f_ϱ/m_ϱ and f_π/m_ϱ ratios and the conformal window

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The f_ϱ/m_ϱ ratio is calculated at N^{3}LO order within perturbative (p)non-relativistic quantum chromodynamics (NRQCD) with N_f flavors of mass degenerate fermions. The massless limit of the ratio is expanded á la Banks-Zaks in $\varepsilon = 16.5 - N_f$ leading to reliable predictions close to the upper end of the conformal window. The comparison of the next-to-next-to leading order (NNLO) and N^{3}LO results indicate that the Banks-Zaks expansion may be reliable down to twelve flavors. Previous lattice calculations combined with the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relations provide us with the same ratio for the range $2 \le N_f \le 10$. Assuming a monotonous dependence on N_f leads to an estimate for the lower end of the conformal window, $N_f^* \simeq 12$, by matching the nonperturbative and our perturbative results. In any case an abrupt change is observed in f_ϱ/m_ϱ at twelve flavors. As a cross-check we also consider the f_π/m_ϱ ratio for which lattice results are also available. The perturbative calculation at present is only at the NNLO level which is insufficient for a reliable and robust matching between the low N_f and high N_f regions. Nonetheless, using the relative size of the N^{3}LO correction of f_ϱ/m_ϱ for estimating the same for f_π/m_ϱ leads to the estimate $N_f^* \simeq 13$.

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I. INTRODUCTION AND SUMMARY

How gauge theories with spontaneous chiral symmetry breaking transition into conformal gauge theories as the massless fermion content is increased á la Banks-Zaks is a nontrivial QFT problem [1]. We propose dimensionless ratios of meson decay constants and masses as promising candidates to shed light on the particulars of the transition. Concretely, we will study f_ϱ/m_ϱ and f_π/m_ϱ in this paper. Our main objective is to estimate or constrain the critical flavor number, N_f^*, in other words the lower end of the conformal window.

The nontrivial problem of finding or constraining N_f^* in gauge theories has been addressed by many different approaches in the past [2–12]. Reviews concerning the nonperturbative lattices studies include [13–15] and references therein.

Clearly, a purely perturbative calculation, even at high orders, is not sufficient to determine N_f^* with any degree of confidence. Some nonperturbative input is required since just below the conformal window the theory is expected to be strongly coupled. In our work we will carry out high order perturbative calculations valid in the high N_f conformal region and combine it with nonperturbative results from the low N_f region in a meaningful way.

Below the conformal window both the nominators and denominators of our ratios have well-defined chiral limits and are both $O(\Lambda)$, the dynamically generated scale. The ratios are finite and can be computed via nonperturbative lattice calculations carefully extrapolated to the infinite volume, chiral and continuum limits. Inside the conformal window both decay constants and masses scale the same with the fermion mass m and the ratios again have a well-defined chiral limit. In this way the ratios can meaningfully be compared across the transition covering the full range of fermion content provided asymptotic freedom is present. This observation is the main motivation for our study. The gauge group will be $SU(3)$ throughout.

Perturbation theory is clearly not applicable below the conformal window, at low N_f, hence the need for nonperturbative lattice simulations there. Continuum and chirally extrapolated lattice results are available for f_π/m_ϱ within the range $2 \le N_f \le 10$ [16–18]. Using a KSRF-relation [19,20] these can be reused for f_ϱ/m_ϱ. This nonperturbative input is essential and will supplement our perturbative results.

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Close to the upper end of the conformal window, at high N_f where the fixed point coupling is small, perturbation theory is unambiguously reliable. This occurs below $N_f = 33/2$ for $SU(3)$, the point at which asymptotic freedom is lost. In this paper we calculate f_ϱ and m_ϱ in perturbation theory at finite fermion mass within the framework of (p) NRQCD to N^{3}LO accuracy, and obtain f_ϱ/m_ϱ, followed by the massless limit. The deviation between the NNLO and N^{3}LO results are very small down to twelve flavors indicating convergence of the perturbative series. Assuming f_ϱ/m_ϱ is monotonous as a function of N_f we attempt to match the nonperturbative low N_f region and the perturbative high N_f region. At twelve flavors an abrupt change occurs which we identify as an estimate of the lower end of the conformal window, $N_f^* \simeq 12$.

The same approach could be applied to the f_π/m_ϱ ratio as well. On the nonperturbative side continuum and chiral extrapolated lattice results are available in the literature as already mentioned. On the perturbative side, inside the conformal window, we are only able to calculate f_π to NNLO order at present, one order lower than it is currently possible for f_ϱ. Nonetheless, if we take the relative size of the N^{3}LO result found for f_ϱ/m_ϱ and estimate the corresponding contribution to f_π/m_ϱ to be about the same, we may extract N_f^* using the same procedure. From f_π/m_ϱ we obtain in this way $N_f^* \simeq 13$ but of course this result should be taken as indicative only, a genuine N^{3}LO calculation should be performed for f_π in the future to validate it.

The organization of the paper is as follows. In the next section we summarize the application of the Banks-Zaks expansion to mesonic bound states in mass perturbed conformal gauge theories. In Sec. III the leading order expressions are presented which are rather straightforward and are given only to fix notation and conventions. Section IV details the (p)NRQCD calculation of the NLO, NNLO and N^{3}LO corrections. These are used in Sec. V to attempt to match the nonperturbative low N_f and perturbative high N_f regions. An assumptions is made explicitly, namely that our decay constant to meson mass ratios are monotonous as a function of N_f, allowing the extraction of an estimate of N_f^*, the flavor number where an abrupt change occurs in the ratios. Finally in Sec. VI we conclude and provide an outlook for future work.

II. BANKS-ZAKS EXPANSION FOR BOUND STATES

In the massless case the theories inside the conformal window are nontrivial interacting conformal gauge theories with some fixed point g_*^2 depending on N_f. At least sufficiently close to $N_f = 33/2$ there is a single relevant $SU(N_f)$-invariant perturbation of this conformal field theory (CFT) given by the flavor singlet fermionic mass term. Its anomalous dimension determines the dependence of all RG invariant dimensionful quantities on the perturbing parameter m. Besides the mass dependence, there is of course dependence on the fixed point coupling (which depends on N_f) and further explicitly on N_f. Schematically, a quantity of dimension k can be written as $m^{k/(1+\gamma)}F(\varepsilon)$ where $\varepsilon = 33/2 - N_f$ and γ is the mass anomalous dimension of the massless theory. A perturbative expansion can then be developed for small ε, combining all N_f-dependence.

Depending on the quantity in question, the function $F(\varepsilon)$ can be determined in perturbation theory through an expansion in g^2. The observables in question for us are quantities related to bound states: mesons with various quantum numbers. A rigorous perturbative treatment of bound states is given in the nonrelativistic effective theory framework (p)NRQCD, which will be our main method.

Inside the conformal window massive fermions are always heavy in the (p)NRQCD language. Hence the setup corresponds to zero flavors of light fermions and N_f flavors of heavy fermions in (p)NRQCD terms. Dimensionful quantities, such as f_π, f_ϱ or m_ϱ are then given as being proportional to m and a double series expansion in $a(\mu) = g^2(\mu)/(16\pi^2)$ with some RG scale μ and in $1/m^2$ once a choice of RG scheme has been made. In the perturbed CFT the natural scale is $\mu = m$ which will be our choice as well. Thus, we will find schematically,

$$f_{\pi,\varrho} = m a^{3/2}(m)(b_0 + b_1 a(m) + \ldots)$$
$$m_\varrho = m(c_0 + c_1 a(m) + \ldots) \tag{1}$$

with some coefficients $b_i(N_f)$ and $c_i(N_f)$ which only depend on N_f and where … refer to higher order as well as nonanalytic terms involving $\log(a(m))$. Naturally, in the massless limit all three quantities are vanishing. But the massless limit can meaningfully can be taken for the ratios,

$$\frac{f_{\pi,\varrho}}{m_\varrho} = a_*^{3/2}(d_0 + d_1 a_* + d_2 a_*^2 + \ldots) \tag{2}$$

where $a(m \to 0) = a_*$ is the fixed point of the massless theory and the coefficients $d_i(N_f)$ again depend only on N_f. The (p)NRQCD calculation will provide all the coefficients above in the $\overline{\mathrm{MS}}$ scheme.

Now the fixed point a_* can trivially be expanded in ε also,

$$a_* = \varepsilon(e_0 + e_1\varepsilon + \ldots) \tag{3}$$

up to 5-loop order [21–26] where the corrections do not contain logarithms only higher orders in ε. Once all the explicit N_f-dependence of the coefficients $d_i(N_f)$ is replaced by $N_f = 33/2 - \varepsilon$ we can expand the final result in ε, leading to,

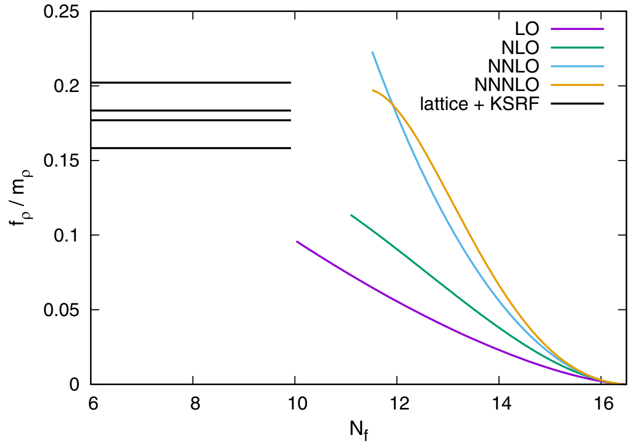

FIG. 1. The f_ϱ/m_ϱ ratio in increasing perturbative order as obtained from the Banks-Zaks expansion in $\varepsilon = 33/2 - N_f$. The nonperturbative result from combined lattice calculations [16–18] and the KSRF-relation is also shown. The smaller error band corresponds to the uncertainty of the lattice calculation, the wider one combines this with a conservative estimate of the uncertainty of the KSRF-relation itself.

$$\frac{f_{\pi,\varrho}}{m_\varrho} = \varepsilon^{3/2}(h_0 + h_1\varepsilon + h_2\varepsilon^2 + \ldots), \tag{4}$$

where again … is short hand for both higher orders in ε as well as powers of $\log\varepsilon$. The coefficients of the above series are constants and will be our main result up to $\mathrm{N^3LO}$ order in (p)NRQCD for f_ϱ/m_ϱ and up to NNLO order for f_π/m_ϱ. The order-by-order results for f_ϱ/m_ϱ are shown in Fig. 1 where the nonperturbative results for $2 \le N_f \le 10$ are also indicated. Clearly, the deviation between the NNLO and $\mathrm{N^3LO}$ approximations is not large down to $N_f = 12$. The same results for f_π/m_ϱ are shown in Fig. 2 but since the $\mathrm{N^3LO}$ correction is not available we cannot conclude firmly

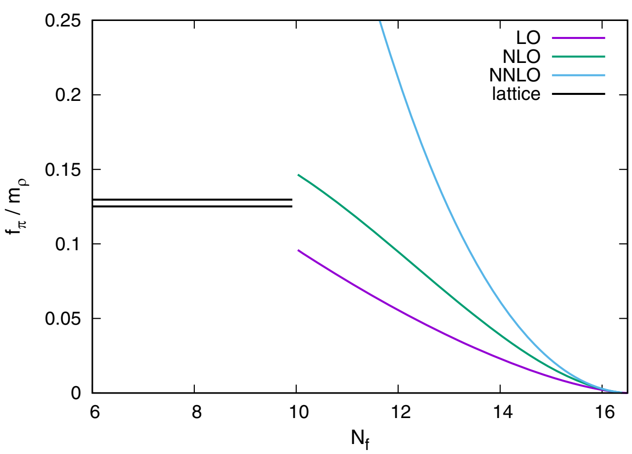

FIG. 2. The f_π/m_ϱ ratio in increasing perturbative order as obtained from the Banks-Zaks expansion in $\varepsilon = 33/2 - N_f$. The nonperturbative result from lattice calculations [16–18] is also shown.

one way or another about the convergence of the perturbative series in this case.

III. LEADING ORDER

The perturbative calculation of f_π in the NRQCD [27,28] and pNRQCD [29–31] formalism is done by first matching the axialvector current of the heavy quark and antiquark pair to NRQCD operators, and then computing the NRQCD matrix elements in pNRQCD in terms of the bound-state wave functions [32,33].

The matching of the decay constant in NRQCD is expressed in terms of NRQCD operator matrix elements which scale with powers of v, the velocity of the heavy quark and antiquark inside the bound state. In the perturbative case, $v \sim g^2$, so that in order to obtain expressions at NNLO accuracy, it suffices to keep corrections up to relative order v^2. To relative order v^2, f_π can be written as [34]

$$f_\pi = \frac{1}{\sqrt{m_\pi}}\left(c_p\langle 0|\chi^\dagger\psi|\pi\rangle - \frac{d_p}{2m^2}\left\langle 0\middle|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi\middle|\pi\right\rangle\right), \tag{5}$$

where ψ and $\chi^\dagger$ are operators that annihilate a heavy quark and antiquark, respectively, $\mathbf{D} = \nabla - ig\mathbf{A}$ is the covariant derivative, $\mathbf{A}$ is the gluon field, $\chi^\dagger\overleftrightarrow{\mathbf{D}}\psi = \chi^\dagger\mathbf{D}\psi - (\mathbf{D}\chi)^\dagger\psi$, $|\pi\rangle$ is the relativistically normalized π state at rest, and $c_p = 1 + O(a)$ and $d_p = 1 + O(a)$ are the matching coefficients that are given by a series in a. The NRQCD matrix elements can be computed in pNRQCD in terms of the bound-state wave function $\psi(r)$ and the binding energy E, which satisfy the Schrödinger equation

$$\left(-\frac{\nabla^2}{m} + V(r)\right)\psi(r) = E\psi(r), \tag{6}$$

where the potential $V(r)$ is obtained by perturbatively matching pNRQCD to NRQCD. The mass of the bound state is given in terms of the binding energy E by

$$m_\pi = 2m + E, \tag{7}$$

and the matrix elements are given by

$$\begin{aligned}\langle 0|\chi^\dagger\psi|\pi\rangle &= \sqrt{2N_c}|\psi(0)|,\\ \left\langle 0\middle|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi\middle|\pi\right\rangle &= \sqrt{2N_c}|\psi(0)|mE,\end{aligned} \tag{8}$$

where $N_c = 3$ is the number of colors. These lead to the following expression for f_π

$$f_\pi = \sqrt{\frac{N_c}{m}}\left[c_p - \left(\frac{c_p}{4} + \frac{d_p}{2}\right)\frac{E}{m}\right]|\psi(0)|, \tag{9}$$

which is valid up to corrections of order a^4.

Analogously, f_ϱ is given in NRQCD by [32,35–37]

$$f_\varrho = \frac{1}{\sqrt{m_\varrho}}\left(c_v\langle 0|\chi^\dagger \epsilon\cdot\sigma\psi|\varrho\rangle - \frac{d_v}{6m^2}\times\left\langle 0\left|\chi^\dagger\epsilon\cdot\sigma\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi\right|\varrho\right\rangle\right), \quad (10)$$

where σ is a Pauli matrix, ϵ is the polarization vector for the ϱ, and $|\varrho\rangle$ is the relativistically normalized ϱ state at rest. Similarly to the case of f_π, the NRQCD matrix elements appearing in the expression for f_ϱ can be computed in pNRQCD, which lead to the following expressions

$$\langle 0|\chi^\dagger\epsilon\cdot\sigma\psi|\varrho\rangle = \sqrt{2N_c}|\psi(0)|,$$
$$\left\langle 0\left|\chi^\dagger\epsilon\cdot\sigma\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi\right|\varrho\right\rangle = \sqrt{2N_c}|\psi(0)|mE, \quad (11)$$

and $m_\varrho = 2m + E$. From these we obtain

$$f_\varrho = \sqrt{\frac{N_c}{m}}\left[c_v - \left(\frac{c_v}{4}+\frac{d_v}{6}\right)\frac{E}{m}\right]|\psi(0)|, \quad (12)$$

which is again valid up to corrections of order a^4.

At leading order in a, it suffices to solve the Schrödinger equation for the Coulomb potential $V(r) = -4\pi a C_F/r$, where $C_F = (N_c^2-1)/(2N_c) = 4/3$. In this case the bound-state solutions are known exactly and we obtain for the ground state

$$\psi(0) = (4m\pi a C_F)^{3/2}/(8\pi)^{1/2}[1+O(a)],$$
$$E = -m(4\pi a C_F)^2/4 + O(a^3), \quad (13)$$

for both the spin-triplet and spin-singlet states. From these we obtain the following expressions for f_π, f_ϱ, and m_ϱ that are valid at leading orders in a.

$$f_\pi = \sqrt{8N_c C_F^3}\pi m a^{3/2}[1+O(a)],$$
$$f_\varrho = \sqrt{8N_c C_F^3}\pi m a^{3/2}[1+O(a)],$$
$$m_\varrho = 2m[1+O(a^2)]. \quad (14)$$

Note that the order-a correction to m_ϱ is absent because E begins at order a^2.

IV. NLO, NNLO, AND N^{3}LO CORRECTIONS

Now we discuss the sources of radiative corrections needed to obtain expressions at NNLO accuracy. We first note that because E begins at order a^2, the leading-order expression for E suffices for obtaining m_ϱ at NNLO accuracy. The corrections at higher orders in a to NNLO accuracy to the decay constants come from the radiative corrections to the matching coefficients c_p and c_v, as well as the corrections to the wave function at the origin. The corrections to c_p have been computed analytically in [34] at NLO and in [38] at NNLO. Likewise, analytical expressions for the radiative corrections to c_v are available in [35,36] at NLO and in [39,40] at NNLO. As is well known from heavy quarkonium phenomenology, the NNLO corrections to c_v and c_p contain logarithms of the NRQCD factorization scale, which must cancel with the logarithms coming from the renormalization of the NRQCD matrix elements [41,42]. The corrections to $|\psi(0)|$ have been computed to NNLO accuracy in [32,43] for the S-wave spin-triplet case. For the spin-singlet case, the corrections to $|\psi(0)|$ to NNLO accuracy can be obtained from the results in Ref. [44]. The NNLO corrections contain the logarithms of the NRQCD factorization scale that cancel against the logarithms coming from c_v and c_p, so that the decay constants are free of dependencies on the factorization scale. These are sufficient ingredients for computing f_π and f_ϱ to NNLO accuracy. We note that the dependence on N_f only comes from the matching coefficients, because all N_f flavors are heavy and are integrated out from the effective field theory. Also note that a nonvanishing imaginary part of the matching coefficients can be discarded at our current level of accuracy.

Additionally, the N^{3}LO correction to c_v has been computed in Refs. [45,46], and the N^{3}LO correction to $|\psi(0)|$ has been computed for the S-wave spin-triplet case in [47,48]. Together with the NLO correction to E available in [43,49] and the NLO correction to d_v available in [50,51], these make possible the computation of f_ϱ and m_ϱ to N^{3}LO accuracy. At N^{3}LO accuracy, in addition to NNLO and N^{3}LO corrections to c_v, the NLO correction to d_v also contains a logarithm of the factorization scale, which cancels against the ultrasoft correction to $|\psi(0)|$ at N^{3}LO accuracy [52]. Because only part of the N^{3}LO correction to c_v is analytically known, we only obtain numerical results for the coefficients of the order-a^3 terms in f_ϱ.

We present below the results of the NLO and NNLO corrections for f_π and also the N^{3}LO correction for m_ϱ and f_ϱ.

A. ϱ mass

From the binding energy E we have, to N^{3}LO accuracy,

$$m_\varrho = c_0 m[1 + c_2 a^2(m) + c_{30}a^3(m) + c_{31}a^3(m)\log a(m) + O(a^4)]. \quad (15)$$

The order-a term in m_ϱ is zero because E begins at order a^2. The first two coefficients are determined by the leading-order binding energy and to NNLO and N^{3}LO we obtain the further coefficients,

$$
\begin{aligned}
c_0 &= 2\\
c_2 &= -2C_F^2\pi^2\\
c_{30} &= \frac{4}{9}\pi^2 C_A C_F^2(66\log(4\pi C_F)-97)\\
c_{31} &= \frac{88}{3}\pi^2 C_A C_F^2,
\end{aligned}
\tag{16}
$$

with $C_A = N_c = 3$.

B. ϱ decay constant

From the corrections to c_v and $|\psi(0)|$ available to NNLO accuracy, we obtain

$$
f_\varrho = b_0^\varrho m a^{3/2}(m)\left(1+\sum_{n=1}^{3}\sum_{k=0}^{n} b_{nk}^\varrho a^n(m)\log^k a(m) + O(a^4)\right).
\tag{17}
$$

The coefficients b_{nl}^ϱ up to relative order a^2 are known analytically and are given by

$$
\begin{aligned}
b_0^\varrho &= \sqrt{8N_c C_F^3}\pi,\\
b_{10}^\varrho &= \frac{161}{6}-\frac{11\pi^2}{3}+33\log\left(\frac{3}{16\pi}\right),\\
b_{11}^\varrho &= -33,\\
b_{20}^\varrho &= \left(-\frac{64\pi^2}{27}+\frac{704}{27}\right)N_f+\frac{9781\zeta(3)}{9}-\frac{27\pi^4}{8}+\frac{1126\pi^2}{81}+\frac{9997}{72}+\frac{1815\log^2\pi}{2}+\frac{1815}{2}\log^2\left(\frac{16}{3}\right)\\
&\quad+\log\left(\frac{16}{3}\right)\left(-\frac{2581}{2}+\frac{605\pi^2}{3}+1815\log(\pi)\right)+\left(\frac{4325\pi^2}{27}-\frac{2581}{2}\right)\log(\pi)-\frac{256}{81}\pi^2\log(8)\\
&\quad-\frac{1120}{27}\pi^2\log\left(\frac{8}{3}\right)-\frac{512}{9}\pi^2\log(2),\\
b_{21}^\varrho &= \frac{4325\pi^2}{27}-\frac{2581}{2}+1815\log\left(\frac{16\pi}{3}\right),\\
b_{22}^\varrho &= \frac{1815}{2}.
\end{aligned}
\tag{18}
$$

The results for the relative order a^3 terms are only obtained numerically because the analytical result for c_v at $\mathrm{N^3LO}$ is only partially known,

$$
\begin{aligned}
b_{30}^\varrho &= 0.8198N_f^2-362.7N_f-1.0901(1)\times10^6,\\
b_{31}^\varrho &= -88.42N_f-7.7493\times10^5,\\
b_{32}^\varrho &= -2.1651\times10^5,\\
b_{33}^\varrho &= -2.3292\times10^4.
\end{aligned}
\tag{19}
$$

C. π decay constant

From the corrections to c_p and $|\psi(0)|$ available to NNLO accuracy, we obtain

$$
f_\pi = b_0^\pi m a^{3/2}(m)\left(1+\sum_{n=1}^{2}\sum_{k=0}^{n} b_{nk}^\pi a^n(m)\log^k a(m) + O(a^3)\right),
\tag{20}
$$

and the coefficients b^{π}_{nk} are given by

$$
\begin{aligned}
b^{\pi}_0 &= \sqrt{8N_c C_F^3}\pi, \\
b^{\pi}_{10} &= \frac{59}{2} - \frac{11\pi^2}{3} + 33\log\left(\frac{3}{16\pi}\right), \\
b^{\pi}_{11} &= -33, \\
b^{\pi}_{20} &= N_f\left(-\frac{32\pi^2}{9} + \frac{344}{9}\right) + 961\zeta(3) - \frac{27\pi^4}{8} + \frac{1310\pi^2}{27} + \frac{23053}{72} + \frac{1815\log^2\pi}{2} + \frac{1815}{2}\log^2\left(\frac{16}{3}\right) \\
&\quad + \log\left(\frac{16}{3}\right)\left(-\frac{2757}{2} + \frac{1271\pi^2}{9} + 1815\log\pi\right) + \left(\frac{1271\pi^2}{9} - \frac{2757}{2}\right)\log\pi - \frac{272}{9}\pi^2\log 2, \\
b^{\pi}_{21} &= \frac{1271\pi^2}{9} - \frac{2757}{2} + \frac{1815}{2}\log\left(\frac{256\pi^2}{9}\right), \\
b^{\pi}_{22} &= \frac{1815}{2}.
\end{aligned} \tag{21}
$$

Unfortunately at present the N^{3}LO corrections for f_π are not available.

D. Banks-Zaks expansion of ratios

Now that all three quantities of interest are available in perturbation theory we may expand the ratios in $\varepsilon = 33/2 - N_f$ as outlined in Sec. II.

Using the 5-loop β-function [21–26] for the expansion of a_* and the perturbative series (15), (17) and (20) we obtain the two meson decay constant to mass ratios in numerical form as,

$$
\begin{aligned}
\frac{f_\varrho}{m_\varrho} &= \varepsilon^{3/2} C_0\left(1 + \sum_{n=1}^{3}\sum_{k=0}^{n} C_{nk}\varepsilon^n \log^k\varepsilon + O(\varepsilon^4)\right) \\
\frac{f_\pi}{m_\varrho} &= \varepsilon^{3/2} C_0\left(1 + \sum_{n=1}^{2}\sum_{k=0}^{n} D_{nk}\varepsilon^n \log^k\varepsilon + O(\varepsilon^3)\right),
\end{aligned}
$$

with the coefficients,

$$
\begin{aligned}
C_0 &= 0.005826678 \\
C_{10} &= 0.4487893 \\
C_{11} &= -0.2056075 \\
C_{20} &= 0.2444502 \\
C_{21} &= -0.1624891 \\
C_{22} &= 0.03522870 \\
C_{30} &= 0.10604(3) \\
C_{31} &= -0.1128420 \\
C_{32} &= 0.03695458 \\
C_{33} &= -0.005633665
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
D_{10} &= 0.4654041 \\
D_{11} &= -0.2056075 \\
D_{20} &= 0.2845697 \\
D_{21} &= -0.1737620 \\
D_{22} &= 0.03528692.
\end{aligned} \tag{23}
$$

Even though the coefficients (16), (18), and (21) are dangerously increasing in the series (15), (17), and (20), the above coefficients of the ratios are much better behaved. This will be important for the reliability and robustness of our findings.

The coefficients (22) and (23) are the main results of this paper.

V. MATCHING ACROSS THE CONFORMAL WINDOW

The perturbative calculations are valid close to the upper end of the conformal window where $\varepsilon = 33/2 - N_f$ is small. Nonperturbative results are available in the low N_f region, specifically for $2 \le N_f \le 10$, all extrapolated to the chiral and continuum limit.

With the perturbative results for f_ϱ/m_ϱ and f_π/m_ϱ up to N^{3}LO and NNLO order, respectively, at hand we attempt to match them to the nonperturbative ones. The latter shows that below the conformal window both of our ratios are constants as a function of N_f to high precision. At $N_f = 33/2$ both ratios are vanishing, and it is natural to expect that both reach zero in a monotonous fashion. Assuming it is indeed the case we may attempt to interpolate.

A. f_ϱ/m_ϱ

In order to match our perturbative f_ϱ/m_ϱ results to the nonperturbative (low N_f) region, continuum and chirally extrapolated lattice results for f_ϱ would be needed. These are not available at the moment, but they are [16–18] for f_π in the range $2 \le N_f \le 10$ and the KSRF-relations [19,20] can be used to relate f_π and f_ϱ. The relation we need is simply $f_\varrho = \sqrt{2} f_\pi$. One does not expect this relation to hold exactly, but even in QCD at finite quark masses it holds to about 12% and toward the chiral limit it is expected to hold to even higher precision. We conservatively assign the 12% uncertainty as the inherent uncertainty of the KSRF-relation throughout. Note that in supersymmetric QCD the KSRF-relations have actually been rigorously derived [53].

Hence by combining the nonperturbative lattice results and the KSRF relations we will have access to f_ϱ/m_ϱ for $2 \le N_f \le 10$. This is shown, together with the Banks-Zaks expansion (22) order-by-order in Fig. 1. The smaller error band displayed for the nonperturbative result corresponds to the uncertainty of the lattice result while the wider one combines this with the estimated 12% error of the KSRF-relation itself. The dominant uncertainty is from the latter. Clearly, the deviation between the NNLO and N^{3}LO results for $N_f \ge 12$ is not substantial. And curiously, close to $N_f = 12$ the perturbative result reaches the nonperturbative one almost exactly. More quantitatively, in the range $11.9 \le N_f \le 12.1$, the deviation between the NNLO and N^{3}LO results is at most 4%, or in the range $11.5 \le N_f \le 12.5$ at most 13%. Hence we conclude that in the region of interest, $N_f \sim 12$, the N^{3}LO result is robust and reliable.

Assuming f_ϱ/m_ϱ is a monotonous function of N_f and that around $N_f \sim 12$ the perturbative result is indeed reliable we are led to conclude that the combination of nonperturbative and perturbative results cover the entire N_f range. And at twelve flavors an abrupt change occurs in the ratio which is tempting to identify with the lower end of the conformal window. Concretely, we obtain $N_f^* = 12.00(4)$ and $N_f^* = 12.08(6)$ from the NNLO and N^{3}LO approximations, respectively, if only the uncertainty of the lattice calculation is taken into account. If the estimated much larger uncertainty of the KSRF-relation is also taken into account we obtain 12.0(3) and 12.0(5) from the NNLO and N^{3}LO approximations, respectively. Clearly, the NNLO and N^{3}LO approximations agree and lead to $N_f^* = 12$ for integer flavor numbers. Our line of reasoning cannot of course determine where exactly the twelve flavor theory lies, whether [54–63] it is just below the conformal window and is hence spontaneously broken or just inside and is hence conformal.

B. f_π/m_ϱ

A similar analysis can be performed for f_π/m_ϱ as well. Here nonperturbative lattice results are available directly without reliance on any further input. The perturbative calculation could unfortunately be only carried out to NNLO order though. The increasing perturbative orders are shown in Fig. 2 which also shows the nonperturbative result obtained from continuum and chirally extrapolated lattice calculations.

The N^{3}LO correction for f_ϱ/m_ϱ was essential to establish the reliability of the perturbative series hence we cannot make a similar statement for f_π/m_ϱ. We may however estimate the size of the N^{3}LO correction by assuming that relative to the NNLO result it is comparable to the case of f_ϱ/m_ϱ. Assuming this is the case we obtain a very similar picture; the perturbative series seems reliable down to $N_f^* = 13$ where it matches the nonperturbative result. The only difference relative to f_ϱ/m_ϱ is the shift in the estimate of the lower end of the conformal window, from $N_f^* \simeq 12$ to $N_f^* \simeq 13$. This latter estimate should of course be checked by a genuine N^{3}LO calculation of f_π in the future.

VI. CONCLUSION AND OUTLOOK

In this paper we introduced two quantities we believe are useful proxies for the transition between chirally broken and conformal gauge theories as the flavor number is varied. A minimal requirement for any such quantity is that it should be well-defined and calculable in the massless limit both outside and inside the conformal window. Outside the conformal window lattice calculations offer a way to obtain results whereas close to the upper end perturbative ones. Our quantities are related to bound states defined in the mass perturbed models and the chiral limit is meaningful for both ratios.

It appears the bridge between the low N_f nonperturbative and high N_f perturbative regions may not be as large as one might have expected. Current lattice results are available up to $N_f = 10$ and the main result from this paper is that at $N_f = 12$, 13 the perturbative series might be reliable if calculations are performed up to N^{3}LO order leaving only the $N_f = 11$ model to be interpolated. Interestingly, at least for the f_ϱ/m_ϱ ratio, the perturbative N^{3}LO result at $N_f = 12$ agrees with the nonperturbative $N_f = 10$ lattice calculation (and the ratio is approximately constant for $2 \le N_f \le 10$). If we assume the ratio is a monotonously decreasing function of N_f, which is a natural assumption based on the behavior at $N_f = 10$ and $N_f = 33/2$, we conclude that a matching between the low N_f nonperturbative and high N_f perturbative regions is possible with an abrupt change at $N_f \simeq 12$. It is tempting to identify this with the lower end of the conformal window $N_f^* \simeq 12$.

Our other ratio, f_π/m_ϱ offers a similar analysis, but unfortunately at the moment only NNLO perturbative results are available. The reliability of the perturbative series cannot be judged from the NLO and NNLO corrections alone, in fact it is clear from the behavior of f_ϱ/m_ϱ that the N^{3}LO correction is mandatory in order to conclude. Such a calculation of f_π within (p)NRQCD seems feasible and will be pursued in the future. Meanwhile, we have estimated the relative size of the unknown N^{3}LO correction for f_π/m_ϱ from that of f_ϱ/m_ϱ. Assuming that this is justified we are led to believe that a matching between the nonperturbative and perturbative regions is possible at $N_f \simeq 13$ with a similarly abrupt change at this value. Hence the estimate shifted to $N_f^* \simeq 13$, however it is important to stress that a genuine N^{3}LO calculation of f_π should be sought first.

Needless to say, we have nothing firm to conclude about the $N_f = 12$ model, whether it is just inside or just outside the conformal window.

In general it is important to remember a key assumption underlying our entire calculation; namely that the only $SU(N_f)$-invariant relevant perturbation of the conformal field theories we discuss is the fermionic mass term. This is certainly correct for small ε but might not hold for a sufficiently strongly coupled CFT, for instance it is conceivable that a 4-fermi term becomes relevant. Addressing this potential situation is beyond the scope of the present paper but we hope to return to it in the future.

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