefore, (bulk) zero-mode wave functions on magnetized blow-up manifolds can be written as

$$\psi^{0,j}_{\text{blow-up}}=\begin{cases}\frac{C'^jz'^\ell}{(R^2+|z'|^2)^{\frac{M'-1}{2}}} & (|z'|\leq\frac{r}{N+1})\\ |g_1(z)|^{m-\ell N}e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2}\tilde{h}^{(j+\alpha_1,\alpha_\tau),M}_{+,m}(z) & (r\leq|z|)\\ \simeq C^jN|z|^{m-\ell N}e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2}z^{\ell N} & \end{cases}. \tag{3.36}$$

To determine the normalization, we first calculate the following inner product,

$$\begin{aligned}G_{ij}&=\int_{\text{blow-up manifold}}dzd\bar{z}\sqrt{|\det(g)|}(\psi^{0,i}_{\text{blow-up}})^*\psi^{0,j}_{\text{blow-up}}\\&=\delta_{i,j}-\int_0^r d|z||z|\int_0^{\frac{2\pi}{N}}d\varphi(C^i)^*C^jN^2|z|^{2m}e^{-\frac{\pi M}{\mathrm{Im}\tau}|z|^2}\\&\quad+\int_0^{\frac{r}{N+1}}d|z'||z'|\int_0^{2\pi}d\varphi\frac{4R^4}{(R^2+|z'|^2)^2}\frac{(C'^i)^*C'^j|z'|^{2\ell}}{(R^2+|z'|^2)^{M'-1}}\\&\simeq\delta_{i,j}+(C^i)^*C^j\pi(r^2)^{m+1}B,\end{aligned} \tag{3.37}$$

with

$$B\simeq\left(\frac{N-1}{2N}(M'-\ell)\right)^{-1}\frac{1-\sum_{p=0}^{\ell}\frac{\Gamma(M'+1)}{\Gamma(M'-p+1)\Gamma(p+1)}\left(\frac{N+1}{2N}\right)^{M'-p}\left(\frac{N-1}{2N}\right)^p}{\frac{\Gamma(M'+1)}{\Gamma(M'-\ell+1)\Gamma(\ell+1)}\left(\frac{N+1}{2N}\right)^{M'-\ell}\left(\frac{N-1}{2N}\right)^\ell}+\left(-\frac{m+1}{N}\right)^{-1}.$$

We next perform the unitary transformation for flavor index j,

$$\begin{aligned}\psi^{0,j'}_{\text{blow-up}}&=U_{j'j}\psi^{0,j}_{\text{blow-up}}\\U&=\prod_J(U^{J(J+1)})\mathrm{diag}(e^{-i\arg(C^j)})\\U^{J(J+1)}&=\begin{pmatrix}1&&&&&\\&\ddots&&&&\\&&\cos\theta_{J(J+1)}&-\sin\theta_{J(J+1)}&&\\&&\sin\theta_{J(J+1)}&\cos\theta_{J(J+1)}&&\\&&&&\ddots&\\&&&&&1\end{pmatrix},\qquad\tan^2\theta_{J(J+1)}=\frac{\sum_{I=1}^J|C^I|^2}{|C^{J+1}|^2}.\end{aligned} \tag{3.38}$$

Then, the inner product $(G)_{i'j'}$ can be rewritten as

$$G \simeq \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1+\sum_j |C^j|^2 \pi (r^2)^{m+1} B \end{pmatrix}. \quad (3.39)$$

Thus, by redefining the normalization factor for the last mode $j' = j'_{\max}$ as $\mathcal{N}'^{0,j'_{\max}}_{T^2/\mathbb{Z}_N,+} = \mathcal{N}^{0,j'_{\max}}_{T^2/\mathbb{Z}_N,+} (1+O((r^2)^{m+1}))^{-1/2}$, all of the above modes can be expressed by orthonormal basis. The detailed calculation of Eq. (3.37) is shown in Appendix B.

Similarly, the same argument can be applied for $z_I^{\rm fp} \neq 0$ by the replacement (3.11). Especially, we obtain the following flux condition:

$$\frac{\pi r_I^2}{N \mathrm{Im}\tau} M + \frac{\xi_I^F}{N} = \frac{N-1}{2N} M'_I. \quad (3.40)$$

This result becomes important for deriving the AS index theorem on the $T^2/\mathbb{Z}_N$ orbifold in the next section.

IV. INDEX THEOREM ON THE BLOW-UP MANIFOLD

This section is the main section of this paper. Our purpose is to establish the AS index theorem on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux background. Due to the existence of singularities on the orbifolds, the AS index theorem cannot be applied directly to the orbifold models. Our strategy is to replace the $T^2/\mathbb{Z}_N$ orbifolds with the blow-up manifolds without singularities and to apply the AS index theorem to them.

A. Index theorem on the blow-up manifold

The AS index theorem on the blow-up manifolds can be obtained as

$$n_+ - n_- = \int_{\text{blow-up manifold}} \frac{F}{2\pi} \quad (4.1)$$

$$= \int_{T^2/\mathbb{Z}_N\,\text{bulk}} \frac{F}{2\pi} + \sum_I \int_{\frac{N-1}{2N}\times S^2} \frac{F'}{2\pi} \quad (4.2)$$

$$= \left(\frac{M}{N} - \sum_I \frac{\pi r_I^2}{N\mathrm{Im}\tau} M\right) + \sum_I \frac{N-1}{2N} M'_I(r_I) \quad (4.3)$$

$$= \left(\frac{M}{N} - \sum_I \frac{\pi r_I^2}{N\mathrm{Im}\tau} M\right) + \sum_I \left(\frac{\pi r_I^2}{N\mathrm{Im}\tau} M + \frac{\xi_I^F}{N}\right) \quad (4.4)$$

$$= \frac{M}{N} + \sum_I \frac{\xi_I^F}{N}. \quad (4.5)$$

There are several comments for the above equations. For Eq. (4.1), we emphasize that the index $n_+ - n_-$ on the blow-up manifolds does not depend on the curvature but only on the flux. It comes from the fact that the AS index theorem on a 2D compact manifold has only the contribution of the flux on the manifold, in general. For the first term of Eq. (4.2) [and Eq. (4.3)], the $T^2/\mathbb{Z}_N$ bulk refers to the region of the $T^2/\mathbb{Z}_N$ orbifold from which the areas near the fixed points are removed. For the second term of Eq. (4.2) [and Eq. (4.3)], it represents each amount of the magnetic flux on the embedded area of S^2 replacing the fixed point. The sum over I is taken for the fixed points of the $T^2/\mathbb{Z}_N$ orbifolds. For Eq. (4.4), we used the relation (3.40).

For the final result (4.5), it should be emphasized that the AS index theorem on the blow-up manifolds does not depend on the blow-up radius r_I, as it should be. In other words, the result of the AS index theorem holds even in the orbifold limit $r_I \to 0$,

$$n_+ - n_- = \int_{T^2/\mathbb{Z}_N} \frac{\tilde{F}}{2\pi} = \frac{M}{N} + \sum_I \frac{\xi_I^F}{N}. \quad (4.6)$$

Here, $\tilde{F}$ is defined in Eq. (3.16) and this term comes from the limit of the right-hand side of Eq. (4.2) as follows: in the $r_I \to 0$ ($R \to 0$) limit, the second term of Eq. (4.2) with the field strength (3.2) can be expressed as

$$\int_{\frac{N-1}{2N}\times S^2} iM'_I \delta(z')\delta(\bar{z}') dz' \wedge d\bar{z}', \quad (4.7)$$

(see Appendix A in Ref. [46]), and it corresponds to Eq. (3.17) by considering Eq. (3.34). Thus, the first term of the rightmost-hand side of Eq. (4.6) represents the contribution of the homogeneous magnetic flux on the $T^2/\mathbb{Z}_N$ orbifolds, which comes from the first term of (3.16), while the second term represents the sum of localized fluxes at each fixed point, which comes from the second term of (3.16). Therefore, Eq. (4.6) becomes the AS index theorem on the $T^2/\mathbb{Z}_N$ orbifold, and the index can be determined by only the contribution of the flux.

From Eq. (3.28), the localized flux ξ_I^F is decided by the localized curvature ξ_I^R and the winding number χ_{+I} at the fixed points, where the winding numbers at the fixed points are investigated in [30]. We can verify that the number of chiral zero modes, which are computed by the zero-mode counting formula in Ref. [30], are completely consistent with the relation (4.6). The results are summarized in Tables I–V of the Appendix. Although it was not clear whether the zero mode counting formula was the AS index theorem, the present results using the blow-up manifolds indicate that it is indeed the case.

B. Reinterpretation of the zero-mode counting formula

We can now reinterpret the zero-mode counting formula (2.48). Using the relation (3.28), the AS index theorem (4.5) can be rewritten in terms of the winding numbers χ_{+I} as

$$n_+ - n_- = \frac{M}{N} + \sum_I \left(\frac{-\chi_{+I}}{N} + \frac{1}{2}\frac{\xi^R}{N} + \ell_I \right) = \frac{M - V_+}{N} + 1 + \sum_I \ell_I. \tag{4.8}$$

Here, we have used the relation

$$\frac{1}{2}\sum_I \frac{\xi_I^R}{N} = 1, \tag{4.9}$$

at the last equality. It can be verified as follows:

$$T^2/\mathbb{Z}_2\colon \sum_I \frac{\xi_I^R}{4} = 4 \times \frac{1}{4} = 1 \qquad \left(z_I^{\rm fp} = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \right), \tag{4.10}$$

$$T^2/\mathbb{Z}_3\colon \sum_I \frac{\xi_I^R}{6} = 3 \times \frac{2}{6} = 1 \qquad \left(z_I^{\rm fp} = 0, \frac{2+\tau}{3}, \frac{1+2\tau}{3} \right), \tag{4.11}$$

$$T^2/\mathbb{Z}_4\colon \sum_{I_{\mathbb{Z}_4}} \frac{\xi_{I_{\mathbb{Z}_4}}^R}{8} + \frac{1}{2}\sum_{I_{\mathbb{Z}_2}} \frac{\xi_{I_{\mathbb{Z}_2}}^R}{4} = 2 \times \frac{3}{8} + \frac{1}{2} \times 2 \times \frac{1}{4} = 1 \quad \left(z_{I_{\mathbb{Z}_4}}^{\rm fp} = 0, \frac{1+\tau}{2}, z_{I_{\mathbb{Z}_2}}^{\rm fp} = \frac{1}{2}, \frac{\tau}{2} \right), \tag{4.12}$$

$$T^2/\mathbb{Z}_6\colon \sum_{I_{\mathbb{Z}_6}} \frac{\xi_{I_{\mathbb{Z}_6}}^R}{12} + \frac{1}{2}\sum_{I_{\mathbb{Z}_3}} \frac{\xi_{I_{\mathbb{Z}_3}}^R}{6} + \frac{1}{3}\sum_{I_{\mathbb{Z}_2}} \frac{\xi_{I_{\mathbb{Z}_2}}^R}{4} = \frac{5}{12} + \frac{2}{6} + \frac{1}{4} = 1 \quad \left(z_{I_{\mathbb{Z}_6}}^{\rm fp} = 0, z_{I_{\mathbb{Z}_3}}^{\rm fp} = \frac{1+\tau}{3}, \frac{2+2\tau}{3}, z_{I_{\mathbb{Z}_2}}^{\rm fp} = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \right). \tag{4.13}$$

Note that $\mathbb{Z}_4$ and $\mathbb{Z}_6$ have subgroups and must include the contributions of their fixed points.

Thus, the zero-mode counting formula (2.48) can be derived from Eq. (4.8) by taking $\ell_I = 0$. The physical meaning of $+1$ in Eq. (2.48), which had been a mystery, is now clear. The factor $+1$ is the contribution of the sum of the localized curvatures at fixed points. When we try to write the index theorem with the winding numbers, $+1$ is needed to remove the contribution of the localized curvature from them, since the winding numbers include the contributions of both localized flux and the localized curvature [see Eq. (3.28)]. This analysis reveals that the zero-mode counting formula includes only the contribution of the flux.

An interesting observation in our analysis is the existence of a new degree of freedom ℓ_I. The AS index theorem says that additional zero modes can appear. In the next section, we study the new zero modes in detail.

V. LOCALIZED ZERO-MODE WAVE FUNCTIONS

As shown in the previous section, the degree of freedom of localized flux means that there exist additional zero modes. In this section, we study wave functions of the new zero modes.

The bulk zero-mode wave functions, in Sec. III, on the bulk region near the fixed point $z = 0$ and the blow-up region are proportional to $z^{\ell N}$ and z'^{ℓ}, respectively. It indicates that the new zero-mode wave functions on the bulk region near $z = 0$ and the blow-up region will be proportional to z^{aN} and z'^{a} for $a = 0, \ldots, \ell - 1$, respectively. Here, the factor $z^{\ell N}$ comes from the fact that the holomorphic function of the following wave function,

$$\psi^{0,N}_{T^2/\mathbb{Z}^1_N,+}(z) \equiv \left(\psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2/\mathbb{Z}^1_N,+}(z)\right)^N = \left(\psi^{0,(\frac{1}{2},\frac{1}{2}),1}_{T^2,+}(z)\right)^N, \tag{5.1}$$

is proportional to z^N near the fixed point though it is $\mathbb{Z}_N$ invariant, because it is made of the wave function with $\mathbb{Z}_N$ charge $m = 1$. Note that its boundary condition is the same as that of wave functions with $M = N$, $(\alpha_1, \alpha_\tau) \equiv (\frac{N}{2} - [\frac{N}{2}], \frac{N}{2} - [\frac{N}{2}])$, and $m = 0$, i.e., $\psi^{0,(j+\frac{N}{2}-[\frac{N}{2}],\frac{N}{2}-[\frac{N}{2}]),N}_{T^2/\mathbb{Z}^0_N,+}(z)$, and then the wave function in Eq. (5.1) can be expanded by these wave functions, where $[x]$ denotes the floor function. Thus, if the other wave function $\psi^{0,N}_{T^2/\mathbb{Z}^0_N}(z)$, which has the same boundary condition of $\psi^{(j+\frac{N}{2}-[\frac{N}{2}],\frac{N}{2}-[\frac{N}{2}]),N}_{T^2/\mathbb{Z}^0_N}(z)$, is constructed from $m = 0$ mode, we can obtain the new wave function whose holomorphic function is proportional to z^{aN} near $z = 0$ by replacing $(\psi^{0,N}_{T^2/\mathbb{Z}^1_N,+}(z))^{\ell-a}$ with $(\psi^{0,N}_{T^2/\mathbb{Z}^0_N,+}(z))^{\ell-a}$. Indeed, the zero-mode number of $\psi^{0,(j+\frac{N}{2}-[\frac{N}{2}],\frac{N}{2}-[\frac{N}{2}]),N}_{T^2/\mathbb{Z}^0_N,+}(z)$ is just two, indicating that there exists the other zero-mode which is different from Eq. (5.1) and can be expanded by $\psi^{0,(j+\frac{N}{2}-[\frac{N}{2}],\frac{N}{2}-[\frac{N}{2}]),N}_{T^2/\mathbb{Z}^0_N,+}(z)$. Then, we can obtain $\psi^{0,N}_{T^2/\mathbb{Z}^1_N,+}(z)$ as

$$\psi^{0,N}_{T^2/\mathbb{Z}^0_N,+}(z)\equiv e^{-\frac{\pi N}{2\mathrm{Im}\tau}|z|^2}h^N_0(z)$$
$$\equiv\begin{cases}\left(\psi^{0,(0,0),1}_{T^2/\mathbb{Z}^0_N,+}(z)\right)^N=\left(\psi^{0,(0,0),1}_{T^2,+}(z)\right)^N & (N=2,4)\\ \left(\psi^{0,(\frac{1}{6},\frac{1}{6}),1}_{T^2/\mathbb{Z}^0_N,+}(z)\right)^N=\left(\psi^{0,(\frac{1}{6},\frac{1}{6}),1}_{T^2,+}(z)\right)^N & (N=3)\\ \left(\psi^{0,(0,0),2}_{T^2/\mathbb{Z}^0_N,+}(0)\psi^{0,(0,0),2}_{T^2/\mathbb{Z}^0_N,+}(z)\right)^{N/2} & (N=6)\end{cases},\tag{5.2}$$

with

$$\psi^{0,(0,0),2}_{T^2/\mathbb{Z}^0_N,+}(z)=\sqrt{\frac{\sqrt{3}+1}{2\sqrt{3}}}e^{-\pi i/8}\psi^{0,(0,0),2}_{T^2,+}(z)+\sqrt{\frac{\sqrt{3}-1}{2\sqrt{3}}}e^{\pi i/8}\psi^{0,(1,0),2}_{T^2,+}(z).$$

Therefore, the ℓ number of new zero-mode wave functions can be expressed as

$$\tilde{\psi}^{0,a,\ell}_{T^2/\mathbb{Z}_N,+}\equiv\mathcal{N}^{0,a}_{T^2/\mathbb{Z}_N,+}\left(\frac{\psi^{0,N}_{T^2/\mathbb{Z}^0_N,+}(z)}{\psi^{0,N}_{T^2/\mathbb{Z}^1_N,+}(z)}\right)^{\ell-a}\tilde{\psi}^{0,(\alpha_1,\alpha_\tau)}_{T^2/\mathbb{Z}^m_N,+}(z)$$
$$\simeq C^aN|z|^{m-\ell N}e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2}z^{aN},\tag{5.3}$$

where the coefficient C^a is given by

$$C^a\equiv\mathcal{N}^a_{T^2/\mathbb{Z}_N}\left(\frac{h^N_0(0)}{(g^{(1)}_1(0))^N}\right)^{\ell-a}\sum_j C^j.\tag{5.4}$$

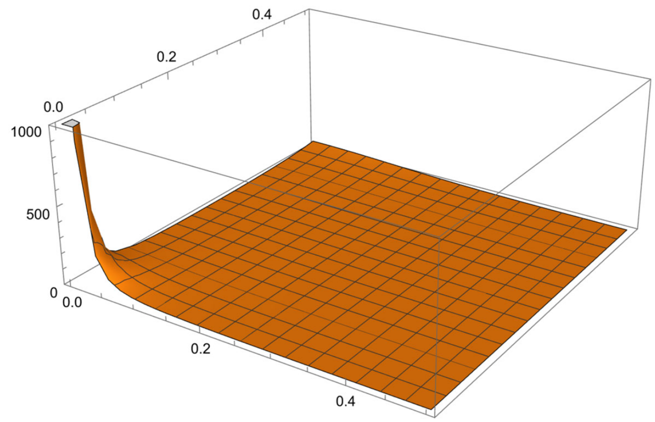

FIG. 4. Probability density of unnormalized zero-mode wave function $|\tilde{\psi}^{a_0=\ell_0-1}_{T^2/Z_4,+,0}|^2$.

Note that the nonholomorphic part of Eq. (5.3) does not change from that of Eq. (3.26). These new zero-modes diverge at the singular point $z=0$, while they are suppressed as they go away from the singular point, as shown in Fig. 4.

That is, these new zero modes correspond to localized modes around the singular point ($z=0$). Although these localized modes diverge at $z=0$, they can be regularized by replacing the cone around $z=0$ with the part of S^2. In other words, to calculate their normalization, we consider their wave functions on the magnetized blow-up manifold. As in Sec. III, through the junction condition in Eq. (3.29), the wave functions on the magnetized blow-up manifold, which correspond to localized modes on the orbifold, can be written as

$$\psi^{0,a}_{\text{blow-up}}=\begin{cases}\frac{C'^a z'^a}{(R^2+|z'|^2)^{\frac{M'-1}{2}}} & \left(|z'|\le\frac{r}{N+1}\right)\\ |g_1(z)|^{m-\ell N}e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2}\mathcal{N}^{0,a}_{T^2/\mathbb{Z}_N,+}\left(\frac{h^N_0(z)}{(g_1(z))^N}\right)^{\ell-a}\sum_j\tilde{h}^j(z) & (r\le|z|)\\ \simeq C^aN|z|^{m-\ell N}e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2}z^{aN}\end{cases},\tag{5.5}$$

where the coefficient C'^a is given by

$$C'^a=C^aNr^{m-(\ell-a)N}e^{-\frac{\pi M}{2\mathrm{Im}\tau}r^2}\left(\frac{r}{N+1}\right)^{M'-1-a}\left(\frac{N-1}{2N}\right)^{-\frac{M'-1}{2}}.\tag{5.6}$$

Furthermore, since these wave functions are suppressed as they go away from the orbifold singular point, it has little effect on the result of the inner product that we use an approximation form in the whole of the bulk region and also expand the integral region to $|z|\to\infty$. Under this approximation, it turns out that the ℓ number of new zero modes are orthogonal to each other and also orthogonal to all of the bulk zero modes by using the following results:

$$\int_0^{\frac{2\pi}{N}}d\arg(z)z^{kN}=0,\qquad\int_0^{2\pi}d\arg(z')z'^k=0,\quad(k\neq0).\tag{5.7}$$

Thus, the normalization of localized modes can be determined in the following way:

$$
\begin{aligned}
1 &= \int_{\text{blow-up manifold}} dz d\bar{z} \sqrt{|\det(g)|}\, |\psi^{0,a}_{\text{blow-up}}|^2 \\
&\simeq \int_r^\infty d|z||z| \int_0^{\frac{2\pi}{N}} d\varphi |C^a|^2 N^2 |z|^{2(m-(\ell-a)N)} e^{-\frac{\pi M}{\mathrm{Im}\tau}|z|^2} + \int_0^{\frac{r}{N+1}} d|z'||z'| \int_0^{2\pi} d\varphi \frac{4R^4}{(R^2+|z'|^2)^2} \frac{|C'^a|^2 |z'|^{2a}}{(R^2+|z'|^2)^{M'-1}} \\
&\simeq |C^a|^2 \pi \left(\frac{1}{r^2}\right)^{(\ell-a)N-(m+1)} \left[N \frac{\left(-\frac{\pi M}{\mathrm{Im}\tau} r^2\right)^{(\ell-a)N-(m+1)}}{[(\ell-a)N-(m+1)]!} E_1\left(\frac{\pi M}{\mathrm{Im}\tau} r^2\right) + L \right], \qquad (5.8)
\end{aligned}
$$

with

$$
L \simeq \left(\frac{N-1}{2N}(M'-a)\right)^{-1} \frac{1-\sum_{p=0}^{a} \frac{M'!}{(M'-p)!p!} \left(\frac{N+1}{2N}\right)^{M'-p} \left(\frac{N-1}{2N}\right)^{p}}{\frac{M'!}{(M'-a)!a!} \left(\frac{N+1}{2N}\right)^{M'-a} \left(\frac{N-1}{2N}\right)^{a}} + \left((\ell-a) - \frac{m+1}{N}\right)^{-1},
$$

where E_1 denotes the exponential integral. The detailed calculation of Eq. (5.8) is shown in Appendix C. Therefore, we obtained normalizable zero-mode wave functions in Eq. (5.5), and they correspond to localized modes under the orbifold limit $r \to 0$. Similarly, the above analysis is valid for localized modes around the other orbifold singular points by just replacement in Eq. (3.11).

VI. YUKAWA COUPLINGS ON MAGNETIZED BLOW-UP MANIFOLDS OF $T^2/\mathbb{Z}_N$ ORBIFOLDS

Finally, we study Yukawa coupling of 4D effective theory derived from the magnetized blow-up manifold. Here, we only replace the cone around $z = 0$ with the part of S^2. Similarly, we can consider the following analysis even at the other orbifold singular points. First, we denote bulk zero-modes and localized zero-modes shortly as B and L, respectively. When we consider the Yukawa coupling X_1-X_2-X_3 ($X = B, L$) in which $M_1 + M_2 = M_3$, $\xi_1^F + \xi_2^F = \xi_3^F$ ($\ell_1 + \ell_2 = \ell_3$, $m_1 + m_2 = m_3$),[4] and $(\alpha_1, \alpha_\tau)_1 + (\alpha_1, \alpha_\tau)_2 \equiv (\alpha_1, \alpha_\tau)_3 (\mathrm{mod}\ 1)$ are satisfied, only three patterns of couplings, (i) B_1-B_2-B_3 coupling, (ii) L_1-L_2-L_3 coupling, and (iii) B_1-L_2-L_3 coupling, are allowed by considering Eq. (5.7). Thus, we have a specific coupling selection rule in our theory. We can calculate their Yukawa coupling by using the results of Eqs. (3.37) and (5.8).

In case (i), the Yukawa coupling in the 4D effective theory can be expressed as

$$
\begin{aligned}
Y^{ijk}_{\text{blow-up}} &= y^{(3)}_{B_1-B_2-B_3} \int_{\text{blow-up manifold}} dz d\bar{z} \sqrt{|\det(g)|} (\psi^{0,k}_{\text{blow-up}})^* \psi^{0,i}_{\text{blow-up}} \psi^{0,j}_{\text{blow-up}} \\
&= Y^{ijk}_{T^2/\mathbb{Z}_N} - y^{(3)}_{B_1-B_2-B_3} \int_0^r d|z||z| \int_0^{\frac{2\pi}{N}} d\varphi (C^k)^* C^i C^j N^3 |z|^{2m_3} e^{-\frac{\pi M_3}{\mathrm{Im}\tau}|z|^2} \\
&\quad + y^{(3)}_{B_1-B_2-B_3} \int_0^{\frac{r}{N+1}} d|z'||z'| \int_0^{2\pi} d\varphi \frac{4R^4}{(R^2+|z'|^2)^2} \frac{(C'^k)^* C'^i C'^j |z'|^{2\ell_3}}{(R^2+|z'|^2)^{M'_3-1}} \\
&= Y^{ijk}_{T^2/\mathbb{Z}_N} + y^{(3)}_{B_1-B_2-B_3} (C^k)^* C^i C^j N \pi (r^2)^{m_3+1} B_3, \qquad (6.1)
\end{aligned}
$$

where $y^{(3)}_{B_1-B_2-B_3}$ denotes the 3-point coupling in higher-dimensional theory, and $Y^{ijk}_{T^2/\mathbb{Z}_N}$ denotes the 4D Yukawa coupling in the orbifold limit. Note that we use wave functions in Eq. (3.36). When we calculate it by orthonormal basis, only $Y^{i'_{\max} j'_{\max} k'_{\max}}_{\text{blow-up}}$ receives the blow-up correction while the others remain $Y^{i'j'k'}_{\text{blow-up}} = Y^{i'j'k'}_{T^2/\mathbb{Z}_N}$.

In case (ii), the Yukawa coupling on the magnetized blow-up manifold can be expressed as

[4] When $m_1 + m_2 = m_3 + N$ and $\ell_1 + \ell_2 = \ell_3 + 1$ are satisfied, correction terms in Eq. (6.1) are vanished, i.e., $Y^{ijk}_{\text{blow-up}} = Y^{ijk}_{T^2/\mathbb{Z}_N}$, while they give corrections for B_1-L_2($b = \ell_2 - 1$)-B_3 coupling, $Y^{i(\ell_2-1)k}_{\text{blow-up}}$, instead.

$$
\begin{aligned}
Y^{abc}_{\text{blow-up}} &= y^{(3)}_{L_1-L_2-L_3} \int_{\text{blow-up manifold}} dz d\bar{z} \sqrt{|\det(g)|} (\psi^{0,c}_{\text{blow-up}})^* \psi^{0,a}_{\text{blow-up}} \psi^{0,b}_{\text{blow-up}} \\
&\simeq y^{(3)}_{L_1-L_2-L_3} \left[\int_r^\infty d|z||z| \int_0^{\frac{2\pi}{N}} d\varphi (C^c)^* C^a C^b N^3 |z|^{2(m_3-(\ell_3-c)N)} e^{-\frac{\pi M_3}{\mathrm{Im}\tau}|z|^2} \right. \\
&\quad \left. + \int_0^{\frac{r}{N+1}} d|z'||z'| \int_0^{2\pi} d\varphi \frac{4R^4}{(R^2+|z'|^2)^2} \frac{(C'^c)^* C'^a C'^b |z'|^{2c}}{(R^2+|z'|^2)^{M'_3-1}} \right] \delta_{a+b,c} \\
&\simeq y^{(3)}_{L_1-L_2-L_3} \frac{C^a C^b}{C^c} N \delta_{a+b,c},
\end{aligned}
\tag{6.2}
$$

where $y^{(3)}_{L_1-L_2-L_3}$ denotes the 3-point coupling in higher-dimensional theory.

The case (iii) is the same as the case (ii) by replacing a_0 and $\delta_{a+b,c}$ with i and $\delta_{\ell_1+b=c}$, respectively, i.e.,

$$
Y^{ibc}_{\text{blow-up}} \simeq y^{(3)}_{B_1-L_2-L_3} \frac{C^i C^b}{C^c} N \delta_{\ell_1+b,c}, \tag{6.3}
$$

where $y^{(3)}_{B_1-L_2-L_3}$ denotes the 3-point coupling in higher-dimensional theory.

As a result, the Yukawa couplings among bulk modes (i) receive the contributions of blow-up radius, which play an important role in realizing the hierarchical structure of fermion masses as well as mixing angles, as demonstrated in Ref. [38]. By contrast, our results exhibit that Yukawa couplings including localized zero modes are determined by the normalization factor depending on the localized flux. Similarly, we can compute higher-dimensional operators. The overall coefficients such as $y^{(3)}_{B_1-B_2-B_3}$, $y^{(3)}_{L_1-L_2-L_3}$, and $y^{(3)}_{B_1-L_2-L_3}$ depend on higher-dimensional theory. They may be unified in supersymmetric Yang-Mills theory on a smooth manifold. All of the couplings originate from the gauge coupling in higher-dimensional supersymmetric Yang-Mills theory, which is a low-energy effective field theory of superstring theory. Obviously, there is no difference between bulk and localized modes in a smooth manifold. It is interesting to understand the flavor structure of localized modes as well as the origin of localized modes from the viewpoint of the string theory, but we leave the detailed study for future work.

VII. CONCLUSION

The main purpose of this paper is to establish the AS index theorem on the $T^2/\mathbb{Z}_N (N = 2, 3, 4, 6)$ orbifolds. In our previous paper [30], we have got the zero-mode counting formula which gives the numbers of the chiral zero modes on $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux background. It is, however, unclear whether the formula can be regarded as the index theorem, because the equality between the left-hand side and the right-hand side of Eq. (2.48) was merely verified in Ref. [30]. Furthermore, it is not obvious why the sum of the winding numbers V_+ appears and what is the physical meaning of the factor $+1$ in the formula (2.48). To confirm the zero-mode counting formula (2.48) as the index theorem and also to reveal the physical and geometrical meanings of the right-hand side of the formula (2.48), we have considered the blow-up manifolds of the magnetized $T^2/\mathbb{Z}_N (N = 2, 3, 4, 6)$ orbifolds, where we have constructed the blow-up manifolds by cutting out around the singularities of the $T^2/\mathbb{Z}_N$ orbifolds and attaching smooth manifolds (parts of S^2) to them.

In Sec. III, we have studied the blow-up manifolds of $T^2/\mathbb{Z}_N$ with magnetic flux more precisely than the previous work in Ref. [37]. The renewed point is introducing the appropriate singular gauge transformation, by which the winding number appeared in the $\mathbb{Z}_N$ twisted boundary condition of wave functions on the magnetized $T^2/\mathbb{Z}_N$ orbifold can be replaced with the localized flux and localized curvature at the fixed point of $T^2/\mathbb{Z}_N$ orbifold. Then, we have obtained zero mode wave functions on the blow-up manifolds of the magnetized $T^2/\mathbb{Z}_N$ orbifold by connecting those on $T^2/\mathbb{Z}_N$ orbifold and those on S^2 smoothly, even if those on $T^2/\mathbb{Z}_N$ orbifold have nonzero winding numbers. In particular, we have found that not only the total curvature but also the total magnetic flux including localized flux are invariant under the blow-up process. This result becomes important for deriving the AS index theorem on the $T^2/\mathbb{Z}_N$ orbifolds. We have also calculated the normalization of zero-mode wave functions on the blow-up manifolds with any winding numbers.

In Sec. IV, we have applied the AS index theorem to the blow-up manifolds of the $T^2/\mathbb{Z}_N$ orbifolds. The numbers of chiral zero modes are given only by the magnetic flux on the blow-up manifolds. Since the total magnetic flux is invariant under the blow-up process, the result is not changed even in the orbifold limit $r_I \to 0$, and the AS index theorem on $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux background is expressed by Eq. (4.6). It shows that the index is decided by the contribution of the homogeneous magnetic flux M and the localized fluxes ξ^F_I at the fixed points. We have verified that the number of chiral zero modes obtained by the zero-mode counting formula (2.48) in [30] is completely consistent with Eq. (4.6). The zero-mode counting formula can be reinterpreted from the viewpoint of the blow-up manifolds. The factor $+1$ in the formula (2.48) is found to be the contribution of the localized curvature at the fixed points and is needed to remove the contribution of the localized curvature from the winding numbers because the winding numbers include the contributions of both the localized flux and the localized curvature. (Remember that the AS index theorem in two dimensions needs only the information of fluxes.) Interestingly, a new degree of freedom of localized flux ℓ in Eq. (4.8), which emerges from the indeterminacy of mod N, suggests that there are new $|\ell|$ number of chiral zero modes.

In Sec. V, we have shown that the new zero modes given by the additional degree of freedom of localized flux correspond to localized zero modes at the orbifold singular point of $T^2/\mathbb{Z}_N$ orbifold. Although they diverge at the singular point, we calculated their normalization on the blow-up manifold to regularize them.

Moreover, in Sec. VI, we have calculated Yukawa coupling among bulk zero modes (discussed in Sec. III) and localized zero modes (discussed in Sec. V), and then it turns out that only three patterns of Yukawa coupling are allowed. We have a specific coupling selection rule. It would be interesting to study phenomenological implications of such coupling selection rules including higher-dimensional operators.

It is interesting to apply our analysis for more general higher-dimensional toroidal orbifolds such as $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$.[5] It is also important to study the relation with string theory. For example, localized modes, i.e., twisted modes should appear massless in heterotic string theory on toroidal-orbifold compactifications with generic gauge background by stringy consistency. It would be important to revisit this aspect from the viewpoint of our analysis of localized gauge fluxes and localized modes. However, that is beyond our scope and we would study them elsewhere.

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APPENDIX A: LOCALIZED FLUX AND INDEX

We compared the value of the index obtained from Eq. (4.6) with the result obtained from the zero-mode counting formula [30] and confirmed that these are consistent in all cases. The results are summarized in Tables I–V.

APPENDIX B: NORMALIZATION OF BULK ZERO MODES

Here, we show the detailed calculation of Eq. (3.37). It consists of three terms. The first term shows the calculation in all regions of the original $T^2/\mathbb{Z}_N$ orbifold. The second term shows the calculation in the region of the cone around $z = 0$ which is cut out from the $T^2/\mathbb{Z}_N$ orbifold. The third term shows the calculation in the region of the part of S^2 which is embedded instead of the cone. In the following, we show the detailed calculation of the second and third terms.

The second term can be calculated as

$$
\begin{aligned}
G_{ij}^{(2)} &\equiv \int_0^r d|z||z| \int_0^{\frac{2\pi}{N}} d\varphi (C^i)^* C^j N^2 |z|^{2m} e^{-\frac{\pi M}{\mathrm{Im}\tau}|z|^2} \\
&= (C^i)^* C^j \pi N \left(\frac{\pi M}{\mathrm{Im}\tau}\right)^{-(m+1)} \int_0^{\frac{\pi M}{\mathrm{Im}\tau}r^2} d\left(\frac{\pi M}{\mathrm{Im}\tau}|z|^2\right) \left(\frac{\pi M}{\mathrm{Im}\tau}|z|^2\right)^m e^{-(\frac{\pi M}{\mathrm{Im}\tau}|z|^2)} \\
&= (C^i)^* C^j \pi N \left(\frac{\pi M}{\mathrm{Im}\tau}\right)^{-(m+1)} \int_0^{\frac{\pi M}{\mathrm{Im}\tau}r^2} dt t^{(m+1)-1} e^{-t} \\
&= (C^i)^* C^j \pi N \left(\frac{\pi M}{\mathrm{Im}\tau}\right)^{-(m+1)} \gamma\left(m+1, \frac{\pi M}{\mathrm{Im}\tau} r^2\right),
\end{aligned}
$$

where $\gamma(m+1, \frac{\pi M}{\mathrm{Im}\tau} r^2)$ denotes the lower incomplete gamma function. It satisfies the following recurrence relation:

[5] See for the higher-dimensional orbifold models with bulk magnetic fluxes [47,48] as well as localized fluxes [49,50].

TABLE I. The values of localized fluxes at fixed points and index ($\ell_I = 0$) on $T^2/\mathbb{Z}_2$.

Flux	Parity	Twist	Localized flux				Index	(2.48)
M	η	(α_1, α_2)	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{\xi_4^F}{N}$	$\frac{M}{N} + \sum_{I=1}^4 \frac{\xi_I^F}{N}$	$\frac{M-V_+}{N} + 1$
$2m+1$	$+1$	(0, 0)	$1/4$	$1/4$	$1/4$	$-1/4$	$(M+1)/2$	$(M+1)/2$
		$(\frac{1}{2}, 0)$	$1/4$	$-1/4$	$1/4$	$1/4$	$(M+1)/2$	$(M+1)/2$
		$(0, \frac{1}{2})$	$1/4$	$1/4$	$-1/4$	$1/4$	$(M+1)/2$	$(M+1)/2$
		$(\frac{1}{2}, \frac{1}{2})$	$1/4$	$-1/4$	$-1/4$	$-1/4$	$(M-1)/2$	$(M-1)/2$
	-1	(0, 0)	$-1/4$	$-1/4$	$-1/4$	$1/4$	$(M-1)/2$	$(M-1)/2$
		$(\frac{1}{2}, 0)$	$-1/4$	$1/4$	$-1/4$	$-1/4$	$(M-1)/2$	$(M-1)/2$
		$(0, \frac{1}{2})$	$-1/4$	$-1/4$	$1/4$	$-1/4$	$(M-1)/2$	$(M-1)/2$
		$(\frac{1}{2}, \frac{1}{2})$	$-1/4$	$1/4$	$1/4$	$1/4$	$(M+1)/2$	$(M+1)/2$
$2m+2$	$+1$	(0, 0)	$1/4$	$1/4$	$1/4$	$1/4$	$M/2+1$	$M/2+1$
		$(\frac{1}{2}, 0)$	$1/4$	$-1/4$	$1/4$	$-1/4$	$M/2$	$M/2$
		$(0, \frac{1}{2})$	$1/4$	$1/4$	$-1/4$	$-1/4$	$M/2$	$M/2$
		$(\frac{1}{2}, \frac{1}{2})$	$1/4$	$-1/4$	$-1/4$	$1/4$	$M/2$	$M/2$
	-1	(0, 0)	$-1/4$	$-1/4$	$-1/4$	$-1/4$	$M/2-1$	$M/2-1$
		$(\frac{1}{2}, 0)$	$-1/4$	$1/4$	$-1/4$	$1/4$	$M/2$	$M/2$
		$(0, \frac{1}{2})$	$-1/4$	$-1/4$	$1/4$	$1/4$	$M/2$	$M/2$
		$(\frac{1}{2}, \frac{1}{2})$	$-1/4$	$1/4$	$1/4$	$-1/4$	$M/2$	$M/2$

TABLE II. The values of localized fluxes at fixed points and index ($\ell_I = 0$) on $T^2/\mathbb{Z}_3$.

Flux	Parity	Twist	Localized flux			Index	(2.48)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{M}{N} + \sum_{I=1}^3 \frac{\xi_I^F}{N}$	$\frac{M-V_+}{N} + 1$
$6m+1$	1	$1/6$	$1/3$	0	$1/3$	$(M+2)/3$	$(M+2)/3$
		$1/2$	$1/3$	$-1/3$	$-1/3$	$(M-1)/3$	$(M-1)/3$
		$5/6$	$1/3$	$1/3$	0	$(M+2)/3$	$(M+2)/3$
	ω	$1/6$	0	$-1/3$	0	$(M-1)/3$	$(M-1)/3$
		$1/2$	0	$1/3$	$1/3$	$(M+2)/3$	$(M+2)/3$
		$5/6$	0	0	$-1/3$	$(M-1)/3$	$(M-1)/3$
	ω^2	$1/6$	$-1/3$	$1/3$	$-1/3$	$(M-1)/3$	$(M-1)/3$
		$1/2$	$-1/3$	0	0	$(M-1)/3$	$(M-1)/3$
		$5/6$	$-1/3$	$-1/3$	$1/3$	$(M-1)/3$	$(M-1)/3$
$6m+2$	1	0	$1/3$	0	0	$(M+1)/3$	$(M+1)/3$
		$1/3$	$1/3$	$-1/3$	$1/3$	$(M+1)/3$	$(M+1)/3$
		$2/3$	$1/3$	$1/3$	$-1/3$	$(M+1)/3$	$(M+1)/3$
	ω	0	0	$-1/3$	$-1/3$	$(M-2)/3$	$(M-2)/3$
		$1/3$	0	$1/3$	0	$(M+1)/3$	$(M+1)/3$
		$2/3$	0	0	$1/3$	$(M+1)/3$	$(M+1)/3$
	ω^2	0	$-1/3$	$1/3$	$1/3$	$(M+1)/3$	$(M+1)/3$
		$1/3$	$-1/3$	0	$-1/3$	$(M-2)/3$	$(M-2)/3$
		$2/3$	$-1/3$	$-1/3$	0	$(M-2)/3$	$(M-2)/3$
$6m+3$	1	$1/6$	$1/3$	$-1/3$	0	$M/3$	$M/3$
		$1/2$	$1/3$	$1/3$	$1/3$	$M/3+1$	$M/3+1$
		$5/6$	$1/3$	0	$-1/3$	$M/3$	$M/3$
	ω	$1/6$	0	$1/3$	$-1/3$	$M/3$	$M/3$
		$1/2$	0	0	0	$M/3$	$M/3$
		$5/6$	0	$-1/3$	$1/3$	$M/3$	$M/3$
	ω^2	$1/6$	$-1/3$	0	$1/3$	$M/3$	$M/3$
		$1/2$	$-1/3$	$-1/3$	$-1/3$	$M/3-1$	$M/3-1$
		$5/6$	$-1/3$	$1/3$	0	$M/3$	$M/3$

TABLE III. The values of localized fluxes at fixed points and index ($\ell_I = 0$) on $T^2/\mathbb{Z}_3$.

Flux	Parity	Twist	Localized flux			Index	(2.48)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{M}{N}+\sum_{I=1}^{3}\frac{\xi_I^F}{N}$	$\frac{M-V_+}{N}+1$
$6m+4$	1	0	1/3	−1/3	−1/3	$(M-1)/3$	$(M-1)/3$
		1/3	1/3	1/3	0	$(M+2)/3$	$(M+2)/3$
		2/3	1/3	0	1/3	$(M+2)/3$	$(M+2)/3$
	ω	0	0	1/3	1/3	$(M+2)/3$	$(M+2)/3$
		1/3	0	0	−1/3	$(M-1)/3$	$(M-1)/3$
		2/3	0	−1/3	0	$(M-1)/3$	$(M-1)/3$
	ω^2	0	−1/3	0	0	$(M-1)/3$	$(M-1)/3$
		1/3	−1/3	−1/3	1/3	$(M-1)/3$	$(M-1)/3$
		2/3	−1/3	1/3	−1/3	$(M-1)/3$	$(M-1)/3$
$6m+5$	1	1/6	1/3	1/3	−1/3	$(M+1)/3$	$(M+1)/3$
		1/2	1/3	0	0	$(M+1)/3$	$(M+1)/3$
		5/6	1/3	−1/3	1/3	$(M+1)/3$	$(M+1)/3$
	ω	1/6	0	0	1/3	$(M+1)/3$	$(M+1)/3$
		1/2	0	−1/3	−1/3	$(M-2)/3$	$(M-2)/3$
		5/6	0	1/3	0	$(M+1)/3$	$(M+1)/3$
	ω^2	1/6	−1/3	−1/3	0	$(M-2)/3$	$(M-2)/3$
		1/2	−1/3	1/3	1/3	$(M+1)/3$	$(M+1)/3$
		5/6	−1/3	0	−1/3	$(M-2)/3$	$(M-2)/3$
$6m+6$	1	0	1/3	1/3	1/3	$M/3+1$	$M/3+1$
		1/3	1/3	0	−1/3	$M/3$	$M/3$
		2/3	1/3	−1/3	0	$M/3$	$M/3$
	ω	0	0	0	0	$M/3$	$M/3$
		1/3	0	−1/3	1/3	$M/3$	$M/3$
		2/3	0	1/3	−1/3	$M/3$	$M/3$
	ω^2	0	−1/3	−1/3	−1/3	$M/3-1$	$M/3-1$
		1/3	−1/3	1/3	0	$M/3$	$M/3$
		2/3	−1/3	0	1/3	$M/3$	$M/3$

$$\gamma\left(m+1,\frac{\pi M}{\mathrm{Im}\tau}r^2\right) = m\gamma\left(m,\frac{\pi M}{\mathrm{Im}\tau}r^2\right) - \left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^m e^{-(\frac{\pi M}{\mathrm{Im}\tau}r^2)}$$
$$\gamma\left(1,\frac{\pi M}{\mathrm{Im}\tau}r^2\right) = 1 - e^{-(\frac{\pi M}{\mathrm{Im}\tau}r^2)},$$

and then by solving this recurrence relation, $\gamma(m+1,\frac{\pi M}{\mathrm{Im}\tau}r^2)$ can be expressed as

$$\gamma\left(m+1,\frac{\pi M}{\mathrm{Im}\tau}r^2\right) = m!e^{-\frac{\pi M}{\mathrm{Im}\tau}r^2}\left[e^{\frac{\pi M}{\mathrm{Im}\tau}r^2} - \sum_{p=0}^{m}\frac{1}{p!}\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^p\right]$$
$$= e^{-\frac{\pi M}{\mathrm{Im}\tau}r^2}\frac{1}{m+1}\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^{m+1}\sum_{p=0}^{\infty}\frac{(m+1)!}{(m+1+p)!}\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^p.$$

Thus, the second term $G_{ij}^{(2)}$ can be expressed as

$$G_{ij}^{(2)} = (C^i)^*C^j\pi(r^2)^{m+1}e^{-\frac{\pi M}{\mathrm{Im}\tau}r^2}\left(\frac{m+1}{N}\right)^{-1}\sum_{p=0}^{\infty}\frac{(m+1)!}{(m+1+p)!}\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^p. \tag{B1}$$

By contrast, the third term can be calculated as

TABLE IV. The values of localized fluxes at fixed points and index ($\ell_I = 0$) on $T^2/\mathbb{Z}_4$.

Flux	Parity	Twist	Localized flux				Index	(2.48)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{\xi_4^F}{N}$	$\frac{M}{N}+\sum_{I=1}^4 \frac{\xi_I^F}{N}$	$\frac{M-V_+}{N}+1$
$4m+1$	1	0	3/8	1/8	1/8	1/8	$(M+3)/4$	$(M+3)/4$
		1/2	3/8	−3/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
	i	0	1/8	−1/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
		1/2	1/8	3/8	1/8	1/8	$(M+3)/4$	$(M+3)/4$
	−1	0	−1/8	−3/8	1/8	1/8	$(M-1)/4$	$(M-1)/4$
		1/2	−1/8	1/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
	$-i$	0	−3/8	3/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
		1/2	−3/8	−1/8	1/8	1/8	$(M-1)/4$	$(M-1)/4$
$4m+2$	1	0	3/8	−1/8	1/8	1/8	$(M+2)/4$	$(M+2)/4$
		1/2	3/8	3/8	−1/8	−1/8	$(M+2)/4$	$(M+2)/4$
	i	0	1/8	−3/8	−1/8	−1/8	$(M-2)/4$	$(M-2)/4$
		1/2	1/8	1/8	1/8	1/8	$(M+2)/4$	$(M+2)/4$
	−1	0	−1/8	3/8	1/8	1/8	$(M+2)/4$	$(M+2)/4$
		1/2	−1/8	−1/8	−1/8	−1/8	$(M-2)/4$	$(M-2)/4$
	$-i$	0	−3/8	1/8	−1/8	−1/8	$(M-2)/4$	$(M-2)/4$
		1/2	−3/8	−3/8	1/8	1/8	$(M-2)/4$	$(M-2)/4$
$4m+3$	1	0	3/8	−3/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
		1/2	3/8	1/8	−1/8	−1/8	$(M+1)/4$	$(M+1)/4$
	i	0	1/8	3/8	−1/8	−1/8	$(M+1)/4$	$(M+1)/4$
		1/2	1/8	−1/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
	−1	0	−1/8	1/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
		1/2	−1/8	−3/8	−1/8	−1/8	$(M-3)/4$	$(M-3)/4$
	$-i$	0	−3/8	−1/8	−1/8	−1/8	$(M-3)/4$	$(M-3)/4$
		1/2	−3/8	3/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
$4m+4$	1	0	3/8	3/8	1/8	1/8	$M/4+1$	$M/4+1$
		1/2	3/8	−1/8	−1/8	−1/8	$M/4$	$M/4$
	i	0	1/8	1/8	−1/8	−1/8	$M/4$	$M/4$
		1/2	1/8	−3/8	1/8	1/8	$M/4$	$M/4$
	−1	0	−1/8	−1/8	1/8	1/8	$M/4$	$M/4$
		1/2	−1/8	3/8	−1/8	−1/8	$M/4$	$M/4$
	$-i$	0	−3/8	−3/8	−1/8	−1/8	$M/4-1$	$M/4-1$
		1/2	−3/8	1/8	1/8	1/8	$M/4$	$M/4$

$$
\begin{aligned}
G_{ij}^{(3)} &\equiv \int_0^{\frac{r}{N+1}} d|z'||z'| \int_0^{2\pi} d\varphi \frac{4R^4}{(R^2+|z'|^2)^2} \frac{(C'^i)^* C'^j |z'|^{2\ell}}{(R^2+|z'|^2)^{M'-1}} \\
&= (C'^i)^* C'^j 4\pi (R^2)^{1-(M'-\ell-1)} \int_{\frac{N+1}{2N}}^1 d\left(\frac{R^2}{R^2+|z'|^2}\right)\left(1-\frac{R^2}{R^2+|z'|^2}\right)^{\ell}\left(\frac{R^2}{R^2+|z'|^2}\right)^{M'-\ell-1} \\
&= (C^i)^* C^j N^2 (r^2)^m e^{-\frac{\pi M}{\mathrm{Im}\tau} r^2}\left(\frac{r^2}{(N+1)^2}\right)^{M'-\ell-1}\left(\frac{2N}{N-1}\right)^{M'-1} 4\pi \left(\frac{r^2}{(N-1)(N+1)}\right)^{2+\ell-M'} \\
&\quad \times \left(\int_0^1 dt t^{(M'-\ell)-1}(1-t)^{(\ell+1)-1} - \int_0^{\frac{N+1}{2N}} dt t^{(M'-\ell)-1}(1-t)^{(\ell+1)-1}\right) \\
&= (C^i)^* C^j \pi (r^2)^{m+1} e^{-\frac{\pi M}{\mathrm{Im}\tau} r^2}\left(\frac{2N}{N+1}\right)^{M'-\ell}\left(\frac{2N}{N-1}\right)^{\ell+1}\left(\beta(M'-\ell,\ell+1) - \beta_{\frac{N+1}{2N}}(M'-\ell,\ell+1)\right),
\end{aligned}
$$

where $\beta(M'-\ell,\ell+1)$ and $\beta_{\frac{N+1}{2N}}(M'-\ell,\ell+1)$ denote the beta function and the incomplete beta function, respectively. They satisfy the following recurrence relations:

TABLE V. The values of localized fluxes at fixed points and index ($\ell_I = 0$) on $T^2/\mathbb{Z}_6$.

Flux	Parity	Twist	Localized flux						Index	(2.48)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{\xi_4^F}{N}$	$\frac{\xi_5^F}{N}$	$\frac{\xi_6^F}{N}$	$\frac{M}{N}+\sum_{I=1}^{6}\frac{\xi_I^F}{N}$	$\frac{M-V_+}{N}+1$
$6m+1$	1	1/2	5/12	−2/12	−2/12	−1/12	−1/12	−1/12	$(M-1)/6$	$(M-1)/6$
	ω	1/2	3/12	2/12	2/12	1/12	1/12	1/12	$(M+5)/6$	$(M+5)/6$
	ω^2	1/2	1/12	0	0	−1/12	−1/12	−1/12	$(M-1)/6$	$(M-1)/6$
	ω^3	1/2	−1/12	−2/12	−2/12	1/12	1/12	1/12	$(M-1)/6$	$(M-1)/6$
	ω^4	1/2	−3/12	2/12	2/12	−1/12	−1/12	−1/12	$(M-1)/6$	$(M-1)/6$
	ω^5	1/2	−5/12	0	0	1/12	1/12	1/12	$(M-1)/6$	$(M-1)/6$
$6m+2$	1	0	5/12	0	0	1/12	1/12	1/12	$(M+4)/6$	$(M+4)/6$
	ω	0	3/12	−2/12	−2/12	−1/12	−1/12	−1/12	$(M-2)/6$	$(M-2)/6$
	ω^2	0	1/12	2/12	2/12	1/12	1/12	1/12	$(M+4)/6$	$(M+4)/6$
	ω^3	0	−1/12	0	0	−1/12	−1/12	−1/12	$(M-2)/6$	$(M-2)/6$
	ω^4	0	−3/12	−2/12	−2/12	1/12	1/12	1/12	$(M-2)/6$	$(M-2)/6$
	ω^5	0	−5/12	2/12	2/12	−1/12	−1/12	−1/12	$(M-2)/6$	$(M-2)/6$
$6m+3$	1	1/2	5/12	2/12	2/12	−1/12	−1/12	−1/12	$(M+3)/6$	$(M+3)/6$
	ω	1/2	3/12	0	0	1/12	1/12	1/12	$(M+3)/6$	$(M+3)/6$
	ω^2	1/2	1/12	−2/12	−2/12	−1/12	−1/12	−1/12	$(M-3)/6$	$(M-3)/6$
	ω^3	1/2	−1/12	2/12	2/12	1/12	1/12	1/12	$(M+3)/6$	$(M+3)/6$
	ω^4	1/2	−3/12	0	0	−1/12	−1/12	−1/12	$(M-3)/6$	$(M-3)/6$
	ω^5	1/2	−5/12	−2/12	−2/12	1/12	1/12	1/12	$(M-3)/6$	$(M-3)/6$
$6m+4$	1	0	5/12	−2/12	−2/12	1/12	1/12	1/12	$(M+2)/6$	$(M+2)/6$
	ω	0	3/12	2/12	2/12	−1/12	−1/12	−1/12	$(M+2)/6$	$(M+2)/6$
	ω^2	0	1/12	0	0	1/12	1/12	1/12	$(M+2)/6$	$(M+2)/6$
	ω^3	0	−1/12	−2/12	−2/12	−1/12	−1/12	−1/12	$(M-4)/6$	$(M-4)/6$
	ω^4	0	−3/12	2/12	2/12	1/12	1/12	1/12	$(M+2)/6$	$(M+2)/6$
	ω^5	0	−5/12	0	0	−1/12	−1/12	−1/12	$(M-4)/6$	$(M-4)/6$
$6m+5$	1	1/2	5/12	0	0	−1/12	−1/12	−1/12	$(M+1)/6$	$(M+1)/6$
	ω	1/2	3/12	−2/12	−2/12	1/12	1/12	1/12	$(M+1)/6$	$(M+1)/6$
	ω^2	1/2	1/12	2/12	2/12	−1/12	−1/12	−1/12	$(M+1)/6$	$(M+1)/6$
	ω^3	1/2	−1/12	0	0	1/12	1/12	1/12	$(M+1)/6$	$(M+1)/6$
	ω^4	1/2	−3/12	−2/12	−2/12	−1/12	−1/12	−1/12	$(M-5)/6$	$(M-5)/6$
	ω^5	1/2	−5/12	2/12	2/12	1/12	1/12	1/12	$(M+1)/6$	$(M+1)/6$
$6m+6$	1	0	5/12	2/12	2/12	1/12	1/12	1/12	$M/6+1$	$M/6+1$
	ω	0	3/12	0	0	−1/12	−1/12	−1/12	$M/6$	$M/6$
	ω^2	0	1/12	−2/12	−2/12	1/12	1/12	1/12	$M/6$	$M/6$
	ω^3	0	−1/12	2/12	2/12	−1/12	−1/12	−1/12	$M/6$	$M/6$
	ω^4	0	−3/12	0	0	1/12	1/12	1/12	$M/6$	$M/6$
	ω^5	0	−5/12	−2/12	−2/12	−1/12	−1/12	−1/12	$M/6-1$	$M/6-1$

$$\beta(M'-\ell,\ell+1) = \frac{\ell_0}{M'-\ell}\beta_{\frac{N+1}{2N}}(M'-\ell+1,\ell)$$

$$\beta(M',1) = \frac{1}{M'},$$

$$\beta_{\frac{N+1}{2N}}(M'-\ell,\ell+1) = \frac{1}{M'-\ell}\left(\ell\beta_{\frac{N+1}{2N}}(M'-\ell+1,\ell) + \left(\frac{N+1}{2N}\right)^{M'-\ell}\left(\frac{N-1}{2N}\right)^{\ell}\right) \quad \beta_{\frac{N+1}{2N}}(M',1) = \frac{1}{M'}\left(\frac{N+1}{2N}\right)^{M'},$$

and then by solving these recurrence relations, they can be expressed as

$$\beta(M'-\ell,\ell-1) = \frac{\Gamma(M'-\ell)\Gamma(\ell+1)}{\Gamma(M'+1)},$$
$$\beta_{\frac{N+1}{2N}}(M'-\ell,\ell-1) = \frac{\Gamma(M'-\ell)\Gamma(\ell+1)}{\Gamma(M'+1)}\sum_{p=0}^{\ell}\frac{\Gamma(M'+1)}{\Gamma(M'-p+1)\Gamma(p+1)}\left(\frac{N+1}{2N}\right)^{M'-p}\left(\frac{N-1}{2N}\right)^{p},$$

respectively. Here, $\Gamma(X)$ denotes the gamma function, which satisfies the recurrence relation

$$\Gamma(X+1) = X\Gamma(X).$$

Thus, the third term $G_{ij}^{(3)}$ can be expressed as

$$G_{ij}^{(3)} = (C^i)^* C^j \pi (r^2)^{m+1} e^{-\frac{\pi M}{\mathrm{Im}\tau}r^2}\left(\frac{N-1}{2N}(M'-\ell)\right)^{-1}\frac{1-\sum_{p=0}^{\ell}\frac{\Gamma(M'+1)}{\Gamma(M'-p+1)\Gamma(p+1)}\left(\frac{N+1}{2N}\right)^{M'-p}\left(\frac{N-1}{2N}\right)^{p}}{\frac{\Gamma(M'+1)}{\Gamma(M'-\ell+1)\Gamma(\ell+1)}\left(\frac{N+1}{2N}\right)^{M'-\ell}\left(\frac{N-1}{2N}\right)^{\ell}}. \tag{B2}$$

By combining these results, we obtain Eq. (3.37).

APPENDIX C: NORMALIZATION OF LOCALIZED ZERO MODES

In this section, we show the detailed calculation of Eq. (5.8). The first term shows the calculation in the bulk region, while the second term shows the calculation in the blow-up region. The first term can be calculated as

$$\begin{aligned}
&\int_r^\infty d|z||z|\int_0^{\frac{2\pi}{N}} d\varphi |C^a|^2 N^2 |z|^{2(m-(\ell-a)N)} e^{-\frac{\pi M}{\mathrm{Im}\tau}|z|^2} \\
&= |C^a|^2 \pi N \left(\frac{\pi M}{\mathrm{Im}\tau}\right)^{(\ell-a)N-(m+1)} \int_{\frac{\pi M}{\mathrm{Im}\tau}r^2}^\infty d\left(\frac{\pi M}{\mathrm{Im}\tau}|z|^2\right)\left(\frac{\pi M}{\mathrm{Im}\tau}|z|^2\right)^{m-(\ell-a)N} e^{-(\frac{\pi M}{\mathrm{Im}\tau}|z|^2)} \\
&= |C^a|^2 \pi N \left(\frac{\pi M}{\mathrm{Im}\tau}\right)^{(\ell-a)N-(m+1)} \int_{\frac{\pi M}{\mathrm{Im}\tau}r^2}^\infty dt t^{m-(\ell-a)N} e^{-t} \\
&= |C^a|^2 \pi N \left(\frac{\pi M}{\mathrm{Im}\tau}\right)^{(\ell-a)N-(m+1)} \Gamma\left(1+m-(\ell-a)N, \frac{\pi M}{\mathrm{Im}\tau}r^2\right),
\end{aligned}$$

where $\Gamma(1+m-(\ell-a)N, \frac{\pi M}{\mathrm{Im}\tau}r^2)$ denotes the upper incomplete gamma function. We note that $1+m-(\ell-a)N<0$. Then, it satisfies the following recurrence relation:

$$\Gamma\left(1+m-(\ell-a)N, \frac{\pi M}{\mathrm{Im}\tau}r^2\right) = \frac{1}{1+m-(\ell-a)N}\left(\Gamma\left(2+m-(\ell-a)N, \frac{\pi M}{\mathrm{Im}\tau}r^2\right) - \left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^{1+m-(\ell-a)N} e^{-(\frac{\pi M}{\mathrm{Im}\tau}r^2)}\right)$$
$$\Gamma\left(0, \frac{\pi M}{\mathrm{Im}\tau}r^2\right) = E_1\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right),$$

where $E_1(\frac{\pi M}{\mathrm{Im}\tau}r^2)$ denotes the exponential integral. Note that if $\frac{\pi M}{\mathrm{Im}\tau}r^2$ is sufficiently large, the exponential integral obeys

$$E_1\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right) \simeq e^{-(\frac{\pi M}{\mathrm{Im}\tau}r^2)}\sum_{p=0}(-1)^p p!\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^{-(p+1)}.$$

By solving this recurrence relation, $\Gamma(1+m-(\ell-a)N, \frac{\pi M}{\mathrm{Im}\tau}r^2)$ can be expressed as

$$\Gamma\left(1+m-(\ell-a)N,\frac{\pi M}{\mathrm{Im}\tau}r^2\right)=\frac{(-1)^{(\ell-a)N-(m+1)}}{[(\ell-a)N-(m+1)]!}\left[E_1\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)-e^{-(\frac{\pi M}{\mathrm{Im}\tau}r^2)}\sum_{p=0}^{(\ell-a)N-(m+2)}(-1)^p p!\left(\frac{\pi M}{\mathrm{Im}\tau}r^2\right)^{-(p+1)}\right].$$

By contrast, the second term is the same as $G_{ij}^{(3)}$ in the previous section by replacing ℓ with a. Thus, by combining these results, we obtain Eq. (5.8).

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[1] M. F. Atiyah and I. M. Singer, The index of elliptic operators on compact manifolds, Bull. Am. Math. Soc. **69**, 422 (1969).
[2] Kazuo Fujikawa, Path-Integral Measure for Gauge-Invariant Fermion Theories, Phys. Rev. Lett. **42**, 1195 (1979).
[3] Kazuo Fujikawa, Erratum: Path integral for gauge theories with fermions, Phys. Rev. D **22**, 1499(E) (1980).
[4] Edward Witten, Constraints on supersymmetry breaking, Nucl. Phys. **B202**, 253 (1982).
[5] Edward Witten and Kazuya Yonekura, Anomaly inflow and the η-invariant, arXiv:1909.08775.
[6] A. V. Ivanov and D. V. Vassilevich, Anomaly inflow for local boundary conditions, J. High Energy Phys. 09 (2022) 250.
[7] Ahmed Abouelsaood, Curtis G. Callan, Jr., C. R. Nappi, and S. A. Yost, Open strings in background gauge fields, Nucl. Phys. **B280**, 599 (1987).
[8] Ralph Blumenhagen, Lars Goerlich, Boris Kors, and Dieter Lust, Noncommutative compactifications of type I strings on tori with magnetic background flux, J. High Energy Phys. 10 (2000) 006.
[9] C. Angelantonj, Ignatios Antoniadis, E. Dudas, and A. Sagnotti, Type I strings on magnetized orbifolds and brane transmutation, Phys. Lett. B **489**, 223 (2000).
[10] Carlo Angelantonj and Augusto Sagnotti, Open strings, Phys. Rep. **371**, 1 (2002); **376**, 407(E) (2003).
[11] Ralph Blumenhagen, Mirjam Cvetic, Paul Langacker, and Gary Shiu, Toward realistic intersecting D-brane models, Annu. Rev. Nucl. Part. Sci. **55**, 71 (2005).
[12] Ralph Blumenhagen, Boris Kors, Dieter Lust, and Stephan Stieberger, Four-dimensional string compactifications with D-branes, orientifolds and fluxes, Phys. Rep. **445**, 1 (2007).
[13] Luis E. Ibanez and Angel M. Uranga, *String Theory and Particle Physics: An Introduction to String Phenomenology* (Cambridge University Press, Cambridge, England, 2012).
[14] Lara B. Anderson, James Gray, Andre Lukas, and Eran Palti, Two hundred heterotic standard models on smooth Calabi-Yau threefolds, Phys. Rev. D **84**, 106005 (2011).
[15] Lara B. Anderson, James Gray, Andre Lukas, and Eran Palti, Heterotic line bundle standard models, J. High Energy Phys. 06 (2012) 113.
[16] Hiroyuki Abe, Tatsuo Kobayashi, Hajime Otsuka, and Yasufumi Takano, Realistic three-generation models from SO(32) heterotic string theory, J. High Energy Phys. 09 (2015) 056.
[17] Hajime Otsuka, SO(32) heterotic line bundle models, J. High Energy Phys. 05 (2018) 045.
[18] Hiroyuki Abe, Kang-Sin Choi, Tatsuo Kobayashi, and Hiroshi Ohki, Three generation magnetized orbifold models, Nucl. Phys. **B814**, 265 (2009).
[19] Tomo-hiro Abe, Yukihiro Fujimoto, Tatsuo Kobayashi, Takashi Miura, Kenji Nishiwaki, Makoto Sakamoto, and Yoshiyuki Tatsuta, Classification of three-generation models on magnetized orbifolds, Nucl. Phys. **B894**, 374 (2015).
[20] D. Cremades, L. E. Ibanez, and F. Marchesano, Computing Yukawa couplings from magnetized extra dimensions, J. High Energy Phys. 05 (2004) 079.
[21] Hiroyuki Abe, Tatsuo Kobayashi, Keigo Sumita, and Yoshiyuki Tatsuta, Gaussian Froggatt-Nielsen mechanism on magnetized orbifolds, Phys. Rev. D **90**, 105006 (2014).
[22] Yukihiro Fujimoto, Tatsuo Kobayashi, Kenji Nishiwaki, Makoto Sakamoto, and Yoshiyuki Tatsuta, Comprehensive analysis of yukawa hierarchies on T^2/Z_N with magnetic fluxes, Phys. Rev. D **94**, 035031 (2016).
[23] Tatsuo Kobayashi, Kenji Nishiwaki, and Yoshiyuki Tatsuta, *CP*-violating phase on magnetized toroidal orbifolds, J. High Energy Phys. 04 (2017) 080.
[24] Wilfried Buchmuller and Julian Schweizer, Flavor mixings in flux compactifications, Phys. Rev. D **95**, 075024 (2017).
[25] Wilfried Buchmuller and Ketan M. Patel, Flavor physics without flavor symmetries, Phys. Rev. D **97**, 075019 (2018).
[26] Shota Kikuchi, Tatsuo Kobayashi, Yuya Ogawa, and Hikaru Uchida, Yukawa textures in modular symmetric vacuum of magnetized orbifold models, Prog. Theor. Exp. Phys. **2022**, 033B10 (2022).
[27] Kouki Hoshiya, Shota Kikuchi, Tatsuo Kobayashi, and Hikaru Uchida, Quark and lepton flavor structure in magnetized orbifold models at residual modular symmetric points, Phys. Rev. D **106**, 115003 (2022).
[28] Edward Witten, Some properties of O(32) superstrings, Phys. Lett. **149B**, 351 (1984).
[29] Michael B. Green, J. H. Schwarz, and Edward Witten, *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies and Phenomenology* (Cambridge University Press, Cambridge, 1988).
[30] Makoto Sakamoto, Maki Takeuchi, and Yoshiyuki Tatsuta, Zero-mode counting formula and zeros in orbifold compactifications, Phys. Rev. D **102**, 025008 (2020).
[31] Hiroyuki Abe, Tatsuo Kobayashi, and Hiroshi Ohki, Magnetized orbifold models, J. High Energy Phys. 09 (2008) 043.

[32] Tomo-Hiro Abe, Yukihiro Fujimoto, Tatsuo Kobayashi, Takashi Miura, Kenji Nishiwaki, and Makoto Sakamoto, Z_N twisted orbifold models with magnetic flux, J. High Energy Phys. 01 (2014) 065.

[33] Tomo-hiro Abe, Yukihiro Fujimoto, Tatsuo Kobayashi, Takashi Miura, Kenji Nishiwaki, and Makoto Sakamoto, Operator analysis of physical states on magnetized T^2/Z_N orbifolds, Nucl. Phys. **B890**, 442 (2014).

[34] Tatsuo Kobayashi and Satoshi Nagamoto, Zero-modes on orbifolds: Magnetized orbifold models by modular transformation, Phys. Rev. D **96**, 096011 (2017).

[35] Makoto Sakamoto, Maki Takeuchi, and Yoshiyuki Tatsuta, Index theorem on $T^2/\mathbb{Z}_N$ orbifolds, Phys. Rev. D **103**, 025009 (2021)

[36] Hiroki Imai, Makoto Sakamoto, Maki Takeuchi, and Yoshiyuki Tatsuta, Index and winding numbers on $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux, Nucl. Phys. **B990**, 116189 (2023).

[37] Tatsuo Kobayashi, Hajime Otsuka, and Hikaru Uchida, Wavefunctions and Yukawa couplings on resolutions of $\mathrm{T}^2/\mathbb{Z}_N$ orbifolds, J. High Energy Phys. 08 (2019) 046.

[38] Tatsuo Kobayashi, Hajime Otsuka, and Hikaru Uchida, Flavor structure of magnetized $T^2/\mathbb{Z}_2$ blow-up models, J. High Energy Phys. 03 (2020) 042.

[39] Hyun Min Lee, Hans Peter Nilles, and Max Zucker, Spontaneous localization of bulk fields: The six-dimensional case, Nucl. Phys. **B680**, 177 (2004).

[40] Wilfried Buchmuller, Markus Dierigl, Fabian Ruehle, and Julian Schweizer, Chiral fermions and anomaly cancellation on orbifolds with Wilson lines and flux, Phys. Rev. D **92**, 105031 (2015).

[41] Wilfried Buchmuller, Markus Dierigl, and Yoshiyuki Tatsuta, Magnetized orbifolds and localized flux, Ann. Phys. (Amsterdam) **401**, 91 (2019).

[42] Hiroyuki Abe, Tatsuo Kobayashi, Shohei Uemura, and Junji Yamamoto, Loop Fayet-Iliopoulos terms in T^2/Z_2 models: Instability and moduli stabilization, Phys. Rev. D **102**, 045005 (2020).

[43] Joseph P. Conlon, Anshuman Maharana, and Fernando Quevedo, Wave functions and yukawa couplings in local string compactifications, J. High Energy Phys. 09 (2008) 104.

[44] J. Polchinski, *String Theory. Vol. 1: An Introduction to the Bosonic String*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2007).

[45] Brian P. Dolan and Aonghus Hunter-McCabe, Ground state wave functions for the quantum Hall effect on a sphere and the Atiyah-Singer index theorem, J. Phys. A **53**, 215306 (2020).

[46] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. **165**, 311 (1994).

[47] Hiroyuki Abe, Tatsuo Kobayashi, Hiroshi Ohki, Keigo Sumita, and Yoshiyuki Tatsuta, Non-Abelian discrete flavor symmetries of 10D SYM theory with magnetized extra dimensions, J. High Energy Phys. 06 (2014) 017.

[48] Shota Kikuchi, Tatsuo Kobayashi, Kaito Nasu, and Hikaru Uchida, Classifications of magnetized T^4 and T^4/Z_2 orbifold models, J. High Energy Phys. 08 (2022) 256.

[49] S. Groot Nibbelink, M. Trapletti, and M. Walter, Resolutions of $C^n/Z(n)$ orbifolds, their U(1) bundles, and applications to string model building, J. High Energy Phys. 03 (2007) 035.

[50] Pompey Leung and Hajime Otsuka, Heterotic stringy corrections to metrics of toroidal orbifolds and their resolutions, Phys. Rev. D **99**, 126011 (2019).