Path integrals of perturbative superstrings on curved backgrounds from string geometry theory

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String geometry theory is one of the candidates for a nonperturbative formulation of string theory. In this paper, from the string geometry theory, we derive path integrals of perturbative superstrings on all string backgrounds, $G_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$, by considering fluctuations around the string background configurations, which are parametrized by the string backgrounds.

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I. INTRODUCTION

String geometry theory is one of the candidates for a nonperturbative formulation of string theory, based on a path integral of string manifolds, which are a class of infinite-dimensional manifolds [1]. String manifolds are defined by patching open sets of the model space defined by introducing a topology to a set of strings. One of the remarkable facts concerning string geometry theory is that the path integral of the perturbative superstrings in the flat background is derived including the moduli of super Riemann surfaces, by considering fluctuations around the flat background in the theory [1–3].

Moreover, configurations of fields in string geometry theory include all configurations of fields in the ten-dimensional supergravities, namely, string backgrounds [4,5]. In particular, it was shown that an infinite number of equations of motion of string geometry theory are consistently truncated to finite numbers of equations of motion of the supergravities; that is, string geometry theory does not include string backgrounds as external fields, as in the perturbative string theories. The dynamics of string backgrounds are a part of the dynamics of the fields in the theory. It is natural to expect to be able to derive the path integral of perturbative strings on the string backgrounds by considering fluctuations around the corresponding configurations in string geometry theory.

For each background, one theory is formulated in the case of a perturbative string theory, whereas perturbative string theories on both flat and nontrivial backgrounds should be derived from a single theory in the case of the nonperturbative formulation of string theory. Actually, the authors of Ref. [6] derived the path integrals of perturbative strings on all string backgrounds, $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $\Phi(x)$, from the bosonic sector of string geometry theory. This paper gives a supersymmetric generalization of this fact.

Such a supersymmetric generalization is necessary in order for the spectrum not to include a tachyon. On the other hand, the supersymmetric action (2.2) is strongly constrained by T symmetry in string geometry theory, which is a generalization of T duality among perturbative vacua in string theory [7]. Moreover, all ten-dimensional supergravities are derived from the supersymmetric action (2.2), as we described above. In this paper, we obtain further evidence that the supersymmetric action (2.2) is correct by deriving path integrals of type II perturbative superstrings on all string backgrounds from the action.

The organization of the paper is as follows. In Sec. II, we briefly review string geometry theory. In Sec. III, we set string background configurations parametrized by the string backgrounds $G_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$, and set the classical part of fluctuations representing strings. In Sec. IV, we consider two-point correlation functions of the quantum part of the fluctuations and derive the path integrals of the perturbative superstrings on the string backgrounds. In the Appendix, we obtain a Green function on the flat superstring manifold.

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II. REVIEW OF STRING GEOMETRY THEORY

String geometry theory is defined by a partition function

$$Z = \int \mathcal{D}\mathbf{G}\mathcal{D}\Phi\mathcal{D}\mathbf{B}\mathcal{D}\mathbf{A}e^{-S}, \tag{2.1}$$

where the action is given by

$$S = \int \mathcal{D}\mathbf{E}\mathcal{D}\bar{\tau}\mathcal{D}\mathbf{X}_{\hat{D}_T}\sqrt{-\mathbf{G}}\bigg(e^{-2\Phi}\bigg(\mathbf{R}+4\nabla_{\mathbf{I}}\Phi\nabla^{\mathbf{I}}\Phi - \frac{1}{2}|\tilde{\mathbf{H}}|^2 - \mathrm{tr}(|\mathbf{F}|^2)\bigg) - \frac{1}{2}\sum_{p=1}^{\infty}|\tilde{\mathbf{F}}_p|^2\bigg), \tag{2.2}$$

where $\mathbf{I} = \{d, (\mu\bar{\sigma}\bar{\theta})\}$, $|\tilde{\mathbf{H}}|^2 := \frac{1}{3!}\mathbf{G}^{\mathbf{I}_1\mathbf{J}_1}\mathbf{G}^{\mathbf{I}_2\mathbf{J}_2}\mathbf{G}^{\mathbf{I}_3\mathbf{J}_3}\tilde{\mathbf{H}}_{\mathbf{I}_1\mathbf{I}_2\mathbf{I}_3}\times\tilde{\mathbf{H}}_{\mathbf{J}_1\mathbf{J}_2\mathbf{J}_3}$, and we use the Einstein notation for the index $\mathbf{I}$. The equations of motion of this model can be consistently truncated to those of all ten-dimensional supergravities, namely, type IIA, type IIB, SO(32) type I, and SO(32) and $E_8 \times E_8$ heterotic supergravities; that is, this model includes all superstring backgrounds. The action (2.2) consists of a scalar curvature $\mathbf{R}$ of a metric $\mathbf{G}_{\mathbf{I}_1\mathbf{I}_2}$, a scalar field Φ, p-forms $\tilde{\mathbf{F}}_p$, and $\tilde{\mathbf{H}} = d\mathbf{B} - \boldsymbol{\omega}_3$, where $\mathbf{B}_{\mathbf{I}_1\mathbf{I}_2}$ is a two-form field, $\boldsymbol{\omega}_3 = \mathrm{tr}(\mathbf{A}\wedge d\mathbf{A} - \frac{2i}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A})$, and $\mathbf{A}$ is a $N\times N$ Hermitian gauge field, whose field strength is given by $\mathbf{F}$. $\tilde{\mathbf{F}}_p$ are defined by $\sum_{p=1}^{\infty}\tilde{\mathbf{F}}_p = e^{-\mathbf{B}_2}\wedge\sum_{k=1}^{\infty}\mathbf{F}_k$, where $\mathbf{F}_k$ are field strengths of $(k-1)$-form fields $\mathbf{A}_{k-1}$. For example, $\tilde{\mathbf{F}}_5 = \mathbf{F}_5 - \mathbf{B}_2\wedge\mathbf{F}_3 + \frac{1}{2}\mathbf{B}_2\wedge\mathbf{B}_2\wedge\mathbf{F}_1$. They are defined on a Riemannian string manifold, whose definition was given in Ref. [1]. The string manifold is constructed by patching open sets in string model space E, whose definition is summarized as follows. First, a global time $\bar{\tau}$ is defined canonically and uniquely on a super-Riemann surface $\bar{\boldsymbol{\Sigma}}$ by the real part of the integral of an Abelian differential uniquely defined on $\bar{\boldsymbol{\Sigma}}$ [8,9]. We restrict $\bar{\boldsymbol{\Sigma}}$ to a constant $\bar{\tau}$ line and obtain $\bar{\boldsymbol{\Sigma}}|_{\bar{\tau}}$. An embedding of $\bar{\boldsymbol{\Sigma}}|_{\bar{\tau}}$ in $\mathbb{R}^d$ represents a many-body state of superstrings in $\mathbb{R}^d$, and is parametrized by coordinates $(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau})$, where $\bar{\mathbf{E}}$ is a supervierbein on $\bar{\boldsymbol{\Sigma}}$ and $\mathbf{X}_{\hat{D}_T}(\bar{\tau})$ is a map from $\bar{\boldsymbol{\Sigma}}|_{\bar{\tau}}$ to $\mathbb{R}^d$. $^-$ represents a representative of the superdiffeomorphism and super-Weyl transformation on the world sheet. Giving a super-Riemann surface $\bar{\boldsymbol{\Sigma}}$ is equivalent to giving a supervierbein $\bar{\mathbf{E}}$ up to superdiffeomorphism and super-Weyl transformations. $\hat{D}_T$ represents all backgrounds except for the target metric, where T runs IIA, IIB, and I, which represent type IIA chart, type IIB chart, and type I chart, respectively. IIA Gliozzi-Scherk-Olive (GSO) projection is taken in the asymptotic region of type IIA chart, and the IIB GSO projection is taken in the asymptotic region of type IIB and type I charts, as in Ref. [1]. $X^{\mu}_{\hat{D}_T}(\bar{\tau}_s) = X^{\mu} + \bar{\theta}^{\alpha}\psi^{\mu}_{\alpha} + \frac{1}{2}\bar{\theta}^2 F^{\mu}$, where $\mu = 0, 1, \ldots d-1$, ψ^{μ}_{α} is a Majorana fermion and F^{μ} is an auxiliary field. We abbreviate $\hat{D}_T$ and $(\bar{\tau}_s)$ of $X^{\mu}_{\hat{D}_T}(\bar{\tau})$, $\psi^{\mu}_{\hat{D}_T\alpha}(\bar{\tau})$ and $F^{\mu}_{\hat{D}_T}(\bar{\tau})$. The string model space E is defined by the collection of string states by considering all $\bar{\boldsymbol{\Sigma}}$, all values of $\bar{\tau}$, and all $\mathbf{X}_{\hat{D}_T}(\bar{\tau})$. How near the two string states is defined by how near the values of $\bar{\tau}$ and $\mathbf{X}_{\hat{D}_T}(\bar{\tau})$. An ϵ-open neighborhood of $[\bar{\boldsymbol{\Sigma}}, \mathbf{X}_{\hat{D}_T s}(\bar{\tau}_s), \bar{\tau}_s]$ is defined by

$$U([\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T s}(\bar{\tau}_s), \bar{\tau}_s], \epsilon) := \left\{[\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau}] \,\middle|\, \sqrt{|\bar{\tau}-\bar{\tau}_s|^2 + \|\mathbf{X}_{\hat{D}_T}(\bar{\tau}) - \mathbf{X}_{\hat{D}_T s}(\bar{\tau}_s)\|^2} < \epsilon\right\}, \tag{2.3}$$

where $\bar{\mathbf{E}}$ is a discrete variable in the topology of string geometry. As a result, $d\bar{\mathbf{E}}$ cannot be a part of a basis that spans the cotangent space in Eq. (2.4), whereas fields are functionals of $\bar{\mathbf{E}}$ as in Eq. (2.5). The precise definition of the string topology was given in Sec. II in Ref. [1]. By this definition, two arbitrary string states on a connected super-Riemann surface in E are connected continuously. Thus, there is a one-to-one correspondence between a super-Riemann surface in $\mathbb{R}^d$ and a curve parametrized by $\bar{\tau}$ from $\bar{\tau} = -\infty$ to $\bar{\tau} = \infty$ on E. That is, curves that represent asymptotic processes on E reproduce the right moduli space of the super-Riemann surfaces in $\mathbb{R}^d$. Therefore, a string geometry theory possesses all-order information of superstring theory. Indeed, the path integral of perturbative superstrings on flat spacetime is derived from the string geometry theory, as in Refs. [1,3]. The consistency of the perturbation theory determines $d = 10$ (the critical dimension). The cotangent space is spanned by[1]

$$d\mathbf{X}^{d}_{\hat{D}_T} := d\bar{\tau}, \qquad d\mathbf{X}^{(\mu\bar{\sigma}\bar{\theta})}_{\hat{D}_T} := d\mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \tag{2.4}$$

where $\mu = 0, \ldots, d-1$. The summation over $(\bar{\sigma}, \bar{\theta})$ is defined by $\int d\bar{\sigma}d^2\bar{\theta}\,\hat{\mathbf{E}}(\bar{\sigma}, \bar{\tau}, \bar{\theta})$. $\hat{\mathbf{E}}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) := \frac{1}{\bar{n}}\bar{\mathbf{E}}(\bar{\sigma}, \bar{\tau}, \bar{\theta})$, where $\bar{n}$ is the lapse function of the two-dimensional metric [see Eq. (4.28)]. This summation is transformed as a scalar under $\bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}))$ and is invariant under a supersymmetry transformation $(\bar{\sigma}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta}))$. As a result, the action (2.2) is invariant under this $\mathcal{N} = (1,1)$ supersymmetry transformation. An explicit form of the line element is given by

[1]The integral over the coordinates in Eq. (2.2) are explicitly given by

$$\int \mathcal{D}\mathbf{E}\mathcal{D}\bar{\tau}\mathcal{D}\mathbf{X}_{\hat{D}_T} = \sum_{\mathbf{E}}\int d\mathbf{X}^{d}_{\hat{D}_T}\prod_{\mu\bar{\sigma}\bar{\theta}}d\mathbf{X}^{(\mu\bar{\sigma}\bar{\theta})}_{\hat{D}_T}.$$

$$ds^2(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau}) = G(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau})_{dd}(d\bar{\tau})^2 + 2d\bar{\tau}\int d\bar{\sigma}d^2\bar{\theta}\,\hat{\mathbf{E}}\sum_{\mu} G(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau})_{d\,(\mu\bar{\sigma}\bar{\theta})} d\mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta})$$
$$+ \int d\bar{\sigma}d^2\bar{\theta}\,\hat{\mathbf{E}}\int d\bar{\sigma}'d^2\bar{\theta}'\hat{\mathbf{E}}'\sum_{\mu,\mu'} G(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau})_{(\mu\bar{\sigma}\bar{\theta})\,(\mu'\bar{\sigma}'\bar{\theta}')} d\mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) d\mathbf{X}^{\mu'}_{\hat{D}_T}(\bar{\sigma}', \bar{\tau}, \bar{\theta}'). \quad (2.5)$$

The inverse metric $\mathbf{G}^{\mathbf{IJ}}(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau})$[2] is defined by $\mathbf{G}_{\mathbf{IJ}}\mathbf{G}^{\mathbf{JK}} = \mathbf{G}^{\mathbf{KJ}}\mathbf{G}_{\mathbf{JI}} = \delta_{\mathbf{I}}^{\mathbf{K}}$, where $\delta_d^d = 1$ and $\delta_{\mu\bar{\sigma}\bar{\theta}}^{\mu'\bar{\sigma}'\bar{\theta}'} = \delta_{\mu}^{\mu'}\delta_{\bar{\sigma}\bar{\theta}}^{\bar{\sigma}'\bar{\theta}'}$, where $\delta_{\bar{\sigma}\bar{\theta}}^{\bar{\sigma}'\bar{\theta}'} = \delta_{(\bar{\sigma}\bar{\theta})(\bar{\sigma}'\bar{\theta}')} = \frac{1}{\hat{\mathbf{E}}}\delta(\bar{\sigma}-\bar{\sigma}')\delta^2(\bar{\theta}-\bar{\theta}')$. The dimensions of string manifolds, which are infinite-dimensional manifolds, are formally given by the traces of the flat metrics, $\delta_M^M = D+1$, where $D \coloneqq \int d\bar{\sigma}d^2\bar{\theta}\,\hat{\mathbf{E}}\,\delta_{(\mu\bar{\sigma}\bar{\theta})}^{(\mu\bar{\sigma}\bar{\theta})}$. Thus, we treat D as a regularization parameter and will take $D \to \infty$ later.

III. SUPERSTRING BACKGROUND CONFIGURATIONS AND FLUCTUATIONS REPRESENTING STRINGS

In this paper, we consider only static configurations, including quantum fluctuations:

$$\begin{aligned}\partial_d\mathbf{G}_{\mathbf{MN}} &= 0,\\ \partial_d\mathbf{B}_{\mathbf{MN}} &= 0,\\ \partial_d\mathbf{\Phi} &= 0,\\ \partial_d\mathbf{A}_{k-1} &= 0.\end{aligned} \quad (3.1)$$

In this section, we set classical backgrounds including string backgrounds and consider fluctuations that represent strings around them. In order to simplify calculations, we consider the classical backgrounds up to the first-order fluctuations around the flat background. Here we fix the charts, where we choose $T = $ IIA, IIB, or I. $\mathbf{A} = 0$ on these charts. The Einstein equation of the action (2.2) is given by

$$\begin{aligned}&\bar{\mathbf{R}}_{\mathbf{MN}} - \frac{1}{4}\bar{\mathbf{H}}_{\mathbf{MAB}}\bar{\mathbf{H}}_{\mathbf{N}}^{\mathbf{AB}} + 2\bar{\nabla}_{\mathbf{M}}\bar{\nabla}_{\mathbf{N}}\bar{\mathbf{\Phi}} - \frac{1}{2}\bar{\mathbf{G}}_{\mathbf{MN}}\\ &\times\left(\bar{\mathbf{R}} - 4\bar{\nabla}_{\mathbf{I}}\bar{\mathbf{\Phi}}\bar{\nabla}^{\mathbf{I}}\bar{\mathbf{\Phi}} + 4\bar{\nabla}_{\mathbf{I}}\bar{\nabla}^{\mathbf{I}}\bar{\mathbf{\Phi}} - \frac{1}{2}|\bar{\mathbf{H}}|^2\right)\\ &-\frac{1}{2}e^{2\bar{\Phi}}\sum_{p=1}^{9}\left[\frac{1}{(p-1)!}\bar{\tilde{\mathbf{F}}}_{\mathbf{ML}_1\cdots\mathbf{L}_{p-1}}\bar{\tilde{\mathbf{F}}}_{\mathbf{N}}^{\mathbf{L}_1\cdots\mathbf{L}_{p-1}}\right.\\ &\left.-\frac{1}{2}\bar{\mathbf{G}}_{\mathbf{MN}}|\bar{\tilde{\mathbf{F}}}_p|^2\right] = 0,\end{aligned} \quad (3.2)$$

where $\bar{\mathbf{R}}$, $\bar{\mathbf{R}}_{\mathbf{MN}}$, $\bar{\mathbf{R}}^{\mathbf{M}}_{\mathbf{NPQ}}$, and $\bar{\nabla}_{\mathbf{M}}$ denote the Ricci scalar, Ricci tensor, curvature tensor, and covariant derivative constructed from the metric $\bar{\mathbf{G}}_{\mathbf{MN}}$. We consider a perturbation with respect to the metric $\bar{\mathbf{G}}_{\mathbf{MN}}$:

$$\bar{\mathbf{G}}_{\mathbf{MN}} = \hat{\mathbf{G}}_{\mathbf{MN}} + \bar{\mathbf{h}}_{\mathbf{MN}}, \quad (3.3)$$

where $\bar{\mathbf{h}}_{\mathbf{MN}}$ denotes a fluctuation around the zeroth-order background $\hat{\mathbf{G}}_{\mathbf{MN}}$. We raise and lower the indices by $\hat{\mathbf{G}}_{\mathbf{MN}}$ in the following. We also consider a perturbation with respect to the two-form $\bar{\mathbf{B}}_{\mathbf{MN}}$, $(k-1)$-form $\bar{\mathbf{A}}_{k-1}$, and the scalar $\bar{\mathbf{\Phi}}$ around the flat background.

First, we generalize the harmonic gauge when a dilaton couples. If we define $\bar{\boldsymbol{\psi}}_{\mathbf{MN}}$ as

$$\bar{\boldsymbol{\psi}}_{\mathbf{MN}} = \bar{\mathbf{h}}_{\mathbf{MN}} - \frac{1}{2}\hat{\mathbf{G}}^{\mathbf{IJ}}\bar{\mathbf{h}}_{\mathbf{IJ}}\hat{\mathbf{G}}_{\mathbf{MN}} + \Lambda\hat{\mathbf{G}}_{\mathbf{MN}}\bar{\mathbf{\Phi}}, \quad (3.4)$$

where Λ is a constant, the Einstein equation (3.2) is written as

$$\begin{aligned}&\hat{\mathbf{R}}_{\mathbf{MN}} - \frac{1}{2}\hat{\mathbf{G}}_{\mathbf{MN}}\hat{\mathbf{R}} + \frac{1}{2}\Big(-\hat{\nabla}_{\mathbf{I}}\hat{\nabla}^{\mathbf{I}}\bar{\boldsymbol{\psi}}_{\mathbf{MN}} + \hat{\mathbf{R}}_{\mathbf{MA}}\bar{\boldsymbol{\psi}}^{\mathbf{A}}_{\mathbf{N}} + \hat{\mathbf{R}}_{\mathbf{NA}}\bar{\boldsymbol{\psi}}^{\mathbf{A}}_{\mathbf{M}}\\ &- 2\hat{\mathbf{R}}_{\mathbf{MANB}}\bar{\boldsymbol{\psi}}^{\mathbf{AB}} + \hat{\nabla}_{\mathbf{M}}\hat{\nabla}_{\mathbf{A}}\bar{\boldsymbol{\psi}}^{\mathbf{A}}_{\mathbf{N}} + \hat{\nabla}_{\mathbf{N}}\hat{\nabla}_{\mathbf{A}}\bar{\boldsymbol{\psi}}^{\mathbf{A}}_{\mathbf{M}}\\ &- \hat{\nabla}^{\mathbf{I}}\hat{\nabla}^{\mathbf{J}}\bar{\boldsymbol{\psi}}_{\mathbf{IJ}}\hat{\mathbf{G}}_{\mathbf{MN}} + \hat{\mathbf{R}}^{\mathbf{IJ}}\bar{\boldsymbol{\psi}}_{\mathbf{IJ}}\hat{\mathbf{G}}_{\mathbf{MN}} - \hat{\mathbf{R}}\,\bar{\boldsymbol{\psi}}_{\mathbf{MN}}\Big)\\ &+ (2-\Lambda)\hat{\nabla}_{\mathbf{M}}\hat{\nabla}_{\mathbf{N}}\bar{\mathbf{\Phi}} - (2-\Lambda)\hat{\mathbf{G}}_{\mathbf{MN}}\hat{\nabla}_{\mathbf{I}}\hat{\nabla}^{\mathbf{I}}\bar{\mathbf{\Phi}} = 0,\end{aligned} \quad (3.5)$$

up to the first order in the fields $\bar{\mathbf{h}}_{\mathbf{IJ}}$, $\bar{\mathbf{B}}_{\mathbf{IJ}}$, $\bar{\mathbf{\Phi}}$, and $\bar{\mathbf{A}}_{k-1}$. $\hat{\mathbf{R}}$, $\hat{\mathbf{R}}_{\mathbf{MN}}$, $\hat{\mathbf{R}}^{\mathbf{M}}{}_{\mathbf{NPQ}}$, and $\hat{\nabla}_{\mathbf{M}}$ denote the Ricci scalar, Ricci tensor, curvature tensor, and covariant derivative constructed from the metric $\hat{\mathbf{G}}_{\mathbf{MN}}$. We set $\Lambda = 2$ so that the Einstein equation includes only $\bar{\boldsymbol{\psi}}_{\mathbf{MN}}$. $\bar{\mathbf{h}}_{\mathbf{MN}}$ is inversely expressed as

$$\bar{\mathbf{h}}_{\mathbf{MN}} = \bar{\boldsymbol{\psi}}_{\mathbf{MN}} + \frac{1}{D-1}\left(-\hat{\mathbf{G}}^{\mathbf{PQ}}\bar{\boldsymbol{\psi}}_{\mathbf{PQ}} + 4\bar{\mathbf{\Phi}}\right)\hat{\mathbf{G}}_{\mathbf{MN}}. \quad (3.6)$$

We impose a generalization of the harmonic gauge,

$$\hat{\nabla}^{\mathbf{M}}\bar{\boldsymbol{\psi}}_{\mathbf{MN}} = 0, \quad (3.7)$$

[2]Similarly, the fields $\mathbf{G}_{\mathbf{IJ}}$, Φ, $\mathbf{B}_{\mathbf{L}_1\mathbf{L}_2}$, and $\mathbf{A}_{\mathbf{L}_1\cdots\mathbf{L}_{p-1}}$ are functionals of the coordinates $\bar{\mathbf{E}}$, $\mathbf{X}_{\hat{D}_T}(\bar{\tau})$, and $\bar{\tau}$.

which reduces to the ordinary harmonic gauge if the dilaton is zero. Then, the Einstein equation (3.5) becomes

$$\hat{\mathbf{R}}_{\mathbf{MN}} - \frac{1}{2}\hat{\mathbf{G}}_{\mathbf{MN}}\hat{\mathbf{R}} + \frac{1}{2}(-\hat{\boldsymbol{\nabla}}_{\mathbf{I}}\hat{\boldsymbol{\nabla}}^{\mathbf{I}}\overline{\boldsymbol{\psi}}_{\mathbf{MN}} + \hat{\mathbf{R}}_{\mathbf{MA}}\overline{\boldsymbol{\psi}}^{\mathbf{A}}_{\mathbf{N}} + \hat{\mathbf{R}}_{\mathbf{NA}}\overline{\boldsymbol{\psi}}^{\mathbf{A}}_{\mathbf{M}} - 2\hat{\mathbf{R}}_{\mathbf{MANB}}\overline{\boldsymbol{\psi}}^{\mathbf{AB}} + \hat{\mathbf{R}}^{\mathbf{IJ}}\overline{\boldsymbol{\psi}}_{\mathbf{IJ}}\hat{\mathbf{G}}_{\mathbf{MN}} - \hat{\mathbf{R}}\,\overline{\boldsymbol{\psi}}_{\mathbf{MN}}) = 0. \tag{3.8}$$

Next, we set the zeroth-order background $\hat{\mathbf{G}}_{\mathbf{MN}}$ as a flat background:

$$\hat{\mathbf{G}}_{\mathbf{MN}} = a_{\mathbf{M}}\eta_{\mathbf{MN}}, \tag{3.9}$$

where $a_d = 1$ and $a_{(\mu\bar{\sigma}\bar{\theta})} = \frac{\bar{e}^3(\bar{\sigma})}{\sqrt{\bar{h}(\bar{\sigma})}}$. Then, the gauge-fixing condition (3.7) becomes

$$\int d\bar{\sigma}\, d^2\bar{\theta}\, \hat{\boldsymbol{E}}\, \partial^{(\mu\bar{\sigma}\bar{\theta})}\overline{\boldsymbol{\psi}}_{(\mu\bar{\sigma}\bar{\theta})\mathbf{M}} = 0, \tag{3.10}$$

the Einstein equation (3.8) becomes a Laplace equation,

$$\int d\bar{\sigma}\, d^2\bar{\theta}\, \hat{\boldsymbol{E}}\, \partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\overline{\boldsymbol{\psi}}_{\mathbf{MN}} = 0, \tag{3.11}$$

and the components of Eq. (3.6) read

$$\begin{aligned}
\bar{\mathbf{h}}_{dd} &= \frac{D-2}{D-1}\overline{\boldsymbol{\psi}}_{dd} + \frac{1}{D-1}\int d\bar{\sigma}''\, d^2\bar{\theta}''\hat{\boldsymbol{E}}''\overline{\boldsymbol{\psi}}^{(\mu''\bar{\sigma}''\bar{\theta}'')}_{(\mu''\bar{\sigma}''\bar{\theta}'')} - \frac{4}{D-1}\overline{\boldsymbol{\Phi}},\\
\bar{\mathbf{h}}_{d(\mu\bar{\sigma}\bar{\theta})} &= \overline{\boldsymbol{\psi}}_{d(\mu\bar{\sigma}\bar{\theta})},\\
\bar{\mathbf{h}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} &= \overline{\boldsymbol{\psi}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} + \frac{\bar{e}^3}{\sqrt{\bar{h}}}\delta_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')}\left(\frac{1}{D-1}\overline{\boldsymbol{\psi}}_{dd} - \frac{1}{D-1}\int d\bar{\sigma}''\, d^2\bar{\theta}''\hat{\boldsymbol{E}}''\overline{\boldsymbol{\psi}}^{(\mu''\bar{\sigma}''\bar{\theta}'')}_{(\mu''\bar{\sigma}''\bar{\theta}'')} + \frac{4}{D-1}\overline{\boldsymbol{\Phi}}\right).
\end{aligned} \tag{3.12}$$

Next, the equation of motion of the scalar of the action (2.2),

$$\bar{\mathbf{R}} - 4\bar{\boldsymbol{\nabla}}_{\mathbf{M}}\overline{\boldsymbol{\Phi}}\partial^{\mathbf{M}}\overline{\boldsymbol{\Phi}} + 4\bar{\boldsymbol{\nabla}}_{\mathbf{M}}\bar{\boldsymbol{\nabla}}^{\mathbf{M}}\overline{\boldsymbol{\Phi}} - \frac{1}{2}|\bar{\mathbf{H}}|^2 = 0, \tag{3.13}$$

is written as

$$\hat{\mathbf{R}} + \hat{\nabla}^{\mathbf{M}}\hat{\nabla}^{\mathbf{N}}\bar{\mathbf{h}}_{\mathbf{MN}} - \hat{\nabla}^{\mathbf{M}}\hat{\nabla}_{\mathbf{M}}\bar{\mathbf{h}}^{\mathbf{N}}_{\mathbf{N}} + 4\hat{\mathbf{G}}^{\mathbf{MN}}\hat{\nabla}_{\mathbf{M}}\hat{\nabla}_{\mathbf{N}}\overline{\boldsymbol{\Phi}} = 0, \tag{3.14}$$

up to the first order in the fields $\bar{\mathbf{h}}_{\mathbf{IJ}}$, $\bar{\mathbf{B}}_{\mathbf{IJ}}$, $\bar{\mathbf{A}}_{k-1}$, and $\overline{\boldsymbol{\Phi}}$. Furthermore, this can be written as

$$\int d\bar{\sigma}\, d^2\bar{\theta}\, \hat{\boldsymbol{E}}\, \partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\overline{\boldsymbol{\Phi}} + \frac{1}{4}\int d\bar{\sigma}\, d^2\bar{\theta}\, \hat{\boldsymbol{E}}\, \partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\overline{\boldsymbol{\psi}}_{dd} - \frac{1}{4}\int d\bar{\sigma}\, d^2\bar{\theta}\, \hat{\boldsymbol{E}}\, \partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\int d\bar{\sigma}'\, d^2\bar{\theta}'\hat{\boldsymbol{E}}'\overline{\boldsymbol{\psi}}^{(\mu'\bar{\sigma}'\bar{\theta}')}_{(\mu'\bar{\sigma}'\bar{\theta}')} = 0 \tag{3.15}$$

around the flat zeroth-order background (3.9) under the static condition (3.1) in the generalized harmonic gauge (3.7). This becomes a Laplace equation,

$$\int d\bar{\sigma}\, d^2\bar{\theta}\, \hat{\boldsymbol{E}}\, \partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\overline{\boldsymbol{\Phi}} = 0, \tag{3.16}$$

if the metric satisfies the Einstein equation (3.11).

On the other hand, the equations of motion of the B field and the $(k-1)$-form fields,

$$\bar{\boldsymbol{\nabla}}_{\mathbf{M}}(e^{-2\bar{\Phi}}\bar{\mathbf{H}}^{\mathbf{MNP}}) + \sum_{p=3}^{9}\sum_{n=0}^{[\frac{p-3}{2}]}\frac{1}{2^{n+1}\cdot(p-2)!}\bar{\mathbf{F}}_{\mathbf{I}_1\cdots\mathbf{I}_{p-2-2n}}\bar{\mathbf{B}}_{\mathbf{J}_1\mathbf{K}_1}\cdots\bar{\mathbf{B}}_{\mathbf{J}_n\mathbf{K}_n}\bar{\tilde{\mathbf{F}}}^{\mathbf{I}_1\cdots\mathbf{I}_{p-2-2n}\mathbf{J}_1\mathbf{K}_1\cdots\mathbf{J}_n\mathbf{K}_n\mathbf{L}_1\mathbf{L}_2} = 0, \tag{3.17}$$

$$\bar{\boldsymbol{\nabla}}_{\mathbf{I}}\bar{\tilde{\mathbf{F}}}^{\mathbf{I}\mathbf{L}_1\cdots\mathbf{L}_{p-1}} + \left(\frac{1}{2}\right)^n\hat{\boldsymbol{\nabla}}_{\mathbf{I}}\left[\bar{\mathbf{B}}_{\mathbf{J}_1\mathbf{K}_1}\cdots\bar{\mathbf{B}}_{\mathbf{J}_n\mathbf{K}_n}\bar{\tilde{\mathbf{F}}}^{\mathbf{J}_1\mathbf{K}_1\cdots\mathbf{J}_n\mathbf{K}_n\mathbf{I}\mathbf{L}_1\cdots\mathbf{L}_{p-1}}\right] = 0, \tag{3.18}$$

are written as

$$\hat{\boldsymbol{\nabla}}_{\mathbf{M}}\bar{\mathbf{H}}^{\mathbf{MNP}} = 0, \tag{3.19}$$

$$\hat{\nabla}_{\mathbf{I}}\bar{\tilde{\mathbf{F}}}^{\mathbf{I}\mathbf{L}_1\cdots\mathbf{L}_{p-1}} = 0, \tag{3.20}$$

up to the first order in the fields $\bar{\mathbf{h}}_{\mathbf{IJ}}$, $\bar{\mathbf{B}}_{\mathbf{IJ}}$, $\bar{\mathbf{A}}_{k-1}$, and $\overline{\mathbf{\Phi}}$. Furthermore, Eq. (3.19) becomes a Laplace equation,

$$\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\mathbf{B}}_{\mathbf{MN}} = 0, \tag{3.21}$$

around the flat zeroth-order background (3.9) under the static condition (3.1) in Lorentz gauge,

$$\hat{\nabla}_{\mathbf{M}}\bar{\mathbf{B}}^{\mathbf{MN}} = 0, \tag{3.22}$$

which is equivalent to

$$\partial_{(\mu\bar{\sigma}\bar{\theta})}\bar{\mathbf{B}}^{(\mu\bar{\sigma}\bar{\theta})\mathbf{N}} = 0. \tag{3.23}$$

It is known that it is too difficult to describe the action of the perturbative strings on the Ramond-Ramond (RR) backgrounds in the Neveu-Schwarz-Ramond (NS-R) formalism. Because string geometry theory is formulated in the NS-R formalism, we should derive the path integrals of the perturbative strings only on the NS-NS backgrounds. Thus, we set RR fields to zero.

We consider classical solutions corresponding to the type II superstring background configurations:

$$\bar{\boldsymbol{\psi}}_{dd} = 0, \tag{3.24}$$

$$\bar{\boldsymbol{\psi}}_{d(\mu\bar{\sigma}\bar{\theta})} = 0, \tag{3.25}$$

$$\bar{\boldsymbol{h}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} = \frac{\bar{e}^3}{\sqrt{\bar{h}}}g_{\mu\nu}(\mathbf{X}_{\hat{D}_T}(\bar{\sigma},\bar{\theta}))\delta_{\bar{\sigma}\bar{\sigma}'}\delta_{\bar{\theta}\bar{\theta}'}, \tag{3.26}$$

$$\bar{\mathbf{B}}_{d(\mu\bar{\sigma}\bar{\theta})} = 0, \tag{3.27}$$

$$\bar{\mathbf{B}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} = \frac{\bar{e}^3}{\sqrt{\bar{h}}}B_{\mu\nu}(\mathbf{X}_{\hat{D}_T}(\bar{\sigma},\bar{\theta}))\delta_{\bar{\sigma}\bar{\sigma}'}\delta_{\bar{\theta}\bar{\theta}'}, \tag{3.28}$$

$$\overline{\mathbf{\Phi}} = \int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\Phi(\mathbf{X}_{\hat{D}_T}(\bar{\sigma},\bar{\theta})), \tag{3.29}$$

$$\bar{\mathbf{A}}_{k-1} = 0, \tag{3.30}$$

$$\bar{\mathbf{A}} = 0, \tag{3.31}$$

where $g_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $\Phi(x)$ satisfy Laplace equations,

$$\begin{aligned}\partial_\rho\partial^\rho g_{\mu\nu}(x) &= 0,\\ \partial_\rho\partial^\rho B_{\mu\nu}(x) &= 0,\\ \partial_\rho\partial^\rho \Phi(x) &= 0,\end{aligned} \tag{3.32}$$

and gauge-fixing conditions,

$$\begin{aligned}\partial^\mu \psi_{\mu\nu}(x) &= 0,\\ \partial^\mu B_{\mu\nu}(x) &= 0,\end{aligned} \tag{3.33}$$

where

$$\psi_{\mu\nu} = g_{\mu\nu} - \frac{1}{2}\delta^{\alpha\beta}g_{\alpha\beta}\delta_{\mu\nu} + 2\delta_{\mu\nu}\Phi, \tag{3.34}$$

which imply Eqs. (3.10), (3.11), (3.16), (3.21), and (3.23). Indeed, these are equivalent to

$$\bar{\mathbf{G}}_{dd} = -1, \tag{3.35}$$

$$\bar{\mathbf{G}}_{d(\mu\bar{\sigma}\bar{\theta})} = 0, \tag{3.36}$$

$$\bar{\mathbf{G}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} = \frac{\bar{e}^3}{\sqrt{\bar{h}}}G_{\mu\nu}(\mathbf{X}_{\hat{D}_T}(\bar{\sigma},\bar{\theta}))\delta_{\bar{\sigma}\bar{\sigma}'}\delta_{\bar{\theta}\bar{\theta}'}, \tag{3.37}$$

$$\bar{\mathbf{B}}_{d(\mu\bar{\sigma}\bar{\theta})} = 0, \tag{3.38}$$

$$\bar{\mathbf{B}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} = \frac{\bar{e}^3}{\sqrt{\bar{h}}}B_{\mu\nu}(\mathbf{X}_{\hat{D}_T}(\bar{\sigma},\bar{\theta}))\delta_{\bar{\sigma}\bar{\sigma}'}\delta_{\bar{\theta}\bar{\theta}'}, \tag{3.39}$$

$$\overline{\mathbf{\Phi}} = \int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\Phi(\mathbf{X}_{\hat{D}_T}(\bar{\sigma},\bar{\theta})), \tag{3.40}$$

$$\bar{\mathbf{A}}_{k-1} = 0, \tag{3.41}$$

$$\bar{\mathbf{A}} = 0, \tag{3.42}$$

where

$$G_{\mu\nu}(\mathbf{X}_{\hat{D}_T}) = \delta_{\mu\nu} + g_{\mu\nu}(\mathbf{X}_{\hat{D}_T}). \tag{3.43}$$

These are the string background configurations themselves [4,5]. Equation (3.32) implies that $\bar{\mathbf{G}}_{\mathbf{MN}}$, $\bar{\mathbf{B}}_{\mathbf{MN}}$, $\bar{\mathbf{A}}_{k-1}$, and $\overline{\mathbf{\Phi}}$ satisfy their equations of motion in string geometry theory[3] [Eqs. (3.11), (3.16), and (3.21)], and $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ also satisfy their equations of motion in the supergravity. Therefore, these string background configurations in string geometry theory represent perturbative string vacua parametrized by the on-shell fields in the supergravity as string backgrounds.

Next, we consider fluctuations around these vacua. The scalar fluctuation $\boldsymbol{\psi}_{dd}$ represents the degrees of freedom of perturbative strings in the case of the flat background, as in Refs. [1–3]. Thus, we also consider the scalar fluctuation $\boldsymbol{\psi}_{dd}$ around the perturbative vacua. We set the classical part of $\boldsymbol{\psi}_{dd}$ as

[3]Under Eq. (3.16), Eq. (3.11) is equivalent to $\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{h}}_{\mathbf{MN}} = 0$, due to Eq. (3.6).

$$\bar{\boldsymbol{\psi}}_{dd}(\mathbf{X}_{\hat{D}_T}) = \int \mathcal{D}\mathbf{X}'_{\hat{D}_T} G(\mathbf{X}_{\hat{D}_T}; \mathbf{X}'_{\hat{D}_T}) \omega(\mathbf{X}'_{\hat{D}_T}), \tag{3.44}$$

where we will choose a particular function $\omega(\mathbf{X}_{\hat{D}_T})$ later and $G(\mathbf{X}_{\hat{D}_T}; \mathbf{X}'_{\hat{D}_T})$ is a Green function on the flat superstring manifold given by

$$G(\mathbf{X}_{\hat{D}_T}; \mathbf{X}'_{\hat{D}_T}) = \mathcal{N}\left[\int d\bar{\sigma}' \, d^2\bar{\theta}' \frac{\bar{e}'^2}{\bar{\mathbf{E}}'}\left(\mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}', \bar{\theta}') - \mathbf{X}'^{\mu}{}_{\hat{D}_T}(\bar{\sigma}', \bar{\theta}')\right)^2\right]^{\frac{2-D}{2}}, \tag{3.45}$$

which satisfies

$$\int d\bar{\sigma} d^2\bar{\theta} \, \bar{\mathbf{E}} \frac{1}{\bar{e}} \frac{\partial}{\partial \mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\theta})} \frac{1}{\bar{e}} \frac{\partial}{\partial \mathbf{X}_{\hat{D}_T \mu}(\bar{\sigma}, \bar{\theta})} G(\mathbf{X}_{\hat{D}_T}; \mathbf{X}'_{\hat{D}_T}) = \delta(\mathbf{X}_{\hat{D}_T} - \mathbf{X}'_{\hat{D}_T}), \tag{3.46}$$

where $\mathcal{N}$ is a normalizing constant. A derivation is given in the Appendix. As a result, $\bar{\boldsymbol{\psi}}_{dd}$ is not on-shell but satisfies

$$\int d\bar{\sigma} d^2\bar{\theta} \, \bar{\mathbf{E}} \frac{1}{\bar{e}} \frac{\partial}{\partial \mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\theta})} \frac{1}{\bar{e}} \frac{\partial}{\partial \mathbf{X}_{\hat{D}_T \mu}(\bar{\sigma}, \bar{\theta})} \bar{\boldsymbol{\psi}}_{dd}(\mathbf{X}_{\hat{D}_T}) = \omega(\mathbf{X}_{\hat{D}_T}). \tag{3.47}$$

Furthermore, we consider the quantum part of $\boldsymbol{\psi}_{dd}$,

$$\tilde{\boldsymbol{\psi}}_{dd} = \frac{D-1}{D-2}\tilde{\phi}, \tag{3.48}$$

where $\frac{D-1}{D-2}$ is introduced for later convenience. In total,

$$\mathbf{G_{MN}} = \hat{\mathbf{G}}_{\mathbf{MN}} + \bar{\mathbf{h}}_{\mathbf{MN}} + \tilde{\mathbf{G}}_{\mathbf{MN}}, \tag{3.49}$$

where $\hat{\mathbf{G}}_{\mathbf{MN}}$ is given by Eq. (3.9), $\bar{\mathbf{h}}_{\mathbf{MN}}$ is given by Eq. (3.12) with Eqs. (3.44), (3.25), (3.26), and (3.29), and $\tilde{\mathbf{G}}_{\mathbf{MN}}$ is given by

$$\tilde{\mathbf{G}}_{dd} = \tilde{\phi}, \qquad \tilde{\mathbf{G}}_{d(\mu\bar{\sigma}\bar{\theta})} = 0,$$
$$\tilde{\mathbf{G}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} = \frac{1}{D-2}\frac{\bar{e}^3}{\sqrt{\bar{h}}}\tilde{\phi}\delta_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')}. \tag{3.50}$$

IV. DERIVING THE PATH INTEGRALS OF THE PERTURBATIVE SUPERSTRINGS ON CURVED BACKGROUNDS

In this section, we derive the path integrals of the perturbative superstrings up to any order from the tree-level two-point correlation functions of the quantum scalar fluctuations of the metric. In order to obtain a propagator, we add a gauge-fixing term corresponding to Eq. (3.7) into the action (2.2) and obtain

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T}\sqrt{-\mathbf{G}}\left[e^{-2\Phi}\left(\mathbf{R} + 4\nabla_{\mathbf{I}}\mathbf{\Phi}\nabla^{\mathbf{I}}\mathbf{\Phi} - \frac{1}{2}|\mathbf{H}|^2\right) - \frac{1}{2}\sum_{p=1}^{\infty}|\tilde{\mathbf{F}}_p|^2 - \frac{1}{2}\left\{\bar{\nabla}^{\mathbf{N}}\left(\tilde{\mathbf{G}}_{\mathbf{MN}} - \frac{1}{2}\bar{\mathbf{G}}^{\mathbf{IJ}}\tilde{\mathbf{G}}_{\mathbf{IJ}}\bar{\mathbf{G}}_{\mathbf{MN}} + 2\bar{\mathbf{G}}_{\mathbf{MN}}\bar{\mathbf{\Phi}}\right)\right\}^2\right], \tag{4.1}$$

where we abbreviate the Faddeev-Popov ghost term because it does not contribute to the tree-level two-point correlation functions of the metrics. By substituting Eqs. (3.49), (3.27)–(3.30) [which do not necessarily satisfy the equations of motion (3.32)] into Eq. (4.1), this is expressed as

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T}\left(c_0 + c_1\tilde{\phi} + \tilde{\phi}c_2\tilde{\phi} + \tilde{\phi}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\int d\bar{\sigma}'\, d^2\bar{\theta}'\hat{\mathbf{E}}' c^{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')}\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial_{(\mu'\bar{\sigma}'\bar{\theta}')}\tilde{\phi}\right), \tag{4.2}$$

where

$$c_0 = -\frac{1}{D-1}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\psi}}_{dd} - \frac{4D}{D-1}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\mathbf{\Phi}} + \frac{1}{D-1}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\int d\bar{\sigma}'\, d^2\bar{\theta}'\hat{\mathbf{E}}'\bar{\boldsymbol{\psi}}^{(\mu'\bar{\sigma}'\bar{\theta}')}_{(\mu'\bar{\sigma}'\bar{\theta}')}, \tag{4.3a}$$

$$c_1 = \frac{1}{2}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\psi}}_{dd} + \frac{1}{2(D-2)}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\int d\bar{\sigma}'\, d^2\bar{\theta}'\hat{\mathbf{E}}'\bar{\boldsymbol{\psi}}^{(\mu'\bar{\sigma}'\bar{\theta}')}_{(\mu'\bar{\sigma}'\bar{\theta}')}, \tag{4.3b}$$

$$c_2 = \frac{1}{4}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\psi}}_{dd} - \frac{1}{4(D-2)^2}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\int d\bar{\sigma}'\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\bar{\boldsymbol{\psi}}^{(\mu'\bar{\sigma}'\bar{\sigma}')}_{(\mu'\bar{\sigma}'\bar{\sigma}')}, \tag{4.3c}$$

$$c^{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} = \left[\frac{D-1}{4(D-2)} + \frac{1}{2}\bar{\boldsymbol{\psi}}_{dd} + \frac{1}{2(D-2)}\int d\bar{\sigma}'' d^2\bar{\theta}''\hat{\mathbf{E}}''\bar{\boldsymbol{\psi}}^{(\mu''\bar{\sigma}''\bar{\theta}'')}_{(\mu''\bar{\sigma}''\bar{\theta}'')} - \frac{2}{D-2}\bar{\boldsymbol{\Phi}}\right]\delta^{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} - \frac{D-1}{4(D-2)}\bar{\boldsymbol{\psi}}^{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')}, \tag{4.3d}$$

up to the first order in the classical fields and the second order in $\tilde{\phi}$. When $D \to \infty$, Eq. (4.2) is

$$\begin{aligned} S = \int \mathcal{D}\bar{\tau}\,\mathcal{D}\mathbf{E}\,\mathcal{D}\mathbf{X}_{\hat{D}_T}\Bigg[&-4\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\Phi}} + \frac{1}{2}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\psi}}_{dd}\tilde{\phi} \\ &+ \tilde{\phi}\frac{1}{4}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\psi}}_{dd}\tilde{\phi} + \tilde{\phi}\left(\frac{1}{4} + \frac{1}{2}\bar{\boldsymbol{\psi}}_{dd}\right)\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\tilde{\phi} \\ &- \frac{1}{4}\tilde{\phi}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\int d\bar{\sigma}'\, d^2\bar{\theta}'\hat{\mathbf{E}}'\bar{\boldsymbol{\psi}}^{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')}\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial_{(\mu'\bar{\sigma}'\bar{\theta}')}\tilde{\phi}\Bigg]. \end{aligned} \tag{4.4}$$

By shifting the field $\tilde{\phi}$ as $\tilde{\phi} = \tilde{\phi}' - \frac{1}{2}$, the first-order term in $\tilde{\phi}'$ vanishes as

$$\begin{aligned} S = \int \mathcal{D}\bar{\tau}\,\mathcal{D}\mathbf{E}\,\mathcal{D}\mathbf{X}_{\hat{D}_T}\Bigg[&\tilde{\phi}'\frac{1}{4}\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\psi}}_{dd}\tilde{\phi}' + \tilde{\phi}'\left(\frac{1}{4} + \frac{1}{2}\bar{\boldsymbol{\psi}}_{dd} + \frac{1}{8}\hat{\mathbf{G}}^{\mathbf{IJ}}\bar{\mathbf{h}}_{\mathbf{IJ}} - \frac{1}{2}\bar{\boldsymbol{\Phi}}\right)\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\tilde{\phi}' \\ &- \frac{1}{4}\tilde{\phi}'\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\int d\bar{\sigma}'\, d^2\bar{\theta}'\hat{\mathbf{E}}'\bar{\boldsymbol{h}}^{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')}\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial_{(\mu'\bar{\sigma}'\bar{\theta}')}\tilde{\phi}'\Bigg], \end{aligned} \tag{4.5}$$

where surface terms are dropped and the gauge-fixing condition in Eq. (3.33) and the relation (3.4) are applied. By normalizing the leading part of the kinetic term as $\tilde{\phi}' = -2(1 - \bar{\boldsymbol{\psi}}_{dd} - \frac{1}{4}\hat{\mathbf{G}}^{\mathbf{IJ}}\bar{\mathbf{h}}_{\mathbf{IJ}} + \bar{\boldsymbol{\Phi}})\tilde{\phi}''$,[4] we have

$$\begin{aligned} S = \int \mathcal{D}\bar{\tau}\mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T}\Bigg[&\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\bar{\boldsymbol{\psi}}_{dd}(\tilde{\phi}'')^2 + \tilde{\phi}''\int d\bar{\sigma} d^2\bar{\theta}\,\hat{\mathbf{E}}\,\partial_{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu\bar{\sigma}\bar{\theta})}\tilde{\phi}'' \\ &- \tilde{\phi}''\int d\bar{\sigma}\, d^2\bar{\theta}\,\hat{\mathbf{E}}\int d\bar{\sigma}'\, d^2\bar{\theta}'\hat{\mathbf{E}}'\,\bar{\boldsymbol{h}}_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')}\partial^{(\mu\bar{\sigma}\bar{\theta})}\partial^{(\mu'\bar{\sigma}'\bar{\theta}')}\tilde{\phi}''\Bigg]. \end{aligned} \tag{4.6}$$

In the following, we consider only the case that the quantum fluctuation is local with respect to the indices $(\bar{\sigma}, \bar{\theta})$ as

$$\tilde{\phi}'' = \int d\bar{\sigma}\, d^2\bar{\theta} f(\mathbf{X}_{\hat{D}_T}(\bar{\sigma}, \bar{\theta})), \tag{4.7}$$

and obtain the component representation of Eq. (4.6). Under the superdiffeomorphism transformation of $\bar{\sigma}$ and $\bar{\theta}$, $\int d\bar{\sigma}\,\bar{e}$ and $\int d\bar{\sigma} d^2\bar{\theta}\,\hat{\mathbf{E}}$ are invariant; then, $\frac{1}{\bar{e}}\delta(\bar{\sigma} - \bar{\sigma}')$ and $\frac{1}{\hat{\mathbf{E}}}\delta(\bar{\sigma} - \bar{\sigma}')\delta^2(\bar{\theta} - \bar{\theta}')$ are scalars, and thus $\frac{\bar{e}}{\hat{\mathbf{E}}}\delta^2(\bar{\theta} - \bar{\theta}')$ is a scalar. Hence, we have

[4] This field redefinition is local with respect to fields in string geometry theory, because the fields are functionals of the coordinates, $\mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta})$.

$$
\begin{aligned}
\frac{\partial}{\partial X^\mu(\bar\sigma)} &= \int d\sigma' d^2\bar\theta' \hat{\boldsymbol{E}}(\bar\sigma',\bar\theta') \frac{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma',\bar\theta')}{\partial X^\mu(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma',\bar\theta')} \\
&= \int d\sigma' d^2\bar\theta' \hat{\boldsymbol{E}}(\bar\sigma',\bar\theta') \frac{1}{\bar e} \delta^\nu_\mu \delta(\bar\sigma-\bar\sigma') \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma',\bar\theta')} \\
&= \int d^2\bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)}, \\
\frac{\partial}{\partial X^\mu(\bar\sigma)} \frac{\partial}{\partial X^\nu(\bar\sigma)} \tilde\phi'' &= \int d^2\bar\theta' \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta')}{\bar e(\bar\sigma)} \int d^2\bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta')} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'' \\
&= \int d^2\bar\theta' \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta')}{\bar e(\bar\sigma)} \int d^2\bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\bar e(\bar\sigma)}{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)} \delta(\bar\theta-\bar\theta') \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'' \\
&= \int d^2\bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'',
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \psi^\mu(\bar\sigma)} &= \int d\sigma' d^2\bar\theta' \hat{\boldsymbol{E}}(\bar\sigma',\bar\theta') \frac{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma',\bar\theta')}{\partial \psi^\mu(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma',\bar\theta')} \\
&= \int d\sigma' d^2\bar\theta' \hat{\boldsymbol{E}}(\bar\sigma',\bar\theta') \frac{1}{\bar e} \delta^\nu_\mu \delta(\bar\sigma-\bar\sigma') \bar\theta' \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma',\bar\theta')} \\
&= \int d^2\bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \bar\theta \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)}, \\
\frac{\partial}{\partial X^\mu(\bar\sigma)} \frac{\partial}{\partial \psi^\nu(\bar\sigma)} \tilde\phi'' &= \int d^2\bar\theta' \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta')}{\bar e(\bar\sigma)} \int d^2\bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta')} \bar\theta \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'' \\
&= \int d^2\bar\theta' \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta')}{\bar e(\bar\sigma)} \int d^2\bar\theta\, \bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\bar e(\bar\sigma)}{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)} \delta(\bar\theta-\bar\theta') \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'' \\
&= \int d^2\bar\theta\, \bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'', \\
\frac{\partial}{\partial X^\mu(\bar\sigma)} \frac{\partial}{\partial F^\nu(\bar\sigma)} \tilde\phi'' &= \int d^2\bar\theta' \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta')}{\bar e(\bar\sigma)} \int d^2\bar\theta \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta')} \frac{1}{2}\bar\theta^2 \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'' \\
&= \int d^2\bar\theta' \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta')}{\bar e(\bar\sigma)} \int d^2\bar\theta \frac{1}{2}\bar\theta^2 \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\bar e(\bar\sigma)}{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)} \delta(\bar\theta-\bar\theta') \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'' \\
&= \int d^2\bar\theta \frac{1}{2}\bar\theta^2 \frac{\hat{\boldsymbol{E}}(\bar\sigma,\bar\theta)}{\bar e(\bar\sigma)} \frac{\partial}{\partial \mathbf{X}^\mu_{\hat D_T}(\bar\sigma,\bar\theta)} \frac{\partial}{\partial \mathbf{X}^\nu_{\hat D_T}(\bar\sigma,\bar\theta)} \tilde\phi'',
\end{aligned}
\tag{4.9}
$$

and

$$
g_{\mu\nu}(\mathbf{X}_{\hat D_T}(\bar\sigma,\bar\theta)) = g_{\mu\nu}(X) + \partial_\rho g_{\mu\nu}(X)\bar\theta\psi^\rho + \frac{1}{2}\bar\theta^2\left(\partial_\rho g_{\mu\nu}(X)F^\rho + \frac{1}{2}\partial_\rho\partial_{\rho'} g_{\mu\nu}(X)\psi^\rho\psi^{\rho'}\right). \tag{4.10}
$$

Collecting the above results, the action (4.6) is expressed as

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}h\mathcal{D}X\tilde{\phi}''\left(\int d\bar{\sigma}\sqrt{\bar{h}}\left((\delta_{\mu\nu}-g_{\mu\nu})\frac{1}{\bar{e}^2}\frac{\partial}{\partial X_\mu}\frac{\partial}{\partial X_\nu}\tilde{\phi}'' - \partial_\rho g_{\mu\nu}(X)\psi^\rho\frac{1}{\bar{e}^2}\frac{\partial}{\partial X_\mu}\frac{\partial}{\partial \psi_\nu}\tilde{\phi}'' \right.\right.$$
$$\left.\left. -\left(\partial_\rho g_{\mu\nu}(X)F^\rho + \frac{1}{2}\partial_\rho\partial_{\rho'}g_{\mu\nu}(X)\psi^\rho\psi^{\rho'}\right)\frac{1}{\bar{e}^2}\frac{\partial}{\partial X_\mu}\frac{\partial}{\partial F_\nu}\tilde{\phi}''\right) + \omega\tilde{\phi}''\right). \tag{4.11}$$

This can be written as

$$S = -2\int \mathcal{D}\bar{\tau}\mathcal{D}h\mathcal{D}X\tilde{\phi}'' H\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial F}, \mathbf{X}_{\hat{D}_T}, \bar{\mathbf{E}}\right)\tilde{\phi}'', \tag{4.12}$$

where

$$\begin{aligned} H\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial F}, \mathbf{X}_{\hat{D}_T}, \bar{\mathbf{E}}\right) &= \frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}(\delta^{\mu\nu}-g^{\mu\nu})\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^\mu}\right)\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^\nu}\right) \\ &+ \int d\bar{\sigma} n^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\mu \bar{e}\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^\mu}\right) + \int d\bar{\sigma}\, i\frac{\sqrt{\bar{h}}}{\bar{e}}\partial_{\bar{\sigma}}X^\nu B_\nu{}^\mu\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^\mu}\right) \\ &- \int d\bar{\sigma}\frac{\bar{h}}{\bar{e}}\bar{\psi}^\nu\frac{i}{2}\left(\frac{1}{\sqrt{\bar{h}}}\Gamma_\nu^{\mu\rho}(X)\gamma^0\left(\frac{\partial}{\partial\psi^\rho}\right) + H_\nu{}^\mu{}_\rho(X)\gamma_5\gamma^0\psi^\rho\right)\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^\mu}\right) \\ &+ \frac{i}{2}\int d\bar{\sigma}\, i\frac{\bar{h}}{\bar{e}}(-2\bar{\chi}_a\gamma^0\gamma^a\psi^\mu)\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^\mu}\right) \\ &+ \int d\bar{\sigma}\sqrt{\bar{h}}\frac{i}{2\bar{e}}\left(\partial_\rho g_{\mu\nu}(X)F^\rho + \frac{1}{2}\partial_\rho\partial_{\rho'}g_{\mu\nu}(X)\psi^\rho\psi^{\rho'}\right)\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X_\mu}\right)\left(\frac{\partial}{\partial F_\nu}\right) - \frac{1}{2}\omega. \end{aligned} \tag{4.13}$$

Here $\Gamma^\rho_{\mu\nu} = \frac{1}{2}G^{\rho\lambda}(\partial_\mu G_{\nu\lambda} + \partial_\nu G_{\mu\lambda} - \partial_\lambda G_{\mu\nu})$, $\bar{\psi}^\mu = \psi^\mu\gamma^0$, $\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and we have added the terms

$$0 = -2\int \mathcal{D}\bar{\tau}\mathcal{D}h\mathcal{D}X\tilde{\phi}''\left[-i\int d\bar{\sigma}\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\mu\frac{\partial}{\partial X^\mu} + \int d\bar{\sigma}\frac{\sqrt{\bar{h}}}{\bar{e}^2}\partial_{\bar{\sigma}}X^\nu B_\nu{}^\mu\frac{\partial}{\partial X^\mu} + \frac{i}{4}\int d\bar{\sigma}\frac{\bar{h}}{\bar{e}^2}i\bar{\psi}^\mu\partial_\nu H_\mu{}^\nu{}_\rho(X)\gamma_5\gamma^0\psi^\rho \right.$$
$$\left. + \frac{i}{2}\int d\bar{\sigma}\frac{\bar{h}}{\bar{e}^2}i\bar{\psi}^\mu H_\mu{}^\nu{}_\rho(X)\gamma_5\gamma^0\psi^\rho\frac{\partial}{\partial X^\nu} + \frac{i}{2}\int d\bar{\sigma}\frac{\bar{h}}{\bar{e}^2}\left(-2\bar{\chi}_a\gamma^0\gamma^a\psi^\mu\frac{\partial}{\partial X^\mu}\right)\right]\tilde{\phi}'', \tag{4.14}$$

which is true because of the gauge-fixing condition (3.33).

The propagator for $\tilde{\phi}$ defined by

$$\Delta_F(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}); \bar{\mathbf{E}}', \mathbf{X}'_{\hat{D}_T}(\bar{\tau}')) = \langle\tilde{\phi}(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}))\tilde{\phi}(\bar{\mathbf{E}}', \mathbf{X}'_{\hat{D}_T}(\bar{\tau}'))\rangle \tag{4.15}$$

satisfies

$$H\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X(\bar{\tau})}, \frac{\partial}{\partial\psi(\bar{\tau})}, \frac{\partial}{\partial F(\bar{\tau})}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\mathbf{E}}\right)\Delta_F(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}); \bar{\mathbf{E}},' \mathbf{X}'_{\hat{D}_T}(\bar{\tau}')) = \delta(\bar{\mathbf{E}} - \bar{\mathbf{E}}')\delta(\mathbf{X}_{\hat{D}_T}(\bar{\tau}) - \mathbf{X}'(\bar{\tau}')). \tag{4.16}$$

In order to obtain a Schwinger representation of the propagator, we use the operator formalism $(\hat{\bar{\mathbf{E}}}, \hat{\mathbf{X}}_{\hat{D}_T}(\bar{\tau}))$ of the first quantization. The eigenstate for $(\hat{\bar{\mathbf{E}}}, \hat{X}(\bar{\tau}))$ is given by $|\bar{\mathbf{E}}, X(\bar{\tau})\rangle$. The conjugate momentum is written as $(\hat{\mathbf{p}}_{\bar{\mathbf{E}}}, \hat{p}_X)$. There is no conjugate momentum for the auxiliary field F^μ, whereas the Majorana fermion ψ^μ_α is self-conjugate and satisfies $\{\hat{\psi}^\mu_\alpha(\bar{\sigma}), \hat{\psi}^\nu_\beta(\bar{\sigma}')\} = \frac{1}{\sqrt{\bar{h}}}\delta_{\alpha\beta}G^{\mu\nu}(X)\delta(\bar{\sigma}-\bar{\sigma}')$. By defining creation and annihilation operators for ψ^μ_α as $\hat{\psi}^{\mu\dagger} \coloneqq \frac{1}{\sqrt{2}}(\hat{\psi}^\mu_1 - i\hat{\psi}^\mu_2)$ and $\hat{\psi}^\mu \coloneqq \frac{1}{\sqrt{2}}(\hat{\psi}^\mu_1 + i\hat{\psi}^\mu_2)$, one obtains an algebra $\{\hat{\psi}^\mu(\bar{\sigma}), \hat{\psi}^{\nu\dagger}(\bar{\sigma}')\} = \frac{1}{\sqrt{\bar{h}}}G^{\mu\nu}(X)\delta(\bar{\sigma}-\bar{\sigma}')$, $\{\hat{\psi}^\mu(\bar{\sigma}), \hat{\psi}^\nu(\bar{\sigma}')\} = 0$, and $\{\hat{\psi}^{\mu\dagger}(\bar{\sigma}), \hat{\psi}^{\nu\dagger}(\bar{\sigma}')\} = 0$. The vacuum $|0\rangle$ for this algebra is defined by $\hat{\psi}^\mu(\bar{\sigma})|0\rangle = 0$. The eigenstate $|\psi\rangle$, which satisfies $\hat{\psi}^\mu(\bar{\sigma})|\psi\rangle = \psi^\mu(\bar{\sigma})|\psi\rangle$, is given by $e^{-\psi\cdot\hat{\psi}^\dagger}|0\rangle \coloneqq e^{-\int d\bar{\sigma}\sqrt{\bar{h}}G_{\mu\nu}(X)\psi^\mu(\bar{\sigma})\hat{\psi}^{\nu\dagger}(\bar{\sigma})}|0\rangle$. Then, the inner product is given by $\langle\psi|\psi'\rangle = e^{\psi^\dagger\cdot\psi'}$, whereas the completeness relation is $\int\mathcal{D}\psi^\dagger\mathcal{D}\psi|\psi\rangle e^{-\psi^\dagger\cdot\psi}\langle\psi| = 1$.

Since Eq. (4.16) means that Δ_F is an inverse of H, Δ_F can be expressed by a matrix element of the operator $\hat{H}^{-1}$ as

$$\Delta_F(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}); \bar{\mathbf{E}}', \mathbf{X}'_{\hat{D}_T}(\bar{\tau}')) = \langle \bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}) | H^{-1}(\hat{p}_X(\bar{\tau}), \times \sqrt{\hat{\bar{h}}} \hat{G}_{\mu\nu} \hat{\psi}^\nu(\bar{\tau}), 0, \hat{\mathbf{X}}_{\hat{D}_T}(\bar{\tau}), \hat{\bar{\mathbf{E}}}) | \bar{\mathbf{E}}', \mathbf{X}'_{\hat{D}_T}(\bar{\tau}') \rangle, \tag{4.17}$$

where we use the fact that the states do not depend on F^μ because it is not an independent variable and is written in terms of the other fields X^μ and ψ^μ.

On the other hand,

$$\hat{H}^{-1} = i \int_0^\infty dT e^{-iT\hat{H}}, \tag{4.18}$$

because

$$\lim_{\epsilon\to 0+} \int_0^\infty dT e^{-T(i\hat{H}+\epsilon)} = \lim_{\epsilon\to 0+} \left[\frac{1}{-(i\hat{H}+\epsilon)} e^{-T(i\hat{H}+\epsilon)} \right]_0^\infty = -i\hat{H}^{-1}. \tag{4.19}$$

This fact and Eq. (4.17) imply

$$\Delta_F(\bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}); \bar{\mathbf{E}}', \mathbf{X}'_{\hat{D}_T}(\bar{\tau}')) = i \int_0^\infty dT \langle \bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}) | e^{-iT\hat{H}} | \bar{\mathbf{E}}', \mathbf{X}'_{\hat{D}_T}(\bar{\tau}') \rangle. \tag{4.20}$$

In order to define two-point correlation functions that are invariant under the general coordinate transformations in the string geometry, we define in and out states as

$$||\mathbf{X}_{\hat{D}_T i}|\mathbf{E}_f, ; \mathbf{E}_i\rangle_{\text{in}} := \int_{\mathbf{E}_i}^{\mathbf{E}_f} \mathcal{D}\mathbf{E}' |\bar{\mathbf{E}}', \mathbf{X}_{\hat{D}_T i}\rangle$$
$$\langle \mathbf{X}_{\hat{D}_T f}|\mathbf{E}_f, ; \mathbf{E}_i||_{\text{out}} := \int_{\mathbf{E}_i}^{\mathbf{E}_f} \mathcal{D}\mathbf{E} \langle \bar{\mathbf{E}}, \mathbf{X}_{\hat{D}_T f}|, \tag{4.21}$$

where $\mathbf{X}_{\hat{D}_T i} := \mathbf{X}_{\hat{D}_T}(\bar{\tau}' = -\infty)$, $\mathbf{X}_{\hat{D}_T f} := \mathbf{X}_{\hat{D}_T}(\bar{\tau} = \infty)$, and $\mathbf{E}_i$ and $\mathbf{E}_f$ represent the vielbein of the supercylinders at $\bar{\tau} = \pm\infty$, respectively. $\int$ in $\int \mathcal{D}\mathbf{E}$ includes $\sum_{\text{compact topologies}}$, where $\mathcal{D}\mathbf{E}$ is the invariant measure of the supervielbein $\mathbf{E}$ on the two-dimensional super-Riemannian manifolds $\mathbf{\Sigma}$. $\mathbf{E}$ and $\bar{\mathbf{E}}$ are related to each other by the superdiffeomorphism and the super-Weyl transformations. When we insert asymptotic states, we integrate out $\mathbf{X}_{\hat{D}_T f}$, $\mathbf{X}_{\hat{D}_T i}$, $\mathbf{E}_f$, and $\mathbf{E}_i$ from the two-point correlation function for these states:

$$\Delta_F(\mathbf{X}_{\hat{D}_T f}; \mathbf{X}_{\hat{D}_T i}|\mathbf{E}_f, ; \mathbf{E}_i) := i \int_0^\infty dT \langle \mathbf{X}_{\hat{D}_T f}|\mathbf{E}_f, ; \mathbf{E}_i||_{\text{out}} e^{-iT\hat{H}} ||\mathbf{X}_{\hat{D}_T i}|\mathbf{E}_f, ; \mathbf{E}_i\rangle_{\text{in}}. \tag{4.22}$$

By inserting

$$1 = \int d\bar{\mathbf{E}}_m d\mathbf{X}_{\hat{D}_T Tm}(\bar{\tau}_m) |\bar{\mathbf{E}}_m, \mathbf{X}_{\hat{D}_T Tm}(\bar{\tau}_m)\rangle \times e^{-\bar{\psi}_m^\dagger \cdot \bar{\psi}_m} \langle \bar{\mathbf{E}}_m, \mathbf{X}_{\hat{D}_T Tm}(\bar{\tau}_m)|,$$
$$1 = \int dp_X^i |p_X^i\rangle\langle p_X^i|, \tag{4.23}$$

this can be written as in Ref. [1],

$$\begin{aligned}
\Delta_F(\mathbf{X}_{\hat{D}_T f}; \mathbf{X}_{\hat{D}_T i}|\mathbf{E}_f, ; \mathbf{E}_i) &:= i \int_0^\infty dT \langle \mathbf{X}_{\hat{D}_T f}|\mathbf{E}_f, ; \mathbf{E}_i||_{\text{out}} e^{-iT\hat{H}} ||\mathbf{X}_{\hat{D}_T i}|\mathbf{E}_f, ; \mathbf{E}_i\rangle_{\text{in}} \\
&= i \int_0^\infty dT \lim_{N\to\infty} \int_{\mathbf{E}_i}^{\mathbf{E}_f} \mathcal{D}\mathbf{E} \int_{\mathbf{E}_i}^{\mathbf{E}_f} \mathcal{D}\mathbf{E}' \prod_{n=1}^N \int d\bar{\mathbf{E}}_n d\mathbf{X}_{\hat{D}_T n}(\bar{\tau}_n) e^{-i\psi_n^\dagger \cdot \psi_n} \\
&\quad \times \prod_{m=0}^N \langle \bar{\mathbf{E}}_{m+1}, \mathbf{X}_{\hat{D}_T m+1}(\bar{\tau}_{m+1}) | e^{-i\frac{1}{N}T\hat{H}} | \bar{\mathbf{E}}_m, \mathbf{X}_{\hat{D}_T m}(\bar{\tau}_m), \rangle \\
&= i \int_0^\infty dT_0 \lim_{N\to\infty} \int dT_{N+1} \int_{\mathbf{E}_i}^{\mathbf{E}_f} \mathcal{D}\mathbf{E} \prod_{m=1}^N \prod_{i=0}^N \int dT_m d\mathbf{X}_{\hat{D}_T m}(\bar{\tau}_m) e^{-\bar{\psi}_m^\dagger \cdot \bar{\psi}_m} \\
&\quad \times \int dp_X^i \langle X_{i+1}|p_X^i\rangle \langle p_X^i| \langle \tilde{\psi}_{i+1}| e^{-\frac{1}{N}T_i \hat{H}} |\tilde{\psi}_i\rangle |X_i\rangle \delta(T_i - T_{i+1}) \\
&= i \int_0^\infty dT_0 \lim_{N\to\infty} \int dT_{N+1} \int_{\mathbf{E}_i}^{\mathbf{E}_f} \mathcal{D}\mathbf{E} \prod_{m=1}^N \prod_{i=0}^N \int dT_m d\mathbf{X}_{\hat{D}_T m}(\bar{\tau}_m) e^{-\bar{\psi}_m^\dagger \cdot \bar{\psi}_m} \\
&\quad \times \int dp_X^i e^{-\frac{1}{N}T_i H(p_X^i, \sqrt{\bar{h}} G_{\mu\nu}(X_i(\bar{\tau}_i)) \psi_i^\nu(\bar{\tau}_i), 0, \mathbf{X}_{\hat{D}_T i}(\bar{\tau}_i), \bar{\mathbf{E}})} e^{\tilde{\psi}_{i+1}^\dagger \cdot \tilde{\psi}_i} \delta(T_i - T_{i+1}) e^{i(p_X^i \cdot (X_{i+1} - X_i))}
\end{aligned}$$

$$
\begin{aligned}
&= i\int_0^\infty dT_0 \lim_{N\to\infty} dT_{N+1} \int_{\mathbf{E}_i}^{\mathbf{E}_f} \mathcal{D}\mathbf{E} \prod_{n=1}^{N} \int dT_n d\mathbf{X}_{\hat{D}_T n}(\bar{\tau}_n) \prod_{m=0}^{N} \int dp_{T_m} dp_{X_m}(\bar{\tau}_m) \\
&\quad \times \exp\Bigg(i \sum_{m=0}^{N} \Delta t \Bigg(p_{T_m} \frac{T_m - T_{m+1}}{\Delta t} + \int d\bar{\sigma}\, \bar{e}\, p_{X_m}(\bar{\tau}_m) \frac{X_m(\bar{\tau}_m) - X_{m+1}(\bar{\tau}_{m+1})}{\Delta t} \\
&\quad + i\psi_m^\dagger \cdot \frac{\psi_m(\bar{\tau}_m) - \psi_{m+1}(\bar{\tau}_{m+1})}{\Delta t} - T_m H(p_{X_m}(\bar{\tau}_m), \sqrt{\bar{h}} G_{\mu\nu}(X_m(\bar{\tau}_m))\psi_m^\nu(\bar{\tau}_m), 0, \mathbf{X}_{\hat{D}_T m}(\bar{\tau}_m), \bar{\mathbf{E}}) \Bigg) \Bigg) \\
&= i\int_{\mathbf{E}_i \mathbf{X}_{\hat{D}_T i}}^{\mathbf{E}_f, \mathbf{X}_{\hat{D}_T f}} \mathcal{D}\mathbf{E} \mathcal{D}\mathbf{X}_{\hat{D}_T}(\bar{\tau}) \int \mathcal{D}T \int \mathcal{D}p_T \mathcal{D}p_X(\bar{\tau}) \\
&\quad \times \exp\Bigg(i\int_{-\infty}^{\infty} dt \Bigg(p_T(t) \frac{d}{dt} T(t) + \int d\bar{\sigma}\, \bar{e}\, p_{X\mu}(\bar{\tau}(t), t) \frac{d}{dt} X^\mu(\bar{\tau}(t), t) \\
&\quad + \int d\bar{\sigma} \frac{i}{2} \sqrt{\bar{h}} G_{\mu\nu}(X(\bar{\tau}(t), t)) \bar{\psi}^\mu(\bar{\tau}(t), t) \gamma^0 \frac{d}{dt} \psi^\nu(\bar{\tau}(t), t) \\
&\quad - T(t) H(p_X(\bar{\tau}(t), t), \sqrt{\bar{h}} G_{\mu\nu}(X(\bar{\tau}(t), t)) \psi^\nu(\bar{\tau}(t), t), 0, \mathbf{X}_{\hat{D}_T}(\bar{\tau}(t), t), \bar{\mathbf{E}}) \Bigg) \Bigg),
\end{aligned} \tag{4.24}
$$

where $\bar{\mathbf{E}}_0 = \bar{\mathbf{E}}'$, $\mathbf{X}_{\hat{D}_T 0}(\bar{\tau}_0) = \mathbf{X}_{\hat{D}_T i}$, $\bar{\tau}_0 = -\infty$, $\bar{\mathbf{E}}_{N+1} = \bar{\mathbf{E}}$, $\mathbf{X}_{\hat{D}_T N+1}(\bar{\tau}_{N+1}) = \mathbf{X}_{\hat{D}_T f}$, $\bar{\tau}_{N+1} = \infty$, and $\Delta t \coloneqq \frac{1}{\sqrt{N}}$. A trajectory of points $[\bar{\mathbf{\Sigma}}, \mathbf{X}_{\hat{D}_T}(\bar{\tau})]$ is necessarily continuous in $\mathcal{M}_D$ so that the kernel $\langle \bar{\mathbf{E}}_{m+1}, \mathbf{X}_{\hat{D}_T m+1}(\bar{\tau}_{m+1}) | e^{-i\frac{1}{N} T_m \hat{H}} | \bar{\mathbf{E}}_m, \mathbf{X}_{\hat{D}_T m}(\bar{\tau}_m)\rangle$ in the fourth line is nonzero when $N \to \infty$.

By integrating out $p_X(\bar{\tau}(t), t)$, we move from the canonical formalism to the Lagrange formalism. Because the exponent of Eq. (4.24) is at most second order in $p_X(\bar{\tau}(t), t)$, integrating out $p_X(\bar{\tau}(t), t)$ is equivalent to substituting into Eq. (4.24) the solution $p_X(\bar{\tau}(t), t)$ of

$$
\begin{aligned}
&- i\bar{e} \frac{d}{dt} X^\mu + iT\bar{e} \left[n^{\bar{\sigma}} \partial_{\bar{\sigma}} X^\mu + i\frac{\sqrt{\bar{h}}}{\bar{e}^2} \partial_{\bar{\sigma}} X^\nu B_\nu{}^\mu - \frac{i}{2} \frac{\bar{h}}{\bar{e}^2} \bar{\psi}^\nu (\Gamma_\nu{}^\mu{}_\rho(X) + H_\nu{}^\mu{}_\rho(X)\gamma_5)\gamma^0 \psi^\rho - \frac{1}{2} \frac{\bar{h}}{\bar{e}^2} (-2\bar{\chi}_a \gamma^0 \gamma^a \psi^\mu) \right] \\
&\quad + iT\sqrt{\bar{h}} p_X^\mu - iT\sqrt{\bar{h}} g^{\mu\nu}(X) p_{\nu X} = 0,
\end{aligned} \tag{4.25}
$$

which is obtained by differentiating the exponent of Eq. (4.24) with respect to $p_X(\bar{\tau}(t), t)$. The solution is given by

$$
p_{\mu X} = \frac{1}{T} \frac{\bar{e}}{\sqrt{\bar{h}}} G_{\mu\nu} \left[\frac{d}{dt} X^\nu - T\left\{ n^{\bar{\sigma}} \partial_{\bar{\sigma}} X^\nu - \frac{1}{2} \frac{\bar{h}}{\bar{e}^2} (-2\bar{\chi}_a \gamma^0 \gamma^a \psi^\nu) \right\} \right] - \frac{i}{\bar{e}} \left[\partial_{\bar{\sigma}} X^\gamma B_{\gamma\mu} - \frac{\sqrt{\bar{h}}}{2} \bar{\psi}^\kappa (\Gamma_{\kappa\mu\rho}(X) + H_{\kappa\mu\rho}(X)\gamma_5)\gamma^0 \psi^\rho \right], \tag{4.26}
$$

up to the first order in the classical backgrounds $g_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$. By substituting this, we obtain

$$
\begin{aligned}
\Delta_F(\mathbf{X}_{\hat{D}_T f}; \mathbf{X}_{\hat{D}_T i} | \mathbf{E}_f; \mathbf{E}_i) &= i\int_{\mathbf{E}_i \mathbf{X}_{\hat{D}_T i}}^{\mathbf{E}_f, \mathbf{X}_{\hat{D}_T f}} \mathcal{D}T \mathcal{D}\mathbf{E} \mathcal{D}\mathbf{X}_{\hat{D}_T}(\bar{\tau}) \mathcal{D}p_T \exp\Bigg(i\int_{-\infty}^{\infty} dt \Bigg(p_T(t) \frac{d}{dt} T(t) \\
&\quad + \int d\bar{\sigma} \sqrt{\bar{h}}\, G_{\mu\nu}(X(\bar{\tau}(t), t)) \Bigg(\frac{1}{2} \bar{h}^{00} \frac{1}{T(t)} \partial_t X^\mu(\bar{\tau}(t), t) \partial_t X^\nu(\bar{\tau}(t), t) \\
&\quad + \bar{h}^{01} \partial_t X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) + \frac{1}{2} \bar{h}^{11} T(t) \partial_{\bar{\sigma}} X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) \Bigg) \\
&\quad + \int d\bar{\sigma} i B_{\mu\nu}(X(\bar{\tau}(t), t)) \partial_t X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) \\
&\quad + \frac{1}{2} \int d\bar{\sigma} \sqrt{\bar{h}} \Bigg(iG_{\mu\nu} \bar{\psi}^\mu \gamma^0 \frac{d}{dt} \psi^\nu + i\bar{\psi}^\nu (\Gamma_{\nu\mu\rho} + H_{\nu\mu\rho}\gamma_5) \gamma^0 \frac{d}{dt} X^\mu(\bar{\tau}(t), t) \psi^\rho
\end{aligned}
$$

$$
\begin{aligned}
&- 2G_{\mu\nu}\bar{\chi}_a\gamma^0\gamma^a\psi^\mu \frac{d}{dt}X^\nu(\bar{\tau}(t),t)\bigg) + \frac{1}{2}T\int d\bar{\sigma}\sqrt{\bar{h}}\bigg(iG_{\mu\nu}\bar{\psi}^\mu\gamma^1\partial_{\bar{\sigma}}\psi^\nu \\
&+ i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^1\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\psi^\rho - 2G_{\mu\nu}\bar{\chi}_a\gamma^1\gamma^a\psi^\mu\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t) + \frac{1}{2}G_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\chi}_a\gamma^b\gamma^a\chi_b \\
&+ \frac{1}{6}R_{\mu\nu\lambda\rho}\bar{\psi}^\mu\psi^\lambda\bar{\psi}^\nu\psi^\rho - \frac{i}{3}H_{\mu\nu\rho}\bar{\chi}_a\gamma^b\gamma^a\bar{\psi}^\mu\psi^\nu\gamma_b\gamma_5\psi^\rho - \frac{1}{4}D_\rho H_{\mu\nu\lambda}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\lambda\gamma_5\psi^\nu + R_{\bar{h}}\frac{\lambda}{2\pi}\bigg)\bigg)\bigg),
\end{aligned}
\tag{4.27}
$$

where we use the Arnowitt-Deser-Misner decomposition of the two-dimensional metric,

$$
\bar{h}_{mn} = \begin{pmatrix} \bar{n}^2 + \bar{n}_{\bar{\sigma}}\bar{n}^{\bar{\sigma}} & \bar{n}_{\bar{\sigma}} \\ \bar{n}_{\bar{\sigma}} & \bar{e}^2 \end{pmatrix}, \qquad \sqrt{\bar{h}} = \bar{n}\,\bar{e}, \qquad \bar{h}^{mn} = \begin{pmatrix} \frac{1}{\bar{n}^2} & -\frac{\bar{n}^{\bar{\sigma}}}{\bar{n}^2} \\ -\frac{\bar{n}^{\bar{\sigma}}}{\bar{n}^2} & \bar{e}^{-2} + \left(\frac{\bar{n}^{\bar{\sigma}}}{\bar{n}}\right)^2 \end{pmatrix},
\tag{4.28}
$$

$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and Eq. (3.47) by choosing a function $\omega(\mathbf{X}_{\hat{D}_T})$ as

$$
\begin{aligned}
\omega(\mathbf{X}_{\hat{D}_T}) = &\int d\bar{\sigma}\sqrt{\bar{h}}\bigg(R_{\bar{h}}\frac{\lambda}{4\pi} + \frac{1}{\bar{e}^2}G_{\mu\nu}\partial_{\bar{\sigma}}X^\mu\partial_{\bar{\sigma}}X^\nu + iG_{\nu\mu}\bar{\psi}^\mu\gamma^1\partial_{\bar{\sigma}}\psi^\nu \\
&- \frac{i}{2}\sqrt{\bar{h}}\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^0\psi^\rho\left\{-2n^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\mu - \frac{\bar{h}}{\bar{e}^2}(-2\bar{\chi}_a\gamma^0\gamma^a\psi^\mu)\right\} \\
&+ i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^1\partial_{\bar{\sigma}}X^\mu\psi^\rho - 2G_{\mu\nu}\bar{\chi}_a\gamma^1\gamma^a\psi^\mu\partial_{\bar{\sigma}}X^\nu - i\sqrt{\bar{h}}(-2\bar{\chi}_a\gamma^0\gamma^a\psi^\mu)\bigg(iG_{\mu\nu}n^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\nu + \frac{\sqrt{\bar{h}}}{\bar{e}^2}\partial_{\bar{\sigma}}X^\lambda B_{\lambda\mu}\bigg) \\
&- \frac{i}{4}\int d\bar{\sigma}\frac{\bar{h}}{\bar{e}^2}i\bar{\psi}^\mu\partial_\nu H_\mu{}^\nu{}_\rho(X)\gamma_5\gamma^0\psi^\rho - \frac{1}{4}\frac{\bar{h}}{e^2}G_{\mu\nu}(-2\bar{\chi}_a\gamma^0\gamma^a\psi^\mu)(-2\bar{\chi}_b\gamma^0\gamma^b\psi^\nu) \\
&+ \frac{1}{2}G_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\chi}_a\gamma^b\gamma^a\chi_b + \frac{1}{6}R_{\mu\nu\lambda\rho}\bar{\psi}^\mu\psi^\lambda\bar{\psi}^\nu\psi^\rho - \frac{i}{3}H_{\mu\nu\rho}\bar{\chi}_a\gamma^b\gamma^a\bar{\psi}^\mu\psi^\nu\gamma_b\gamma_5\psi^\rho - \frac{1}{4}D_\rho H_{\mu\nu\lambda}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\lambda\gamma_5\psi^\nu\bigg),
\end{aligned}
\tag{4.29}
$$

where $R_{\bar{h}}$ is the scalar curvature of the two-dimensional metric $\bar{h}_{mn}$ and λ is a constant, which will be identified with the logarithm of the string coupling constant g_s. In the following, we consider only constant dilaton backgrounds because a world-sheet theory of superstrings in nonconstant dilaton backgrounds is not known.

In this way, the Green function can generate all of the terms without $\bar{\tau}$ derivatives in the string action as in Eq. (3.47), but cannot generate those with $\bar{\tau}$ derivatives, which need to be derived nontrivially, because the coordinates $X^\mu(\bar{\tau})$ in string geometry theory are defined on the constant-$\bar{\tau}$ lines. We should note that the time derivative in Eq. (4.27) is in terms of t, not $\bar{\tau}$ at the moment. In the following, we will see that t can be fixed to $\bar{\tau}$ by using a reparametrization of t that parametrizes a trajectory.

By inserting $\int \mathcal{D}c\mathcal{D}b e^{\int_0^1 dt(\frac{db(t)}{dt}\frac{dc(t)}{dt})}$, where $b(t)$ and $c(t)$ are bc ghosts, we obtain

$$
\begin{aligned}
\Delta_F(\mathbf{X}_{\hat{D}_Tf};\mathbf{X}_{\hat{D}_Ti}|\mathbf{E}_f;\mathbf{E}_i) = Z_0\int_{\mathbf{E}_i\mathbf{X}_{\hat{D}_Ti}}^{\mathbf{E}_f,\mathbf{X}_{\hat{D}_Tf}} &\mathcal{D}T\mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T}(\bar{\tau})\mathcal{D}p_T\mathcal{D}c\mathcal{D}b\exp\bigg(-\int_{-\infty}^{\infty}dt\bigg(-ip_T(t)\frac{d}{dt}T(t) + \frac{db(t)}{dt}\frac{d(T(t)c(t))}{dt} \\
&+ \int d\bar{\sigma}\sqrt{\bar{h}}\,G_{\mu\nu}(X(\bar{\tau}(t),t))\bigg(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_tX^\mu(\bar{\tau}(t),t)\partial_tX^\nu(\bar{\tau}(t),t) \\
&+ \bar{h}^{01}\partial_tX^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t) + \frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)\bigg) \\
&+ \int d\bar{\sigma}iB_{\mu\nu}(X(\bar{\tau}(t),t))\partial_tX^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t) \\
&+ \frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\bigg(iG_{\mu\nu}\bar{\psi}^\mu\gamma^0\frac{d}{dt}\psi^\nu + i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^0\frac{d}{dt}X^\mu(\bar{\tau}(t),t)\psi^\rho
\end{aligned}
$$

$$-2G_{\mu\nu}\bar{\chi}_a\gamma^0\gamma^a\psi^\mu\frac{d}{dt}X^\nu(\bar{\tau}(t),t)\bigg)+\frac{1}{2}T\int d\bar{\sigma}\sqrt{\bar{h}}\bigg(iG_{\mu\nu}\bar{\psi}^\mu\gamma^1\partial_{\bar{\sigma}}\psi^\nu+i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^1\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\psi^\rho$$
$$-2G_{\mu\nu}\bar{\chi}_a\gamma^1\gamma^a\psi^\mu\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}G_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\chi}_a\gamma^b\gamma^a\chi_b+\frac{1}{6}R_{\mu\nu\lambda\rho}\bar{\psi}^\mu\psi^\lambda\bar{\psi}^\nu\psi^\rho$$
$$-\frac{i}{3}H_{\mu\nu\rho}\bar{\chi}_a\gamma^b\gamma^a\bar{\psi}^\mu\psi^\nu\gamma_b\gamma_5\psi^\rho-\frac{1}{4}D_\rho H_{\mu\nu\lambda}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\lambda\gamma_5\psi^\nu+R_{\bar{h}}\frac{\lambda}{2\pi}\bigg)\bigg)\bigg), \tag{4.30}$$

where we redefine $c(t)\to T(t)c(t)$, and Z_0 represents an overall constant factor. In the following, we rename it $Z_1, Z_2, \cdots$ when the factor changes. The integrand variable $p_T(t)$ plays the role of the Lagrange multiplier providing the following condition:

$$F_1(t)\coloneqq\frac{d}{dt}T(t)=0, \tag{4.31}$$

which can be understood as a gauge-fixing condition. Indeed, by choosing this gauge in

$$\Delta_F(\mathbf{X}_{\hat{D}_Tf};\mathbf{X}_{\hat{D}_Ti}|\mathbf{E}_f;\mathbf{E}_i)=Z_1\int_{\mathbf{E}_i\mathbf{X}_{\hat{D}_Ti}}^{\mathbf{E}_f,\mathbf{X}_{\hat{D}_Tf}}\mathcal{D}T\mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T}(\bar{\tau})\exp\bigg(-\int_{-\infty}^{\infty}dt\bigg(\int d\bar{\sigma}\sqrt{\bar{h}}\,G_{\mu\nu}(X(\bar{\tau}(t),t))$$
$$\times\bigg(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_tX^\mu(\bar{\tau}(t),t)\partial_tX^\nu(\bar{\tau}(t),t)+\bar{h}^{01}\partial_tX^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)$$
$$+\frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)\bigg)+\int d\bar{\sigma}iB_{\mu\nu}(X(\bar{\tau}(t),t))\partial_tX^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)$$
$$+\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\bigg(iG_{\mu\nu}\bar{\psi}^\mu\gamma^0\frac{d}{dt}\psi^\nu+i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^0\frac{d}{dt}X^\mu(\bar{\tau}(t),t)\psi^\rho$$
$$-2G_{\mu\nu}\bar{\chi}_a\gamma^0\gamma^a\psi^\mu\frac{d}{dt}X^\nu(\bar{\tau}(t),t)\bigg)$$
$$+\frac{1}{2}T(t)\int d\bar{\sigma}\sqrt{\bar{h}}\bigg(iG_{\mu\nu}\bar{\psi}^\mu\gamma^1\partial_{\bar{\sigma}}\psi^\nu+i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^1\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\psi^\rho$$
$$-2G_{\mu\nu}\bar{\chi}_a\gamma^1\gamma^a\psi^\mu\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}G_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\chi}_a\gamma^b\gamma^a\chi_b+\frac{1}{6}R_{\mu\nu\lambda\rho}\bar{\psi}^\mu\psi^\lambda\bar{\psi}^\nu\psi^\rho$$
$$-\frac{i}{3}H_{\mu\nu\rho}\bar{\chi}_a\gamma^b\gamma^a\bar{\psi}^\mu\psi^\nu\gamma_b\gamma_5\psi^\rho-\frac{1}{4}D_\rho H_{\mu\nu\lambda}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\lambda\gamma_5\psi^\nu+R_{\bar{h}}\frac{\lambda}{2\pi}\bigg)\bigg)\bigg), \tag{4.32}$$

we obtain Eq. (4.30). Equation (4.32) has a manifestly one-dimensional diffeomorphism symmetry with respect to t, where $T(t)$ is transformed as an einbein [10].

Under $\frac{d\bar{\tau}}{d\bar{\tau}'}=T(t)$, which implies

$$\begin{aligned}\bar{h}^{00}&=T^2\bar{h}'^{00},\\ \bar{h}^{01}&=T\bar{h}'^{01},\\ \bar{h}^{11}&=\bar{h}'^{11},\\ \sqrt{\bar{h}}&=\frac{1}{T}\sqrt{\bar{h}'},\end{aligned} \tag{4.33}$$

$T(t)$ disappears from Eq. (4.32) and we obtain

$$\begin{aligned}\Delta_F(\mathbf{X}_{\hat{D}_T f};\mathbf{X}_{\hat{D}_T i}|\mathbf{E}_f;\mathbf{E}_i) = Z_2 \int_{\mathbf{E}_i\mathbf{X}_{\hat{D}_T i}}^{\mathbf{E}_f,\mathbf{X}_{\hat{D}_T f}} &\mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T}(\bar{\tau}) \\ &\times \exp\Bigg(-\int_{-\infty}^{\infty} dt\Bigg(\int d\bar{\sigma}\sqrt{\bar{h}}\,G_{\mu\nu}(X(\bar{\tau}(t),t))\Bigg(\frac{1}{2}\bar{h}^{00}\partial_t X^\mu(\bar{\tau}(t),t)\partial_t X^\nu(\bar{\tau}(t),t) \\ &+\bar{h}^{01}\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\bar{h}^{11}\partial_{\bar{\sigma}} X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t),t)\Bigg) \\ &+\int d\bar{\sigma} iB_{\mu\nu}(X(\bar{\tau}(t),t))\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t),t) \\ &+\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\,(iG_{\mu\nu}\bar{\psi}^\mu\gamma^0\partial_{\bar{\tau}}\psi^\nu + i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^0\partial_{\bar{\tau}}X^\mu(\bar{\tau}(t),t)\psi^\rho \\ &-2G_{\mu\nu}\bar{\chi}_a\gamma^0\gamma^a\psi^\mu\partial_{\bar{\tau}}X^\nu(\bar{\tau}(t),t)) \\ &+\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\Bigg(iG_{\mu\nu}\bar{\psi}^\mu\gamma^1\partial_{\bar{\sigma}}\psi^\nu + i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^1\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\psi^\rho \\ &-2G_{\mu\nu}\bar{\chi}_a\gamma^1\gamma^a\psi^\mu\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}G_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\chi}_a\gamma^b\gamma^a\chi_b+\frac{1}{6}R_{\mu\nu\lambda\rho}\bar{\psi}^\mu\psi^\lambda\bar{\psi}^\nu\psi^\rho \\ &-\frac{i}{3}H_{\mu\nu\rho}\bar{\chi}_a\gamma^b\gamma^a\bar{\psi}^\mu\psi^\nu\gamma_b\gamma_5\psi^\rho-\frac{1}{4}D_\rho H_{\mu\nu\lambda}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\lambda\gamma_5\psi^\nu+R_{\bar{h}}\frac{\lambda}{2\pi}\Bigg)\Bigg)\Bigg). \end{aligned} \tag{4.34}$$

This action is still invariant under the diffeomorphism with respect to t if $\bar{\tau}$ transforms in the same way as t.

If we choose a different gauge,

$$F_2(t) \coloneqq \bar{\tau}(t) - t = 0, \tag{4.35}$$

in Eq. (4.34), we obtain

$$\begin{aligned}\Delta_F(\mathbf{X}_{\hat{D}_T f};\mathbf{X}_{\hat{D}_T i}|\mathbf{E}_f;\mathbf{E}_i) = Z_3 \int_{\mathbf{E}_i\mathbf{X}_{\hat{D}_T i}}^{\mathbf{E}_f,\mathbf{X}_{\hat{D}_T f}} &\mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T}(\bar{\tau})\mathcal{D}\alpha\mathcal{D}c\mathcal{D}b\exp\Bigg(-\int_{-\infty}^{\infty} dt\Bigg(\alpha(t)(\bar{\tau}-t)+b(t)c(t)\Bigg(1-\frac{d\bar{\tau}(t)}{dt}\Bigg) \\ &+\int d\bar{\sigma}\sqrt{\bar{h}}\,G_{\mu\nu}(X(\bar{\tau}(t),t))\Bigg(\frac{1}{2}\bar{h}^{00}\partial_t X^\mu(\bar{\tau}(t),t)\partial_t X^\nu(\bar{\tau}(t),t) \\ &+\bar{h}^{01}\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\bar{h}^{11}\partial_{\bar{\sigma}} X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t),t)\Bigg) \\ &+\int d\bar{\sigma} iB_{\mu\nu}(X(\bar{\tau}(t),t))\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t),t) \\ &+\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\Bigg(iG_{\mu\nu}\bar{\psi}^\mu\gamma^a\partial_a\psi^\nu + i\bar{\psi}^\nu(\Gamma_{\nu\mu\rho}+H_{\nu\mu\rho}\gamma_5)\gamma^a\partial_a X^\mu(\bar{\tau}(t),t)\psi^\rho \\ &-2G_{\mu\nu}\bar{\chi}_a\gamma^b\gamma^a\psi^\mu\partial_b X^\nu(\bar{\tau}(t),t)+\frac{1}{2}G_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\chi}_a\gamma^b\gamma^a\chi_b+\frac{1}{6}R_{\mu\nu\lambda\rho}\bar{\psi}^\mu\psi^\lambda\bar{\psi}^\nu\psi^\rho \\ &-\frac{i}{3}H_{\mu\nu\rho}\bar{\chi}_a\gamma^b\gamma^a\bar{\psi}^\mu\psi^\nu\gamma_b\gamma_5\psi^\rho-\frac{1}{4}D_\rho H_{\mu\nu\lambda}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\lambda\gamma_5\psi^\nu+R_{\bar{h}}\frac{\lambda}{2\pi}\Bigg)\Bigg)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= Z \int_{\mathbf{E}_i,\mathbf{X}_{\hat{D}_T i}}^{\mathbf{E}_f,\mathbf{X}_{\hat{D}_T f}} \mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T} \exp\bigg(- \int_{-\infty}^{\infty} d\bar{\tau} \int d\bar{\sigma} \sqrt{\bar{h}}\, G_{\mu\nu}(X(\bar{\tau}(t),t)) \bigg(\frac{1}{2} \bar{h}^{00} \partial_{\bar{\tau}} X^\mu(\bar{\sigma},\bar{\tau}) \partial_{\bar{\tau}} X^\nu(\bar{\sigma},\bar{\tau}) \\
&\quad + \bar{h}^{01} \partial_{\bar{\tau}} X^\mu(\bar{\sigma},\bar{\tau}) \partial_{\bar{\sigma}} X^\nu(\bar{\sigma},\bar{\tau}) + \frac{1}{2} \bar{h}^{11} \partial_{\bar{\sigma}} X^\mu(\bar{\sigma},\bar{\tau}) \partial_{\bar{\sigma}} X^\nu(\bar{\sigma},\bar{\tau}) \bigg) + \int d\bar{\sigma} i B_{\mu\nu}(X(\bar{\sigma},\bar{\tau})) \partial_{\bar{\tau}} X^\mu(\bar{\sigma},\bar{\tau}) \partial_{\bar{\sigma}} X^\nu(\bar{\sigma},\bar{\tau}) \\
&\quad + \frac{1}{2} \int d\bar{\sigma} \sqrt{\bar{h}} \bigg(i G_{\mu\nu} \bar{\psi}^\mu \gamma^a \partial_a \psi^\nu + i \bar{\psi}^\nu (\Gamma_{\nu\mu\rho} + H_{\nu\mu\rho}\gamma_5) \gamma^a \partial_a X^\mu(\bar{\tau}(t),t) \psi^\rho \\
&\quad - 2 G_{\mu\nu} \bar{\chi}_a \gamma^b \gamma^a \psi^\mu \partial_b X^\nu(\bar{\tau}(t),t) + \frac{1}{2} G_{\mu\nu} \bar{\psi}^\mu \psi^\nu \bar{\chi}_a \gamma^b \gamma^a \chi_b + \frac{1}{6} R_{\mu\nu\lambda\rho} \bar{\psi}^\mu \psi^\lambda \bar{\psi}^\nu \psi^\rho \\
&\quad - \frac{i}{3} H_{\mu\nu\rho} \bar{\chi}_a \gamma^b \gamma^a \bar{\psi}^\mu \psi^\nu \gamma_b \gamma_5 \psi^\rho - \frac{1}{4} D_\rho H_{\mu\nu\lambda} \bar{\psi}^\mu \psi^\rho \bar{\psi}^\lambda \gamma_5 \psi^\nu + R_{\bar{h}} \frac{\lambda}{2\pi} \bigg) \bigg).
\end{aligned} \tag{4.36}
$$

The path integral is defined over all possible two-dimensional super-Riemannian manifolds with fixed punctures in the manifold $\mathcal{M}$ defined by the metric $G_{\mu\nu}$, as in Fig. 1. The super-diffeomorphism times super-Weyl invariance of the action in Eq. (4.36) implies that the correlation function is given by

$$
\Delta_F(\mathbf{X}_{\hat{D}_T f}; \mathbf{X}_{\hat{D}_T i} | \mathbf{E}_f; \mathbf{E}_i) = Z \int_{\mathbf{E}_i,\mathbf{X}_{\hat{D}_T i}}^{\mathbf{E}_f,\mathbf{X}_{\hat{D}_T f}} \mathcal{D}\mathbf{E}\mathcal{D}\mathbf{X}_{\hat{D}_T} e^{-\lambda\chi} e^{-S_s}, \tag{4.37}
$$

where

$$
\begin{aligned}
S_s &= \frac{1}{2} \int_{-\infty}^{\infty} d\tau \int d\sigma \sqrt{h(\sigma,\tau)} \bigg((h^{mn}(\sigma,\tau) G_{\mu\nu}(X(\sigma,\tau)) + i\varepsilon^{mn}(\sigma,\tau) B_{\mu\nu}(X(\sigma,\tau))) \partial_m X^\mu(\sigma,\tau) \partial_n X^\nu(\sigma,\tau) \\
&\quad + i G_{\mu\nu} \bar{\psi}^\mu \gamma^a \partial_a \psi^\nu + i \bar{\psi}^\nu (\Gamma_{\nu\mu\rho} + H_{\nu\mu\rho}\gamma_5) \gamma^a \partial_a X^\mu(\sigma,\tau) \psi^\rho - 2 G_{\mu\nu} \bar{\chi}_a \gamma^b \gamma^a \psi^\mu \partial_b X^\nu(\sigma,\tau) + \frac{1}{2} G_{\mu\nu} \bar{\psi}^\mu \psi^\nu \bar{\chi}_a \gamma^b \gamma^a \chi_b \\
&\quad + \frac{1}{6} R_{\mu\nu\lambda\rho} \bar{\psi}^\mu \psi^\lambda \bar{\psi}^\nu \psi^\rho - \frac{i}{3} H_{\mu\nu\rho} \bar{\chi}_a \gamma^b \gamma^a \bar{\psi}^\mu \psi^\nu \gamma_b \gamma_5 \psi^\rho - \frac{1}{4} D_\rho H_{\mu\nu\lambda} \bar{\psi}^\mu \psi^\rho \bar{\psi}^\lambda \gamma_5 \psi^\nu \bigg),
\end{aligned} \tag{4.38}
$$

and χ is the Euler number of the two-dimensional Riemannian manifold. For regularization, by renormalizing $\tilde{\phi}$, we divide the correlation function by the constant factor Z and by the volume of the super-diffeomorphism times the super-Weyl transformation $V_{\text{diff}\times\text{Weyl}}$. Equation (4.37) are the path integrals of perturbative superstrings on an arbitrary background that possess the supermoduli in the type IIA, type IIB, and SO(32) type I superstring theories for $T = $ IIA, IIB, and I, respectively [11,12,13]. In particular, in string geometry, the consistency of the perturbation theory around the background consisting of Eqs. (3.3), (3.27)–(3.30) determines $d = 10$ (the critical dimension).

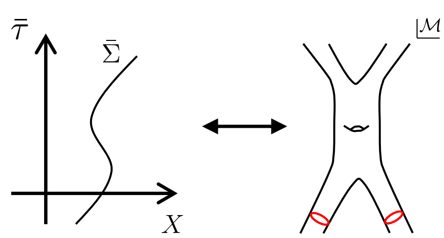

FIG. 1. A path and a super-Riemann surface. The line on the left is a trajectory in the path integral. The trajectory parametrized by $\bar{\tau}$ from $-\infty$ to ∞ represents a super-Riemann surface, with fixed punctures in $\mathcal{M}$ on the right.

V. CONCLUSION AND DISCUSSION

In this paper, in the string geometry theory, we fixed the classical part of the scalar fluctuation of the metric around the string background configurations, which are parametrized by the superstring backgrounds $G_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$. We showed that the two-point correlation functions of the quantum parts of the scalar fluctuation are path integrals of the perturbative superstrings on the string backgrounds. In this derivation, we moved from the second quantization formalism to the first one, where the coordinates of the two fields in the correlation functions become the asymptotic fields that represent the initial state $\mathbf{X}^\mu(\tau = -\infty, \sigma, \theta)$ and final state $\mathbf{X}^\mu(\tau = \infty, \sigma, \theta)$, respectively. All paths on the string manifolds from $\mathbf{X}^\mu(\tau = -\infty, \sigma, \theta)$ to $\mathbf{X}^\mu(\tau = \infty, \sigma, \theta)$ are summed up in the first quantization representation of the two-point correlation functions. Because the paths on the string manifolds are world sheets with genera as shown in Sec. VI of Ref. [1], they reproduce the path integrals of the perturbative strings up to any order, although the correlation functions are at tree level.

In this paper, we considered the classical backgrounds up to the first-order fluctuations around the flat background. Our next task is to consider the second- and higher-order classical fluctuations. Another task is to derive path integrals of the perturbative heterotic strings on all string backgrounds, $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $A_\mu(x)$, from the string geometry theory by considering the heterotic string manifolds.

Let us consider classical instanton effects in string geometry theory. If we substitute the string background configurations into the action, the terms that include RR fields are zero because of the Poincaré duality [5]. In the case of a constant dilaton $\bar{\phi}$, which is related to the string coupling constant g_s by $g_s = e^{\bar{\phi}}$, the action is given as $S = e^{-2\bar{\phi}}\bar{S}$, where $\bar{S} = \int \mathcal{D}\mathbf{E}\mathcal{D}\bar{\tau}\mathcal{D}\mathbf{X}_{\hat{D}_T}\sqrt{-\mathbf{G}}(\mathbf{R} + 4\nabla_{\mathbf{I}}\Phi\nabla^{\mathbf{I}}\Phi - \frac{1}{2}|\mathbf{H}|^2)$. Then, the partition function $Z = e^{-\frac{\bar{S}}{g_s^2}}$ has a correct dependence on the string coupling constant in the role of the nonperturbative effects in string theory, where $\bar{S}$ is identified with an instanton action itself. This fact, and the fact that one can derive the perturbative superstrings up to any order in the string coupling constant from string geometry theory at the tree level, suggest that string geometry theory can be defined by a classical limit of the path integral (2.1).

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APPENDIX: GREEN FUNCTION ON STRING GEOMETRY

In this appendix, we show that Eq. (3.45) is indeed a Green function on the flat superstring manifold. If $\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) \not\equiv \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta})$, we have

$$\begin{aligned}
&\frac{1}{\bar{e}'}\frac{\partial}{\partial \mathbf{X}_{\hat{D}_T\nu}(\bar{\sigma}',\bar{\theta}')}\mathcal{N}\left[\int d\bar{\sigma}d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}))^2\right]^{\frac{2-D}{2}} \\
&= (2-D)\mathcal{N}\left[\int d\bar{\sigma}d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}))^2\right]^{-\frac{D}{2}}\frac{\bar{e}'}{\bar{\mathbf{E}}'}\left(\mathbf{X}^\nu_{\hat{D}_T}(\bar{\sigma}',\bar{\theta}') - \mathbf{X}'^\nu_{\hat{D}_T}(\bar{\sigma}',\bar{\theta}')\right),
\end{aligned} \tag{A1}$$

and then,

$$\begin{aligned}
&\frac{1}{\bar{e}''}\frac{\partial}{\partial \mathbf{X}^\nu_{\hat{D}_T}(\bar{\sigma}'',\bar{\theta}'')}\frac{1}{\bar{e}'}\frac{\partial}{\partial \mathbf{X}_{\hat{D}_T\nu}(\bar{\sigma}',\bar{\theta}')}\mathcal{N}\left[\int d\bar{\sigma}\, d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}\left(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta})\right)^2\right]^{\frac{2-D}{2}} \\
&= d(2-D)\frac{1}{\bar{\mathbf{E}}'}\frac{\bar{e}'}{\bar{e}''}\mathcal{N}\left[\int d\bar{\sigma}\, d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}\left(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta})\right)^2\right]^{-\frac{D}{2}}\delta(\bar{\sigma}'-\bar{\sigma}'')\delta(\bar{\theta}'-\bar{\theta}'') \\
&\quad - D(2-D)\frac{1}{\bar{\mathbf{E}}'}\frac{1}{\bar{\mathbf{E}}''}\bar{e}'\bar{e}''\mathcal{N}\left[\int d\bar{\sigma}\, d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}\left(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta})\right)^2\right]^{-\frac{D+2}{2}} \\
&\quad \times \left(\mathbf{X}^\nu_{\hat{D}_T}(\bar{\sigma}',\bar{\theta}') - \mathbf{X}'^\nu_{\hat{D}_T}(\bar{\sigma}',\bar{\theta}')\right)\left(\mathbf{X}_{\hat{D}_T\nu}(\bar{\sigma}'',\bar{\theta}'') - \mathbf{X}'_{\hat{D}_T\nu}(\bar{\sigma}'',\bar{\theta}'')\right).
\end{aligned} \tag{A2}$$

Thus,

$$\begin{aligned}
&\int d\bar{\sigma}'\, d\bar{\theta}'\bar{\mathbf{E}}'\frac{1}{\bar{e}'}\frac{\partial}{\partial \mathbf{X}^\nu_{\hat{D}_T}(\bar{\sigma}',\bar{\theta}')}\frac{1}{\bar{e}'}\frac{\partial}{\partial \mathbf{X}_{\hat{D}_T\nu}(\bar{\sigma}',\bar{\theta}')}\mathcal{N}\left[\int d\bar{\sigma}d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}\left(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta})\right)^2\right]^{\frac{2-D}{2}} \\
&= d\int d\bar{\sigma}'\, d\bar{\theta}'\boldsymbol{\delta}(0)(2-D)\mathcal{N}\left[\int d\bar{\sigma}\, d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}\left(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta})\right)^2\right]^{-\frac{D}{2}} \\
&\quad - D(2-D)\mathcal{N}\left[\int d\bar{\sigma}d\bar{\theta}\frac{\bar{e}^2}{\bar{\mathbf{E}}}\left(\mathbf{X}^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta}) - \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\sigma},\bar{\theta})\right)^2\right]^{-\frac{D+2}{2}}\int d\bar{\sigma}'\, d\bar{\theta}'\frac{\bar{e}'^2}{\bar{\mathbf{E}}'}\left(\mathbf{X}^\nu_{\hat{D}_T}(\bar{\sigma}',\bar{\theta}') - \mathbf{X}'^\nu_{\hat{D}_T}(\bar{\sigma}',\bar{\theta}')\right)^2 = 0,
\end{aligned} \tag{A3}$$

where we use $D = d\int d\bar{\sigma}'\, d\bar{\theta}'\boldsymbol{\delta}(0)$. Hence, we find

$$\int d\bar{\sigma}' d\bar{\theta}' \bar{\mathbf{E}}' \frac{1}{\bar{e}'} \frac{\partial}{\partial \mathbf{X}^{\nu}_{\hat{D}_T}(\bar{\sigma}', \bar{\theta}')} \frac{1}{\bar{e}'} \frac{\partial}{\partial \mathbf{X}_{\hat{D}_T \nu}(\bar{\sigma}', \bar{\theta}')} \mathcal{N} \left[\int d\bar{\sigma} d\bar{\theta} \frac{\bar{e}^2}{\bar{\mathbf{E}}} (\mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\theta}) - \mathbf{X}'^{\mu}_{\hat{D}_T}(\bar{\sigma}, \bar{\theta}))^2 \right]^{\frac{2-D}{2}} = \delta(\mathbf{X}_{\hat{D}_T} - \mathbf{X}'_{\hat{D}_T}), \qquad \text{(A4)}$$

where $\mathcal{N}$ is a normalizing constant.

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