Gauge dependence of the quark gap equation: An exploratory study

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We study the gauge dependence of the quark propagator in quantum chromodynamics by solving the gap equation with a nonperturbative quark-gluon vertex which is constrained by longitudinal and transverse Slavnov-Taylor identities, the discrete charge conjugation and parity symmetries and which is free of kinematic singularities in the limit of equal incoming and outgoing quark momenta. We employ gluon propagators in renormalizable R_ξ gauges obtained in lattice QCD studies. We report the dependence of the nonperturbative quark propagator on the gauge parameter, in particular we observe an increase, proportional to the gauge-fixing parameter, of the mass function in the infrared domain, whereas the wave renormalization decreases within the range $0 \leq \xi \leq 1$ considered here. The chiral quark condensate reveals a mild gauge dependence in the region of ξ investigated. We comment on how to build and improve upon this exploratory study in future in conjunction with generalized gauge covariance relations for QCD.

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I. INTRODUCTION

Gauge symmetry and its innumerable consequences have played a fundamental role in the development of modern quantum field theory during the past century. Amongst its celebrated implications, the Ward-Takahashi identities (WTIs) [1,2] in quantum electrodynamics (QED) and the corresponding Slavnov-Taylor identities (STIs) [3,4] of quantum chromodynamics (QCD) relate different n-point Green functions to each other. In particular, the WTI and the STI connecting the divergence of the fermion-boson vertex to the fermion propagator help us identify the longitudinal part [5–7] of this three-point vertex, whether it be the quark-photon or the quark-gluon vertex. These are exact nonperturbative relations which are observed order by order in perturbation theory. There exists a plethora of work within the nonperturbative exploration of Dyson-Schwinger equations (DSEs) which incorporates these identities in studying dynamical chiral symmetry breaking (DCSB) via the electron/quark gap equation.

Whereas the usual WTI or STI relates the divergence of the three-point fermion-boson vertex to the inverse fermion propagator, there exist transverse Takahashi identities (TTIs) and transverse Slavnov-Taylor identities (TSTIs) which play a similar role for the curl of the fermion-boson vertex [8–12]. The TTIs and TSTIs are richer and more complicated in their structure, and they shed light on the transverse part of the fermion-boson vertex. They have been employed to compute the critical coupling and study its gauge independence, as well as the quark condensate and pion decay constant [13–17].

While the longitudinal and transverse gauge identities relate different n-point Green functions with each other, another important consequence of gauge covariance in QED are the Landau-Khalatnikov-Fradkin transformations (LKFTs), which describe how the individual Green functions respond to an arbitrary gauge transformation [18,19]. The LKFTs are a well-defined set of transformations which leave the DSEs and related WTIs of the fermion-boson vertex form invariant. While the STIs are the QCD generalization of the WTIs, the equivalent generalization

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of the LKFTs has only recently been derived by two groups employing two different methods: (i) ABG in Ref. [20] through direct generalization of the method employed by Landau and Khalatnikov, and (ii) MDSDB in Refs. [21,22] based on the introduction of a gauge invariant transverse gauge field. Moreover, the Nielsen identities allow us to study the variation of Green functions as derivatives with respect to the gauge parameter [22,23]. Nevertheless, even with these formal results available, starting from the knowledge of Green functions in a given gauge, their explicit extraction in another gauge remains a nontrivial problem in QCD even at one loop in perturbation theory. This is already the case for two-point Green functions, let alone their nonperturbative transformation and that of three- and higher n-point Green functions.

These local transformations are in no simple manner amenable to straightforward comparisons of Green functions in different gauges, especially in momentum space, nor to explicitly prove the gauge invariance of physical observables. However, it has explicitly been shown in QED, and verified in QED3 by constructing a nonperturbative vertex [24], studying gauge covariance relations [25,26] and generating numerical solutions of the electron gap equation, that the electron condensate is manifestly gauge independent once its propagator is correctly LKF transformed from a given covariant gauge to other covariant gauges [27,28]. Moreover, the ABG and the MDSDB generalizations of LKFT in QCD confirm the formal gauge-invariance of the quark condensate after the quark propagator has been adequately gauge transformed.

We are not able to directly verify this result by gauge transforming every Green function computed in the Landau gauge to repeat the calculation in any other gauge—it remains a prohibitively difficult task for the time being. We thus follow a more modest approach to study the gauge dependence of the quark propagator in QCD and solve the associated DSE, employing quenched gluon propagators in R_ξ gauges from lattice QCD [29], and the dressed quark-gluon vertex constructed in Ref. [17], invoking the STI and TSTI. Our approach not only allows for a comparison of the DSE solutions in R_ξ gauges, i.e., the mass and wave renormalization functions of the quark for gauge parameters in the range $\xi \in [0, 0.5]$, extrapolated to the Feynman gauge, $\xi = 1$, but also to compute the quark condensate as a function of the gauge parameter within this interval. We emphasize that this is an initial, exploratory study which we expect to shed light on how far we are from obtaining the gauge independence of the condensate. This invariance would be obtained through formal incorporation of the local gauge transformations of every Green function in the gap equation, i.e., the gluon propagator, the quark-gluon vertex, as well as the ghost propagator and the ghost-quark scattering kernel.

Within the limitations of this hybrid approach, we demonstrate how the numerical solutions of the quark propagator vary as function of the gauge parameter in R_ξ gauges, and that this behavior leads to a slightly gauge-dependent quark condensate up to Feynman gauge within the error estimates involved. Nonetheless, to our knowledge, we present the first DSE solutions for the quark obtained with all 12 vector structures of the nonperturbative quark-gluon vertex and the gluon and ghost propagators in covariant R_ξ gauges. Our findings are encouraging and represent an important first step as they incorporate the gauge covariance into the DSEs through the formal implementation of gauge identities for the Green functions. This is a necessary, though not sufficient, requirement for calculating gauge-independent bound-state properties.

This article is organized as follows: in Sec. II we describe our functional approach to QCD and discuss the quark-gluon vertex, the lattice-extracted gluon, and ghost propagators and the quark-ghost form factor in different gauges. In Sec. III, we solve the quark gap equation and obtain the mass and wave renormalization functions in different gauges and calculate the quark condensate. Final remarks are given in Sec. IV.

II. DYSON-SCHWINGER EQUATION IN R_ξ GAUGES

The DSEs are the relativistic equations of motion in quantum field theory, see, e.g., Ref. [30] for a review. For a given flavor and in the R_ξ gauge, the DSE of the inverse quark propagator in Euclidean space reads

$$S_\xi^{-1}(p) = Z_2 i\gamma\cdot p + Z_4 m(\mu) + Z_1 4\pi\alpha_s^\xi \int^\Lambda \frac{d^4k}{(2\pi)^4}\Delta_{\mu\nu}^{ab}(q)\gamma_\mu t^a S_\xi(k)\Gamma_\nu^{b\xi}(k,p), \tag{1}$$

where $m(\mu)$ is the renormalized current-quark mass, $Z_1(\mu,\Lambda)$, $Z_2(\mu,\Lambda)$, and $Z_4(\mu,\Lambda)$ are the vertex, wave function, and mass renormalization constants, respectively, while $\Gamma_\mu^{a\xi}(k,p) = \Gamma_\mu^\xi(k,p)t^a$ is the dressed quark-gluon vertex and $t^a = \lambda^a/2$ are the SU(3) group generators in the fundamental representation. The gluon propagator in R_ξ gauge with momentum $q = k - p$,

$$\Delta_{\mu\nu}^{ab}(q) = \delta^{ab}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Delta_\xi(q^2) + \delta^{ab}\xi\frac{q_\mu q_\nu}{q^4} \tag{2}$$

is characterized by a nonperturbative transverse dressing function, $\Delta_\xi(\mu^2) = 1/\mu^2$, and was studied with different lattice and functional approaches, e.g., Refs. [29,31,32].

The most general Poincaré-covariant form of the solutions to Eq. (1) is written in terms of covariant scalar and vector amplitudes:

$$S_\xi(p) = \frac{1}{i\gamma\cdot pA_\xi(p^2) + B_\xi(p^2)} = \frac{Z_\xi(p^2)}{i\gamma\cdot p + M_\xi(p^2)}. \tag{3}$$

In the self-energy integral, Λ is a Poincaré-invariant ultraviolet cutoff and μ is the renormalization scale imposed such that $\Lambda \gg \mu$. This scale is implicit in our convenient notation: $A(p^2) \equiv A(p^2, \mu^2)$ and $B(p^2) \equiv B(p^2, \mu^2)$, as is a flavor index f for these quantities and for all renormalization constants. The flavor- and gauge-dependent mass and wave renormalization functions are, respectively,

$$M_\xi(p^2) = B_\xi(p^2, \mu^2)/A_\xi(p^2, \mu^2), \tag{4}$$

$$Z_\xi(p^2, \mu^2) = 1/A_\xi(p^2, \mu^2). \tag{5}$$

We choose the renormalization scale, $\mu = 4.3$ GeV, for a twofold reason: (i) we consistently renormalize the DSE at the scale at which the transverse dressing function $\Delta_\xi(q^2)$ in R_ξ gauge is renormalized [29], and (ii) we can compare the dressed functions $M_\xi(p^2)$ and $Z_\xi(p^2, \Lambda^2)$ with the solutions of lattice regularized QCD at this scale [33,34]. We therefore impose [35,36] $Z_\xi(\mu^2) = 1$ and $M_\xi(\mu^2) \equiv m(\mu^2) = 25$ MeV.

For the dressed quark-gluon vertex we employ the decomposition detailed, e.g., in Ref. [16],

$$\Gamma^\xi_\mu(k,p) = \Gamma^{L\xi}_\mu(k,p) + \Gamma^{T\xi}_\mu(k,p) = \sum_{i=1}^{4} \lambda^\xi_i(k,p) L^i_\mu(k,p) + \sum_{i=1}^{8} \tau^\xi_i(k,p) T^i_\mu(k,p), \tag{6}$$

where the transverse vertex $\Gamma^{T\xi}_\mu(k,p)$ in Eq. (6) is naturally defined by $iq \cdot \Gamma^{T\xi}(k,p) = 0$. The usual STI [3,4] constrains the longitudinal vertex, $\Gamma^{L\xi}_\mu(k,p)$, to four independent structures but leaves the transverse components undetermined. The remaining eight tensor structures can be explored with TSTI derived from the symmetry transformation that involves the Lorentz transformation acting on the usual infinitesimal gauge transformation.

The TSTI [13] constrain but also couple the vector and axial vector vertices. However, as shown in Refs. [14,16], these can be decoupled and merely one identity for the vector vertex is sufficient to obtain analytic expressions for the transverse form factors $\tau^\xi_i(k,p)$. The full vertex is thus described by the form factors derived in Refs. [17,37],

$$\lambda^\xi_1(k,p) = \frac{1}{2} G(q^2) X^\xi_0(q^2)[A_\xi(k^2) + A_\xi(p^2)], \tag{7}$$

$$\lambda^\xi_2(k,p) = G(q^2) X^\xi_0(q^2) \frac{A_\xi(k^2) - A_\xi(p^2)}{k^2 - p^2}, \tag{8}$$

$$\lambda^\xi_3(k,p) = G(q^2) X^\xi_0(q^2) \frac{B_\xi(k^2) - B_\xi(p^2)}{k^2 - p^2}, \tag{9}$$

$$\lambda^\xi_4(k,p) = 0, \tag{10}$$

for the longitudinal part and by

$$\tau^\xi_1(k,p) = -\frac{Y_1}{2(k^2 - p^2)\nabla(k,p)}, \tag{11}$$

$$\tau^\xi_2(k,p) = -\frac{Y_5 - 3Y_3}{4(k^2 - p^2)\nabla(k,p)}, \tag{12}$$

$$\tau^\xi_3(k,p) = \frac{1}{2} G(q^2) X^\xi_0(q^2) \left[\frac{A_\xi(k^2) - A_\xi(p^2)}{k^2 - p^2}\right] + \frac{Y_2}{4\nabla(k,p)} - \frac{(k+p)^2(Y_3 - Y_5)}{8(k^2 - p^2)\nabla(k,p)}, \tag{13}$$

$$\tau^\xi_4(k,p) = -\frac{6Y_4 + Y^A_6}{8\nabla(k,p)} - \frac{(k+p)^2 Y^S_7}{8(k^2 - p^2)\nabla(k,p)}, \tag{14}$$

$$\tau^\xi_5(k,p) = -G(q^2) X^\xi_0(q^2) \left[\frac{B_\xi(k^2) - B_\xi(p^2)}{k^2 - p^2}\right] - \frac{2Y_4 + Y^A_6}{2(k^2 - p^2)}, \tag{15}$$

$$\tau^\xi_6(k,p) = \frac{(k-p)^2 Y_2}{4(k^2 - p^2)\nabla(k,p)} - \frac{Y_3 - Y_5}{8\nabla(k,p)}, \tag{16}$$

$$\tau^\xi_7(k,p) = \frac{q^2(6Y_4 + Y^A_6)}{4(k^2 - p^2)\nabla(k,p)} + \frac{Y^S_7}{4\nabla(k,p)}, \tag{17}$$

$$\tau^\xi_8(k,p) = -G(q^2) X^\xi_0(q^2) \left[\frac{A_\xi(k^2) - A_\xi(p^2)}{k^2 - p^2}\right] - \frac{2Y^A_8}{k^2 - p^2}, \tag{18}$$

for the transverse components, where the Gram determinant is defined as $\nabla(k,p) = k^2 p^2 - (k \cdot p)^2$.

The form factors $\lambda_i(k,p)$, $i = 1, 2, 3$, and $\tau_i(k,p)$, $i = 3, 5, 8$ are proportional to the ghost-dressing function $G(q^2)$ defined by the propagator

$$D^{ab}(q^2) = -\delta^{ab} \frac{G(q^2)}{q^2}, \tag{19}$$

and are renormalized as $G(\mu^2) = 1$; $X^\xi_0(q^2)$ is the leading form factor of the quark-ghost scattering amplitude [38,39], $H^{a\xi}(k,p) = H^\xi(k,p) t^a$, which can most generally be decomposed as

$$H^\xi(k,p) = X^\xi_0(k,p) \mathbb{1}_D + i X^\xi_1(k,p) \gamma \cdot k + i X^\xi_2(k,p) \gamma \cdot p + X^\xi_3(k,p) [\gamma \cdot k, \gamma \cdot p]. \tag{20}$$

Calculated in one-loop dressed approximation [35,40], $X^\xi_0(k,p)$ can be projected out from the integral equation for the quark-ghost scattering amplitude. We do so in the simplified kinematic configuration $k = -p = q/2$, thus omitting an angular dependence between k and p, which is expressed by the following integral:

$$X_0^\xi(q^2) = \frac{1}{4}\mathrm{Tr}_{\mathrm{CD}} H^\xi(q/2, -q/2),$$
$$= 1 + \frac{C_A}{4} g^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \Delta^\xi_{\mu\nu}(\ell) D(\ell + q)$$
$$\times \mathrm{Tr}_D \left[G_\nu S_\xi\left(\ell + \frac{q}{2}\right) \Gamma^\xi_\mu\left(\ell + \frac{q}{2}, -\frac{q}{2}\right)\right]. \quad (21)$$

Here, $C_A = 3$ is the Casimir operator in the adjoint representation,

$$G_\nu = i(\ell_\nu + q_\nu)H_1 - i\ell_\nu H_2 \quad (22)$$

is the dressed ghost-gluon vertex for which the form factor H_1 was calculated in Ref. [41], ℓ is the gluon momentum exchanged between the quark and the ghost, and in this configuration q coincides with the gluon momentum in the DSE (1). The trace is over color and Dirac indices. Beyond Landau gauge, the integral in Eq. (21) diverges since the gluon propagator (2) is not transverse anymore and does therefore not project out the ℓ_μ terms of the ghost-gluon vertex (22). The divergence is mildly logarithmic and given that we employ lattice-QCD propagators for momenta $p \lesssim 8$ GeV, no renormalization constant is necessary in the numerical integration, even for large Λ. Nonetheless, we multiply the left-hand side of Eq. (21) with a renormalization factor Z_H and impose $X_0(\mu^2) = 1$.

We neglect the remaining $X_i^\xi(k, p)$ form factors as they are suppressed with respect to $X_0^\xi(k, p)$ [6] in the Landau gauge. One may ask whether this approximation is justified in other gauges as the form factors may vary in strength with the gauge parameter. This dependence is depicted in Fig. 1 for $X_0^\xi(q^2)$ and discussed in more detail in Sec. III. It turns out that setting $X_0^\xi(q^2) = 1$ has only modest quantitative effects on $M_\xi(p^2)$ and $Z_\xi(p^2)$, at least for $\xi \in [0, 1]$. Therefore, we make bold to assume that our observation is valid for the other form factors and will analyze their behavior as a function of ξ in a forthcoming study. We will see in Sec. III that the contribution of $X_0(q^2)$ to DCSB is small compared with the effect of the transverse vertex.

The $Y_i^{(A,S)}$ functions represent the form-factor decomposition of the Fourier transform of a four-point function in coordinate space in the TSTI. The latter involves a line integral over a nonlocal vector vertex and a Wilson line to preserve gauge invariance. As its matrix elements are rather complicated expressions, we refer to the discussion in Ref. [13]. We note that the $Y_i^{(A,S)}$ functions are *a priori* unknown and have been constrained by us [16] with the vertex ansatz in Ref. [42]; i.e., we equated our transverse form factors (11)–(18) with those of Ref. [42] and derived expressions for the $Y_i(k, p)$ that depend on $A(p^2)$, $B(p^2)$, and $X_0(p^2)$. On the other hand, if the $Y_i^{(A,S)}$ form factors were known exactly, so would be the vertex. However, additional input, e.g., from the generalized LKFT that describes the vertex itself, would be required for its full determination. We remind that we do not solve the BSE of the quark-gluon vertex; hence we do not truncate the vertex form factors. Therefore, the expressions in Eqs. (7)–(18) represent a most complete form factor decomposition of the vertex that satisfies SU(3) gauge symmetry and the C and P symmetries of the bare vertex, and which is not plagued by kinematic singularities in the limit $k \to p$.

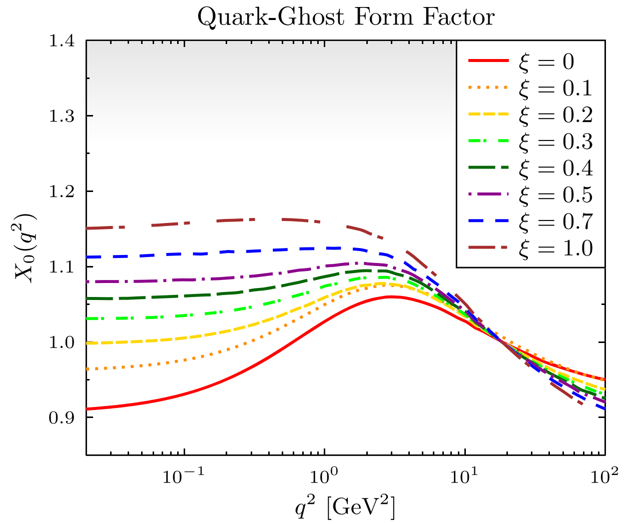

FIG. 1. Gauge-parameter dependence of the form factor $X_0^\xi(q^2)$ associated with the quark-ghost scattering amplitude.

In principle, one must also solve the gluon and ghost DSE which are coupled to the gap equation (1). We deliberately abstain from doing so, as our interest is in employing quenched gluon and ghost propagators in R_ξ gauge from lattice QCD [29]. For practical reasons in their numerical implementation, we fit the gluon dressing functions with the parametrization [43]

$$\Delta_\xi(q^2) = \frac{Z(q^2 + M_1^2)}{q^4 + M_2^2 q^2 + M_3^4}\left[1 + \omega \ln\left(\frac{q^2 + M_0^2}{\Lambda^2_{\mathrm{QCD}}}\right)\right]^{-\gamma_{\mathrm{gl}}}, \quad (23)$$

where $\omega = 11 N_c \alpha_s(\mu)/12\pi$, $\Lambda_{\mathrm{QCD}} = 0.425$ GeV, and $\gamma_{\mathrm{gl}} = (13 - 3\xi)/22$ is the 1-loop anomalous gluon dimension. The renormalization scale is $\mu = 4.3$ GeV at which the strong coupling is chosen to be $\alpha_s = 0.29$ in Landau gauge [44]. This parametrization is motivated by the refined Gribov-Zwanziger tree-level gluon propagator in the infrared domain and by the one-loop renormalization group behavior for large momenta, which amounts to a renormalization-group improved Padé approximation. We collect the values Z, M_0, M_1, M_2, and M_3 as a function of the gauge parameter ξ in Table I. It turns out that all fit

TABLE I. Parameters of the gluon-dressing parametrization (23) as a function of the gauge parameter ξ. Note that the bare gluon propagators of Ref. [29] were fitted which must be renormalized as $\Delta_\xi(\mu^2) = 1/\mu^2$. The values $\xi = 0.7$ and $\xi = 1$ are obtained from a linear extrapolation.

ξ	Z	M_0^2 [GeV^2]	M_1^2 [GeV^2]	M_2^2 [GeV^2]	M_3^4 [GeV^4]	$\chi^2/d.o.f.$
0.0	8.407 ± 0.031	0.071 ± 0.023	2.783 ± 0.128	0.532 ± 0.055	0.380 ± 0.012	1.209
0.1	8.321 ± 0.023	0.071 ± 0.018	2.712 ± 0.099	0.517 ± 0.043	0.374 ± 0.009	0.734
0.2	8.238 ± 0.018	0.074 ± 0.015	2.677 ± 0.076	0.515 ± 0.033	0.371 ± 0.007	0.421
0.3	8.160 ± 0.012	0.069 ± 0.010	2.617 ± 0.051	0.499 ± 0.023	0.367 ± 0.005	0.205
0.4	8.072 ± 0.011	0.076 ± 0.010	2.595 ± 0.046	0.499 ± 0.020	0.363 ± 0.004	0.178
0.5	7.996 ± 0.009	0.075 ± 0.009	2.570 ± 0.041	0.500 ± 0.018	0.362 ± 0.004	0.162
0.7	7.831	0.077	2.488	0.487	0.354	…
1.0	7.585	0.080	2.378	0.473	0.344	…

parameters depend linearly on ξ and we take this feature to our advantage to extrapolate the parametrization to $\xi = 1$.

The bare lattice data [45] for the ghost propagator is parametrized with an analogous expression,

$$G(q^2) = \frac{Z(q^4 + M_2^2 q^2 + M_1^4)}{q^4 + M_4^2 q^2 + M_3^4}\left[1 + \omega \ln\left(\frac{q^2 + \frac{m_1^4}{q^2 + m_0^2}}{\Lambda_{\mathrm{QCD}}^2}\right)\right]^{\gamma_{\mathrm{gh}}}, \tag{24}$$

independent of ξ, as supported by a preliminary study in lattice QCD [46]. The anomalous ghost dimension is $\gamma_{\mathrm{gh}} = -9/44$, while ω, Λ_{QCD} and μ are as in Eq. (23). A least-squares fit yields $\chi^2/d.o.f. = 0.247$ and the parameters: $Z = 5.068 \pm 0.012$, $M_1^4 = 19.281 \pm 0.552$ GeV^4, $M_2^2 = 27.721 \pm 0.696$ GeV^2, $M_3^4 = 7.695 \pm 0.329$ GeV^4, $M_4^2 = 24.340 \pm 0.565$ GeV^2, $m_0^2 = 0.527 \pm 1.263$ GeV^2, $m_1^4 = 0.018 \pm 0.035$ GeV^4; this fit must be normalized by a factor $N = 1/4.706$ to ensure the renormalization condition $G(4.3\ \mathrm{GeV}) = 1$.

III. GAUGE DEPENDENCE OF THE QUARK PROPAGATOR AND INVARIANCE OF THE QUARK CONDENSATE

With all the calculational tools and elements at hand in the gauge-parameter interval between the Landau and Feynman gauges, we can evaluate the gauge dependence of the quark's mass and wave-renormalization functions as well as that of the quark condensate.

A. Gauge dependent mass and wave-renormalization functions

Since the gauge propagators and the quark-gluon vertex are now determined we can proceed to solve the DSE (1). After taking the color trace, and with the renormalization procedure detailed in Ref. [17], the DSE becomes

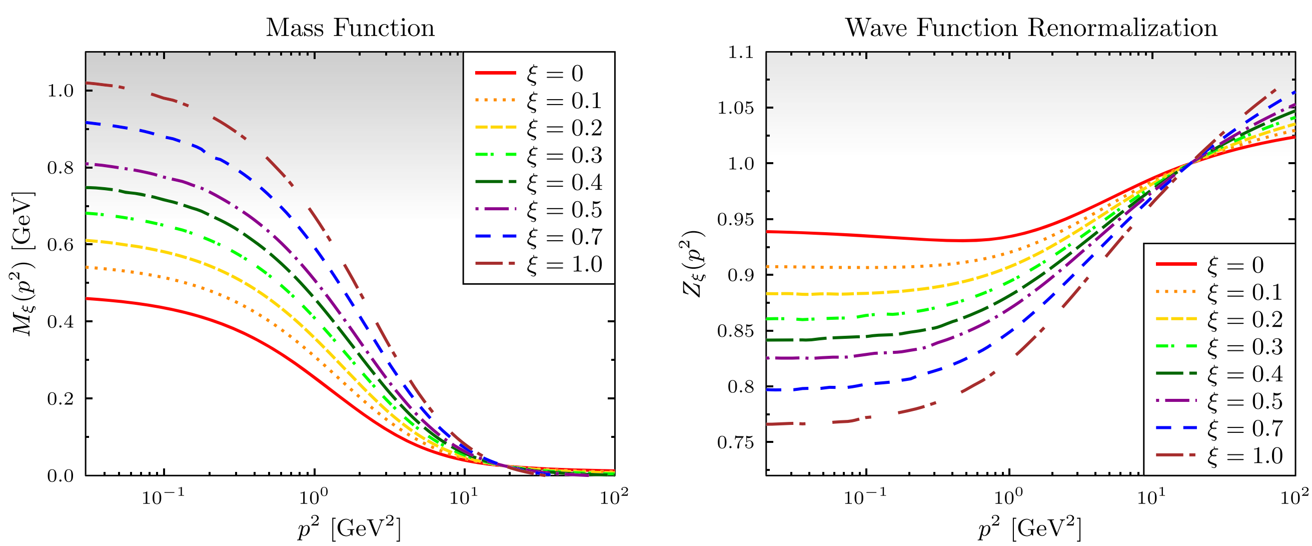

FIG. 2. Mass function $M_\xi(p^2)$ and wave renormalization $Z_\xi(p^2)$ as functions of the gauge parameter ξ obtained by solving the DSE (25) with the gluon and ghost propagators in Eqs. (23) and (24), respectively, and the vertex $\Gamma_\mu^\xi(k, p)$ of Eq. (6) with corresponding form factors $\lambda_i^\xi(k, p)$ and $\tau_i^\xi(k, p)$.

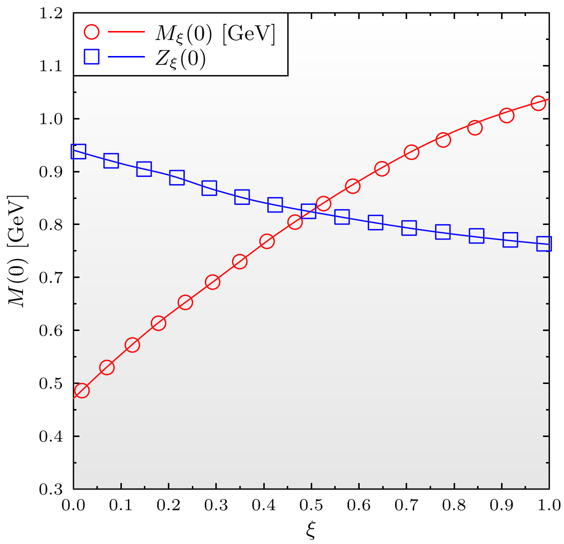

FIG. 3. Increasing and decreasing dependence of the mass function $M_\xi(0)$ and wave renormalization $Z_\xi(0)$, respectively, on the gauge parameter ξ.

$$S_\xi^{-1}(p) = Z_2 i\gamma \cdot p + Z_4 m(\mu) + Z_2 \frac{16\pi\alpha_s^\xi}{3} \int^\Lambda \frac{d^4k}{(2\pi)^4} \Delta_{\mu\nu}^\xi(q)\gamma_\mu S_\xi(k)\Gamma_\nu^\xi(k,p). \tag{25}$$

We plot the solutions of this DSE in Fig. 2, which clearly exposes that the mass function $M_\xi(p^2)$ increases with ξ below the renormalization point $\mu = 4.3$ GeV, whereas the wave renormalization $Z_\xi(p^2)$ is suppressed. In fact, $M_\xi(0)$ increases rather continuously with ξ, whereas $Z_\xi(0)$ slowly decreases as a function of ξ, as can be read from Fig. 3. We remark that the gauge dependence is threefold in Eq. (25), namely ξ enters directly via the longitudinal component and indirectly via the dressing function of the transverse component, $\Delta_\xi(q^2)$, of the gluon propagator (2) but also via the gauge-dependent strong coupling α_s^ξ for which we use the parametrization [44]

$$\alpha_s^\xi = 0.29 + 0.098\xi - 0.064\xi^2. \tag{26}$$

The mass-function enhancement in Fig. 2 is due to the feedback of the gauge dependent $A_\xi(p^2)$ and $B_\xi(p^2)$ functions in the dressed quark-gluon vertex, namely Eqs. (7)–(18), as $M_\xi(p^2)$ also increases when we keep the coupling $\alpha_s = 0.29$ constant for all values of ξ. Moreover, $M_\xi(p^2)$ slightly decreases with ξ in the rainbow truncation, $\Gamma_\mu^\xi(k,p) = \gamma_\mu$, while the DSE solutions employing $\Gamma_\mu^{L\xi}(k,p)$ with Eqs. (7)–(10) show an enhancement of the mass as a function of ξ. We emphasize that the overwhelming contribution to mass generation is due to the massive terms in the expressions for $\lambda_i^\xi(k,p)$ and $\tau_i^\xi(k,p)$. While $X_0^\xi(q^2)$ is gauge dependent, as becomes clear from Fig. 1 where an enhancement of this form factor with increasing values of ξ in the low-momentum region is observed, setting $X_0^\xi(q^2) = 1, \xi \in [0,1]$, leads to a 10% reduction of $M_{\xi=1}(0)$ in the Feynman gauge and to almost no variation of $M_{\xi=0}(p^2)$ in the Landau gauge. Neglecting $X_0^\xi(q^2)$ does not qualitatively alter our results; in particular our conclusions about the quark condensate remain the same.

We also illustrate the gauge-variation impact on the strength of the quark-gluon vertex in Fig. 4 with the form factors $\lambda_1^\xi(k,p)$ and $\lambda_3^\xi(k,p)$ obtained in the symmetric limit $k = -p$ and with the solutions for $M_\xi(p^2)$ and $Z_\xi(p^2)$ in Fig. 2. Both form factors are enhanced, though this is more so the case for $\lambda_3^\xi(-p,p)$. The latter is proportional to the mass function $B_\xi(p^2)$ whose variation with ξ is more prominent.

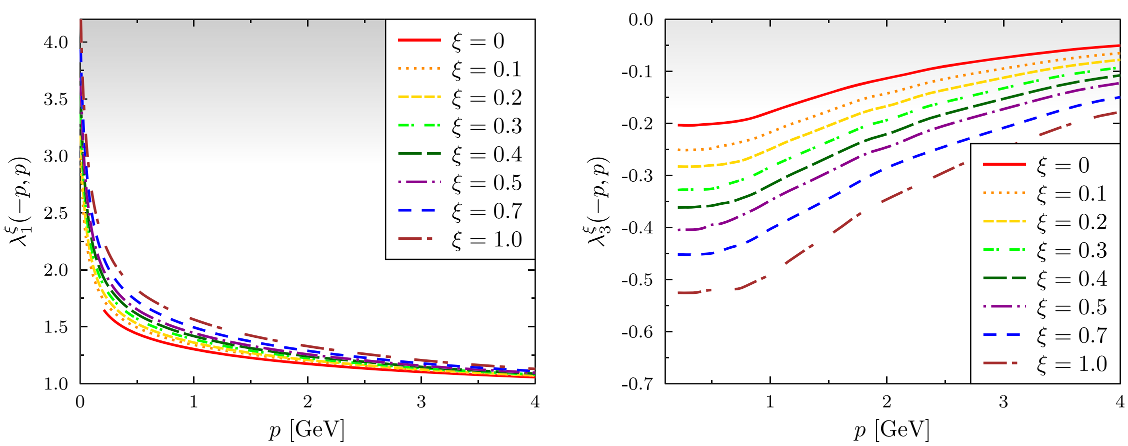

FIG. 4. The form factors $\lambda_1^\xi(-p,p)$ and $\lambda_3^\xi(-p,p)$ as functions of the gauge parameter ξ, obtained with Eqs. (7) and (9) in the symmetric limit $k = -p$. Note that the dimensionless quantity $p\lambda_3^\xi$ is plotted.

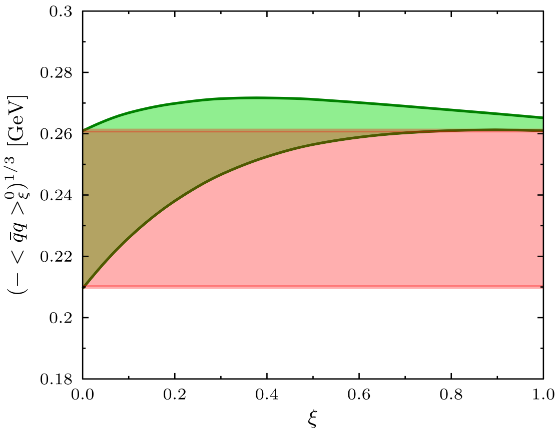

FIG. 5. Gauge dependence of the quark condensate. The horizontal pink-shaded band indicates the admissible region of a gauge-independent chiral quark condensate as implied by the LKFT in QCD.

B. Quark condensate and gauge invariance

As a practical application, we calculate the quark condensate which is an order parameter for DCSB and was shown to be a manifestly gauge-invariant quantity in any SU(N) theory using the generalized LKFT (ABG) transformations [20]:

$$-\langle\bar{q}q\rangle^0_\xi \equiv Z_4 N_c \int^\Lambda \frac{d^4k}{(2\pi)^4}\mathrm{tr}_D[S^0_\xi(k)]. \tag{27}$$

To this end, we obtain $M_\xi(p^2)$ and $Z_\xi(p^2)$ in the limit $m(\mu)\to 0$ with which we compute $(-\langle\bar{q}q\rangle^0_\xi)^{1/3}$ as a function of ξ. The result is presented in Fig. 5, where the central value exhibits a moderate dependence on ξ, i.e., a maximum deviation from the Landau-gauge value of 7% for $\xi \gtrsim 0.3$ after which its dependence on ξ appears to become milder. The green-shaded band depicts an error estimate due to the statistical error of $\pm 10\%$ of the gluon propagator. This is because the lattice calculations of the gluon propagator in R_ξ gauges were performed using smaller physical volumes, $V \simeq (3.2\ \mathrm{fm})^4$, and smaller gauge ensembles than for the ghost propagator for which the fit reproduces the simulation results in a large physical volume, $V \simeq (8.1\ \mathrm{fm})^4$, and with an ensemble of gauge configurations that is about three times larger. Therefore, the main error source is due to the gluon propagator and we neglect a statistical error associated with the ghost.

Our error estimate only enters via the transverse part of the gluon propagator (2), yet with increasing values of the gauge parameter the contribution of the longitudinal component is more important. This is the reason why the error band is wider in Landau gauge and narrows towards the Feynman gauge. We did not include the uncertainty of the strong coupling (26) and the systematic error due to our treatment of the nonlocal four-point functions in the TSTI.

IV. FINAL REMARKS

We have calculated the quark propagator's dependence on the gauge parameter ξ in covariant gauges. It turns out that the gluon propagator parameterization and $M_\xi(0)$ can be fitted with linear functions of ξ within the gauge-parameter interval, $\xi \in [0, 0.5]$, while $Z_\xi(0)$ decreases in a proportionate manner. Furthermore, the strength of the quark-gluon interaction increases with ξ which illustrates the gauge dependence of the quark dynamics. Having said that, studying the local gauge transformation of the nonperturbative quark propagator directly from its solution in a given gauge is illuminating, as were similar QED3 studies [28] that explicitly demonstrated the gauge independence of the fermion condensate. Both the Nielsen identities and the generalized LKFTs (ABG and MDSDB) [20–23] formally provide the basis for this endeavor. However, it is not a straightforward exercise in QCD, even at the perturbative level [47].

While Green functions are not physical observables, they are essential objects for the elucidation and understanding of strong interactions. They enter each and every hadron observable computed from QCD's elementary degrees of freedom, namely quarks and gluons (and ghosts within covariant gauges). How the dependence on ξ is washed out in the building of physical observables is a nontrivial problem that constrains the truncated kernels of Bethe-Salpeter and Faddeev equations and deserves further attention. The quark mass function inserted in a bound-state equation conspires with an intricate interaction kernel involving quarks and gluons at all energy scales in a manner that preserves the axial vector Ward-Takahashi identity in addition to all other gauge identities and transformations mentioned in the text in detail, yielding gauge-independent physical observables. Any variation of these observables with the gauge parameter is a measure of our departure from the full implementation of the generalized LKFT and the Nielsen identities.

Our analysis of the DSE relies on the dressed vertex (6) described by form factors $\lambda^\xi_i(k,p)$ and $\tau^\xi_i(k,p)$, which are plotted for $\xi = 0$ in Ref. [37] and show to be in reasonable agreement with those obtained in lattice QCD simulations [48]. We are thus encouraged by our exploratory study on the gauge dependence of the quark propagator, as they provide a valuable initial step in the numerical representation of the generalized LKFT and Nielsen identities in QCD despite the current limitations. However, further efforts must be made to include all $X^\xi_i(k,p)$ form factors of the quark-ghost scattering amplitude, along with their angular dependence, to derive a true analytic expression for the four-point function in the TSTI and to explore the dependence on these hitherto ignored features in the current analysis.

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