Seebeck and Nernst coefficients of a magnetized hot QCD medium with a number conserving kernel

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(Received 11 December 2022; accepted 4 April 2023; published 25 April 2023)

We study the thermoelectric response of a hot and magnetized QCD medium created in the noncentral events at heavy-ion collider experiments. The collisional aspects of the medium have been embedded in the relativistic Boltzmann transport equation (RBTE) using the Bhatnagar-Gross-Krook (BGK) collision integral, which insures the particle number conservation, unlike the commonly used relaxation time approximation (RTA). We have incorporated the thermal medium effects in the guise of a quasiparticle model, where the interaction among the quarks and gluons is assimilated in the medium dependent masses of the quarks, which have been evaluated using imaginary-time formalism of thermal QCD with a background magnetic field. In the absence of B, the Seebeck coefficient for individual quark flavors gets slightly reduced in the BGK term in comparison to naive RTA, while it gets enhanced for the composite partonic medium. In the strong magnetic field (B), the BGK term enhances the Seebeck coefficient for the individual flavors, as well as that for the medium. The medium Seebeck coefficient rises with the strength of quark chemical potential (μ) in the absence, as well as that in the strong B. We observe chirality dependence in the transport coefficients in the weak B as the masses of chiral modes become nondegenerate. In the case of the L modes, the BGK collision term causes a slight reduction in the Seebeck coefficient, while for R modes both the collision integrals produce the same results. The Nernst coefficient gets reduced (enhanced) for L (R) chiral modes in the BGK term.

DOI: 10.1103/PhysRevD.107.074034

I. INTRODUCTION

A transition from the hadronic matter to a deconfined phase of quark gluon plasma (QGP) takes place in heavy-ion collision experiments at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). In noncentral collisions, a magnetic field (around m_π^2 at the RHIC [1] and $15m_\pi^2$ at the LHC [2]) is also produced, which persists in the medium for a considerable amount of time, due to the finite electrical conductivity of the medium. This magnetic field leads to the modification in the thermodynamical [3,4] and transport properties [5–11] of the hot and dense quark matter and also induces novel phenomena *such as* the chiral magnetic effect [1,12], magnetic and inverse magnetic catalysis [13–16], axial magnetic effect [17,18], chiral vortical effect in rotating QGP [19,20], the conformal anomaly, and production of soft photons [21,22]. In addition to this, the dilepton production rate [23–25], dispersion relations [26], refractive indices, and decay constants [21,27] have been explored in the magnetic field background.

Transport coefficients are crucial input parameters needed in the dissipative hydrodynamics and transport simulation to describe the evolution of the partonic medium created postcollision at the RHIC and LHC. Shear viscosity quantifies the response of the medium to the transverse momentum gradients, while bulk viscosity to the pressure gradients. Both shear and bulk viscosities have been studied in the magnetic field extensively in different models [5,7–10]. Electrical and thermal conductivities measure the response of the system to the electromagnetic fields and thermal gradients in the medium, respectively. Electrical conductivity plays an important role in the elongation of the lifetime of the magnetic field created in noncentral collisions, while thermal conductivity controls the attenuation of sound through the Prandtl number. Both electrical and thermal conductivities have been extensively studied in phenomenological models, as well as using perturbative methods [6,28–33]. On the other hand, the transport coefficient corresponding to the thermoelectric response is known as the Seebeck coefficient, which measures the ability of any material to convert the thermal gradient into the electric current. Thermoelectric properties of the materials have been mainly studied in the context of condensed matter physics over the years. There have been numerous

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studies regarding the thermoelectric properties of the various condensed matter systems *such as* superconductors [34–37], the graphene superconductor junction [38], correlated quantum dots coupled to superconducting electrodes [39], high temperature cuprates [40], a ferromagnet-superconductor hybrid junction [41], and low dimensional correlated organic metals [42].

We have recently explored the charge, heat, and momentum transport coefficients [43,44]. Motivated by our earlier studies, we are now interested in thermoelectric effects in the strongly interacting matter produced in the heavy-ion collisions, where a thermal gradient is present between the central and peripheral regions of the fireball. In addition to the temperature gradient, a finite baryon chemical potential is also needed to observe the thermoelectric effect in strongly interacting matter, unlike the condensed matter systems where only one type of the charge carriers participate in the transport process. Contrary to that, in a strongly interacting medium, both positive and negative charge carriers take part in the transport phenomena. In the absence of the quark chemical potential, both particles and antiparticles have equal numbers, so no net thermoelectric effect is observed. The Seebeck effect, in the absence of a magnetic field, has been studied recently for a hot hadron gas in a hadron resonance gas model [45] and for the QGP phase in the ambit of the Nambu–Jona Lasinio (NJL) model [46]. In the magnetic field background, the thermoelectric response of the hot QCD medium has been explored earlier in [47–52]. In the earlier works, authors have used the relativistic kinetic theory approach, where the collisional effects of the medium have been incorporated with the relaxation time approximation (RTA). But, the widely used RTA collision integral has a drawback in that it violates the conservation of the particle number and current. Taking this fact into consideration, we have used a more realistic Bhatnagar-Gross-Krook (BGK)-type collision integral, which insures the particle number and current conservation in the medium. The BGK collision integral has been used earlier to study dielectric functions, dispersion relations, and damping rates of longitudinal and transverse modes of a photon in the electromagnetic plasma [53]. The authors noticed a small shift in the dispersion relations towards the lower energies for the collisional case in comparison to the collisionless case. Schenke *et al.* [54] have studied the effects of the collisions using the BGK kernel on the collective modes of a gluon in the anisotropic QCD medium and have observed that incorporation of the BGK collision integral slows down the growth of the unstable modes. The gluonic collective modes have been also studied in anisotropic medium within the effective fugacity model [55], and suppression of the instabilities were reported there also. The effects of the collision have been investigated using the BGK term on the square of the refractive index (n^2) and Depine-Lakhtakia index (n_{DL}) for the QGP medium in Ref. [56]. It was noticed that the real and imaginary parts of the n^2 gets changed dramatically compared to the collisionless case. For a small collision rate, n_{DL} becomes negative in a certain frequency range and as the collision rate increases, the frequency range for $n_{\mathrm{DL}} < 0$ becomes narrower. The wake phenomenon has been explored for both isotropic [57] as well as anisotropic mediums [58], and it is observed that the wake structure becomes less pronounced in both the cases in comparison to the collisionless plasma. The effect of collisions on the heavy quark energy loss has been investigated via the BGK kernel, and it is found that for the same momentum and collision frequency, energy loss gets increased in the BGK case in comparison to the collisionless case for both charm and bottom quarks and further increases as the collision rate is increased [59]. Authors in [60] perform a similar study using the effective fugacity model and consider the RTA as well as BGK collision terms. They observed that the energy loss gets reduced in the BGK case as compared to RTA. In addition to these works, the response of stationary and homogenous quark gluon plasma to the background electromagnetic field has been studied in [61]. It was found that the late-time magnetic field is mainly determined by the static electrical conductivity of the medium. A similar kind of study was made for the electron positron plasma with time and space dependent magnetic fields [62]. The electric charge transport in a weakly magnetized hot QCD medium in the presence of a time varying electric field has been investigated in [63]. Both Ohmic and Hall conductivities get enhanced in the BGK term as compared to RTA. Similar observations were noticed in the strong magnetic field, where longitudinal electrical conductivity becomes larger in the BGK term [43]. The momentum transport coefficients have been studied in a strong magnetic field with the BGK kernel by us in Ref. [44], and we notice that the shear viscosity gets enhanced, while bulk viscosity is reduced slightly in comparison to RTA.

Motivated by the earlier works, our main objective here is to investigate how the current conserving BGK collision integral modifies the thermoelectric transport coefficients *namely* Seebeck and Nernst coefficients of the hot QCD medium. We include the medium effects in the framework of a quasiparticle model [64], where medium dependence enters through the dispersion relations of the quark and gluon quasiparticles. Quasiparticle models are widely used to study the thermodynamical and transport properties of the hot QCD medium. The masses of the quarks have been computed from the pole of the propagator resummed using the Dyson-Schwinger equation. We have employed the perturbative thermal QCD in magnetic field background to calculate the self-energy of the quark. We compare the BGK results with those obtained using RTA. We have explored two regimes of the magnetic field, the strong ($|q_iB| \gg T^2 \gg m_i^2$) and the weak magnetic field regime ($T^2 \gg |q_iB| \gg m_i^2$). In the magnetic field, the motion of the quarks is quantized in the transverse direction leading to the discrete energy spectrum in terms of the Landau levels. When the strength of the magnetic field is large, the energy separation between the consecutive Landau levels become large (of the order $\sqrt{|qB|}$), consequently, the quarks get confined in the lowest Landau level (LLL) only. Moreover, in the weak magnetic field case, the magnetic field dependence enters through the cyclotron frequency. We found that in the absence of the magnetic field, the magnitude of the Seebeck coefficient for the individual u, d, and s quarks gets reduced in the BGK collision integral, while for the composite medium, it gets enhanced. In the strong B, it gets enhanced in the BGK collision term for individual flavors, as well as for medium. In the case of the weak magnetic field, the Seebeck coefficient is not very sensitive to the collision integral and found to be almost similar in both collision terms. In addition, a hall type Nernst coefficient also appears, which quantifies the thermoelectric response in the transverse direction. The Nernst coefficient gets reduced (enhanced) in the BGK term in the case of L (R) modes for individual flavors, as well as for the medium.

The present manuscript has been organized as follows: In Sec. II, we discuss the quasiparticle model and thermal mass of the quarks in the thermal and magnetic field backgrounds obtained using the perturbative thermal QCD. In Secs. III A and III B, we calculate the Seebeck coefficient without and with a magnetic field background, respectively. We discuss the results in Sec. IV, and finally we conclude in Sec. V.

II. QUASIPARTICLE MODEL

The central feature of quasiparticle models is that a strongly interacting system of massless quarks and gluons can be described in terms of the massive weakly interacting quasiparticles originated due to the collective excitations in the medium. There are many quasiparticle models, such as the NJL and Polyakov-Nambu-Jona-Lasinio models [65–68], which are based on the respective effective QCD models, the effective fugacity model [69], and the model based on the Gribov-Zwanziger approach [70–72]. Such a kind of model was first proposed by Goloviznin and Satz [73] to study the gluonic plasma and then by Peshier *et al.* [74,75] to study the equation of state of QGP obtained from lattice QCD at finite temperature. At the same time, authors in Refs. [76–79] used a quasiparticle picture to explain the lattice data by using a suitable quasiparticle description for QGP with temperature and chemical potential dependent masses. These results suggest that the high-temperature QGP phase is suitably described by a thermodynamically consistent quasiparticle model. In the present study, we have used the quasiparticle model by Bannur [64], where the total effective mass of the ith quark flavor with bare quark mass $m_{i,0}$ has been parametrized as [64,80,81]

$$m_i^2 = m_{i,0}^2 + \sqrt{2} m_{i,0} m_{i,T} + m_{i,T}^2, \tag{1}$$

to explain the lattice data with finite bare quark masses. The thermal mass ($m_{i,T}$) of the quark in Eq. (1) can be calculated using the hard-thermal-loop perturbation theory as [82]

$$m_{iT}^2 = \frac{g'^2 T^2}{6}\left(1 + \frac{\mu^2}{\pi^2 T^2}\right), \tag{2}$$

where $g' = \sqrt{4\pi\alpha_s}$ refers to the coupling constant which depends on the temperature as

$$\alpha_s(T) = \frac{g'^2}{4\pi} = \frac{6\pi}{(33 - 2N_f)\ln\left(\frac{Q}{\Lambda_{\rm QCD}}\right)}, \tag{3}$$

and Q is set at $2\pi\sqrt{T^2 + \frac{\mu^2}{\pi^2}}$.

Now, we will include the strong magnetic field in the quasiparticle description. The quasiparticle mass in the presence of strong B can be generalized as

$$m_{i,s}^2 = m_{i0}^2 + \sqrt{2} m_{i0} m_{iB,T} + m_{iB,T}^2, \tag{4}$$

where $m_{iB,T}$ is the medium dependent quark mass, which is obtained from the pole of the resummed propagator. We know from the Dyson-Schwinger equation

$$S^{-1}(p_\parallel) = \gamma^\mu p_{\parallel\mu} - \Sigma(p_\parallel), \tag{5}$$

where $\Sigma(p_\parallel)$ refers to the self-energy of the quark at finite T and strong B, which has been calculated as [83]

$$\Sigma(p_\parallel) = \frac{g^2|q_iB|}{3\pi^2}\left[\frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2\zeta(3)}{8\pi^2T^2} - \frac{31\mu^4\zeta(5)}{32\pi^4T^4}\right] \times \left[\frac{\gamma^0 p_0}{p_\parallel^2} + \frac{\gamma^3 p_z}{p_\parallel^2} + \frac{\gamma^0\gamma^5 p_z}{p_\parallel^2} + \frac{\gamma^3\gamma^5 p_0}{p_\parallel^2}\right], \tag{6}$$

where $g = \sqrt{4\pi\alpha_s}$ is the running coupling which depends on T, B, and μ as

$$\alpha_s(\Lambda^2, eB) = \frac{g^2}{4\pi} = \frac{\alpha_s(\Lambda^2)}{1 + b_1\alpha_s(\Lambda^2)\ln\left(\frac{\Lambda^2}{\Lambda^2 + eB}\right)}, \tag{7}$$

with

$$\alpha_s(\Lambda^2) = \frac{1}{b_1 \ln\left(\frac{\Lambda^2}{\Lambda_{\overline{\rm MS}}^2}\right)}, \tag{8}$$

and Λ is set at $2\pi\sqrt{T^2 + \frac{\mu^2}{\pi^2}}$ for quarks, $b_1 = \frac{11N_c - 2N_f}{12\pi}$ and $\Lambda_{\overline{\rm MS}} = 0.176$ GeV.

Due to the heat bath and magnetic field, the Lorentz (boost) and rotational invariance of the system get broken.

In such a nontrivial background, the covariant form of the quark self-energy $\Sigma(p_\parallel)$ can be written as [4,84]

$$\Sigma(p_\parallel) = A_1\gamma^\mu u_\mu + A_2\gamma^\mu b_\mu + A_3\gamma^5\gamma^\mu u_\mu + A_4\gamma^5\gamma^\mu b_\mu. \quad (9)$$

Here $u^\mu(1,0,0,0)$ and $b^\mu(0,0,0,-1)$ correspond to the heat bath and the magnetic field, respectively. A_1, A_2, A_3, and A_4 refer to the structure functions, which are given in the LLL approximation as [83]

$$A_1 = \frac{1}{4}\mathrm{Tr}[\Sigma\gamma^\mu u_\mu] = \frac{g^2|q_iB|}{3\pi^2}\left[\frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2\zeta(3)}{8\pi^2T^2} - \frac{31\mu^4\zeta(5)}{32\pi^4T^4}\right]\frac{p_0}{p_\parallel^2}, \quad (10)$$

$$A_2 = -\frac{1}{4}\mathrm{Tr}[\Sigma\gamma^\mu b_\mu] = \frac{g^2|q_iB|}{3\pi^2}\left[\frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2\zeta(3)}{8\pi^2T^2} - \frac{31\mu^4\zeta(5)}{32\pi^4T^4}\right]\frac{p_z}{p_\parallel^2}, \quad (11)$$

$$A_3 = \frac{1}{4}\mathrm{Tr}[\gamma^5\Sigma\gamma^\mu u_\mu] = -\frac{g^2|q_iB|}{3\pi^2}\left[\frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2\zeta(3)}{8\pi^2T^2} - \frac{31\mu^4\zeta(5)}{32\pi^4T^4}\right]\frac{p_z}{p_\parallel^2}, \quad (12)$$

$$A_4 = -\frac{1}{4}\mathrm{Tr}[\gamma^5\Sigma\gamma^\mu b_\mu] = -\frac{g^2|q_iB|}{3\pi^2}\left[\frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2\zeta(3)}{8\pi^2T^2} - \frac{31\mu^4\zeta(5)}{32\pi^4T^4}\right]\frac{p_0}{p_\parallel^2}, \quad (13)$$

where $\zeta(3)$ and $\zeta(5)$ correspond to the Riemann zeta functions. We can further cast the quark self-energy (9) using the chirality projection operators as

$$\Sigma(p_\parallel) = P_R[(A_1 - A_2)\gamma^\mu u_\mu + (A_2 - A_1)\gamma^\mu b_\mu]P_L + P_L[(A_1 + A_2)\gamma^\mu u_\mu + (A_2 + A_1)\gamma^\mu b_\mu]P_R, \quad (14)$$

where P_R and P_L are the right- and left-handed chiral projection operators, respectively,

$$P_R = \frac{(1+\gamma^5)}{2}, \quad (15)$$

$$P_L = \frac{(1-\gamma^5)}{2}. \quad (16)$$

We obtain the resummed quark propagator in terms of P_R and P_L from (5),

$$S(p_\parallel) = \frac{1}{2}\left[P_L\frac{\gamma^\mu X_\mu}{X^2/2}P_R + P_R\frac{\gamma^\mu Y_\mu}{Y^2/2}P_L\right], \quad (17)$$

where

$$\gamma^\mu X_\mu = \gamma^\mu p_{\parallel\mu} - (A_2 - A_1)\gamma^\mu b_\mu - (A_1 - A_2)\gamma^\mu u_\mu, \quad (18)$$

$$\gamma^\mu Y_\mu = \gamma^\mu p_{\parallel\mu} - (A_2 + A_1)\gamma^\mu b_\mu - (A_1 + A_2)\gamma^\mu u_\mu, \quad (19)$$

and

$$\frac{X^2}{2} = X_1^2 = \frac{1}{2}[p_0 - (A_1 - A_2)]^2 - \frac{1}{2}[p_z + (A_2 - A_1)]^2, \quad (20)$$

$$\frac{Y^2}{2} = Y_1^2 = \frac{1}{2}[p_0 - (A_1 + A_2)]^2 - \frac{1}{2}[p_z + (A_2 + A_1)]^2. \quad (21)$$

The static limit ($p_0 = 0,\ p_z \to 0$) of the poles of the propagator (17) (of either X_1^2 or Y_1^2) gives the mass of the quark as

$$m_{i,B}^2 = \frac{g^2|q_iB|}{3\pi^2}\left[\frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2\zeta(3)}{8\pi^2T^2} - \frac{31\mu^4\zeta(5)}{32\pi^4T^4}\right], \quad (22)$$

which depends on the magnetic field, temperature, and quark chemical potential.

The effective quark mass for the ith flavor in the case of a weak magnetic field can be parametrized like the earlier cases as

$$m_{i,w}^2 = m_{i0}^2 + \sqrt{2}m_{i0}m_{i,L/R} + m_{i,L/R}^2, \quad (23)$$

where $m_{i,L/R}$ refers to the thermal mass for the left- or right-handed chiral mode of the ith flavor, which can be evaluated from the Dyson-Schwinger equation,

$$S^{*-1}(P) = \not{P} - \Sigma(P). \quad (24)$$

Here $\Sigma(P)$ represents the self-energy of the quark in the weakly magnetized thermal medium, which can be written in the covariant form at finite T and B as [85]

$$\Sigma(P) = -a_1\not{P} - a_2\not{u} - a_3\gamma_5\not{u} - a_4\gamma_5\not{b}, \quad (25)$$

where a_1, a_2, a_3, a_4 are the structure functions, which can be evaluated by taking the appropriate contractions of Eq. (25) as [85],

$$a_1(p_0,|\mathbf{p}|) = \frac{m_{\rm th}^2}{|\mathbf{p}|^2} Q_1\left(\frac{p_0}{|\mathbf{p}|}\right), \tag{26}$$

$$a_2(p_0,|\mathbf{p}|) = -\frac{m_{\rm th}^2}{|\mathbf{p}|}\left[\frac{p_0}{|\mathbf{p}|} Q_1\left(\frac{p_0}{|\mathbf{p}|}\right) - Q_0\left(\frac{p_0}{|\mathbf{p}|}\right)\right], \tag{27}$$

$$a_3(p_0,|\mathbf{p}|) = -4g^2 C_F M^2 \frac{p_z}{|\mathbf{p}|^2} Q_1\left(\frac{p_0}{|\mathbf{p}|}\right), \tag{28}$$

$$a_4(p_0,|\mathbf{p}|) = -4g^2 C_F M^2 \frac{1}{|\mathbf{p}|} Q_0\left(\frac{p_0}{|\mathbf{p}|}\right), \tag{29}$$

where

$$M^2(T,\mu,B) = \frac{|q_i B|}{16\pi^2}\left(\frac{\pi T}{2m_{i0}} - \ln 2 + \frac{7\mu^2\zeta(3)}{8\pi^2 T^2}\right), \tag{30}$$

and Q_0 and Q_1 are given by

$$Q_0(t) = \frac{1}{2}\ln\left(\frac{t+1}{t-1}\right), \tag{31}$$

$$Q_1(t) = \frac{t}{2}\ln\left(\frac{t+1}{t-1}\right) - 1 = tQ_0(t) - 1. \tag{32}$$

Self-energy (25) can be written in the basis of right- and left-hand chiral projection operators as

$$\Sigma(P) = -P_R(a_1\not{P} + (a_2+a_3)\not{u} + a_4\not{b})P_L - P_L(a_1\not{P} + (a_2-a_3)\not{u} - a_4\not{b})P_R. \tag{33}$$

We calculate the effective quark propagator from (24),

$$S^*(P) = \frac{1}{2}\left[P_L \frac{\not{L}}{L^2/2} P_R + P_R \frac{\not{R}}{R^2/2} P_L\right], \tag{34}$$

where

$$L^2 = (1+a_1)^2 P^2 + 2(1+a_1)(a_2+a_3)p_0 - 2a_4(1+a_1)p_z + (a_2+a_3)^2 - a_4^2, \tag{35}$$

$$R^2 = (1+a_1)^2 P^2 + 2(1+a_1)(a_2-a_3)p_0 + 2a_4(1+a_1)p_z + (a_2-a_3)^2 - a_4^2. \tag{36}$$

Now in order to get the quark thermal mass in weakly magnetized thermal QCD medium, we take the static limit ($p_0 = 0$, $|\mathbf{p}| \to 0$) of $L^2/2$ and $R^2/2$ modes,[1] and we get

$$\left.\frac{L^2}{2}\right|_{p_0=0,|\mathbf{p}|\to 0} = m_{\rm th}^2 + 4g^2 C_F M^2, \tag{37}$$

$$\left.\frac{R^2}{2}\right|_{p_0=0,|\mathbf{p}|\to 0} = m_{\rm th}^2 - 4g^2 C_F M^2. \tag{38}$$

The masses of the left- and right-handed modes are given by

$$m_L^2 = m_{\rm th}^2 + 4g^2 C_F M^2, \tag{39}$$

$$m_R^2 = m_{\rm th}^2 - 4g^2 C_F M^2, \tag{40}$$

respectively. We will use these medium generated masses in the dispersion relation of the quarks to calculate the Seebeck and Nernst coefficients in the forthcoming sections.

III. THERMOELECTRIC RESPONSE OF A THERMAL QCD MEDIUM

In the kinetic theory approach, the evolution of the phase space distribution function is given by the relativistic Boltzmann transport equation (RBTE), which reads as

$$p^\mu \frac{\partial f}{\partial x^\mu} + qF^{\rho\sigma} p_\sigma \frac{\partial f}{\partial p^\rho} = C[f], \tag{41}$$

where $f = f_{\rm eq} + \delta f$; δf is a small deviation from the equilibrium, and $F^{\rho\sigma}$ corresponds to the electromagnetic field strength tensor. Note that $C[f]$ corresponds to the collision integral, which provides microscopic input to the RBTE. In general, $C[f]$ is nonlinear in f, but Anderson and Witting proposed a simple collision integral, which is known as RTA,

$$C[f] = -\frac{p^\mu u_\mu}{\tau}(f - f_{\rm eq}), \tag{42}$$

where τ is the relaxation time. The RTA collision term violates the particle number and current conservation. This shortcoming is the artifact of the linearization of the collision term otherwise, in principle, the full collision term respects all the conservation laws. Later this shortcoming was circumvented by BGK by modifying the RTA as [54,86]

[1] We have expanded the Legendre functions appearing in the structure functions in the power series of $\frac{|\mathbf{p}|}{p_0}$ and have considered only up to $\mathcal{O}(g^2)$.

$$C[f] = -\frac{p^\mu u_\mu}{\tau}\left(f - \frac{n}{n_{\rm eq}} f_{\rm eq}\right), \tag{43}$$

where n and $n_{\rm eq}$ are the out of equilibrium and equilibrium number densities, respectively. The collision term (43) respects the conservation of the particle number, i.e.,

$$\int \frac{d^3p}{(2\pi)^3} C[f] = 0. \tag{44}$$

In what follows, we will apply the framework discussed herewith to examine the thermoelectric response of the thermal medium of quarks and gluons with and without external magnetic field background.

A. Seebeck coefficient in the absence of the magnetic field

In this subsection, we will evaluate the Seebeck coefficient of the thermal QCD medium composed of u, d, and s quarks (and their antiparticles). In the presence of the thermal gradient, the charge carriers will move from the hotter regions to the colder ones. As a result, a current is induced in the medium, which can be written as

$$J_\mu = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{p_\mu}{\omega_i} (q_i \delta f_i(x,p) + \overline{q_i}\delta\overline{f}_i(x,p)), \tag{45}$$

where $\delta f_i(x,p)$ $(\delta\bar{f}_i(x,p))$ refers to the infinitesimal deviation in the phase space density of quarks (antiquarks) of the ith flavor, and g_i corresponds to the degeneracy factor.

The Boltzmann transport equation (41) in the presence of the temperature gradient with the BGK collision integral can be written as

$$\vec{p}.\frac{\partial f_i}{\partial \vec{r}} + q_i \vec{E}.\vec{p}\frac{\partial f_i}{\partial p^0} + q_i p_0 \vec{E}.\frac{\partial f_i}{\partial \vec{p}} = -p^\mu u_\mu \nu_i (f_i - n_i n_{{\rm eq},i}^{-1} f_{{\rm eq},i}), \tag{46}$$

where $f_i = f_{{\rm eq},i} + \delta f_i$, and

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3}(f_{{\rm eq},i} + \delta f_i), \tag{47}$$

$$n_{{\rm eq},i} = g_i \int \frac{d^3p}{(2\pi)^3} f_{{\rm eq},i}. \tag{48}$$

Note that $f_{{\rm eq},i}$ is the Fermi-Dirac distribution function, and ν_i is the collision frequency, which is estimated by inverse of the relaxation time [87]

$$\tau_i(T) = \frac{1}{5.1 T \alpha_s^2 \log\left(\frac{1}{\alpha_s}\right)[1 + 0.12(2N_f + 1)]}, \tag{49}$$

where α_s is the running coupling constant (3).

The RBTE (46) can be recast after some simplification as (see Appendix A)

$$\delta f_i - g_i n_{{\rm eq},i}^{-1} f_{{\rm eq},i}\int_p \delta f_i = \frac{\vec{p}}{\omega_i}.(\omega_i - \mu)\tau_i\left(-\frac{1}{T^2}\right) f_{{\rm eq},i}(1 - f_{{\rm eq},i})\nabla_{\vec{r}} T(\vec{r}) + 2q_i\beta\tau_i \frac{\vec{E}\cdot\vec{p}}{\omega_i} f_{{\rm eq},i}(1 - f_{{\rm eq},i}), \tag{50}$$

which can be further solved for δf_i as

$$\delta f_i = \delta f_i^{(0)} + g_i n_{{\rm eq},i}^{-1} f_{\rm eq,i}\int_{p'} \delta f_i^{(0)}, \tag{51}$$

where

$$\delta f_i^{(0)} = \frac{\vec{p}}{\omega_i}.(\omega_i - \mu)\tau_i\left(-\frac{1}{T^2}\right) f_{{\rm eq},i}(1 - f_{{\rm eq},i})\nabla_{\vec{r}} T(\vec{r}) + \frac{2\beta q_i \tau_i}{\omega_i}\vec{E}.\vec{p} f_{{\rm eq},i}(1 - f_{{\rm eq},i}). \tag{52}$$

Following similar steps, $\delta\bar{f}_i$ can be calculated as

$$\delta\bar{f}_i = \delta\bar{f}_i^{(0)} + g_i n_{{\rm eq},i}^{-1}\bar{f}_{\rm eq,i}\int_{p'} \delta\bar{f}_i^{(0)}, \tag{53}$$

where

$$\delta\bar{f}_i^{(0)} = \frac{\vec{p}}{\omega_i}.(\omega_i + \mu)\tau_i\left(-\frac{1}{T^2}\right)\bar{f}_{{\rm eq},i}(1 - \bar{f}_{{\rm eq},i})\nabla_{\vec{r}} T(\vec{r}) + 2\beta\bar{q}_i\tau_i\frac{\vec{E}.\vec{p}}{\omega_i}\bar{f}_{{\rm eq},i}(1 - \bar{f}_{{\rm eq},i}). \tag{54}$$

Now substituting δf_i and $\delta\bar{f}_i$ in Eq. (45) to obtain the space part of the induced current, due to a single quark flavor, it reads

$$\begin{aligned}J_{k,i} = q_i g_i \tau_i \int \frac{d^3p}{(2\pi)^3}\Bigg[\left\{\frac{p_k^2}{\omega_i^2}(\omega_i-\mu)\left(\frac{-1}{T^2}\right)f_{\mathrm{eq},i}(1-f_{\mathrm{eq},i})\nabla_{\vec{r}}T(\vec{r}) + 2\frac{p_k^2}{\omega_i^2}q_i E_k\beta f_{\mathrm{eq},i}(1-f_{\mathrm{eq},i})\right\}\\ + \frac{g_i}{n_{\mathrm{eq},i}}\frac{p_k}{\omega_i}f_{\mathrm{eq},i}\int_{p'}\left\{\frac{p_k}{\omega_i}(\omega_i-\mu)\left(\frac{-1}{T^2}\right)f_{\mathrm{eq},i}(1-f_{\mathrm{eq},i})\nabla_{\vec{r}}T(\vec{r}) + 2\frac{p_k}{\omega_i}q_i E_k\beta f_{\mathrm{eq},i}(1-f_{\mathrm{eq},i})\right\}\Bigg]\\ + \bar{q}_i g_i\tau_i\int\frac{d^3p}{(2\pi)^3}\Bigg[\left\{\frac{p_k^2}{\omega_i^2}(\omega_i+\mu)\left(\frac{-1}{T^2}\right)\bar{f}_{\mathrm{eq},i}(1-\bar{f}_{\mathrm{eq},i})\nabla_{\vec{r}}T(\vec{r}) + 2\frac{p_k^2}{\omega_i^2}\bar{q}_i E_k\beta\bar{f}_{\mathrm{eq},i}(1-\bar{f}_{\mathrm{eq},i})\right\}\\ + \frac{g_i}{n_{\mathrm{eq},i}}\frac{p_k}{\omega_i}\bar{f}_{\mathrm{eq},i}\int_{p'}\left\{\frac{p_k}{\omega_i}(\omega_i+\mu)\left(-\frac{1}{T^2}\right)\bar{f}_{\mathrm{eq},i}(1-\bar{f}_{\mathrm{eq},i})\nabla_{\vec{r}}T(\vec{r}) + 2\frac{p_k}{\omega_i}\bar{q}_i E_k\beta\bar{f}_{\mathrm{eq},i}(1-\bar{f}_{\mathrm{eq},i})\right\}\Bigg]. \qquad (55)\end{aligned}$$

In the state of equilibrium, the resultant current due to the ith quark flavor becomes zero, i.e., $\vec{J}_i = 0$. Putting the induced current (55) to zero, we get a relation between the thermal gradient in the medium and electric field as[2]

$$\begin{aligned}\vec{E} &= \frac{1}{2Tq}\left(\frac{L_1+L_2}{L_3+L_4}\right)\nabla_{\vec{r}}T(\vec{r}),\\ &\equiv S\nabla_{\vec{r}}T(\vec{r}). \qquad (56)\end{aligned}$$

Here S is the Seebeck coefficient, which is given by

$$S = \frac{1}{2Tq}\left(\frac{L_1+L_2}{L_3+L_4}\right), \qquad (57)$$

where

$$L_1 = \int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\omega^2}(\omega-\mu)f_{\mathrm{eq}}(1-f_{\mathrm{eq}}) + \frac{g}{n_{\mathrm{eq}}}\frac{p}{\omega}f_{\mathrm{eq}}\int_{p'}\frac{p'}{\omega'}(\omega'-\mu)f_{\mathrm{eq}}(1-f_{\mathrm{eq}})\right\}, \qquad (58)$$

$$L_2 = -\int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\omega^2}(\omega+\mu)\bar{f}_{\mathrm{eq}}(1-\bar{f}_{\mathrm{eq}}) + \frac{g}{\bar{n}_{\mathrm{eq}}}\frac{p}{\omega}\bar{f}_{\mathrm{eq}}\int_{p'}\frac{p'}{\omega'}(\omega'+\mu)\bar{f}_{\mathrm{eq}}(1-\bar{f}_{\mathrm{eq}})\right\}, \qquad (59)$$

$$L_3 = \int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\omega^2}f_{\mathrm{eq}}(1-f_{\mathrm{eq}}) + \frac{g}{n_{\mathrm{eq}}}\frac{p}{\omega}f_{\mathrm{eq}}\int_{p'}\frac{p'}{\omega'}f_{\mathrm{eq}}(1-f_{\mathrm{eq}})\right\}, \qquad (60)$$

$$L_4 = \int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\omega_i^2}\bar{f}_{\mathrm{eq}}(1-\bar{f}_{\mathrm{eq}}) + \frac{g}{\bar{n}_{\mathrm{eq}}}\frac{p}{\omega}\bar{f}_{\mathrm{eq}}\int_{p'}\frac{p'}{\omega'}\bar{f}_{\mathrm{eq}}(1-\bar{f}_{\mathrm{eq}})\right\}. \qquad (61)$$

Up to this point, we have only considered a single quark flavor, we will now focus on the hot QCD medium with multiple quark flavors. In our case, we have considered three flavor (u,d, and s quarks and their antiparticles) quark gluon plasmas. The total induced current can be written as the sum of the currents because of individual flavors as

$$\begin{aligned}\vec{J} = \sum_i \vec{J}_i = \left(\frac{q_1^2 g_1\tau_1}{T}(L_3+L_4)_1 + \frac{q_2^2 g_2\tau_2}{T}(L_3+L_4)_2 + \cdots\right)\vec{E}\\ - \left(\frac{q_1 g_1\tau_1}{T^2}(L_1+L_2)_1 + \frac{q_2 g_2\tau_2}{T^2}(L_1+L_2)_2 + \cdots\right)\nabla_{\vec{r}}T(\vec{r}). \qquad (62)\end{aligned}$$

[2]We have omitted the flavor label i here for simplicity as we are interested in the Seebeck coefficient, due to a single quark flavor. It is, again, taken when the Seebeck coefficient of the medium is considered in Eq. (63).

In the steady state condition, the total induced current vanishes, i.e., $\vec{J} = 0$. As a result, we get

$$\vec{E} = \frac{1}{2T}\frac{\sum_i q_i g_i \tau_i (L_1 + L_2)_i}{\sum_i q_i^2 g_i \tau_i (L_3 + L_4)_i} \nabla_{\vec{r}} T(\vec{r}), \tag{63}$$

which gives the Seebeck coefficient for the medium as

$$S_{\rm tot} = \frac{1}{2T}\frac{\sum_i q_i g_i \tau_i (L_1 + L_2)_i}{\sum_i q_i^2 g_i \tau_i (L_3 + L_4)_i}. \tag{64}$$

Since all the flavors have the same relaxation time and degeneracy factor, the total Seebeck coefficient for the medium can be expressed in terms of the Seebeck coefficient of the individual flavor as

$$S_{\rm tot} = \frac{\sum_i S_i q_i^2 (L_3 + L_4)_i}{\sum_i q_i^2 (L_3 + L_4)_i}. \tag{65}$$

In the next subsection, we will explore how the presence of the background magnetic field modulates the thermoelectric response of the hot QCD medium.

B. Seebeck and Nernst coefficient in the presence of the magnetic field

Now we will calculate the Seebeck coefficient in the magnetic field background. Firstly, we will consider the strong field regime, where the energy of the quark is quantized via Landau quantization. Then, we will explore the weak field limit, where the magnetic field dependence in the transport coefficients enters through the cyclotron frequency, which manifests a classical description of the motion of the charged particle in the magnetic field.

1. The strong magnetic field case

In the presence of strong B, the quark energy gets quantized as [88]

$$\omega_i = \sqrt{p_3^2 + m_i^2 + 2n|q_i B|}, \tag{66}$$

where $n = 0, 1, 2....$ correspond to the discrete Landau levels, and the phase space factor takes the form [88]

$$\int \frac{d^3 p}{(2\pi)^3} \to \sum_{n=0}^{\infty} \frac{|q_i B|}{2\pi} \int \frac{dp_3}{2\pi} (2 - \delta_{n0}). \tag{67}$$

Since we are interested in the strong magnetic field limit with scale hierarchy ($|qB| \gg T^2 \gg m_i{}^2$), the quarks are confined to the lowest Landau levels, i.e., $n = 0$. A dimensional reduction in the quark dynamics takes place from $3+1$ to $1+1$ dimensions rendering the induced current along the z direction as

$$J_3 = \sum_i g_i \frac{|q_i B|}{4\pi^2} \int dp_3 \frac{p_3}{\omega_i} (q_i \delta f_i^B + \bar{q}_i \delta \bar{f}_i^B), \tag{68}$$

where δf_i^B and $\delta \bar{f}_i^B$ are the deviations in the quark and antiquark distribution functions, respectively. The RBTE (46) in the strong B becomes

$$\begin{aligned} &p^0 \frac{\partial f_i^B}{\partial x^0} + p^3 \frac{\partial f_i^B}{\partial x^3} + q_i F^{03} p_3 \frac{\partial f_i^B}{\partial p^0} + q_i F^{30} p_0 \frac{\partial f_i^B}{\partial p^3} \\ &= -p^\mu u_\mu \nu_i^B (f_i^B - n_i^B n_{{\rm eq},i}^{B-1} f_{{\rm eq},i}^B), \end{aligned} \tag{69}$$

where $f_i^B = f_{{\rm eq},i}^B + \delta f_i^B$; $f_{{\rm eq},i}^B$ is given by

$$f_{{\rm eq},i}^B = \frac{1}{e^{\beta\omega_i} + 1}. \tag{70}$$

Here $\omega_i = \sqrt{p_3^2 + m_i^2}$; m_i is the quasiparticle mass (4), and ν_i^B is computed by the inverse of the relaxation time [89],

$$\tau_i^B(p_3; T, |qB|) = \frac{\omega_i (e^{\beta\omega_i} - 1)}{\alpha_s C_F m_i^2 (e^{\beta\omega_i} + 1)} \left(\int \frac{dp_3'}{\omega_i' (e^{\beta\omega_i'} + 1)} \right)^{-1}. \tag{71}$$

In Eq. (69), $n_{{\rm eq},i}^B$ and n_i^B are the equilibrium and nonequilibrium number densities of quarks in strong B, which are given as

$$n_{{\rm eq},i}^B = \frac{g_i |q_i B|}{4\pi^2} \int dp_3 \, f_{\rm eq,i}^B, \tag{72}$$

$$n_i^B = \frac{g_i |q_i B|}{4\pi^2} \int dp_3 \, (f_{\rm eq,i}^B + \delta f_i^B). \tag{73}$$

In order to obtain δf_i^B, we simplify Eq. (69) as

$$\begin{aligned} &\delta f_i^B - g_i n_{{\rm eq},i}^B{}^{-1} f_{\rm eq,i}^B \int_{p_3} \delta f_i^B \\ &= \tau_i^B \frac{p_3}{\omega_i} (\omega_i - \mu) \left(-\frac{1}{T^2} \right) f_{\rm eq,i}^B (1 - f_{\rm eq,i}^B) (\nabla T)_3 \\ &\quad + 2 q_i \beta \tau_i^B \frac{p_3 E_3}{\omega_i} f_{\rm eq,i}^B (1 - f_{\rm eq,i}^B), \end{aligned} \tag{74}$$

which is further solved for δf_i^B up to first order in iteration as

$$\delta f_i^B = \delta f_i^{B(0)} + g_i n_{{\rm eq},i}^B{}^{-1} f_{\rm eq,i}^B \int_{p_3'} \delta f_i^{B(0)}, \tag{75}$$

where

$$\delta f_i^{B(0)} = \frac{p_3}{\omega_i}\tau_i^B(\omega_i-\mu)\left(-\frac{1}{T^2}\right)f^B_{\rm eq,i}(1-f^B_{\rm eq,i})(\nabla T)_3 + \frac{2q_i\beta\tau_i^B}{\omega_i}p_3E_3f^B_{\rm eq,i}(1-f^B_{\rm eq,i}). \tag{76}$$

Similarly, we can write for the antiquarks as

$$\delta\bar{f}_i^B = \delta\bar{f}_i^{B(0)} + g_i n^B_{\mathrm{eq},i}{}^{-1}\bar{f}^B_{\rm eq,i}\int_{p_3'}\delta\bar{f}_i^{B(0)}, \tag{77}$$

where

$$\delta\bar{f}_i^{B(0)} = \frac{p_3}{\omega_i}\tau_i^B(\omega_i+\mu)\left(-\frac{1}{T^2}\right)\bar{f}^B_{\rm eq,i}(1-\bar{f}^B_{\rm eq,i})(\nabla T)_3 + \frac{2\bar{q}_i\beta\tau_i^B}{\omega_i}p_3E_3\bar{f}^B_{\rm eq,i}(1-\bar{f}^B_{\rm eq,i}). \tag{78}$$

Now substituting δf_i^B and $\delta\bar{f}_i^B$ in Eq. (68) to obtain the induced current in the strong magnetic field due to a single ith quark flavor,

$$\begin{aligned} J_{3,i} = {}& q_ig_i\frac{|q_iB|}{4\pi^2}\int dp_3\left[\left\{\frac{p_3^2}{\omega_i^2}\tau_i^B(\omega_i-\mu)\left(\frac{-1}{T^2}\right)f^B_{\rm eq,i}(1-f^B_{\rm eq,i})(\nabla T)_3 + \frac{p_3^2}{\omega_i^2}2\beta E_3q_i\tau_i^B(p_3)\right.\right.\\ &\left.\times f^B_{\rm eq,i}(1-f^B_{\rm eq,i})\right\} + \frac{g_i}{n^B_{\mathrm{eq},i}}\frac{p_3}{\omega_i}f^B_{\rm eq,i}\int_{p_3'}\left\{\frac{p_3'}{\omega_i'}\tau_i^B(\omega_i-\mu)\left(\frac{-1}{T^2}\right)f^B_{\rm eq,i}(1-f^B_{\rm eq,i})(\nabla T)_3\right.\\ &\left.\left.+\frac{p_3'}{\omega_i'}2\beta E_3q_i\tau_i^B(p_3')f^B_{\rm eq,i}(1-f^B_{\rm eq,i})\right\}\right] + \bar{q}_ig_i\frac{|q_iB|}{4\pi^2}\int dp_3\left[\left\{\frac{p_3^2}{\omega_i^2}\tau_i^B(\omega_i+\mu)\left(\frac{-1}{T^2}\right)\right.\right.\\ &\left.\times\bar{f}^B_{\rm eq,i}(1-\bar{f}^B_{\rm eq,i})(\nabla T)_3 + \frac{p_3^2}{\omega_i^2}2\beta E_3\bar{q}_i\tau_i^B(p_3)\bar{f}^B_{\rm eq,i}(1-\bar{f}^B_{\rm eq,i})\right\} + \frac{g_i}{\bar{n}^B_{\mathrm{eq},i}}\frac{p_3}{\omega_i}\bar{f}^B_{\rm eq,i}\\ &\left.\times\int_{p_3'}\left\{\frac{p_3'}{\omega_i'}\tau_i^B(\omega_i'+\mu)\left(\frac{-1}{T^2}\right)\bar{f}^B_{\rm eq,i}(1-\bar{f}^B_{\rm eq,i})(\nabla T)_3 + \frac{p_3'}{\omega_i'}2\beta E_3\bar{q}_i\tau_i^B(p_3')\bar{f}^B_{\rm eq,i}(1-\bar{f}^B_{\rm eq,i})\right\}\right]. \end{aligned} \tag{79}$$

In the state of equilibrium $J_{3,i}$ becomes zero, and we get the relation (omitting label i for simplicity),

$$E_3 = \frac{1}{2qT}\left(\frac{H_1+H_2}{H_3+H_4}\right)(\nabla T)_3, \tag{80}$$

$$\equiv S_B(\nabla T)_3. \tag{81}$$

S_B here corresponds to the Seebeck coefficient in the strong B background, which reads

$$S_B = \frac{1}{2qT}\left(\frac{H_1+H_2}{H_3+H_4}\right), \tag{82}$$

where the integrals H_1, H_2, H_3, and H_4 are given by

$$H_1 = \frac{|qB|}{4\pi^2}\int dp_3\left\{\frac{p_3^2}{\omega^2}\tau^B(\omega-\mu)f^B_{\rm eq}(1-f^B_{\rm eq}) + g\frac{p_3}{\omega}\frac{f^B_{\rm eq}}{n^B_{\rm eq}}\int_{p_3'}\frac{p_3'}{\omega'}\tau^B(\omega'-\mu)f^B_{\rm eq}(1-f^B_{\rm eq})\right\}, \tag{83}$$

$$H_2 = -\frac{|qB|}{4\pi^2}\int dp_3\left\{\frac{p_3^2}{\omega^2}\tau^B(\omega+\mu)\bar{f}^B_{\rm eq}(1-\bar{f}^B_{\rm eq}) + g\frac{p_3}{\omega}\frac{\bar{f}^B_{\rm eq}}{\bar{n}^B_{\rm eq}}\int_{p_3'}\frac{p_3'}{\omega'}\tau^B(\omega'+\mu)\bar{f}^B_{\rm eq}(1-\bar{f}^B_{\rm eq})\right\}, \tag{84}$$

$$H_3 = \frac{|qB|}{4\pi^2}\int dp_3\left\{\frac{p_3^2}{\omega^2}\tau^B f^B_{\rm eq}(1-f^B_{\rm eq}) + g\frac{p_3}{\omega}\frac{f^B_{\rm eq}}{n^B_{\rm eq}}\int_{p_3'}\frac{p_3'}{\omega'}\tau^B f^B_{\rm eq}(1-f^B_{\rm eq})\right\}, \tag{85}$$

$$H_4 = \frac{|qB|}{4\pi^2}\int dp_3\left\{\frac{p_3^2}{\omega^2}\tau^B\bar{f}^B_{\rm eq}(1-\bar{f}^B_{\rm eq}) + g\frac{p_3}{\omega}\frac{\bar{f}^B_{\rm eq}}{\bar{n}^B_{\rm eq}}\int_{p_3'}\frac{p_3'}{\omega'}\tau^B\bar{f}^B_{\rm eq}(1-\bar{f}^B_{\rm eq})\right\}. \tag{86}$$

For the hot QCD medium consisting of u, d, and s quarks, the third component of the induced current can be written as the vector sum of the individual currents as

$$J_3 = \sum_i J_{3,i} \tag{87}$$

$$= \left(\frac{q_1^2 g_1 |q_1 B|}{T}(H_3+H_4)_1 + \frac{q_2^2 g_2 |q_2 B|}{T}(H_3+H_4)_2 + \cdots\cdots\right)E_3 - \left(\frac{q_1 g_1 |q_1 B|}{T^2}(H_1+H_2)_1 + \frac{q_2 g_2 |q_2 B|}{T^2}(H_1+H_2)_2 + \cdots\cdots\right)(\nabla T)_3, \tag{88}$$

which vanishes in the steady state, i.e., $J_3 = 0$ and gives

$$E_3 = \frac{1}{2T}\frac{\sum_i q_i g_i (H_1+H_2)_i}{\sum_i q_i^2 g_i (H_3+H_4)_i}(\nabla T)_3. \tag{89}$$

We extract the Seebeck coefficient for the composite medium as

$$S^B_{\mathrm{tot}} = \frac{1}{2T}\frac{\sum_i q_i |q_i B| (H_1+H_2)_i}{\sum_i q_i^2 |q_i B| (H_3+H_4)_i}, \tag{90}$$

which can be expressed in terms of the Seebeck coefficient of the single quark flavor as

$$S^B_{\mathrm{tot}} = \frac{\sum_i S_i |q_i|^3 (H_3+H_4)_i}{\sum_i |q_i|^3 (H_3+H_4)_i}. \tag{91}$$

In the strong magnetic field, there is no current in the transverse direction due to the one-dimensional (LLL) quark dynamics. Hence the Nernst coefficient, which measures the thermoelectric response in the transverse direction, vanishes. In the next subsection, we will explore the weak magnetic field regime, where the quark dynamics is not affected by the Landau quantization, rather the magnetic field dependence enters via the cyclotron frequency, which manifests the semiclassical description. In this scenario, the Nernst coefficient would also appear.

2. The weak magnetic field case

In the weak magnetic field, the dispersion relation of the charged particle is not directly affected by the magnetic field, rather B acts as a perturbation. So the $1+1$ dimensional Landau level kinematics is not applicable. The induced four current in the medium is given by

$$J_\mu = \sum_i g_i \int \frac{d^3p}{(2\pi)^3}\frac{p_\mu}{\epsilon_i}[q_i \delta f_i + \bar{q}_i \delta \bar{f}_i], \tag{92}$$

where $\epsilon_i = \sqrt{p^2 + m_i^2}$. The RBTE (41) in the presence of the Lorenz force can be written as (see Appendix)

$$\frac{\partial f_i}{\partial t} + \vec{v}.\frac{\partial f_i}{\partial \vec{r}} + \vec{F}.\frac{\partial f_i}{\partial \vec{p}} = -\frac{1}{\tau_i}\left(f_i - \frac{n_i}{n_{0,i}} f_{0,i}\right), \tag{93}$$

where $\vec{F} = q_i(\vec{E} + \vec{v}\times\vec{B})$. Since the magnetic field is not a dominant energy scale here, we have used the same relaxation time calculated in the absence of B (49). The magnetic field and chemical potential dependence in τ_i will enter through the strong coupling constant (7). Note that n_i and $n_{0,i}$ are given by Eqs. (47) and (48), respectively, except the mass in the dispersion relation will be replaced by the thermomagnetic mass calculated in weak magnetic field in Sec. III.

Without the loss of generality, let us consider the electric field in the xy plane, i.e., $\vec{E} = E_x\hat{x} + E_y\hat{y}$ and the magnetic field in z direction, i.e., $\vec{B} = B\hat{z}$. Then for QCD medium, which is homogeneous in time, Eq. (93) takes the form

$$q_i B \tau_i \left(v_x \frac{\partial f_i}{\partial p_y} - v_y \frac{\partial f_i}{\partial p_x}\right) - \tau_i \vec{v}.\frac{\partial f_i}{\partial \vec{r}} - \tau_i q_i \vec{E}.\frac{\partial f_i}{\partial \vec{p}} = \left(\delta f_i - g_i n_{0,i}^{-1} f_{0,i}\int \delta f_i\right), \tag{94}$$

which can be solved for δf_i up to first order as

$$\delta f_i = \delta f_i^{(a)} + g_i n_{0,i}^{-1} f_{0,i}\int \delta f_i^{(a)}, \tag{95}$$

where

$$\delta f_i^{(a)} = q_i B \tau_i \left(v_x \frac{\partial f_i}{\partial p_y} - v_y \frac{\partial f_i}{\partial p_x}\right) - \tau_i \vec{v}.\frac{\partial f_i}{\partial \vec{r}} - \tau_i q_i \vec{E}.\frac{\partial f_i}{\partial \vec{p}}. \tag{96}$$

In order to solve Eq. (95), we take an ansatz [90],

$$\delta f_i = \delta f_i^{(b)} + g_i n_{0,i}^{-1} f_{0,i}\int_{p'} \delta f_i^{(b)}, \tag{97}$$

where

$$\delta f_i^{(b)} = f_i - f_{0,i} = -\tau_i q_i \vec{E}.\frac{\partial f_{0,i}}{\partial \vec{p}} - \vec{\lambda}.\frac{\partial f_{0,i}}{\partial \vec{p}}. \tag{98}$$

Equating Eqs. (95) and (97), we get

$$\vec{\lambda}.\frac{\partial f_{0,i}}{\partial \vec{p}} - q_i B\tau_i\left(v_y\frac{\partial f_i}{\partial p_x} - v_x\frac{\partial f_i}{\partial p_y}\right) - \tau_i\vec{v}.\frac{\partial f_i}{\partial \vec{r}} = g_i n_{0,i}^{-1} f_{0,i}\left(\int_{p'}\delta f_i^{(a)} + \int_{p'}\delta f_i^{(b)}\right). \tag{99}$$

We calculate $\frac{\partial f_i}{\partial p_y}$ and $\frac{\partial f_i}{\partial p_x}$ using the ansatz (97) as

$$\left(v_x\frac{\partial f_i}{\partial p_y} - v_y\frac{\partial f_i}{\partial p_x}\right) = (v_y\lambda_x + v_y\tau_i q_i E_x - v_x\lambda_y - v_x\tau_i q_i E_y)\frac{\partial f_{0,i}}{\partial \epsilon}\frac{1}{\epsilon_i}, \tag{100}$$

where we have retained terms which are linear in the velocity only.

Now substituting Eq. (100) in (99), and after doing some rearrangement, we get

$$v_x\left[\frac{\lambda_x}{\tau_i} - \omega_c\tau_i q_i E_y - \omega_c\lambda_y + \left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial x}\right] + v_y\left[\frac{\lambda_y}{\tau_i} + \omega_c\tau_i q_i E_x + \omega_c\lambda_x + \left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial y}\right] + \frac{g_i T}{n_{0,i}\tau_i}\left(\int_{p'}\delta f_i^{(a)} + \int_{p'}\delta f_i^{(b)}\right) = 0, \tag{101}$$

where $\omega_c = \frac{qB}{\epsilon}$ is the cyclotron frequency. Equating the coefficient of v_x and v_y on both sides gives

$$\frac{\lambda_x}{\tau_i} - \omega_c\tau_i q_i E_y - \omega_c\lambda_y + \left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial x} = 0, \tag{102}$$

$$\frac{\lambda_y}{\tau_i} + \omega_c\tau_i q_i E_x + \omega_c\lambda_x + \left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial y} = 0. \tag{103}$$

We solve the above Eqs. (102) and (103) to get λ_x and λ_y as

$$\lambda_x = -\frac{\omega_c^2\tau_i^3}{1+\omega_c^2\tau_i^2}q_i E_x - \frac{\tau_i}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial x} + \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}q_i E_y - \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial y}, \tag{104}$$

$$\lambda_y = -\frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}q_i E_x + \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial x} - \frac{\omega_c^2\tau_i^3}{1+\omega_c^2\tau_i^2}q_i E_y - \frac{\tau_i}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)\frac{\partial T}{\partial y}. \tag{105}$$

Now we substitute λ_x and λ_y in (97) to obtain δf, which reads

$$\delta f_i = \delta f_i^{(b)} + g_i n_{0,i}^{-1} f_{0,i}\int_{p'}\delta f_i^{(b)}, \tag{106}$$

where

$$\begin{aligned}\delta f_i^{(b)} = {}&\frac{\partial f_{0,i}}{\partial \epsilon}\left[-\frac{\tau_i}{1+\omega_c^2\tau_i^2}q_i v_x + \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}q_i v_y\right]E_x + \frac{\partial f_0}{\partial \epsilon}\left[-\frac{\tau_i}{1+\omega_c^2\tau_i^2}q_i v_y - \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}q_i v_x\right]E_y\\ &+ \frac{\partial f_0}{\partial \epsilon}\left[\frac{\tau_i}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_x - \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_y\right]\frac{\partial T}{\partial x}\\ &+ \frac{\partial f_{0,i}}{\partial \epsilon}\left[\frac{\tau_i}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_y + \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_x\right]\frac{\partial T}{\partial y}.\end{aligned} \tag{107}$$

Similarly, deviation in the antiquark distribution function can be evaluated as [replacing q_i with $-q_i$ and ω_c with $-\omega_c$ in Eq. (107)]

$$\delta\bar{f}_i = \delta\bar{f}_i^{(b)} + g_i\bar{n}_{0,i}^{-1}\bar{f}_{0,i}\int_{p'}\delta\bar{f}_i^{(b)}, \tag{108}$$

where

$$\delta\bar{f}_i^{(b)} = \frac{\partial\bar{f}_{0,i}}{\partial\epsilon}\left[\frac{\tau_i}{1+\omega_c^2\tau_i^2}q_i v_x + \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}q_i v_y\right]E_x + \frac{\partial\bar{f}_{0,i}}{\partial\epsilon}\left[\frac{\tau_i}{1+\omega_c^2\tau_i^2}q_i v_y - \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}q_i v_x\right]E_y$$
$$+ \frac{\partial\bar{f}_{0,i}}{\partial\epsilon}\left[\frac{\tau_i}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_x + \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_y\right]\frac{\partial T}{\partial x}$$
$$+ \frac{\partial\bar{f}_{0,i}}{\partial\epsilon}\left[\frac{\tau_i}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_y - \frac{\omega_c\tau_i^2}{1+\omega_c^2\tau_i^2}\left(\frac{\epsilon_i-\mu}{T}\right)v_x\right]\frac{\partial T}{\partial y}. \tag{109}$$

We substitute δf_i and $\delta\bar{f}_i$ in (92) to get the x and y components of the induced current density, due to ith quark flavor,

$$J_{x,i} = q_i g_i\left[(q_i\beta I_{1,i})E_x + (q_i\beta I_{2,i})E_y + (\beta^2 I_{3,i})\frac{\partial T}{\partial x} + (\beta^2 I_{4,i})\frac{\partial T}{\partial y}\right], \tag{110}$$

$$J_{y,i} = q_i g_i\left[(-q_i\beta I_{2,i})E_x + (q_i\beta I_{1,i})E_y + (-\beta^2 I_{4,i})\frac{\partial T}{\partial x} + (\beta^2 I_{3,i})\frac{\partial T}{\partial y}\right]. \tag{111}$$

The integrals I_1, I_2, I_3, and I_4 in the above equations are given by (omitting label i for simplicity)

$$I_1 = I_{1q} + I_{1\bar{q}}, \tag{112}$$

$$I_2 = I_{2q} + I_{2\bar{q}}, \tag{113}$$

$$I_3 = I_{3q} + I_{3\bar{q}}, \tag{114}$$

$$I_4 = I_{4q} + I_{4\bar{q}}, \tag{115}$$

where

$$I_{1q} = \int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\tau}{(1+\omega_c^2\tau^2)}f_0(1-f_0) + \frac{g}{n_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\tau}{(1+\omega_c^2\tau^2)}f_0(1-f_0)\right\},$$
$$I_{1\bar{q}} = \int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\tau}{(1+\omega_c^2\tau^2)}\bar{f}_0(1-\bar{f}_0) + \frac{g}{\bar{n}_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\tau}{(1+\omega_c^2\tau^2)}\bar{f}_0(1-\bar{f}_0)\right\},$$
$$I_{2q} = \int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}f_0(1-f_0) + \frac{g}{n_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}f_0(1-f_0)\right\},$$
$$I_{2\bar{q}} = -\int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}\bar{f}_0(1-\bar{f}_0) + \frac{g}{\bar{n}_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}\bar{f}_0(1-\bar{f}_0)\right\},$$
$$I_{3q} = -\int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\tau}{(1+\omega_c^2\tau^2)}(\epsilon-\mu)f_0(1-f_0) + \frac{g}{n_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\tau}{(1+\omega_c^2\tau^2)}(\epsilon'-\mu)f_0(1-f_0)\right\},$$
$$I_{3\bar{q}} = \int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\tau}{(1+\omega_c^2\tau^2)}(\epsilon+\mu)\bar{f}_0(1-\bar{f}_0) + \frac{g}{\bar{n}_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\tau}{(1+\omega_c^2\tau^2)}(\epsilon'+\mu)\bar{f}_0(1-\bar{f}_0)\right\},$$
$$I_{4q} = -\int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}(\epsilon-\mu)f_0(1-f_0) + \frac{g}{n_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}(\epsilon'-\mu)f_0(1-f_0)\right\},$$
$$I_{4\bar{q}} = -\int\frac{d^3p}{(2\pi)^3}\left\{\frac{p^2}{3\epsilon^2}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}(\epsilon+\mu)\bar{f}_0(1-\bar{f}_0) + \frac{g}{\bar{n}_0}\frac{p}{\epsilon}f_0\int_{p'}\frac{p'}{\epsilon'}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}(\epsilon'+\mu)\bar{f}_0(1-\bar{f}_0)\right\}.$$

In the state of equilibrium, the components of the induced current density along the x and y directions vanish, i.e., $J_{x,i} = J_{y,i} = 0$. We can write from Eqs. (110) and (111)

$$C_1E_x + C_2E_y + C_3\frac{\partial T}{\partial x} + C_4\frac{\partial T}{\partial y} = 0, \tag{116}$$

$$-C_2E_x + C_1E_y - C_4\frac{\partial T}{\partial x} + C_3\frac{\partial T}{\partial y} = 0, \qquad (117)$$

provided $C_1 = qI_1$, $C_2 = qI_2$, $C_3 = \beta I_3$, and $C_4 = \beta I_4$. Thermoelectric transport coefficients are related to the electric field components and temperature gradients via a matrix equation

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} S & N|B| \\ -N|B| & S \end{pmatrix}\begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{pmatrix}. \qquad (118)$$

We solve Eqs. (116) and (117) for E_x and E_y as

$$E_x = \left(-\frac{C_1C_3 + C_2C_4}{C_1^2 + C_2^2}\right)\frac{\partial T}{\partial x} + \left(-\frac{C_2C_3 - C_1C_4}{C_1^2 + C_2^2}\right)\frac{\partial T}{\partial y}, \qquad (119)$$

$$E_y = \left(-\frac{C_1C_3 + C_2C_4}{C_1^2 + C_2^2}\right)\frac{\partial T}{\partial y} - \left(-\frac{C_2C_3 - C_1C_4}{C_1^2 + C_2^2}\right)\frac{\partial T}{\partial x}, \qquad (120)$$

which give the Seebeck and Nernst coefficients

$$S = -\frac{(C_1C_3 + C_2C_4)}{C_1^2 + C_2^2}, \qquad (121)$$

$$N|B| = \frac{(C_2C_3 - C_1C_4)}{C_1^2 + C_2^2}, \qquad (122)$$

respectively. The integrals C_2 and C_4 vanish in the absence of the magnetic field, as a result the Nernst coefficient would also vanish.

Now we will compute the Seebeck and Nernst coefficients for the medium. In the medium composed of u and d light quarks, the x and y components of the current can be written as the sum of the individual contributions as

$$J_x = \sum_{i=u,d}\left[q_i(I_1)_iE_x + q_i(I_2)_iE_y + \beta(I_3)_i\frac{\partial T}{\partial x} + \beta(I_4)_i\frac{\partial T}{\partial y}\right], \qquad (123)$$

$$J_y = \sum_{i=u,d}\left[q_i(I_2)_iE_x + q_i(I_1)_iE_y - \beta(I_4)_i\frac{\partial T}{\partial x} + \beta(I_3)_i\frac{\partial T}{\partial y}\right]. \qquad (124)$$

The Seebeck and Nernst coefficients of the medium can be extracted by imposing the steady state condition (i.e., putting $J_x = J_y = 0$) as

$$S^{B'}_{\rm tot} = -\frac{(K_1K_3 + K_2K_4)}{K_1^2 + K_2^2}, \qquad (125)$$

$$N|B| = \frac{(K_2K_3 - K_1K_4)}{K_1^2 + K_2^2}, \qquad (126)$$

where

$$K_1 = \sum_{i=u,d} q_i(I_1)_i, \qquad K_2 = \sum_{i=u,d} q_i(I_2)_i, \qquad (127)$$

$$K_3 = \sum_{i=u,d} \beta(I_3)_i, \qquad K_4 = \sum_{i=u,d} \beta(I_4)_i. \qquad (128)$$

IV. RESULTS AND DISCUSSION

In this section, we will discuss the results obtained in the previous sections numerically. In Fig. 1(a), we display the variation of the Seebeck coefficient with T in the BGK and RTA collision terms for u quarks at $\mu = 60$ MeV. It was found that the magnitude of the Seebeck coefficient decreases with T in both the collision terms. We have computed the ratio of the Seebeck coefficients in the BGK collision term to that calculated with the RTA (BGK/RTA) to get the numerical estimates of the relative competition between the two collision integrals. The ratio is found to be around ~ 0.98 for the individual flavors in the temperature domain 160–400 MeV, which indicates that the Seebeck coefficient is slightly reduced in the BGK term. The sign of the Seebeck coefficient for d and s quarks gets reversed due to their negative charges [see Fig. 1(b) and Fig. 2(a)]. In the case of the composite medium, the Seebeck coefficient ($S_{\rm tot}$) [see Fig. 2(b)] has been found to be positive. We notice a considerable enhancement in the magnitude of $S_{\rm tot}$ in the BGK collision term, which is around 12% at lower T (160 MeV) and 26% at high T (400 MeV) [see Fig. 3(b)]. We further study the effects of the quark chemical potential (μ) on the medium Seebeck coefficient in the BGK collision term in Fig. 3(a), taking the strength of $\mu = 40$, 60, and 80 MeV and have found that $S_{\rm tot}$ increases as we raise μ. Similar results of μ dependence in the Seebeck coefficient have been found in Ref. [50] with RTA collision integral. We can conclude that both the BGK collision term and baryon asymmetry in the medium enhance its ability to convert the temperature gradient into the current.

In Figs. 4 and 5, we explore the effects of the BGK collision integral on the Seebeck coefficient of a strongly magnetized hot QCD medium. We have chosen the strength of the magnetic field as $eB = 15m_\pi^2$ and $10m_\pi^2$ with $\mu = 60$ MeV. The Seebeck coefficient for the individual quark flavors, as well as for the combined medium, gets enhanced in the BGK collision integral considerably. The enhancement is around 18%–25% (for u quarks) and

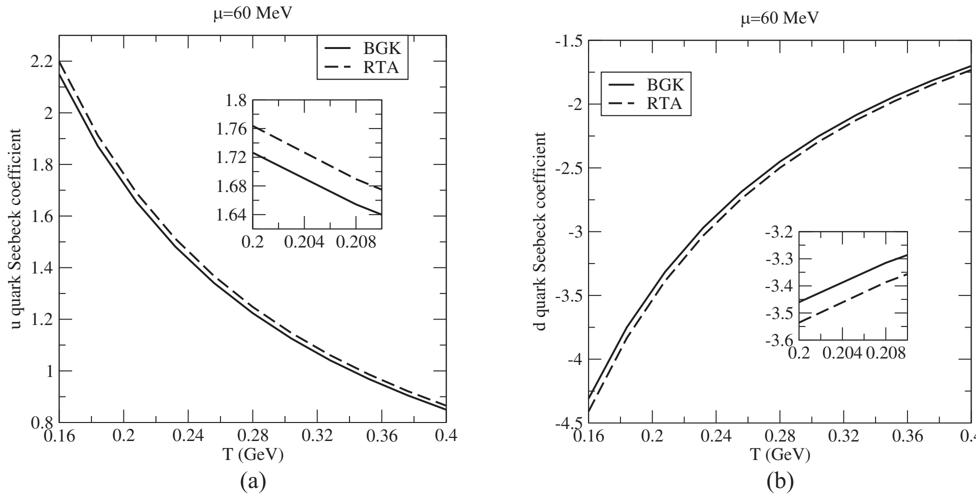

FIG. 1. Temperature dependence of the Seebeck coefficient in the $B = 0$ case: (a) for u quarks and (b) for d quarks.

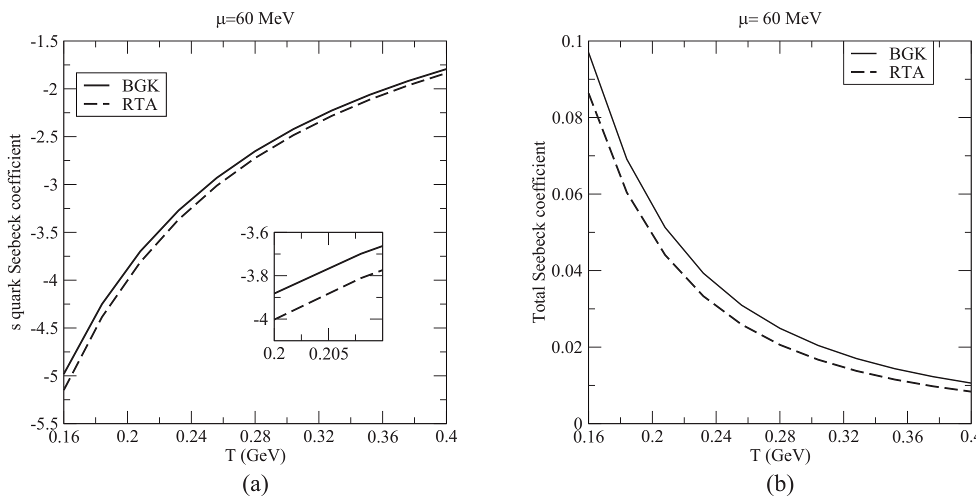

FIG. 2. Temperature dependence of the Seebeck coefficient for the $B = 0$ case: (a) for s quarks and (b) for composite medium.

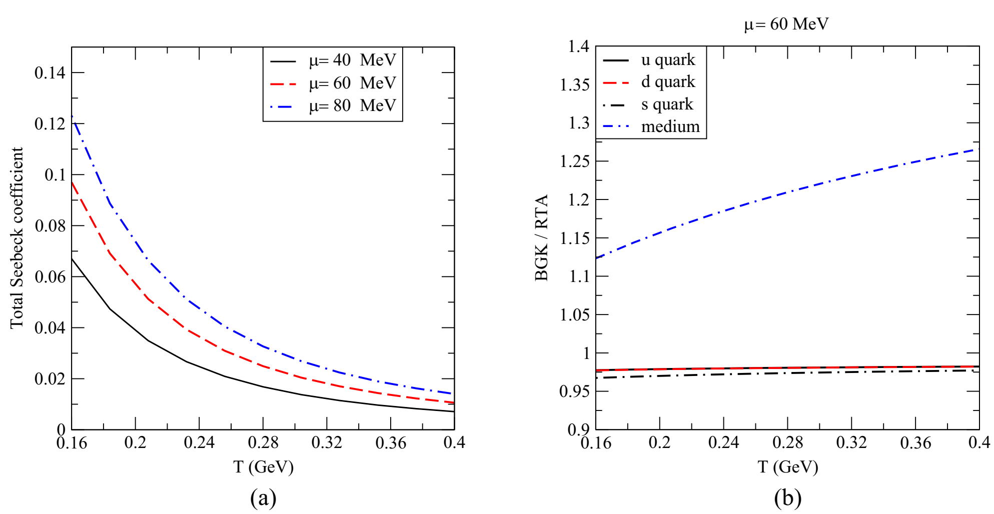

FIG. 3. Left panel: temperature dependence of the Seebeck coefficient for the medium with the BGK collision term in absence of magnetic field for different strengths of μ. Right panel: ratio of Seebeck coefficients in BGK to that in RTA collision integral with temperature.

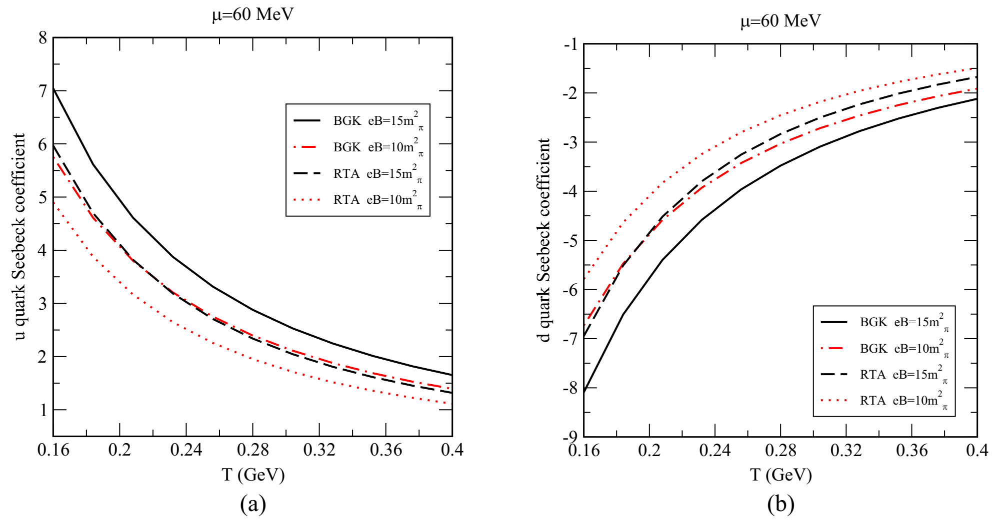

FIG. 4. Temperature dependence of the Seebeck coefficient in the strong B: (a) for u quarks and (b) for d quarks.

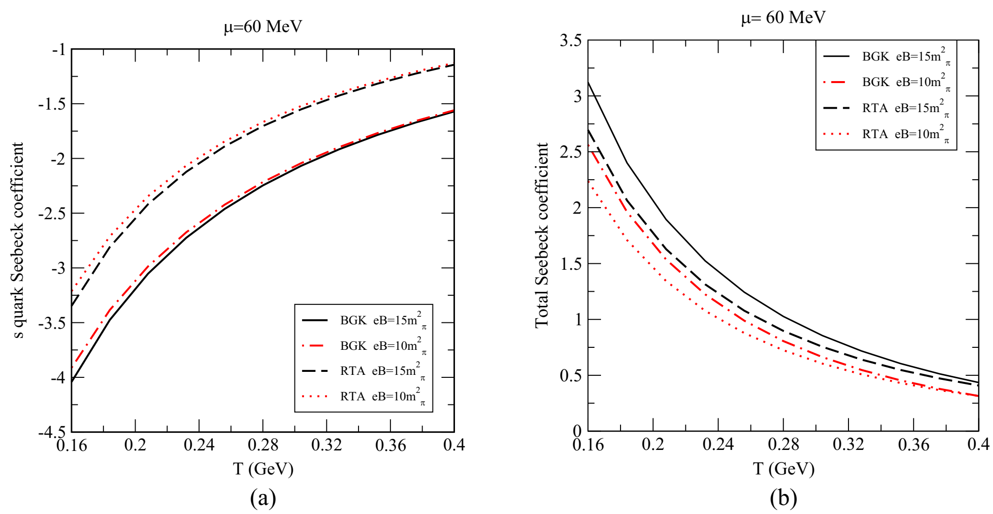

FIG. 5. Temperature dependence of the Seebeck coefficient in strong B: (a) for s quarks and (b) for composite medium.

16%–27% (for d quarks) in the temperature range $160 < T < 400$ MeV. For the s quark, the enhancement is between 21% to 37% in the same domain of T. In the case of the medium, it is around 16% near T_c but decreases as we go towards higher temperatures [see Fig. 6(b)]. $S^B_{\rm tot}$ increases with the strength of μ like the $B = 0$ case [see Fig. 6(a)], which is in agreement with the study made in Ref. [51] in the RTA framework.

In Fig. 7, we investigate how the BGK collision term modifies the thermoelectric response in the presence of weak magnetic field ($eB = 0.3m_\pi^2$) for u (left panel) and d (right panel) quarks. We notice that the magnitude of the Seebeck coefficient depends on the chirality of the quark quasiparticles. For the left-handed chiral modes, the ratio BGK/RTA is less than 1 for individual quark flavors, as well as for composite medium. The ratio is found in the range 0.97–0.98 for the u quarks in the temperature range $160 < T < 400$ MeV. In case of d quarks and medium, this ratio is around ~ 0.98 [seen in Fig. 9(a)]. This concludes that the BGK collision integral causes reduction in the Seebeck coefficients in comparison to the RTA [52]. On the other hand, BGK to the RTA ratio is very close to unity in the case of right-handed modes, which manifests that both the collision terms produce almost similar results [seen in Fig. 9(b)]. We have also studied the effects of the baryon asymmetry on the thermoelectric phenomenon in Fig. 8(b) for $\mu = 40$, 60, and 80 MeV and have noticed an increase in the Seebeck coefficient with μ for both L and R modes.

Since the BGK collision integral shows an improvement over RTA, it gives more realistic estimates of the transport coefficients *like* electrical conductivity ($\sigma_{\rm el}$),

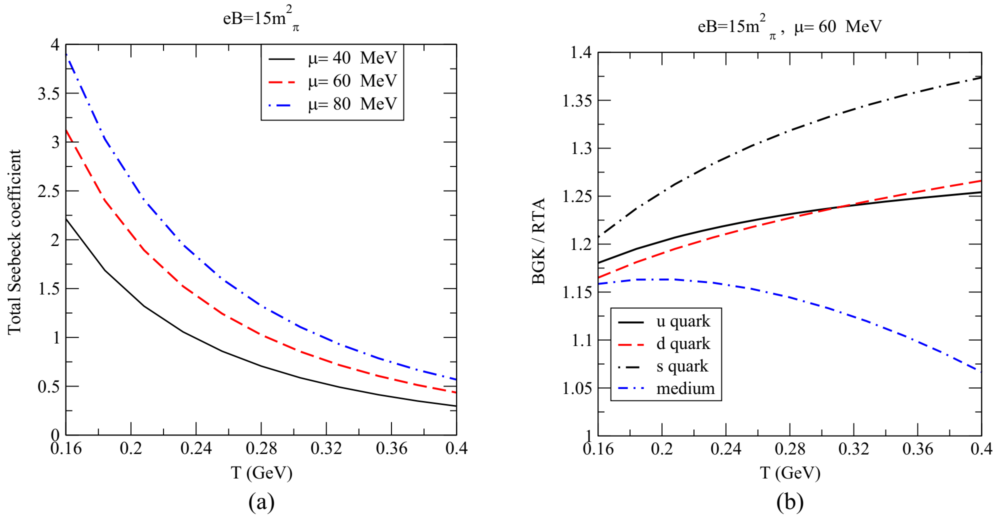

FIG. 6. Left panel: temperature dependence of the Seebeck coefficient for the medium with the BGK collision term in strong magnetic field for different strengths of μ. Right panel: ratio of the Seebeck coefficient in BGK to that in RTA collision integral with temperature.

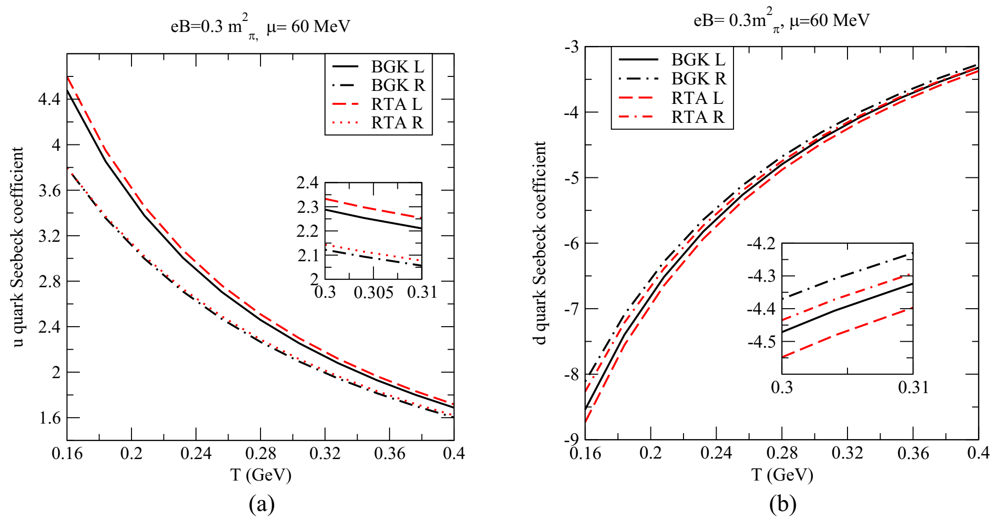

FIG. 7. Variation of the Seebeck coefficient with T in the weak magnetic field: (a) for u quarks and (b) for d quarks.

thermal conductivity (κ), shear (η), and bulk (ζ) viscosities as compared to RTA [43,44]. The nonzero value of the Seebeck coefficient modifies the electric current ($J = \sigma_{\rm el} E - \sigma_{\rm el} S \nabla T$) and thermal conductivity ($\kappa = \kappa_0 - T \sigma_{\rm el} S^2$) of the medium. Therefore, the BGK collision term will indirectly influence the charge and heat transport in the medium. Electrical conductivity plays an important role in the time evolution of the electromagnetic fields produced in the noncentral collisions. Hence, the estimation of the electrical conductivity with realistic collision integrals is of paramount importance to understand the strength and lifespan of the magnetic field during the various stages of its evolution in the medium. The magnetic field influences the particle production, dynamics of the heavy quarks and their bound states (quarkonium), and many aspects of the QCD phase diagram [91,92]. Similarly, a more accurate understanding of the thermal conductivity is necessary to study the dynamics of the first-order phase transition [93] and the chiral critical point in the heavy ion collisions [94]. It also governs the attenuation of the sound in the medium via the Prandtl number. In addition to charge and heat transport, momentum transport also gets affected by the BGK collision integral so the hydrodynamic evolution of the medium may get influenced as shear and bulk viscosities act as input to the dissipative hydrodynamical equations. In principle, the BGK collision term can affect the phenomenology of the heavy ion collision in many ways.

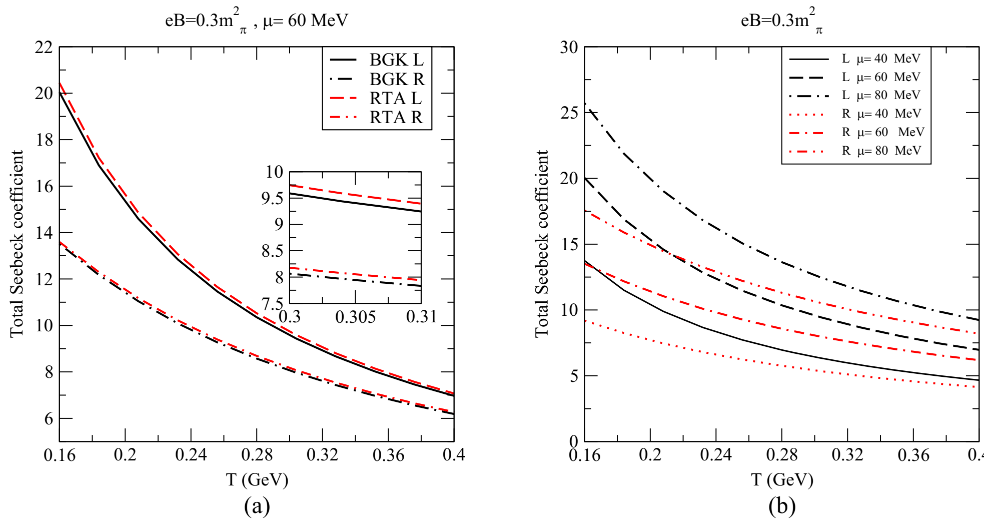

FIG. 8. (a) Variation of the Seebeck coefficient with T for the composite medium. (b) The Seebeck coefficient with respect to T in the BGK collision term for different strengths of μ.

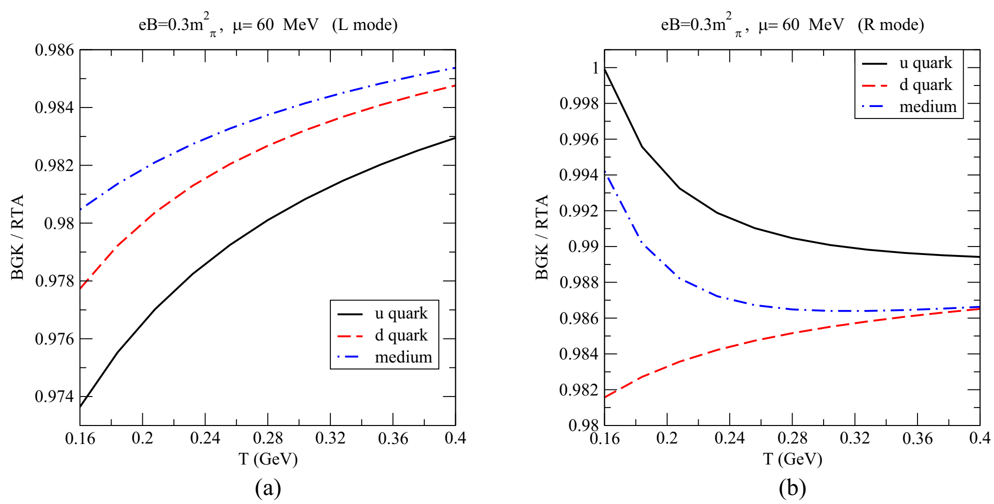

FIG. 9. (a) Ratio of the Seebeck coefficient in BGK to that in RTA with T for L modes. (b) Ratio of the Seebeck coefficient in BGK to that in RTA with T for R modes.

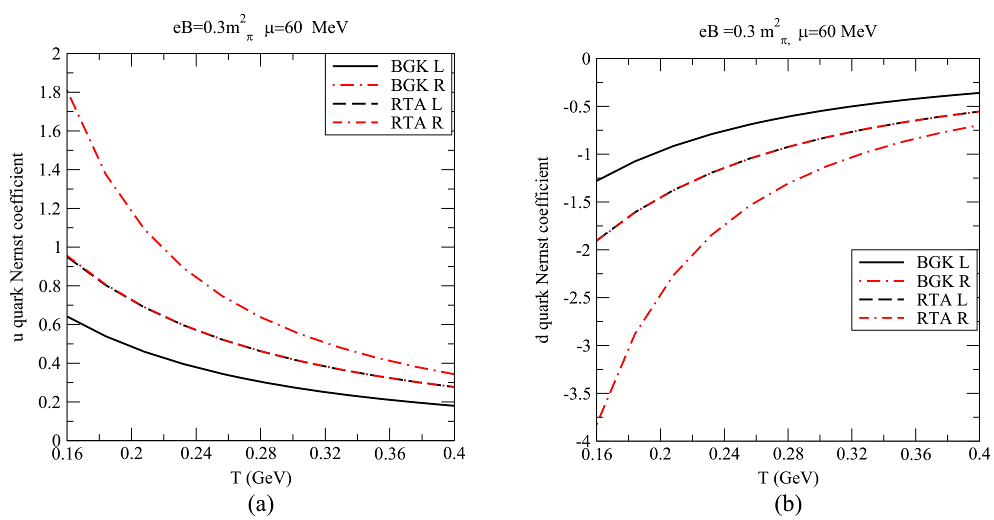

FIG. 10. Variation of the Nernst coefficient with respect to T in the weak magnetic field: (a) for u quarks and (b) for d quarks.

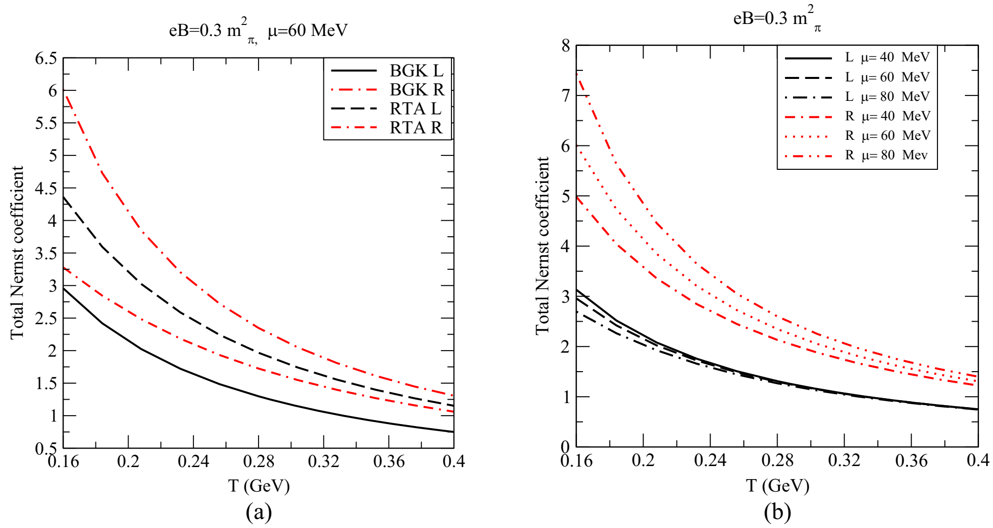

FIG. 11. (a) Variation of Nernst coefficient with respect to T for the composite medium. (b) Nernst coefficient with resect to T in the BGK collision term for different strengths of μ.

In Fig. 10, we have examined the collision integral dependence of the Nernst coefficient for u (left panel) and d (right panel) quarks. We observe that the magnitude of the Nernst coefficient gets changed drastically in the BGK collision term in comparison to RTA. The ratio BGK/RTA for the Nernst coefficient is less than 1 in the case of left-handed modes, which shows that the BGK collision integral causes a reduction in the magnitude of the Nernst coefficient [see Fig. 12(a)]. The ratio is in the range 0.67–0.65 for the u quarks, while in the range 0.67–0.64 and 0.68–0.65 for the d quarks and for the medium, respectively. In the case of R modes, ratio becomes greater than unity, and its value is found to be in the range 1.89–1.23 for u quarks, 2.00–1.26 for d quarks, and 1.83–1.23 for the medium [see Fig. 12(b)]. One important observation we notice is that there is no difference in the Nernst coefficient corresponding to the L and R modes in RTA for u and d quarks, but in the BGK the magnitude of the Nernst coefficient is greater for R modes. In Fig. 11(b), we study the μ dependance of the medium Nernst coefficient in the BGK collision integral. It is not very visible in the case of L modes except near the transition temperature, where it gets slightly reduced as μ increases. In the case of R modes, the Nernst coefficient increases with μ at a fixed value of temperature.

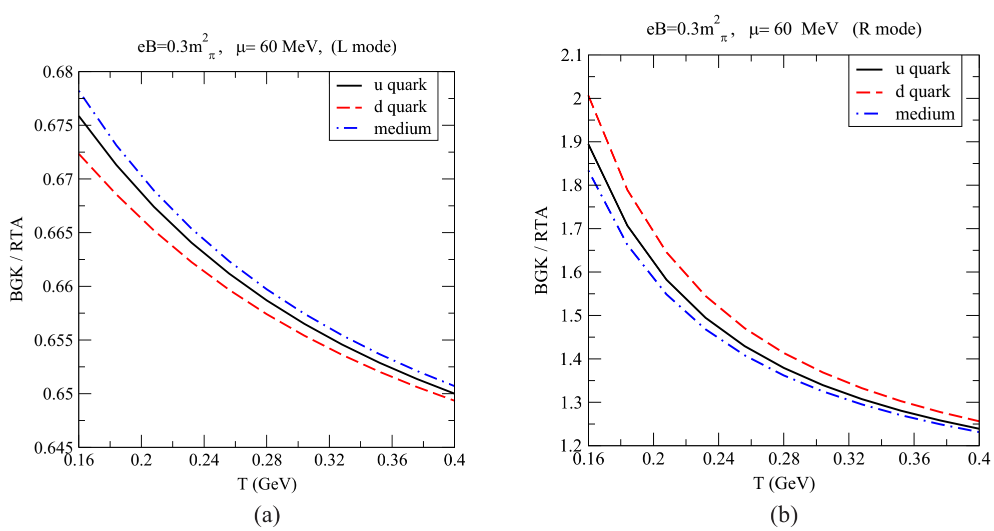

FIG. 12. (a) Ratio of Nernst coefficient in BGK to that in RTA with temperature for L modes. (b) Ratio of Nernst coefficient in BGK to that in RTA with temperature for R modes.

V. CONCLUSION

To conclude, we have investigated the thermoelectric response of a hot and magnetized QCD medium produced in the noncentral collisions at the RHIC and LHC. We have employed the Boltzmann transport equation linearized by the BGK collision integral, which conserves the particle number and current instantaneously. We incorporate the medium effects via dispersion relation wherein T, μ, and B dependent masses have been calculated using the imaginary-time formalism of the finite temperature QCD. In the absence of B, the Seebeck coefficient gets reduced in the BGK collision term for u, d, and s quarks, whereas it gets enhanced for the composite medium. In the strong B background, the magnitude of the individual, as well as the medium Seebeck coefficient, gets lifted in the BGK collision term. The Seebeck coefficient of the medium gets enhanced with the quark chemical potential in both the cases. In addition to the Seebeck coefficient, the Nernst coefficient also appears in the weak B. The Seebeck coefficient gets slightly reduced in the BGK collision integral for L modes, while for R modes, both the collision integrals give the same results. On the other hand, the Nernst coefficient gets changed drastically in the BGK collision term, and its magnitude gets reduced (enhanced) for L (R) modes in comparison to RTA. Both Seebeck and Nernst coefficients increase with μ for both L and R modes.

A nonvanishing Seebeck coefficient will modify the electric, as well as heat, current in the medium. The electric current in the presence of the Seebeck effect becomes $J = \sigma_{\rm el}E - \sigma_{\rm el}S\nabla T$, while thermal conductivity gets modified as $\kappa = \kappa_0 - T\sigma_{\rm el}S^2$. Both electrical and thermal conductivities should take positive values in accordance with the second law of thermodynamics, i.e. $T\partial_\mu S^\mu > 0$. Hence, a positive Seebeck coefficient will always reduce the electric current and the thermal conductivity. It will be also interesting to take the thermoelectric effects into account in the calculation of the entropy production, which has been completely neglected in [95,96]. Moreover, thermoelectric coefficients could also be relevant in the context of the spin Hall effect (SHE). In the SHE, a transverse spin current is generated due to the external electric field but the lifetime of such an electric field produced in the heavy ion collisions could be too small to observe the SHE. The electric field produced due to the temperature gradients in the medium may induce spin Hall effect in a hot and dense strongly interacting matter produced in heavy-ion collisions [97]. So, the study of the various implications of thermoelectric effects in the hot and dense medium needs further investigation.

ACKNOWLEDGMENTS

S. A. K. would like to thank Debarshi Dey and Pushpa Panday for various discussions.

APPENDIX A: DERIVATION OF EQUATION (50)

The BGK collision term is given by (46) as

$$\begin{aligned} C[f_i] &= -p^\mu u_\mu \nu_i\left(f_i - \frac{n_i}{n_{{\rm eq},i}}f_{{\rm eq},i}\right)\\ &= -p^\mu u_\mu \nu_i\left(f_i - \frac{g_i\int_p (f_{{\rm eq},i}+\delta f_i)}{n_{\rm eq}}f_{\rm eq}\right)\\ &= -p^\mu u_\mu \nu_i\left(f - \frac{(g_i\int_p f_{{\rm eq},i} + g_i\int_p \delta f_i)}{n_{{\rm eq},i}}f_{{\rm eq},i}\right)\\ &= -p^\mu u_\mu \nu_i\left(\delta f_i - g_i n^{-1}_{{\rm eq},i} f_{{\rm eq},i}\int_p \delta f_i\right). \end{aligned} \tag{A1}$$

APPENDIX B: BOLTZMANN EQUATION IN THE WEAK MAGNETIC FIELD

The RBTE (41) can be written with the BGK collision integral as

$$p^\mu \frac{\partial f_i}{\partial x^\mu} + q_i F'^\sigma \frac{\partial f_i}{\partial p^\sigma} = -p^\mu u_\mu \nu_i\left(f_i - \frac{n_i}{n_{{\rm eq},i}}f_{{\rm eq},i}\right), \tag{B1}$$

where $F'^\sigma = qF^{\sigma\rho}p_\rho = (p^0\vec{v}.\vec{F}, p^0\vec{F})$ is the covariant form of the Lorenz force $\vec{F} = q_i(\vec{E} + \vec{v}\times\vec{B})$. We can write Eq. (B1) using $F^{0i} = -E^i$ and $2F_{ij} = \epsilon_{ijk}B^k$ (ϵ_{ijk} is the antisymmetric Levi-Civita tensor) as

$$\frac{\partial f_i}{\partial t} + \vec{v}.\frac{\partial f_i}{\partial \vec{r}} + \frac{\vec{F}.\vec{p}}{p^0}\frac{\partial f_i}{\partial p^0} + \vec{F}.\frac{\partial f_i}{\partial \vec{p}} = -\nu_i\left(f_i - \frac{n_i}{n_{{\rm eq},i}}f_{{\rm eq},i}\right), \tag{B2}$$

considering p^0 as an independent variable,

$$\frac{\partial}{\partial \vec{p}} \to \frac{\partial p^0}{\partial \vec{p}}\frac{\partial}{\partial p^0} + \frac{\partial}{\partial \vec{p}} = \frac{\vec{p}}{p^0}\frac{\partial}{\partial p^0} + \frac{\partial}{\partial \vec{p}}. \tag{B3}$$

Equation (B2) takes the form

$$\vec{v}.\frac{\partial f_i}{\partial \vec{r}} + \vec{F}.\frac{\partial f_i}{\partial \vec{p}} = -\nu_i\left(f_i - \frac{n_i}{n_{{\rm eq},i}}f_{{\rm eq},i}\right). \tag{B4}$$

APPENDIX C: SEEBECK COEFFICIENT IN RELAXATION TIME APPROXIMATION

The Seebeck coefficient in the RTA collision term in the $B = 0$ case has been calculated as [50]

$$S = \frac{1}{2Tq}\left(\frac{L_1}{L_2}\right), \tag{C1}$$

$$L_1 = \int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3\omega^2}\bigg\{(\omega-\mu)f_{\rm eq}(1-f_{\rm eq}) + (\omega+\mu)\bar{f}_{\rm eq}(1-\bar{f}_{\rm eq})\bigg\}, \quad \text{(C2)}$$

$$L_2 = \int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3\omega^2}\bigg\{f_{\rm eq}(1-f_{\rm eq}) + \bar{f}_{\rm eq}(1-\bar{f}_{\rm eq})\bigg\}, \quad \text{(C3)}$$

and for the case of strong B as [50]

$$S_B = \frac{1}{2qT}\left(\frac{H_1}{H_2}\right), \quad \text{(C4)}$$

where

$$H_1 = \int dp_3 \frac{p_3^2}{\omega^2}\tau^B\bigg\{(\omega-\mu)f^B_{\rm eq}(1-f^B_{\rm eq}) + (\omega+\mu)\bar{f}^B_{\rm eq}(1-\bar{f}^B_{\rm eq})\bigg\}, \quad \text{(C5)}$$

$$H_2 = \int dp_3 \frac{p_3^2}{\omega^2}\tau^B\bigg\{f^B_{\rm eq}(1-f^B_{\rm eq}) + \bar{f}^B_{\rm eq}(1-\bar{f}^B_{\rm eq})\bigg\}. \quad \text{(C6)}$$

In other work [51], the thermoelectric response of the hot QCD medium has been studied in the weak magnetic field, where Seebeck and Nernst coefficients are found to be

$$S = -\frac{(C_1C_3 + C_2C_4)}{C_1^2 + C_2^2}, \quad \text{(C7)}$$

$$N|B| = \frac{(C_2C_3 - C_1C_4)}{C_1^2 + C_2^2}, \quad \text{(C8)}$$

provided $C_1 = qI_1$, $C_2 = qI_2$, $C_3 = \beta I_3$, and $C_4 = \beta I_4$. The integrals I_1, I_2,I_3, and I_4 are given by

$$I_1 = \int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3\epsilon^2}\frac{\tau}{(1+\omega_c^2\tau^2)}\bigg\{f_0(1-f_0) + \bar{f}_0(1-\bar{f}_0)\bigg\}, \quad \text{(C9)}$$

$$I_2 = \int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3\epsilon^2}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}\bigg\{f_0(1-f_0) - \bar{f}_0(1-\bar{f}_0)\bigg\}, \quad \text{(C10)}$$

$$I_3 = -\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3\epsilon^2}\frac{\tau}{(1+\omega_c^2\tau^2)}\bigg\{(\epsilon-\mu)f_0(1-f_0) - (\epsilon+\mu)\bar{f}_0(1-\bar{f}_0)\bigg\}, \quad \text{(C11)}$$

$$I_4 = -\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3\epsilon^2}\frac{\omega_c\tau^2}{(1+\omega_c^2\tau^2)}\bigg\{(\epsilon-\mu)f_0(1-f_0) + (\epsilon+\mu)\bar{f}_0(1-\bar{f}_0)\bigg\}. \quad \text{(C12)}$$

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