Entanglement islands in generalized two-dimensional dilaton black holes

Ming-Hui Yu* and Xian-Hui Ge†

Department of Physics, Shanghai University, Shanghai, 200444, China

(Received 26 September 2022; accepted 6 March 2023; published 22 March 2023)

The Fabbri–Russo model is a generalized model of a two-dimensional dilaton gravity theory with various parameters "n" describing various specific gravities. Particularly, the Russo–Susskind–Thorlacius gravity model fits the case $n = 1$. In the Fabbri–Russo model, we investigate Page curves and the entanglement island. Islands are considered in eternal and evaporating black holes. Surprisingly, in any black hole, the emergence of islands causes the rise of the entanglement entropy of the radiation to decelerate after the Page time, satisfying the principle of unitarity. For eternal black holes, the fine-grained entropy reaches a saturation value that is twice the Bekenstein–Hawking entropy. For evaporating black holes, the fine-grained entropy finally reaches zero. At late times and large distance limit, the impact of the parameter "n" is a subleading term and is exponentially suppressed. As a result, the shape of Page curves is "n" independent in the leading order, which also indicates the universality of the Page curve in generalized two-dimensional models. Furthermore, we discuss the relationship between islands and firewalls. We show that the island is a better candidate than firewalls for encountering the quantum entanglement-monogamy problem. Finally, we briefly review the gravity/ensemble duality as a possible resolution to the state paradox resulting from the island formula.

DOI: 10.1103/PhysRevD.107.066020

I. INTRODUCTION

One of the most important and urgent objectives of modern physics is studying the theory of quantum gravity. Black holes serve as fantastic study tools for this theory. When the quantum field theory is introduced into curved spacetime, black holes can emit Hawking radiation [1]. However, a series of issues also follow. One of the most famous issues is the black hole information loss paradox, or the information paradox for short, which was proposed by Stephen Hawking in 1976 [2]. The entanglement entropy of the radiation increases with time and is divergent at late times due to the continuous occurrence of the Hawking radiation. This is inconsistent with the unitary evolution of quantum mechanics (QM). Therefore, the information paradox challenges QM, general relativity (GR), thermodynamics, and many others fundamental fields of modern physics. A specific solution to this issue is to obtain the Page curve directly without the unitary evaporation hypothesis during the evaporation of a black hole [3,4]. Undoubtedly, the discovery of the anti–de Sitter (AdS)/conformal field theory (CFT) duality opens up a wider field of investigation for the information paradox [5]. Based on the view that the evolution of the pure high-energy state in the CFT is unitary, we insist that the AdS/CFT duality is right, implying the evolution of black holes in pure states in AdS spacetime is also unitary. Thus, the general consensus is that information is conserved for AdS black holes, although the precise specifics and methods are yet unknown. Moreover, quantum entanglement provides a starting point for addressing this issue. The Ryu–Takayanagi (RT) formula for the holographic entanglement entropy is not only a valid demonstration of the AdS/CFT dual conjecture, but also paves the way for the microscopic origin of the black hole entropy [6]. After considering the von Neumann entropy contributed by CFT at the boundary of the bulk spacetime, the RT formula is extended to the quantum RT formula [7] and the quantum extremal surface (QES) prescription with high-order quantum corrections [8]. Surprisingly, one obtains the opposite result of Hawking's calculation when using the QES prescription to calculate the entanglement entropy for AdS black holes [9]. At late times, the (quantum extremal) island locates near the event horizon. This construction renders the degree of freedom (d.o.f.) inside the island belonging to the d.o.f. of outside radiation. Therefore, the entanglement entropy decreases rather than increases at late times, which reproduces the unitary Page curve [10,11]. Physicists summarize this approach as the island paradigm [10,12]. The ansatz for

*yuminghui@shu.edu.cn
†Corresponding author.
gexh@shu.edu.cn

calculating the fine-grained entropy is called the island formula [12],

$$S_{\mathrm{rad}} = \mathrm{Min}\left\{\mathrm{Ext}\left[\frac{\mathrm{Area}(\partial I)}{4G_N} + S_{\mathrm{matter}}(I \cup R)\right]\right\}. \tag{1.1}$$

In the above equation, the lhs is the fine-grained entropy of the radiation; on the rhs, the terms inside the square bracket are called the generalized entropy of radiation, where ∂I represents the boundary of the island, and $I \cup R$ is the union of the island and the radiation. This equation instructs us to first evaluate the generalized entropy before extremizing it. The smallest value among candidates is the correct answer.

Besides, the island formula can also be derived equivalently from the gravitational path integral [13,14]. The nth order Renyi entropy is obtained by using the replica trick,

$$S_A^{(n)} = \frac{1}{1-n}\log[\mathrm{Tr}(\rho_A^{(n)})], \tag{1.2}$$

where ρ_A is the reduced density matrix. The von Neumann or fine-grained entropy is given by

$$S_{\mathrm{von}} = \lim_{n\to 1} S_A^{(n)} = -\mathrm{Tr}(\rho_A \log \rho_A). \tag{1.3}$$

Along with the original Hawking saddle, we also need to consider the replica wormholes saddle provided by the island while performing the explicit computation. The unitary result is produced at late times when the replica wormholes dominate.

The Page curve is first obtained by evaporating black holes in Jackiw–Teitelboim (JT) gravity [10,15,16]. Later work extends it to eternal black holes [17]. Furthermore, there are many interesting developments, such as some other two-dimensional (2D) asymptotically flat or AdS black holes [18–26], higher-dimensional black holes [27–54], charged black holes [55–61], and some models of the Universe [62–67]. Interestingly, there is also some work on complexity related to entanglement islands [68–70]. Here is an incomplete list because the field is always developing. Note that most works are performed in the background of eternal black holes. Although we can more easily understand the information paradox in eternal black holes, the actual evaporating black hole is the key to obtaining Page curves. Therefore, the Page curve for 2D evaporating black holes is the major focus of this paper.

A contender for the quantum gravity theory, the 2D gravity theory is crucial for studying black holes. On the one hand, 2D models are widely studied as they are symmetric and simpler to calculate analytically. On the other hand, they come from the four-dimensional theory through the dimensional reduction, and we hope that the keen insight from these models can be applied to higher-dimensional problems. One of the most famous models—the Callan–Giddings–Harvey–Strominger (CGHS) model [71] or the Russo–Susskind–Thorlacius (RST) model with considerations for backreaction effects [72]—has the advantage of being analytically solvable and offers a semiclassical theory of the backreaction of the Hawking radiation. It can describe the entire process of a black hole forming and evaporating (including the backreaction) in classical physics. Up to now, the Page curves for evaporating black holes has been reproduced only in special gravitational models [9,19–21]. A strong motivation is to study generalized 2D gravitational models to obtain the universality of Page curves. Therefore, this paper focuses on the generalized 2D theory of gravity—the Fabbri-Russo (FR) model [73,74]. This model represents a one-parameter family of exactly solvable models and includes the RST model as the special case. We use the island paradigm to reproduce the Page curve of this model and provide some more general conclusions for 2D models. We also revisit the firewall paradox from the perspective of the island paradigm. We show that the island is a better candidate than the firewall for solving the entanglement monogamy issue. Finally, we review the latest report on the state paradox and the gravity/ensemble duality.

This paper is organized as follows. In Sec. II, we review the FR model and give the formula for the entanglement entropy in 2D CFT. In Sec. III, we calculate the entanglement entropy of the radiation for evaporating black holes by the island formula (1.1). The results show that islands emerged at late times and led to a unitary Page curve. In Sec. IV, the same procedure is applied to the eternal black hole. The corresponding result is similar to the evaporating black hole. In Sec. V, we demonstrate the Page curves and the scrambling time based on the previous results. We also investigate the effect of the parameter n on these results. In Sec. VI, we discuss the relations between the island and the firewall and show that islands are a better candidate for solving the firewall paradox. Furthermore, based on the latest report, we briefly review the paradox related to the state of the Hawking radiation. The final discussion and conclusion are presented in Sec. VII. The Planck units are applied, $\hbar = c = k_B = 1$, throughout.

II. EXACTLY SOLVABLE 2D FR MODEL

We are inspired by the special 2D RST action [72] to obtain the generalized 2D gravity; we consider the following action [73]:

$$S_{\mathrm{cl}} = S_0 + S_{\mathrm{CFT}}, \tag{2.1a}$$

$$S_0 = \frac{1}{2\pi}\int d^2x\sqrt{-g}\left\{e^{-\frac{2\phi}{n}}\left[R + \frac{4}{n}(\nabla\phi)^2\right] + 4\lambda^2 e^{-2\phi}\right\}, \tag{2.1b}$$

$$S_{\mathrm{CFT}} = -\frac{1}{4\pi}\sum_{i=1}^{N}\int d^2x\sqrt{-g}(\nabla f_i)^2, \tag{2.1c}$$

where ϕ is a dilaton field, λ is the cosmological constant, f_i is a set of N massless scalar fields, and n is a parameter characterizing the different theory. Similar to the RST model [72], we choose the conformal gauge:

$$g_{\pm\pm} = 0, \qquad g_{\pm\mp} = -\frac{1}{2}e^{2\rho}. \tag{2.2}$$

The metric in this gauge is given by

$$ds^2 = -e^{2\rho}dx^+dx^-, \tag{2.3}$$

with the light cone coordinate $x^\pm = x^0 \pm x^1$.

After a field redefinition, the action (2.1a) can be written in the "free field" form [73],

$$S_{\rm cl} = \frac{1}{\pi}\int d^2x\left[\frac{1}{\kappa}(-\partial_+\chi\partial_-\chi + \partial_+\Omega\partial_-\Omega) + \lambda^2 e^{2(\chi-\Omega)/\kappa} + \frac{1}{2}\sum_{i=0}^{N}\partial_+ f_i\partial_- f_i\right], \tag{2.4}$$

where

$$\kappa = \frac{N}{12}, \tag{2.5a}$$

$$\chi = \kappa\rho + e^{-\frac{2\phi}{n}} + \left(\frac{1}{2n} - 1\right)\kappa\phi, \tag{2.5b}$$

$$\Omega = e^{-2\phi/n} + \frac{\kappa}{2n}\phi, \tag{2.5c}$$

where n is a real positive number. The special case $n = 1$ corresponds to the RST model [72]. For simplicity, in this paper, we only consider the parameter range[1] $0 < n < 2$, while for the case of critical values $n = 0$ and $n = 2$, we discuss them in the Appendix.

One should also note that for real ϕ, Ω has a lower value,

$$\Omega \geq \Omega_{\rm crit} = \frac{\kappa}{4} - \frac{\kappa}{4}\log\frac{\kappa}{4}. \tag{2.6}$$

Here, we only give a qualitative physical interpretation. As the 2D gravity theory results from the dimensional reduction of the four-dimensional (4D) theory, Ω corresponds to the area of the transverse two-sphere surface. The condition (2.6) allows us to patch the vacuum at the endpoint of evaporation. Therefore, the curve (2.6) is converted from a timelike curve to a spacelike curve, i.e., this critical curve can be regarded as the boundary of spacetime, which is similar to the curve $r = 0$ in the spherically symmetric reduction of 4D Minkowski spacetime. The explicit calculation is given in the Appendix.

[1]For $n < 0$, the geometry does not exist. When $n > 2$, the corresponding geometry is very different, where the singularity is null. One can refer to [74] for details.

The equations of motion (EOM) derived by (2.4) are

$$\partial_+\partial_-(\chi - \Omega) = 0, \qquad \partial_+\partial_-\chi = -\lambda^2 e^{2(\chi-\Omega)/\kappa}. \tag{2.7}$$

We choose the "Kruskal" gauge, where

$$\chi = \Omega. \tag{2.8}$$

Then, it implies that

$$\phi = \rho_K, \tag{2.9}$$

and

$$\Omega = e^{-2\rho_K/n} + \frac{\kappa}{2n}\rho_K, \tag{2.10}$$

where ρ is denoted as ρ_K in the Kruskal coordinate. In this gauge, the most general static solution for (2.7) is

$$\Omega = \chi = -\lambda^2 x^+x^- + Q\log(\lambda^2 x^+x^-) + \frac{M}{\lambda}, \quad Q, M = \text{constant}. \tag{2.11}$$

Afterward, we set $\lambda \equiv 1$ for convenience.

A. Vacuum solution

The simplest case of the above solution is the vacuum, where the Ricci scalar R and the integrate constant M vanish. Accordingly, the conformal factor is[2]

$$e^{-2\rho/n} = e^{-2\phi/n} = -x^+x^-, \tag{2.12a}$$

$$\Omega_{\rm vac} = -x^+x^- - \frac{\kappa}{4}\log(-x^+x^-). \tag{2.12b}$$

The most general theory that differs from and incorporates the RST model may be found. Referring to the RST model [72], the general theory obeys the following requirements. The action should contain an RST-like term and a conformal anomaly term to describe the backreaction effect, resulting in the effective action for the large N limit, as indicated by the sum [73],

$$\begin{aligned} S_{\rm eff} &= S_{\rm cl} + S_{\rm RST} + S_{\rm anom} \\ &= \frac{1}{2\pi}\int d^2x\sqrt{-g}\left[e^{-2\phi/n}\left(R + \frac{4}{n}(\nabla\phi)^2\right) + 4e^{-2\phi} \right. \\ &\quad \left. - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2 + \kappa\left(\frac{1-2n}{2n}\phi R + \frac{n-1}{n}(\nabla\phi)^2 - \frac{1}{4}R\Box^{-1}R\right)\right]. \end{aligned} \tag{2.13}$$

[2]For convenience, we omit the subscript "K" for the Kruskal coordinate, henceforth.

In the last line, the first two terms are one-loop quantum correction terms similar to the RST model ($n = 1$), and the last term is a nonlocal Polyakov term, where the symbol $\Box^{-1}$ is the scalar Green function. In the conformal gauge (2.2), it has a simple form: $\Box^{-1}R = 2\rho$.

We first consider the classical limit, in which $\hbar$ is turned to zero. By recovering the effective action (2.13) with $\hbar$, the last three terms vanish. Then, we obtain the action (2.1b). The EOM in the conformal gauge (2.2) are derived by variation of the metric,

$$-\frac{4}{n}\partial_+\phi\partial_-\phi + \frac{2}{n}\partial_+\phi\partial_-\phi - e^{\frac{2-2n}{n}\phi+2\rho} = 0, \tag{2.14a}$$

$$\frac{n}{2}\partial_+\phi\partial_-\phi + \frac{4}{n^2}\partial_+\phi\partial_-\phi - \frac{4}{n}\partial_+\phi\partial_-\phi + e^{\frac{2-2n}{n}\phi+2\rho} = 0, \tag{2.14b}$$

$$\partial_+\partial_- f_i = 0. \tag{2.14c}$$

The constraint equation is

$$e^{-2\phi/n}\left[\frac{4}{n}\left(1-\frac{1}{n}\right)\partial_\pm\phi\partial_\pm\phi + \frac{2}{n}\partial_\pm^2\phi - \frac{4}{n}\partial_\pm\rho\partial_\pm\phi\right] - \frac{1}{2}\sum_{i=0}^{N}\partial_\pm f_i\partial_\pm f_i = 0. \tag{2.15}$$

The expressions (2.14a) and (2.14b) imply that

$$\frac{2}{n}\partial_+\partial_-(\rho-\phi) = 0. \tag{2.16}$$

Thus, we can still preserve the Kruskal gauge (2.9), for which $\rho = \phi$. The left expressions take the following form:

$$\partial_+\partial_-(e^{-2\phi/n}) = -1, \qquad \partial_\pm^2(e^{-2\phi/n}) = -\frac{1}{2}\sum_{i=0}^{N}\partial_\pm f_i\partial_\pm f_i. \tag{2.17}$$

At last, the general solution is obtained by [see (2.11)]

$$e^{-2\phi/n} = e^{-2\rho/n} = -x^+x^- + Q\log(-x^+x^-) + M. \tag{2.18}$$

In fact, the constant M is related to the mass of the black hole $M_{\rm BH}$, $M = \pi M_{\rm BH}$. The constant Q can be interpreted as the incoming and outgoing energy flux. If we substitute (2.15) into (2.16), then the stress tensor is given by $T_{\pm\pm} = \frac{Q}{(x^\pm)^2}$.

B. Static black holes

For a static black hole with $Q = 0$, which corresponds to an eternal black hole with unchanged temperature,[3] or a black hole in thermal equilibrium with the thermal bath, the solution takes the following form:

$$e^{-2\phi/n} = e^{-2\rho/n} = M - x^+x^-, \tag{2.19a}$$

$$\Omega_{\rm ete} = -x^+x^- + M - \frac{\kappa}{4}\log(-x^+x^- + M). \tag{2.19b}$$

According to (2.6), to ensure the singularity at $\Omega = \Omega_{\rm crit}$ inside the apparent horizon, the mass parameter M is required to be $M > \frac{\kappa}{4}$. Besides, the event horizon at $x^+x^- = 0$. Because the factor Ω can be viewed as the area of a black hole in 2D gravity, we can obtain the location of the apparent horizon by $\partial_+\Omega = 0$, which leads to[4] $x^- = 0$. Therefore, for an eternal black hole, the apparent horizon and the event horizon overlap at $x^+x^- = 0$.

C. Dynamical black holes

Let us now return to the dynamical black hole that forms due to the collapse of a spherical shell of photons. Using the constraint equation (2.15), we can obtain the general solution regarding some physical quantities, such as the Kruskal momentum and energy [72],

$$P_+(x^+) = \int_0^{x^+} dx^+ T_{++}(x^+), \tag{2.20}$$

and

$$M(x^+) = \int_0^{x^+} dx^+ x^+ T_{++}(x^+). \tag{2.21}$$

Then we obtain

$$e^{-2\phi/n} = e^{-2\rho/n} = -x^+[x^- + P_+(x^+)] + M(x^+). \tag{2.22}$$

The above equation provides a singularity at $M(x^+) - x^+(x^- + P_+(x^+)) = 0$. The event horizon is located at $x_H^- + P_+(\infty) = 0$. Now, we consider an incoming shock wave with the stress tensor $T_{++} = \frac{1}{2}\sum_{i=0}^{N}\partial_+ f_i \partial_- f_i = a\delta(x^+ - x_0^+)$ at x_0^+. Then, geometries can be patched along this null trajectory. The corresponding solution is satisfied by

$$e^{-2\phi/n} = e^{-2\rho/n} = -\frac{M}{x_0^+}(x^+ - x_0^+)\theta(x^+ - x_0^+) - x^+x^-, \tag{2.23a}$$

$$\Omega_{\rm eva} = -x^+x^- - \frac{M}{x_0^+}(x^+ - x_0^+)\theta(x^+ - x_0^+) - \frac{\kappa}{4}\log(-x^+x^-), \tag{2.23b}$$

[3]The temperature of the FR model is special and always a constant (2.29).

[4]The other solution is $x^+x^- = M - \frac{\kappa}{4}$. However, under the limitation of a large mass, this location is near the singularity, which is unphysical and can be discarded.

where θ is a step function, and the parameter a is given by $a = \frac{M}{x_0^+}$. Therefore, it is clear that below the region $x^+ < x_0^+$, the geometry is vacuum (2.12b). In contrast, on the region $x^+ > x_0^+$, the geometry corresponds to the static black hole (2.19b) discussed before.

Similarly, this solution has an apparent horizon, which is defined by $\partial_+\Omega = 0$. Thus, the curve of the apparent horizon is given by[5]

$$(M + x^-)x^+ = -\frac{\kappa}{4}. \tag{2.24}$$

We now calculate the coordinate of the endpoint of evaporation, which is the intersection of the critical curve (2.6) and the curve of the apparent horizon (2.24). In addition, since the critical curve (2.6) as the boundary of spacetime, the coordinate of the end point also satisfies equation (2.6), we combine these two conditions and obtain

$$(M + x^-_{\rm end})x^+_{\rm end} = -\frac{\kappa}{4}, \tag{2.25}$$

and

$$-x^+_{\rm end}x^-_{\rm end} - M(x^+_{\rm end} - 1) - \frac{\kappa}{4}\log(-x^+_{\rm end}x^-_{\rm end}) = \frac{\kappa}{4} - \frac{\kappa}{4}\log\frac{\kappa}{4}. \tag{2.26}$$

At large mass limit, the coordinates of the endpoint are given by

$$x^+_{\rm end} = \frac{(e^{\frac{4M}{\kappa}} - 1)\kappa}{4M} \simeq \frac{\kappa e^{\frac{4M}{\kappa}}}{4M}, \tag{2.27a}$$

$$x^-_{\rm end} = \frac{Me^{\frac{4M}{\kappa}}}{1 - e^{\frac{4M}{\kappa}}} \simeq -M. \tag{2.27b}$$

The singularity becomes naked after the endpoint, which contradicts the cosmic censorship hypothesis. Spacetime as a whole is ill-defined. However, we ignore these considerations and only focus on the geometry before evaporation. Moreover, one should note that the curve (2.27b) also represents the event horizon because the apparent and event horizons are coincident at the endpoint.

D. Coarse-grained and fine-grained entropy

We briefly discuss two types of entropies and the related expressions.[6] The thermodynamic entropy, or the Bekenstein–Hawking entropy, is just the coarse-grained entropy. For 2D gravity, the Bekenstein–Hawking entropy can be derived by the Wald method,

$$S_{\rm BH} = \frac{4\pi}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial R}\bigg|_{\rm horizon} = e^{-2\phi_H/n} = 2M, \tag{2.28}$$

where $\mathcal{L}$ is the Lagrangian of the action (2.1b), and ϕ_H is denoted by the value of dilaton at the event horizon. Therefore, the Hawking temperature is given by

$$T_H = \frac{1}{2\pi}. \tag{2.29}$$

Now recall the island formula (1.1), in which its complement is the fine-grained entropy of the *black hole* based on the complementary of von Neumann entropy. In this sense, the generalized entropy of a black hole is the Bekenstein–Hawking entropy plus the entropy contributed by matter fields surrounding the black hole's spacetime. For the matter field, the entanglement entropy, or equivalently, the von Neumann entropy in vacuum CFT in flat spacetime $ds^2_{\rm flat} = -dx^+dx^-$ is given by[7] [75]

$$S_{\rm matter} = \frac{N}{3}\log[d(A, B)], \tag{2.30}$$

where $d(A, B)$ is the proper distance between points A and B in flat spacetime, which is written by

$$d(A, B) = \sqrt{[x^+(A) - x^+(B)][x^-(B) - x^-(A)]}. \tag{2.31}$$

For the curved 2D metric $ds^2_{2D} = -e^{2\rho(x^+,x^-)}dx^+dx^-$, the vacuum expectation value of normal ordering stress tensor can be written in terms of function $t_\pm$,

$$\langle\psi|{:}T_{\pm\pm}(x^\pm){:}|\psi\rangle = -\frac{1}{12}t_\pm(x^\pm). \tag{2.32}$$

Taking a conformal reparametrization for $x^\pm \to y^\pm$, the conformal factor transforms into

$$\rho(y^\pm) = \rho(x^\pm) - \frac{1}{2}\log\frac{dy^+}{dx^+}\frac{dy^-}{dx^-}, \tag{2.33}$$

and the function $t_\pm$ transforms a Weyl rescaling,

$$t_\pm(y^\pm) = \left(\frac{dx^\pm}{dy^\pm}\right)t_\pm(x^\pm) + \frac{1}{2}\{x^\pm, y^\pm\}, \tag{2.34}$$

[5]In the large mass limit, we can set $x_0^+ = 1$ to rescale the parameter a.

[6]Strictly speaking, the fine-grained entropy at the finest level should be a constant in time in a black hole without a bath. In this sense, the fine-grained entropy here is only more fine than the coarse-grained entropy and not so fine-grained. We thank Hao Geng for providing this comment.

[7]Here CFT with the central charge $c = N$ is minimally coupled to gravity. In addition, note that this expression is renormalized. For 2D CFT, the entanglement entropy for a subsystem in a constant-time slice follows the logarithmic law: $S = \frac{c}{3}\log\frac{\ell}{\epsilon_{\rm uv}}$, where ℓ is the length of the subsystem, and $\epsilon_{\rm uv}$ is the UV cutoff. In what follows, we will take the finite part of this expression for convenience.

with the Schwarzian derivative,

$$\{x^{\pm}, y^{\pm}\} \equiv \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2. \tag{2.35}$$

Then, we can map the vacuum state from the flat metric ds^2_{flat} to the curved metric ds^2_{2D} by the Weyl transformation. Thus, (2.32) vanishes, then the function $t_{\pm}(x^{\pm})$ is zero. We further examine the behavior of von Neumann entropy under this transformation [9],

$$S_{W^{-2}g} = S_g - \frac{N}{6}\sum_{\text{endpoints}} \log W, \tag{2.36}$$

where $W^{-2} = e^{-2\rho}$, and g is the metric. Finally, we obtain the general expression for entanglement entropy of the single interval $[A, B]$ in 2D spacetime [19],

$$S_{\text{matter}} = \frac{N}{6}\log[d^2(A,B)e^{\rho(A)}e^{\rho(B)}]|_{t_{\pm}=0}. \tag{2.37}$$

Actually, similar to (2.30), this expression is also renormalized. Consequently, we still maintain the finite part and use it to calculate the entanglement entropy with and without islands in Secs. III and IV.

III. ENTANGLEMENT ENTROPY FOR EVAPORATING BLACK HOLES

In this section, we use the island formula (1.1) to calculate the entanglement entropy of Hawking radiation for the evaporating black hole. We first consider the geometry without islands and then focus on the construction with an island. The Penrose diagram of evaporating black holes is shown in Fig. 1.

A. Without island

In the absence of islands, the matter part of entanglement entropy S_{matter} is only contributed by radiation. We assume that a black hole is formed by a collapsing shell of photons in pure states. Then, according to the complementary, we have $S_{\text{matter}}(R) = S_{\text{matter}}(C)$. In the case without islands, we choose the reference point $O(x_o^+, x_o^-)$ at the boundary. The corresponding entropy $S_{\text{matter}}(C)$ can be calculated by (2.37)

$$S_{\text{matter}}(C) = \frac{N}{6}\log[d^2_{\text{in}}(A,O)e^{\rho_{\text{in}}(A)}e^{\rho_{\text{in}}(O)}]. \tag{3.1}$$

The meaning of the subscript "in" will be explained later. For the convenience of calculation, we first introduce the following "ingoing" and "outgoing" coordinate frames:

$$\text{near } \mathcal{J}^-: \; x^+ = +e^{+\omega^+}, \qquad x^- = -e^{-\omega^-}, \tag{3.2a}$$

$$\text{near } \mathcal{J}^+: \; x^+ = +e^{\sigma^+}, \qquad x^- = -e^{-\sigma^-} - M. \tag{3.2b}$$

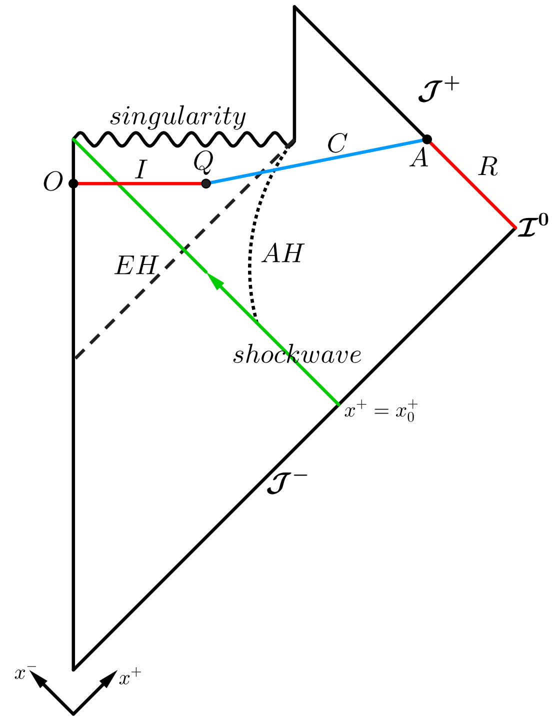

FIG. 1. The Penrose diagram of an evaporating black hole formed by null shell collapsing at $x^+ = x_0^+ (= 1)$. "EH" and "AH" represent the event and apparent horizons, respectively. $\mathcal{I}^0$ and $\mathcal{J}^{\pm}$ are denoted as the spacelike infinity and the future/past null infinity. The island is at point Q. The boundary of the radiation, i.e., the cutoff surface, is A. In the late time and the large distance limit, A is approximately at $\mathcal{J}^+$, where the asymptotic observer collects the Hawking radiation. The region C is the complementary of the union region $I \cup R$.

In the region $x^+ < 1$, the vacuum solution (2.12a) in the frame $\{\omega^{\pm}\}$ takes the form

$$ds^2 = -(e^{\omega^+ - \omega^-})^{1-n} d\omega^+ d\omega^-. \tag{3.3}$$

We can see that it corresponds to the Minkowski spacetime for $n = 1$. However, for $n \neq 1$, it corresponds to the Rindler spacetime.[8]

In the region $x^+ > 1$, the metric (2.23a) in the frame $\{\sigma^{\pm}\}$ can be written as

$$ds^2 = -\frac{e^{\sigma^+ - \sigma^-}}{(M + e^{\sigma^+ - \sigma^-})^n} d\sigma^+ d\sigma^-. \tag{3.4}$$

Again, near the $\mathcal{J}^+$ ($\sigma^+ \to \infty$), we find that the above metric approaches the Rindler metric $ds^2(\sigma^+ \to \infty) \simeq -(e^{\sigma^+ - \sigma^-})^{1-n} d\sigma^+ d\sigma^-$ unless $n = 1$.

[8] We can take a coordinate transformation: $\omega^{\pm} = \eta \pm \xi$. Then the metric (3.3) becomes $ds^2 = -e^{2(1-n)\xi}(d\eta^2 - d\xi^2)$, which is the standard Rindler metric for $n \neq 1$.

Following [76], we impose a reflecting condition $f_i|_{\Omega=\Omega_{\rm crit}} = 0$ on the matter fields f_k, where $\Omega = \Omega_{\rm crit}$ represents the boundary of spacetime.[9] In this way, one can consider the quantum state of the incoming matter as a coherent state that built on the vacuum state at $\mathcal{J}^-$. Therefore, the entanglement entropy for the matter field in this state is identical to the von Neumann entropy of the vacuum state. Thus, the subscript in means that these quantities can be evaluated in the ingoing frame $\{\omega^\pm\}$ (see Fig. 2). Then the formula (3.1) can be calculated in the ingoing frame, in which

$$e^{2\rho_{\rm in}(o)} = (e^{\omega^+(o)-\omega^-(o)})^{1-n} = (-x_o^+x_o^-)^{1-n}, \tag{3.5}$$

$$e^{2\rho_{\rm in}(A)} = \frac{-x_A^+x_A^-}{(M - x_A^+x_A^- - Mx_A^+)^n}, \tag{3.6}$$

$$\begin{aligned} d^2(A,O) &= (\omega^+(A) - \omega^+(O))(\omega^-(O) - \omega^-(A)) \\ &= \log\frac{x_A^+}{x_O^+}\log\frac{x_A^-}{x_O^-}. \end{aligned} \tag{3.7}$$

Substituting these equations into the expression (3.1), we obtain

$$\begin{aligned} S_{\rm rad}(\text{no island}) &= \frac{N}{12}\log\left[\left(\log\frac{x_A^+}{x_O^+}\log\frac{x_A^-}{x_O^-}\right)^2(-x_O^+x_O^-)^{1-n}\frac{-x_A^+x_A^-}{(M - x_A^+x_A^- - Mx_A^+)^n}\right] \\ &\simeq \frac{N}{12}\log\left[\frac{-x_A^+x_A^-}{(M - x_A^+x_A^- - Mx_A^+)^n}\right] \\ &= \frac{N}{12}\log(1 + Me^{\sigma_A^-}) + \frac{N}{12}\log[(\sigma_A^+ - \sigma_A^-)^{1-n}] \\ &\simeq \frac{N}{12}\sigma_A^- + (1-n)\frac{N}{12}\log(\sigma_A^+). \end{aligned} \tag{3.8}$$

As the cutoff surface is near $\mathcal{J}^+$ ($x_A^+ \to \infty$), all dependence on $x_O^\pm$ can be neglected. In addition, we rewrite the result in the outgoing coordinate σ_A^- that represents the affine retarded time for the asymptotic observer. Interestingly, we find that the second term in the above result (3.8) is vanishing for the special RST ($n = 1$) model. The remanent first term is consistent with [19,20]. On the contrary, the second term is logarithmically divergent for the usual FR model ($n \neq 1$). Nevertheless, notice that all expressions for the entanglement entropy in calculations is renormalized [see (2.30) and (2.37)]. Thus, this divergent indeed can be absorbed in the UV cutoff term $\log\frac{1}{\epsilon_{\rm uv}}$. Then if we rescale the cutoff $\epsilon_{\rm uv}$, then the second n-dependence term can be removed [20]. For this reason, we only focus on the behavior of the first term.

Now, the entanglement entropy increases linearly with the retarded time $S \propto \frac{N}{12}\sigma^-$. It does not cause any issue at early times since the black hole has just formed, and only a small amount of radiation is emitted. However, at late times, as more and more Hawking pairs appear, the entanglement entropy without island violates the unitary principle. At late times, the fine-grained entropy should decrease. In addition, in order to maintain the entropy bound, it can never exceed the Bekenstein–Hawking entropy $S_{\rm BH}$ [77]. In particular, on the one hand, $S_{\rm BH}$ is very small because the area of the event horizon of a black hole is small at late times. On the other hand, the entropy in the black hole interior is much larger than $S_{\rm BH}$ because the amount of external radiation that entangled with the interior is large. Due to the difference between the two kinds of

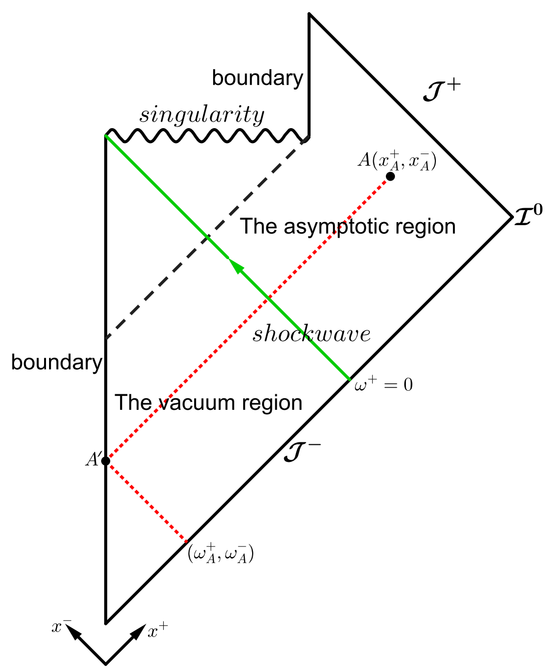

FIG. 2. The diagrammatic sketch of the reflecting condition. The coordinate (x_A^+, x_A^-) of a point A on the asymptotic region where $x^+ > 1$ is reflected to $\mathcal{J}^-$ by the timelike boundary $\Omega = \Omega_{\rm cirt}$. In such way, the coordinate of A is mapped to the ingoing frame (3.2a).

[9]Although this method is only mentioned in the RST model [19,20], subsequent calculations show that this method is also applicable to the FR model.

entropy, one cannot attribute the entropy inside the black hole to the Bekenstein–Hawking entropy of the black hole, which is known as the "bag of gold" paradox [12].

B. With island

We now introduce a modified semiclassical theory—the island paradigm—to reconcile the contradiction between the result (3.8) and the fundamental principles of QM. We use the island formula (1.1) to recalculate the entanglement entropy of the radiation for the construction with an island and eventually reproduce a unitary Page curve.

We still insist that the cutoff surface A is near $\mathcal{J}^+$ and assume that the island Q is located at the asymptotic region where $x^+ > 1$. For this construction, the union region $I \cup C \cup R$ (see Fig. 1) is considered the full Cauchy slice. According to the island formula (1.1), the corresponding entropy of matter fields is determined by the union $I \cup R$, which is written as follows [78]:

$$S_{\mathrm{matter}}(R\cup I) = \frac{N}{6}\log\left[e^{\rho_{\mathrm{in}}(A)}e^{\rho_{\mathrm{in}}(Q)}\frac{d^2(A,Q)d(A,A')d(Q,Q')}{d(A',Q)d(A,Q')}\right], \tag{3.9}$$

where A' and Q' are the reflection points on the boundary. Although the expression looks complicated, we only focus on the behavior of the Page curve at late times. At late times and large distances, the following approximate relation exists:

$$d(A,A') \simeq d(Q,Q') \simeq d(A,Q) \simeq d(A',Q') \gg d(A,Q) \simeq d(A',Q'). \tag{3.10}$$

Then, we obtain

$$S_{\mathrm{matter}}(R\cup I) \simeq \frac{N}{6}\log[e^{\rho_{\mathrm{in}}(A)}e^{\rho_{\mathrm{in}}(Q)}d^2(A,Q)] = \frac{N}{12}\log\left[\left(\log\frac{x_A^+}{x_Q^+}\log\frac{x_A^-}{x_Q^-}\right)^2\frac{x_A^+x_A^-}{(M-x_A^+x_A^- - Mx_A^+)^n}\frac{x_Q^+x_Q^-}{(M-x_Q^+x_Q^- - Mx_Q^+)^n}\right]. \tag{3.11}$$

We then recall the Bekenstein–Hawking entropy S_{BH} (2.28), which corresponds to a quarter of the area of the black hole in 4D spacetime. Thus, we obtain the analogy of the gravity part,

$$S_{\mathrm{grav}} = \frac{\mathrm{Area}(\partial I)}{4G_N} = 2(\Omega_{\mathrm{eva}}(Q) - \Omega_{\mathrm{crit}}). \tag{3.12}$$

Here, we introduce Ω_{crit} to ensure that the area of the island vanishes at the boundary. Correspondingly, the generalized entropy of the radiation is

$$\begin{aligned}S_{\mathrm{gen}} &= 2(\Omega_{\mathrm{eva}}(Q) - \Omega_{\mathrm{crit}}) + S_{\mathrm{matter}}(R\cup I)\\ &= 2M\left[\left(1-\frac{x_Q^+x_Q^-}{M}-x_Q^+\right)-\frac{\kappa}{4M}\log(-x_Q^+x_Q^-)\right]\\ &\quad+\frac{N}{12}\log\left[\left(\log\frac{x_A^+}{x_Q^+}\log\frac{x_A^-}{x_Q^-}\right)^2\frac{x_A^+x_A^-}{(M-x_A^+x_A^- - Mx_A^+)^n}\frac{x_Q^+x_Q^-}{(M-x_Q^+x_Q^- - Mx_Q^+)^n}\right].\end{aligned} \tag{3.13}$$

Extremizing the above expression with respect to x_Q^+ and x_Q^- yields the following equations:

$$\frac{\partial S_{\mathrm{gen}}}{\partial x_Q^-} = -2M\left(1+\frac{x_Q^-}{M}\right) - \frac{N(M+(n-1)(M+x_Q^-)x_Q^+)}{12x_Q^+(-M+(M+x_Q^-)x_Q^+)} - \frac{N}{6\,\log(\frac{x_A^+}{x_Q^+})x_Q^+} = 0, \tag{3.14a}$$

$$\frac{\partial S_{\mathrm{gen}}}{\partial x_Q^+} = -2Mx_Q^+ - \frac{N(M-(M+(1-n)x_Q^-)x_Q^+)}{12x_Q^-(-M+(M+x_Q^-)x_Q^+)} - \frac{N}{6\log(\frac{x_A^-}{x_Q^-})x_Q^-} = 0. \tag{3.14b}$$

As the point A nears $\mathcal{J}^+$, i.e., $x_A^+ \to \infty$, the last term in (3.14a) can be omitted. We also make the near event horizon limitation assumption to obtain the location of the island, where $x_Q^- \simeq -M$. Solving these two equations in the near event horizon limit, we obtain the following equation from (3.14a):

$$x_Q^- \simeq -M + \frac{N}{24x_Q^+}, \qquad x_Q^+ \simeq \frac{N}{24(M+x_Q^-)}. \tag{3.15}$$

The large mass limit suppresses all n-dependent terms in the above solution. In this assumption, we also discover that the location of the island $x_Q^- \simeq -M + \mathcal{O}(\frac{1}{x_Q^+})$ is indeed very close to the event horizon and inside the event horizon [see (2.27b)]. Then, substituting the solution (3.15) into (3.14b), we obtain

$$\log\left(\frac{x_A^-}{x_Q^-}\right) = \frac{2(M + x_Q^-)}{M}, \qquad x_A^- = -M - \frac{N}{24x_Q^+}. \tag{3.16}$$

On the other side, comparing (3.16) with the outgoing coordinate $x_A^- = -e^{\sigma_A^-} - M$ (3.2b), the following approximations are obtained as

$$\sigma_A^- = \log\frac{24x_Q^+}{N} \sim \frac{M}{N}, \qquad x_Q^+ \sim Ne^{\frac{M}{N}}. \tag{3.17}$$

The first approximation is made possible because the lifetime of black holes and asymptotic observer's retarded time σ_A^- are of the same order. From (2.27b), the lifetime is

$$\sigma_A^- \sim \sigma_{\text{end}}^- = -\log(-x_{\text{end}}^- - M) = \frac{4M}{\kappa} - \log M \simeq \frac{48M}{N} \sim \frac{M}{N}. \tag{3.18}$$

Finally, the generalized entropy after the extremization is given by (3.13), (3.15), (3.16), and (3.17),

$$\begin{aligned} S_{\text{gen}} &\simeq 2M - \frac{N}{12} - \frac{N}{24}\log\left(NMe^{\frac{M}{N}} - \frac{N}{24}\right) + \mathcal{O}(\log(M^{2-2n})), \\ &\simeq 2M - \frac{N}{24}\sigma_A^- = S_{\text{BH}} - \frac{N}{24}\sigma_A^-, \end{aligned} \tag{3.19}$$

where the parameter n is exponentially suppressed at the large mass and late times limit. Therefore, we obtain the entanglement entropy with the island as a function of the retarded time σ^-. Combining the result (3.8), we have

$$\begin{aligned} S_{\text{rad}} &= \text{Min}[S_{\text{gen}}(\text{no island}), S_{\text{gen}}(\text{island})] \\ &= \text{Min}\left(\frac{N}{12}\sigma^-, 2M - \frac{N}{24}\sigma^-\right), \end{aligned} \tag{3.20}$$

which leads to a unitary Page curve.

In conclusion, we reproduce the Page curve of an evaporating black hole using the island paradigm. At early times, the QES is a trivial surface lying on the boundary, which results in an increase in entanglement entropy curve that is linearly proportional to the retarded time (3.8). However, the island appears inside the event horizon after the Page time. The QES now is a nontrivial surface, leading to a transition in the entropy. At late times, the entanglement entropy is approximately equal to the Bekenstein–Hawking entropy and drops to zero when the black hole is evaporated. This result is also consistent with the Page curve derived from the Page theorem [79].

IV. ENTANGLEMENT ENTROPY FOR ETERNAL BLACK HOLES

In this section, we repeat the calculation procedure in Sec. III to calculate the Page curve for an eternal black hole. The information paradox for eternal black holes is a deformed version of the black hole information paradox. The corresponding solution is more strict: at late times, a black hole in the Hartle–Hawking state has infinite entanglement with the outside thermal Hawking radiation, which far exceeds the entropy bound of a black hole [77]. Similarly, we first calculate the entanglement entropy without the island and then consider the island.

A. Without island

The Penrose diagram for the eternal black hole is shown in the Fig. 3. We only calculate the entanglement entropy of the union $R \cup R'$ for the no-island construction. According to the complementary of von Neumann entropy, the entropy for the matter part is

$$S_{\text{matter}}(R \cup R') = \frac{N}{6}\log[e^{\rho(A)}e^{\rho(A')}d^2(A, A')]. \tag{4.1}$$

Then, we transform to the asymptotically flat frame and make the following coordinate transformation:

$$\text{The right wedge: } x^+ = +e^{y+t}, \qquad x^- = -e^{y-t}, \tag{4.2a}$$

$$\text{The left wedge: } x^+ = -e^{y-t}, \qquad x^- = +e^{y+t}, \tag{4.2b}$$

where t and y are time and spatial coordinates for the asymptotic observer near $\mathcal{J}^+$, respectively. For the eternal solution, we obtain the following expression of the entanglement entropy from (2.19a):

$$\begin{aligned} S_{\text{rad}}(\text{no island}) &= S_{\text{matter}}(R \cup R') \\ &= \frac{N}{12}\log\left[\frac{(x_A^+ - x_{A'}^+)^2(x_A^- - x_{A'}^-)^2}{(M - x_A^+ x_A^-)^n(M - x_{A'}^+ x_{A'}^-)^n}\right] \\ &= \frac{N}{6}\log(4\cosh^2 t_A) \\ &\quad + \frac{N}{6}\log\left[\frac{e^{2y_A}}{(e^{2y_A} + M)^{\frac{1+n}{2}}}\right]. \end{aligned} \tag{4.3}$$

Similarly, we only consider the behavior of the entanglement entropy at late times. So at late times and large distance limits, the second n-dependence term can be neglected. Then the expression becomes

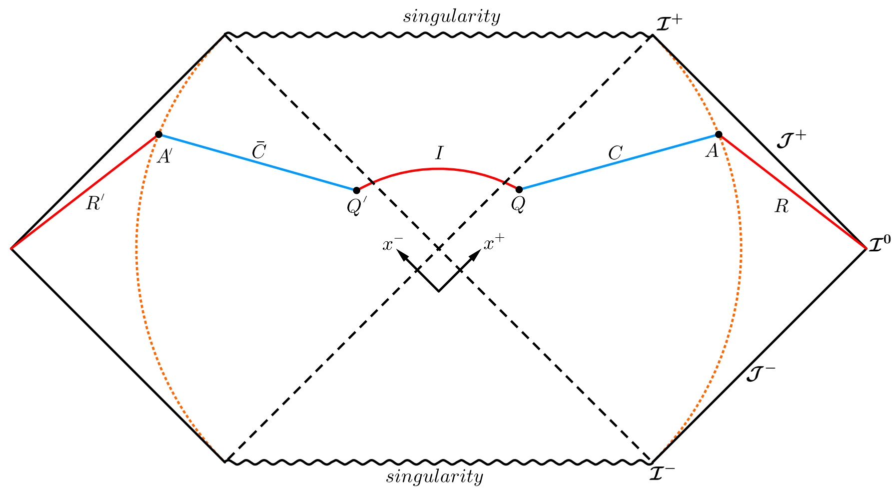

FIG. 3. The Penrose diagram for eternal black holes. The black dashed line represents the event horizon, and the orange dashed line represents the cutoff surface. The boundaries of the island and the radiation are denoted as Q and A, respectively. Their symmetry points are denoted as Q' and A'. The regions C and $\bar{C}$ are complementary.

$$S_{\rm rad}(\text{no island}) \simeq \frac{N}{3}t_A + \frac{N}{3}y_A - \frac{N}{12}(1+n)\log(e^{2y_A}+M) \simeq \frac{N}{3}t_A. \tag{4.4}$$

One can find that the contribution from the parameter n is a subleading term and is exponentially suppressed. Thus, the entropy without the island grows linearly with time $S \propto \frac{N}{3}t$ at late times. Then, the information paradox is sharpened here, the entanglement entropy never stops increasing and becomes divergent at late times.

B. With island

We still expect the island paradigm to protect the bound of the entanglement entropy at late times. For the configuration with an island, the generalized entropy of the radiation is twice the area term plus the von Neumann entropy of the CFT, which is

$$S_{\rm gen} = 2S_{\rm grav}(Q) + S_{\rm matter}(I \cup R). \tag{4.5}$$

At late times and large distances, we still preserve the approximation (3.10), which allows us to evaluate the entropy of the matter sector only in the interval $[A', Q'] \cup [A, Q]$. Furthermore, due to the symmetry, we only need to calculate one of the intervals. Thus, we have

$$\begin{aligned} S_{\rm gen} &\simeq 4(\Omega_{\rm ete}(Q) - \Omega_{\rm crit}) + 2S_{\rm matter}(C) \\ &= 4(-x_Q^+x_Q^- + M) + \frac{N}{6}\log\frac{(x_A^+ - x_Q^+)^2(x_A^- - x_Q^-)^2}{(M - x_A^+x_A^-)^n(M - x_Q^+x_Q^-)^n}. \end{aligned} \tag{4.6}$$

We extremize the expression (4.6) with respect to $x_Q^\pm$, which gives

$$\frac{\partial_{\rm gen}}{\partial x_Q^+} = -4x_Q^- - \frac{N}{3(x_A^+ - x_Q^+)} + \frac{Nnx_Q^-}{6(M - x_Q^+x_Q^-)} = 0, \tag{4.7a}$$

$$\frac{S_{\rm gen}}{\partial x_Q^-} = -4x_Q^+ - \frac{N}{3(x_A^- - x_Q^-)} + \frac{Nnx_Q^+}{6(M - x_Q^+x_Q^-)} = 0. \tag{4.7b}$$

Then, putting A into $\mathcal{J}^+$ ($x_A^+ \to \infty$), we obtain

$$\frac{x_Q^+}{x_Q^-} = \frac{x_A^+}{x_A^-}, \qquad x_Q^\pm \simeq \frac{-N}{12x_A^\mp}, \tag{4.8}$$

which implies that the island is near and outside the horizon as $x_Q^+ \cdot x_Q^- \lesssim 0$. Finally, the entropy with the island is obtained by substituting (4.8) into (4.6)

$$\begin{aligned} S_{\rm gen} &\simeq 4M - \frac{N^2}{36x_A^+x_A^-} + \frac{N}{6}\log\frac{(-x_A^+x_A^-)^{2-n}}{M^n} \\ &\simeq 4M + \frac{N}{3}y(2-n) - \frac{nN}{6}\log\ M \\ &\simeq 2S_{\rm BH}. \end{aligned} \tag{4.9}$$

Clearly, at late times, the emergence of islands curbs the growth of the entanglement entropy. Therefore, we reproduce the Page curve for eternal black holes from the results of (4.4) and (4.9). We also find that the eternal case is very similar to the evaporating case. The only difference is that the entanglement entropy with an island for eternal black holes reaches a saturation value at late times and does not change with time. This is because the coupled auxiliary bath or incident boundary conditions continuously replenish the energy to the black hole. In addition, it is surprising that the location of the island is outside the event horizon. However, this result is consistent with the quantum focusing conjecture [80]. Accordingly, there is a quantum

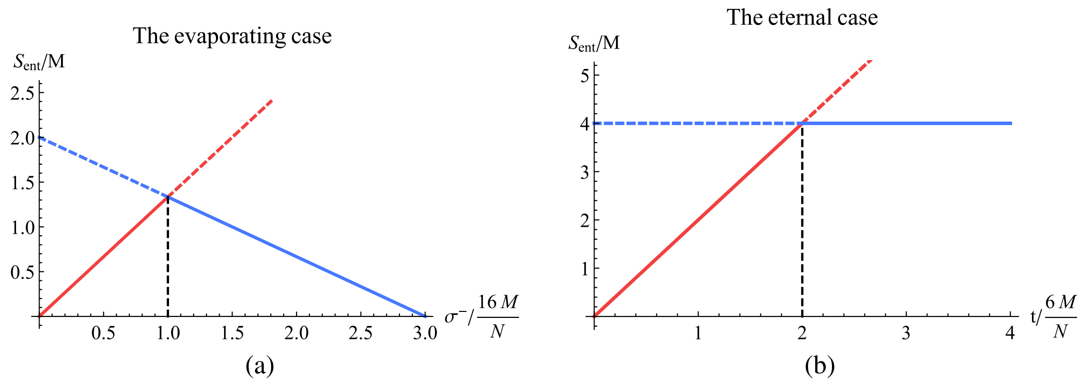

FIG. 4. Page curves for black holes. Red lines represent the entanglement entropy without islands, while blue lines represent the entropy by considering the island. On the left, the Page time for 2D evaporating black holes is approximately one-third of the lifetime. On the right, the entropy for eternal black holes grows to twice the Bekenstein–Hawking entropy after the Page time.

teleportation protocol by which we can extract the information from the island, and one can refer to [17,81] for details.

V. PAGE CURVE AND SCRAMBLING TIME

In this section, we plot the Page curve and derive the scrambling time. Then, we provide some remarks on the Page curve for 2D gravity.

A. Evaporating case

For evaporating black holes, the lifetime is defined by (3.18), which is $\sigma_{\rm life} \simeq \frac{4M}{\kappa} = \frac{48M}{N}$. The fine-grained entropy of the radiation is

$$S_{\rm rad} = {\rm Min}\left[\frac{N}{12}\sigma^-, 2M - \frac{N}{24}\sigma^-\right] = \frac{N}{24}{\rm Min}[2\sigma^-, \sigma_{\rm life} - \sigma^-]. \tag{5.1}$$

We obtain the Page time as

$$t_{\rm Page}^{\rm eva} = \frac{1}{3}\sigma_{\rm life} = \frac{16M}{N}. \tag{5.2}$$

The corresponding Page curve is shown in Fig. 4(a).

We now calculate the scrambling time. According to the Hayden–Preskill experiment, if Alice throws a quantum diary into the black hole after the Page time, then she must wait for the so-called scrambling time to recover this information from the Hawking radiation [82]. As suggested by the entanglement wedge reconstruction, the scrambling time corresponds to the time when the infalling information hits the boundary of the island [11].

After the Page time, the observer on the cutoff surface sends a light signal to the black hole at $x^+ = x_L^+$. We assume that the cutoff surface has the same order as the event horizon, which is[10]

$$\Omega_{\rm near} = (1+\alpha)\Omega_H \simeq (1+\alpha)[-x_H^+ x_H^- - M x_H^+ + M] = (1+\alpha)M = \frac{1}{2}(1+\alpha)S_{\rm BH},, \tag{5.3}$$

where α is a constant with the order $\mathcal{O}(1)$. Ω_H is the conformal factor at the event horizon, $x_H = -M$. Correspondingly, the infalling information at the cutoff surface satisfies

$$-x_L^+(x_L^- + M) + \frac{1}{2}S_{\rm BH} = \frac{1}{2}(1+\alpha)S_{\rm BH}. \tag{5.4}$$

In the outgoing coordinate (3.2b), the initial time is

$$\sigma_L^- \equiv \log\left(\frac{1}{-x_L^- - M}\right) = \log\left(\frac{2x_L^+}{\alpha S_{\rm BH}}\right). \tag{5.5}$$

When the information reaches the island, the corresponding retarded time is

$$\sigma_Q^- \equiv \log\left(\frac{1}{-x_Q^- - M}\right) = \log\left(\frac{24x_Q^+}{N}\right) = \log\left(\frac{24x_L^+}{N}\right). \tag{5.6}$$

Then, the scrambling time is

$$t_{\rm scr} = \sigma_Q^- - \sigma_L^- = \log\left(\frac{12\alpha S_{\rm Bh}}{N}\right) \simeq \log\frac{S_{\rm BH}}{N} \sim \frac{\beta}{2\pi}\log S_{\rm BH}. \tag{5.7}$$

[10]We define the region inside the cutoff surface as the "near black hole" region. In the initial Hayden–Preskill experiment, this region is about 2 or 3 Schwarzschild radii for 4D Schwarzschild black holes [82].

Here, β is the inverse temperature $\beta = \frac{1}{T_H} = 2\pi$ (2.29). This result is consistent with [83,84].

B. Eternal case

The calculation for the eternal case is simpler. Comparing (4.4) with (4.9), we obtain the fine-grained entropy of the radiation in the whole process as

$$S_{\rm rad} = {\rm Min}\left[\frac{N}{3}t, 4M\right]. \quad (5.8)$$

So the Page time is

$$t^{\rm ete}_{\rm Page} = \frac{12M}{N}. \quad (5.9)$$

Compared with (5.2), we can still consider that it has the same order of lifetime, even though the lifetime of an eternal black hole is indeed infinite. The corresponding Page curve is shown in Fig. 4(b).

Similarly, if we emit a light signal from the cutoff surface at time t_0, then the signal approaches the island at time t_1. The scrambling time is determined by

$$t_{\rm scr} = t_1 - t_0 \simeq \log\left(\frac{x_Q^+}{x_L^+}\right) = \log\left(\frac{\alpha S_{\rm BH}}{6N}\right) \sim \frac{\beta}{2\pi}\log S_{\rm BH}, \quad (5.10)$$

where α has the same meaning as in (5.3), and x^+ is the asymptotic coordinate (4.2a). Here, we assume the spatial coordinate of the cutoff surface is approximately equal to the island, namely $y_L \simeq y_Q$.

Let us now make some remarks. The different parameter n for this generalized 2D model corresponds to different theories. However, at late times, we can find that the effect of n is neglectable as the subleading term in both evaporating (5.1) and eternal cases (5.8). This also suggests that the Page curve for the FR model is similar to the RST model [19,20] at the leading order. The fundamental reason is that the Hawking temperature (2.29) is the same in both models because the original Page curve is derived from the Page theorem with the unitary assumption [79]: At early times, the dimensions of the Hilbert space of the radiation were small. However, the dimensionality of the black hole was very big. The entanglement entropy of the radiation is approximate to the thermodynamic entropy, which is proportional to the Hawking temperature T_H. At late times, the radiation dominates, hence, the radiation and the black hole exchange the status. The entanglement entropy is finally approximated to the Bekenstein–Hawking entropy determined by the temperature T_H. We can provide a simplified version of the proof by the second law of thermodynamics. The total entropy change during the whole evaporation process is

$$\Delta S_{\rm tot} = \Delta S_{\rm BH} + \Delta S_{\rm rad} = \int \frac{dM_{\rm BH}}{T_H} = \int \frac{dM}{\pi T_H} = 2M. \quad (5.11)$$

We fix the constant of integration by requesting that $S_{\rm BH} = 0$ as $M_{\rm BH} = 0$. After the evaporation, the black hole entropy changes to $\Delta S_{\rm BH} = -S_{\rm BH} = -2M$. Then, we obtain

$$\frac{\Delta S_{\rm rad}}{|\Delta S_{\rm BH}|} = \frac{\Delta S_{\rm tot} - \Delta S_{\rm BH}}{|\Delta S_{\rm BH}|} = 2. \quad (5.12)$$

The entropy of the radiation increases twice as fast as the decreases of the black hole entropy, which leads to the Page time being one-third of the lifetime. This result is consistent with our calculation by the island paradigm. However, one should note that in the final evaporation stage, the semiclassical approximation is invalid. The quantum gravity effects are expected to play a major role. Therefore, the shape of the last part of the Page curve for evaporating black holes may not be suitable.

VI. REVISIT THE FIREWALLS AND STATE PARADOX

Up to now, we have reproduced the Page curve for evaporating and eternal black holes through the island formula (1.1). It is beneficial and natural to extend the study of the entanglement island beyond the Page curves and focus on issues of the black hole interior related to the information paradox. Note that the most important step is that when we calculate the von Neumann entropy from matter fields, the contribution from the island region inside the black hole[11] is considered. This is different from most previous theories that describe black holes from the *outside* of event horizons, which seems to derive a theory about the *interior* of the black hole. The interior of black holes plays an essential role in modern physics; the essence of the information issue stems from our ignorance of the interior of black holes. Another issue that has been closely related to the interior of black holes over the past few years is the famous firewall paradox [85]. Therefore, in this section, we first review the firewall paradox and reconsider the relationship between the firewall and the island from the perspective of the entanglement island. We will show that the island can be a better candidate for solving this paradox. Next, based on the latest reports and for a more complete description of entanglement islands, we review the state paradox [86] and its solution, the gravity/ensemble duality [87].

[11]Note that we use the cutoff surface to divide the regions inside and outside black holes. Therefore, although the island is outside the event horizon for eternal black holes, it is inside the region of black holes.

A. Firewalls paradox

The firewall paradox is a variant version of the information paradox proposed by Almheiri, Marolf, Polchinski, and Sully (AMPS) in 2012 [85]. The emergence of the firewall further intensifies the contradiction between the classical GR and QM near the event horizon. On the one hand, classical GR does not expect that there exists such strange objects like firewalls near the event horizon. On the other hand, the principle of quantum entanglement monogamy has to be maintained by introducing a firewall [88].

Let us now review the firewall paradox in detail. This paradox is based on four postulates [85]:

Postulate 1 (unitarity): For a distant observer, the formation and evaporation of black holes can be described by the standard quantum theory. In particular, there is a unitary scattering S-matrix that can describe such a process from falling matter to outgoing Hawking radiation.

Postulate 2 (the semiclassical equation): Physics beyond the stretched event horizon of massive black holes can be approximated by semiclassical field equations.

Postulate 3 (d.o.f.): For distant observers, black holes can be regarded as quantum systems with discrete energy levels. For a black hole of mass M, the number of microscopic states is $\exp S(M)$, where $S(M)$ is the Bekenstein–Hawking entropy.

Postulate 4 (effective field theory): A free-falling observer experiences nothing unusual as he crosses the event horizon.

The first three postulates consist of the black hole complementary (BHC) [89]. The last postulate is supplemented by AMPS. Then they argued that the BHC (the Postulates 1–3) and Postulate 4 are inconsistent for an old black hole after the Page time if the theorem of the monogamy of entanglement is satisfied [88]. We will see that this theorem leads to firewalls.

For simplicity and clarity, we follow the opposite logic in the original argument of [85]. Consider a black hole after the Page time. We choose a full Cauchy slice Σ_1 and divide it into three regions, A, B, and R, as shown in Fig. 5. We assume that the black hole is formed by pure quantum states. Then, we have the following equation:

$$S(A \cup B \cup R) \equiv S_{ABR} = 0. \tag{6.1}$$

This is based on the fact that the combined region $A \cup B \cup R$ is the Cauchy slice, which has zero entropy, because the Cauchy slice contains all information in spacetime (see Fig. 5). Furthermore, we assume that there is a firewall at the event horizon of the black hole to ensure that the monogamy holds. Namely, the black hole interior (the subsystem A) and the outside of black hole (the subsystem BR) are not entangled. Thus, the mutual information between the A and BR must be zero:

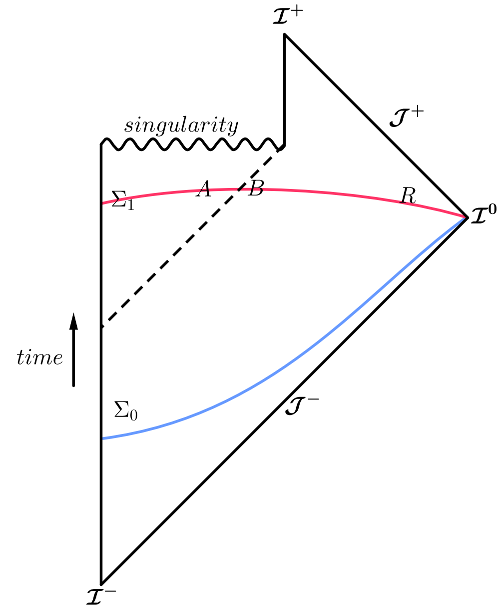

FIG. 5. The schematic diagram of the firewall. Σ_0 and Σ_1 are the Cauchy slices at early and late times, respectively. We divide Σ_1 into three parts, where A represents the ingoing Hawking mode inside the horizon, i.e., the black hole region or the nonemitted Hawking radiation, B represents the late Hawking radiation, and R is the early Hawking radiation or the region of the radiation.

$$I(A;BR) \equiv S_A + S_{BR} - S_{ABR} = 0, \tag{6.2}$$

where $S_{ABR} = 0$ (6.1). Based on the non-negativity of the entanglement entropy, we obtain the only result,

$$S_A = S_{BR} = 0. \tag{6.3}$$

It is clear to see that this result leads to a conflicting interpretation: Although the above equation (6.3) implies the purity of late Hawking radiation ($S_{BR} = 0$), it violates the unitary hypothesis 1 in the BHC. Because of the unitary evaporation hypothesis, the just-emitted Hawking radiation B is highly entangled with the mode A inside the black hole. Thus, the mutual information should satisfy $I(A;BR) \gg 1$. In the next part, the cost of introducing firewalls is the collapse of classical GR at the event horizon. In this scenario, there is no entanglement between the inside and outside black holes; the reduced density matrix of the total state ρ_{ABR} in the Cauchy slice Σ_1 can be decomposed into a direct product form: $\rho_{ABR} = \rho_A \otimes \rho_{BR}$. Therefore, the wave function of the radiation ψ has a large discontinuity near the event horizon and behaves as

$$\partial_x \psi|_{x_H} \sim \frac{1}{\epsilon^2}, \tag{6.4}$$

where x_H is the location of the event horizon, and ϵ is the cutoff. Then, the Hamiltonian of wave function has the following form:

$$H \sim \int (\partial_x \psi)^2 dx dy \sim \frac{\text{Area}}{\epsilon^3}. \tag{6.5}$$

The energy at the event horizon is so high that it is called the "firewall." This obviously destroys the smoothness of horizons in classical GR.

On the other hand, even if the firewall cannot exist, we still inevitably encounter other issues. Because there are no firewalls, A and B are highly entangled. Simultaneously, the Hawking radiation R at early times is maximally entangled with the Hawking radiation B at late times from the unitary Page theorem. This violates the principle of entanglement monogamy mentioned above.

In brief, the firewall leads to conflict between GR and QM. Although many different solutions are proposed, all have some flaws [90,91]. Now we revisit the issue from the point of view of the island paradigm. Based on the entanglement wedge reconstruction [11], the *interior* of black holes belongs to the *external* radiation, which means that the ingoing Hawking mode is identified with a part of the outgoing Hawking radiation. In this sense, the Hilbert space of subsystem A is regarded as a subset of the Hilbert space of subsystem BR, namely $\mathcal{H}_A \subset \mathcal{H}_{BR}$, then the corresponding fine-grained entropy are determined by

$$\begin{aligned} S(A \cup B \cup R) &\equiv S_{ABR}, \\ S(B \cup R) &\equiv S_{BR}, \\ S_{ABR} &= S_{BR}. \end{aligned} \tag{6.6}$$

In this time, we can obtain the following relation from the strong subadditivity inequality by substituting the above relation[12]:

$$\begin{aligned} &S_{ABR} = S_{BR} \leq S_A + S_{BR}, \\ &S_{ABR} = S_{BR} \geq |S_A - S_{BR}|, \\ &\text{Cauchy slice: } S_{ABR} = 0. \end{aligned} \tag{6.7}$$

Finally, we obtain

$$S_A = 0, \qquad S_{BR} = 0. \tag{6.8}$$

At first glance, this result looks the same as Eq. (6.3), but the physical interpretation is completely different. From the perspective of the island, the interior of black holes is automatically contained in outgoing Hawking radiation. So this result not only guarantees that the radiation is in a pure state but also consistent with the unitarity. Besides, this is not contrary to the entanglement monogamy and does not lead to the firewall since the interior is identified as a part of the early Hawking radiation. Thus, it also ensures the smoothness of the horizon. Therefore, in terms of physical interpretation, the island seems to be a better candidate than firewalls. Moreover, we should note that whether islands or firewalls, they are singular objects at the horizon, and they exist to avoid the occurrence of the paradox. We only need reserve one of them, but not both, in the description of the evolution of black holes [92].

[12]Here we assume that this inequality can be held also in the vicinity of event horizons.

B. State paradox

While the island paradigm is an almost perfect solution to the firewall paradox, one still needs to be careful when using it. In the following content, we review the state paradox and a possible solution to it, the gravity/ensemble duality. In order to understand the island formula more completely, we go back to the island formula (1.1) and rewrite it by the entanglement wedge proposal,

$$S_{\text{rad}} = S_{\text{gen}}[\text{EW}(\text{rad})] = \frac{\text{Area}(\partial \text{EW})}{4G_N} + S(\text{EW}), \tag{6.9}$$

where "EW" is denoted as the entanglement wedge. We can see that "rad" exists on both sides of the equation. The radiation is in the *full* quantum description on the left. However, on the right, the radiation is still in the *semi-classical* frame, i.e., the Hawking's view (a mixed state). In particular, we still take Hawking's initial state, which leads to the Hawking's monotonically increasing curve as the input when we calculate the fine-grained entropy of the radiation. By contrast, we only contain the island in the entanglement wedge of the radiation after the Page time, which leads to the Page curve (a pure state). As the S-matrix that describes the evolution of black holes is an observable quantity, the corresponding state of the Hawking radiation should not be ambiguous. Accordingly, the fine-grained entropy of the state cannot take two different values. Such a conflict is called the state paradox [86].

A possible logical solution to the conflict is to introduce the gravity/ensemble duality [87]. We assume that the theory of gravity corresponds to a series of field theories. Once we introduce the ensemble, we need to reclassify the entropy firstly. The first entropy is the ensemble average of the entropy $\langle S(\rho) \rangle$. The second entropy is the entropy of the ensemble-averaged state of the radiation $S(\langle \rho \rangle)$. In general, these two entropies are not equal because the fine-grained entropy is a nonlinear function of states. Based on this proposal, therefore, the entropy of radiation on both sides of Eq. (6.9) are two different quantities and can take different values. The left side is $\langle S(\rho) \rangle$, while on the right side, $S(\langle \rho \rangle)$ dominates the "EW." More specifically, in the gravity/ensemble duality, the single pure state evolves into different pure states under every Hamiltonian. If one first evaluates the entanglement entropy $S(\rho_i)$ that corresponds to each pure state, then one implements the ensemble average for this entropy to obtain $\langle S(\rho) \rangle$. Since every entropy $S(\rho_i)$ follows the unitary evolution of the pure state, the result is self-averaging. The corresponding result reproduces the Page curve. On the contrary, if one first performs the ensemble average for all pure states ρ_i to obtain $\langle\rho_i\rangle$, then, in general, each state ρ_i is different in different theories. Their ensemble average is a thermal (mixed) state: $\langle\rho_i\rangle = \rho_{\text{thermal}}$. At last, the Hawking's result $S(\langle\rho\rangle)$ is obtained by calculating the entanglement entropy for this mixed state. Accordingly, the island formula (6.9) can be modified into the following form based on the above discussion [87]:

$$\langle S[\rho(t)]\rangle = S_{\text{gen}}[\text{EW}(\langle\rho(t)\rangle)]. \tag{6.10}$$

In such way, the ensemble average of the entanglement entropy $\langle S(\rho)\rangle$ follows unitary Page curves, while the entanglement entropy of the ensemble average $S(\langle\rho\rangle)$ follows Hawking's curves.

At last, one should note that introducing the theory of an ensemble is not valid for all cases, such as the duality between type II B in $\text{AdS}_5\times S^5$ and the $\mathcal{N}=4$ Yang–Mills gauge theory in 4D. There is no known ensemble theory, and in this case the state paradox may still exist. More specific content is beyond the scope of this paper; one can refer to [87] for more information.

VII. DISCUSSION AND CONCLUSION

In this paper, we investigated the information paradox for the generalized 2D gravity. Unlike the famous RST model, the FR model represents a family of exactly solvable theories, and the corresponding geometries are also interesting. We calculate the entanglement entropy of the Hawking radiation by using the island paradigm in this generalized 2D background. No matter whether it is the evaporating black hole or the eternal black hole, the island is always absent at early times, which results in the entropy of the radiation growing proportional to the time [(3.8) and (4.4)]. The behavior of the entanglement entropy satisfies the Hawking curve. However, a Page transition occurs at the Page time when the entropy increases to the Bekenstein–Hawking entropy bound. In our calculations, the minimality condition requires us to attribute the island region inside the black hole to the entanglement wedge of the outside radiation. After the Page time, the area term is dominant. The island is close to and inside the event horizon for the evaporating black hole. The entanglement entropy drops to zero eventually (3.20). The corresponding Page time is about a third of the lifetime of black holes (Fig. 4(a)). The island is close to and outside the event horizon for the eternal black hole in the Hartle–Hawking state. The entanglement entropy finally reaches a saturation value of twice the Bekenstein–Hawking entropy (4.9). The Page curve of this case is shown in Fig. 4(b). However, once we consider the late times and the large distance limits, the contribution of the parameter is only a subleading term. It is exponentially suppressed in both the evaporating and the eternal case. Although this result leads to Page curves being the same as the RST model in the leading order, it also seems to imply the universality of Page curves in generalized 2D models. The essence is that the Hawking temperature of the two models is the same and does not depend on the parameter n. The generalized second law can also verify this, and the result is consistent with the original derivation of Page curves through the Page theorem. Therefore, the calculations in this paper are meaningful and extend the scope of the application of the island paradigm.

Besides, we also discuss the connection between the firewall and the island. The price of firewalls is the collapse of the effective field theory or the classical GR near the horizon. Although this is still an open question, we show that islands appear to be a better candidate to replace the firewall from the perspective of the entanglement wedge reconstruction when dealing with the entanglement monogamy issue [(6.3) and (6.8)]. At last, we briefly review the state paradox. The island provides a d.o.f. that can purify the thermal radiation, which somewhat solves the information paradox. However, the island formula does not modify any calculations of the original Hawking's calculation but introduces a nonlocal d.o.f. such as islands in the black hole interior. Therefore, there are still some issues called the state paradox. The gravity/ensemble duality is an attempt to solve the issue. By introducing the ensemble average to reclassify the entropy, we finally obtain the modified island formula (6.10).

At last, some of the following issues require our attention in the future:

(1) The aspect of the calculation. First, we only consider the large mass limit for evaporating black holes, hence, the n-dependent term is discarded. But at the end of evaporation, the n-dependent term is no longer the subleading term and significantly impacts the result. In particular, the semiclassical approximation is broken down when the mass approaches the Planck scale. The anti-Page curve may appear on the Planck scale [93]. The physical details at this point are expected to be dominated by quantum gravity, which we do not yet know. Second, although the island formula is derived from the AdS black hole with couples the auxiliary bath, its applicability extends far beyond AdS spacetime. However, baths are not introduced in our model. The black hole in the FR model is asymptotically flat. Therefore, the Hawking radiation naturally propagates to the null infinity. Nonetheless, we insist on the importance of baths. The associated calculations may fail without baths [22]. Third, so far, most related work that studies the island of evaporating black holes focuses on 2D gravity. On the one hand, the calculation corresponding to higher-dimensional black holes is difficult, especially as the entanglement entropy

formula is divergent in high dimensions. On the other hand, there are no known analytical solutions for higher-dimensional asymptotically flat evaporating black holes. We expect some related work in the future, such as the classical Vaidya spacetime. Fourth, the island formula can be obtained by the gravitational path integral, in which a new saddle point is needed to consider, called the replica wormholes saddle. We can also investigate the dynamic evolution of the replica wormhole geometry in the Sachdev–Ye–Kitaev (SYK) model based on the JT/SYK dual. There are also some interesting reports in [94–96].

(2) The aspect of quantum information. According to "$ER = EPR$" [97], the entanglement between the d.o.f. of two subsystems produces a connected geometry–wormholes that bridge them to each other. However, the island formula only provides the evolution of Page curves. Still, it does not explain how the quantum information escapes from the black hole to the Hawking radiation. We can only treat the island formula as a black box operation.

(3) The aspect of the ensemble theory. Some study demonstrates that the ensemble that corresponds to the gravity comes from the baby universe inside the black hole [98–103]. However, this is still debated. The core issue is understanding the dynamics inside the black hole. However, there is currently no effective means to detect these dynamics.

In conclusion, the current theories describing the evaporation process of black holes are imperfect, especially in explaining what happens when a black hole is near the end of evaporation. Perhaps the discovery of quantum gravity or some other new physical mechanism in the future can explain this conundrum.

ACKNOWLEDGMENTS

We would like to thank Haiming Yuan for discussions related to quantum field theories, Cheng Ran for discussions related to the ensemble theory, and Wencong Gan for discussions related to the Vaidya spacetime. We would also like to thank Hao Geng for useful comments. The study was partially supported by NSFC, China (Grants No. 12275166 and No. 11875184).

APPENDIX: BRIEF REVIEW OF THE FR MODEL

In this appendix, we display more details about the FR model and briefly discuss the geometry for the special case of $n = 0$ and $n = 2$.

The metric of static black hole spacetime can be read from (2.19a),

$$ds^2 = -e^{2\rho} dx^+ dx^- = -\frac{1}{(M - x^+ x^-)^n} dx^+ dx^-. \tag{A1}$$

The Ricci curvature scalar R is

$$R = 8e^{-2\rho}\partial_+\partial_-\rho = 4Mn(M - x^+x^-)^{n-2}. \tag{A2}$$

It is convenient to recast the metric (A1) in the chiral form by the Eiddton transformation,

$$x^+ = V, \qquad x^+x^- = r, \tag{A3}$$

which is followed by

$$V = e^v, \qquad x = \frac{(M - r)^{1-n}}{2(1-n)} = \frac{M - x^+x^-}{2(1-n)}, \tag{A4}$$

which yields the chiral form

$$\begin{aligned} ds^2 &= \{2(1-n)x - M[2(1-n)x]^{\frac{n}{n-1}}\}dv^2 + 2dvdx \\ &= -f(x)dv^2 + 2dvdx, \end{aligned} \tag{A5}$$

where $f(x)$ is the metric function, whose derivative is

$$\begin{aligned} f'(x) &= 2(1-n)\left\{1 - M\left(\frac{n}{n-1}\right)[2(1-n)x]\right\}^{\frac{1}{1-n}}, \\ f''(x) &= -4Mn[2(1-n)x]^{\frac{2-n}{n-1}}. \end{aligned} \tag{A6}$$

Then, the Ricci curvature scalar R is simply in the chiral coordinates,
$R = -f''(x) = 4Mn[2(1-n)x]^{\frac{2-n}{n-1}} = 4Mn(M - x^+x^-)^{n-2}$, and the dilaton ϕ gives the coupling,

$$e^{2\phi/n} = [2(1-n)x]^{\frac{1}{n-1}} = \frac{1}{M - x^+x^-}. \tag{A7}$$

Now, we rewrite the metric in the Schwarzschild gauge $ds^2 = -f(x)dt^2 + f^{-1}(x)dr^2$. Then, the event horizon can be obtained by $f(x_H) = 0$, which obtains

$$x_H = \frac{(M - x_H^+ x_H^-)^{1-n}}{2(1-n)} = \left(\frac{1}{M}\right)^{n-1} \frac{1}{2(1-n)}, \tag{A8a}$$

$$x_H^+ x_H^- = 0. \tag{A8b}$$

Thus, the Hawking temperature is given by

$$T_H = \frac{f'(x_H)}{4\pi} = \frac{1}{2\pi}. \tag{A9}$$

The singularity is located as

$$x_s^+ x_s^- = M, \tag{A10}$$

for $0 < n < 2$. For $n = 0$, we find that the spacetime is flat according to (A2). However, it is not a trivial result because the coupling (A7) is nontrivial and becomes singular. One should rescale the dilaton ϕ to $\tilde{\phi} = n\phi$ first and then take the limit $n \to 0$. Accordingly, the classical action (2.1b) takes the following form:

$$\tilde{S}_0 = \frac{1}{2\pi} \int d^2x \sqrt{-g} (e^{-2\tilde{\phi}} R + 4\lambda^2). \tag{A11}$$

If we redefine the metric tensor by $g_{\mu\nu} \to e^{2\phi} g_{\mu\nu}$, then this action becomes the CGHS action [71]. Therefore, the special case $n = 0$ represents an unusual black hole with a spacelike singularity in the coupling, but the Ricci curvature vanishes. For the concrete content of this case, one can refer to [73]. For the other case $n = 2$, the curvature scalar is a constant (A2). However, a similar interpretation for the $n = 0$ case applies here. The dilaton is singular again. However, in the following discussion, we see that the curvature of the $n = 2$ case is no longer constant when considering the quantum effects.

Next, we return to the dynamical black hole with the backreaction. The backreaction effects are well explained in the RST model [72]. Nevertheless, some nontrivial quantities still have parameter-dependent evolution in the FR model. Therefore, we give some discussions of the differences.

The usual time-dependent geometry that describes the collapse of a massless shock wave and then evaporates is in terms of Ω,

$$\Omega = \chi = -x^+[x^- + P_+(x^+)] - \frac{\kappa}{4} \log(-x^+x^-) + M(x^+). \tag{A12}$$

Moreover, the Ricci curvature is

$$R = 4ne^{-2\rho} \frac{1}{e^{-2\phi/n} - \frac{\kappa}{4}} \left(1 - \frac{4}{n^2} \partial_+\phi \partial_-\phi e^{-2\phi/n}\right). \tag{A13}$$

We find there exists a singularity at the following line:

$$\phi = \phi_{\rm crit} = -\frac{n}{2} \log \frac{\kappa}{4}. \tag{A14}$$

When $T_{++} < \frac{\kappa}{4}(x^+)^2$, the line is timelike. However, it turns to spacelike if $T_{++} > \frac{\kappa}{4}(x^+)^2$. Substituting (A14) to the conformal factor Ω by (2.9) and (2.10), we obtain (2.6) eventually. We should note that in the case $n = 2$, it represents the spacetime with a constant curvature at the classical level. However, once the one-loop correction is considered, the curvature is divergent at $\phi_{\rm crit}$ rather than a constant because the coupling becomes strong. Besides, the metric also deviates from the classical theory in the strong-coupling region. For more information, one can refer to [73,74], and we end our discussion here.

[1] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975); **46**, 206(E) (1976).

[2] S. W. Hawking, Breakdown of predictability in gravitational collapse, Phys. Rev. D **14**, 2460 (1976).

[3] D. N. Page, Information in Black Hole Radiation, Phys. Rev. Lett. **71**, 3743 (1993).

[4] D. N. Page, Time dependence of Hawking radiation entropy, J. Cosmol. Astropart. Phys. 09 (2013) 028.

[5] J. Maldacena, The large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. **38**, 1113 (1999).

[6] S. Ryu and T. Takayanagi, Holographic Derivation of Entanglement Entropy from AdS/CFT, Phys. Rev. Lett. **96**, 181602 (2006).

[7] A. Lewkowycz and J. Maldacena, Generalized gravitational entropy, J. High Energy Phys. 08 (2013) 090.

[8] N. Engelhardt and A. Wall, Quantum extremal surfaces: Holographic entanglement entropy beyond the classical regime, J. High Energy Phys. 01 (2015) 073.

[9] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole, J. High Energy Phys. 12 (2019) 063.

[10] A. Almheiri, R. Mahajan, J. Maldacena, and Y. Zhao, The page curve of Hawking ridiation from semiclassical geometry, J. High Energy Phys. 03 (2020) 149.

[11] G. Penington, Entanglement wedge reconstruction and the information paradox, J. High Energy Phys. 09 (2020) 002.

[12] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulianm, and A. Tajdini, The entropy of Hawking radiation, Rev. Mod. Phys. **93**, 035002 (2021).

[13] G. Penington, S. Shenker, D. Stanford, and Z. Yang, Replica wormholes and the black hole interior, arXiv:1911.11977.

[14] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, Replica wormholes and the entropy of Hawking radiation, J. High Energy Phys. 05 (2020) 013.

[15] T. Hollowood and S. Kumar, Islands and page curves for evaporating black holes in JT gravity, arXiv:2004.14944.

[16] K. Goto, T. Hartman, and A. Tajdini, Replica wormholes for an evaporating 2D black hole, J. High Energy Phys. 04 (2021) 289.

[17] A. Almheiri, R. Mahajan, and J. Maldacena, Islands outside the horizon, arXiv:1910.11077.

[18] T. Anegawa and N. Iizuka, Notes on islands in asymptotically flat 2d dilaton black holes, J. High Energy Phys. 07 (2020) 036.

[19] F. Gautason, L. Schneiderbauer, W. Sybesma, and L. Thorlacius, Page curve for an evaporating black hole, J. High Energy Phys. 05 (**2020**) 091.

[20] T. Hartman, E. Shaghoulian, and A. Strominger, Islands in asymptotically flat 2D gravity, J. High Energy Phys. 07 (2020) 022.

[21] X. Wang, R. Li, and J. Wang, Islands and Page curves for a family of exactly solvable evaporating black holes, Phys. Rev. D **103**, 126026 (2021).

[22] R. Li, X. Wang, and J. Wang, Island may not save the information paradox of Liouville black holes, Phys. Rev. D **104**, 106015 (2021).

[23] J. Tian, Islands in generalized dilaton theories, arXiv: 2204.08751.

[24] J. F. Pedraze, A. Svesko, W. Sybesma, and M. R. Visser, Microcanonical action and the entropy of Hawking radiation, Phys. Rev. D **105**, 126010 (2022).

[25] J. F. Pedraze, A. Svesko, W. Sybesma, and M. R. Visser, Semi-classical thermodynamics of quantum extremal surfaces in Jackiw-Teitelboim gravity, J. High Energy Phys. 12 (2021) 134.

[26] A. R. Chowdhury, A. Saha, and S. Gangopadhyay, The role of mutual information in the Page curve, Phys. Rev. D **106**, 086019.

[27] A. Almheiri, R. Mahajan, and J. E. Santos, Entanglement islands in higher dimensions, SciPost Phys. **9**, 001 (2020).

[28] S. He, Y. Sun, L. Zhao, and Y. X. Zhang, The universality of islands outside the horizon, J. High Energy Phys. 05 (2022) 047.

[29] M. Alishahiha, A. Astaneh, and A. Naseh, Island in the presence of higher derivative terms, J. High Energy Phys. 02 (2021) 035.

[30] K. Hashimoto, N. Iizuka, and Y. Matsuo, Islands in Schwarzschild black holes, J. High Energy Phys. 06 (2020) 085.

[31] Y. Matsuo, Islands and stretched horizon, J. High Energy Phys. 07 (2021) 051.

[32] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, Inconsistency of islands in theories with long-range gravity, J. High Energy Phys. 01 (2022) 182.

[33] H. Geng and A. Karch, Massive islands, J. High Energy Phys. 09 (2020) 121.

[34] C. Krishnan, Critical islands, J. High Energy Phys. 01 (2021) 179.

[35] C. Krishnan, V. Patil, and J. Pereira, Page curve and the information paradox in flat space, arXiv:2005.02993.

[36] F. Omidi, Entropy of Hawking radiation for two-sided hyperscaling violating black branes, J. High Energy Phys. 04 (2022) 022.

[37] C. F. Uhlemann, Islands and Page curves in 4d from Type IIB, J. High Energy Phys. 08 (2021) 104.

[38] C. F. Uhlemann, Information transfer with a twist, J. High Energy Phys. 01 (2022) 126.

[39] A. Saha, S. Gangopadhyay, and J. P. Saha, Mutual information, Islands in black holes and the Page curve, Eur. Phys. J. C **82**, 476 (2022).

[40] G. Karananas, A. Kehagias, and J. Taskas, Islands in linear dilaton black holes, J. High Energy Phys. 03 (2021) 253.

[41] Y. Lu and J. Lin, Islands in Kaluza-Klein black holes, Eur. Phys. J. C **82**, 132 (2022).

[42] T. Li, J. Chu, and Y. Zhou, Reflected entropy for an evaporating black hole, J. High Energy Phys. 11 (2020) 155.

[43] F. Deng, J. Chu, and Y. Zhou, Defect extremals surface as the holographic counterpart of Island formula, J. High Energy Phys. 03 (2021) 008.

[44] J. Chu, F. Deng, and Y. Zhou, Page curve from defect extremal surface and Island in higher dimensions, J. High Energy Phys. 10 (2021) 149.

[45] T. Li, M. Yuan, and Y. Zhou, Defect extremal surface for reflected entropy, J. High Energy Phys. 01 (2022) 018.

[46] C. Chou, H. Lao, and Y. Yang, Page curve of effective Hawking radiation, Phys. Rev. D **106**, 066008 (2022).

[47] M. Yu, C. Lu, X. Ge, and S. Sin, Island, Page curve and superradiance of rotating BTZ black holes, Phys. Rev. D **105**, 066009 (2022).

[48] W. Gan, D. Du, and F. Shu, Island and Page curve for one-sided asymptotically flat black hole, J. High Energy Phys. 07 (2022) 020.

[49] D. Ageev and I. Aref'eva, Thermal density matrix breaks down the Page curve, arXiv:2206.04094.

[50] D. Du, W. Gan, F. Shu, and J. Sun, Unitary constraints on semiclassical Schwarzschild black holes in the presence of Island, arXiv:2206.10339.

[51] K. Goswami and K. Narayan, Small Schwarzschild de Sitter black holes, quantum extremal surfaces and islands, J. High Energy Phys. 10 (**2022**) 031.

[52] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, Information transfer with a gravitating bath, SciPost Phys. **10**, 103 (2021).

[53] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, Entanglement phase structure of a holographic BCFT in a black hole background, J. High Energy Phys. 05 (2022) 153.

[54] H. Geng, S. Lust, R. K. Mishra, and D. Wakeham, Holographic BCFTs and communicating black holes, J. High Energy Phys. 08 (2021) 003.

[55] Y. Ling, Y. Liu, and Z. Xian, Island in charged black holes, J. High Energy Phys. 03 (2021) 251.

[56] X. Wang, R. Li, and J. Wang, Islands and Page curves of Reissner-Nordström black holes, J. High Energy Phys. 04 (2021) 103.

[57] W. Kim and M. Nam, Entanglement entropy of asymptotically flat non-extremal and extremal black holes with an island, Eur. Phys. J. C **81**, 869 (2021).

[58] B. Ahn, S. Bak, H. Jeong, K. Y. Kim, and Y. W. Sun, Islands in charged linear dilaton black holes, Phys. Rev. D **105**, 046012 (2022).

[59] M. Yu and X. Ge, Page curves and Islands in charged dilaton black holes, Eur. Phys. J. C **82**, 14 (2022).

[60] G. Yadav, Page curves of Reissner-Nordström black hole in HD Gravity, arXiv:2204.11882.

[61] J. Vuyst and T. Mertens, Operational Islands and black hole dissipation in JT gravity, arXiv:2207.03351.

[62] T. Hartman, Y. Jiang, and E. Shaghoulian, Islands in cosmology, J. High Energy Phys. 11 (2020) 111.

[63] V. Balasubramanian, A. Kar, and T. Ugajin, Islands in de Sitter space, J. High Energy Phys. 02 (2021) 072.

[64] Y. Chen, V. Gorbenko, and J. Maldacena, Bra-ket wormholes in gravitationally prepared states, J. High Energy Phys. 02 (2020) 009.

[65] M. Raamsdonk, Comments on wormholes, ensembles, and cosmology, J. High Energy Phys. 12 (2021) 156.

[66] S. Azarnia, R. Fareghbal, A. Naseh, and H. Zolfi, Islands in flat-space cosmology, Phys. Rev. D **104**, 126017.

[67] H. Geng, Y. Nomura, and H. Sun, An information paradox and its resolution in de Sitter holography, Phys. Rev. D **103**, 126004.

[68] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, Islands and complexity of eternal black hole and radiation subsystems for a doubly holographic model, J. High Energy Phys. 05 (**2021**) 135.

[69] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, Partial islands and subregion complexity in geometric secret-sharing model, J. High Energy Phys. 12 (2021) 091.

[70] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, Bath deformations, islands and holographic complexity, Phys. Rev. D **105**, 066019.

[71] C. Callan, S. Giddings, J. Harvey, and A. Strominger, Evanescent black holes, Phys. Rev. D **45**, R1005 (1992).

[72] J. Russo, L. Susskind, and L. Thorlacius, The endpoint of Hawking evaporation, Phys. Rev. D **46**, 3444 (1993).

[73] A. Fabbri and J. Russo, Soluble models in 2d dilaton gravity, Phys. Rev. D **53**, 6995 (1996).

[74] A. Fabbri and R. Balbinot, Two-dimensional black holes in accelerated frames: Spacetime structure, Classical Quantum Gravity **13**, 2457 (1996).

[75] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. **B424**, 443 (1994).

[76] T. Fiola, J. Preskill. A. Strominger, and S. Trivedi, Black hole thermodynamics and information loss in two dimensions, Phys. Rev. D **50**, 3987 (1994).

[77] J. D. Bekenstein, A universal upper bound on the entropy to energy ratio for bounded systems, Phys. Rev. D **23**, 287 (1981).

[78] H. Casini, C. D. Fosco, and M. Huerta, Entanglement and alpha entropies for a massive Dirac field in two dimensions, J. Stat. Mech. (2005) P07007.

[79] D. Page, Average Entropy of a Subsystem, Phys. Rev. Lett. **71**, 1291 (1993).

[80] R. Bousso, Z. Fisher, S. Leichenauer, and A. Wall, Quantum focusing conjecture, Phys. Rev. D **93**, 064044 (2016).

[81] P. Gao, D. Jafferis, and A. Wall, Traversable wormholes via a double trace deformation, J. High Energy Phys. 12 (2017) 151.

[82] P. Hayden and J. Preskill, Black holes as mirrors: Quantum information in random subsystems, J. High Energy Phys. 09 (2007) 120.

[83] Y. Sekino and L. Susskind, Fast scramblers, J. High Energy Phys. 10 (2008) 065.

[84] Q. Wang, M. Yu, and X. Ge, Scrambling time for analogue black holes embedded in AdS space, Eur. Phys. J. C **82**, 468 (2022).

[85] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, Black holes: Complementarity or firewalls?, J. High Energy Phys. 02 (2013) 062.

[86] R. Bousso and M. Tomašević, Unitarity from a smooth horizon?, Phys. Rev. D **102**, 106019 (2020).

[87] R. Bousso and E. Wildenhain, Gravity/ensemble duality, Phys. Rev. D **102**, 066005 (2020).

[88] M. Koashi and A. Winter, Monogamy of quantum entanglement and other correlations, Phys. Rev. A **69**, 022309 (2004).

[89] L. Susskind, L. Thorlacius, and J. Uglum, The stretched horizon and black hole complementarity, Phys. Rev. D **48**, 3743 (1993).

[90] D. Harlow and P. Hayden, Quantum computation vs. firewalls, J. High Energy Phys. 06 (2013) 085.

[91] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford, and J. Sully, An apologia for firewalls, J. High Energy Phys. 09 (2013) 018.

[92] X. Wang, K. Zhang, and J. Wang, What can we learn about islands and state paradox from quantum information theory?, arXiv:2107.09228.

[93] I. Aref'eva and I. Volovich, A note on Islands in Schwarzschild black holes, arXiv:2110.04233.

[94] Y. Chen, X. Qi, and P. Zhang, Replica wormhole and information retrieval in the SYK model coupled to Majorana chains, J. High Energy Phys. 06 (2020) 121.

[95] P. Saad, S. Shenker, D. Stanford, and S. Yao, Wormholes without averaging, arXiv:2103.16754.

[96] A. Almheiri and H. Lin, The entanglement wedge of unknown couplings, J. High Energy Phys. 08 (2022) 062.

[97] J. Maldacena and L. Susskind, Cool horizons for entangled black holes, Fortschr. Phys. **61**, 781 (2013).

[98] D. Marolf and H. Maxfield, Transcending the ensemble: Baby universes, spacetime wormholes, and the order and disorder of black hole information, J. High Energy Phys. 08 (2020) 044.

[99] J. McNamara and C. Vafa, Baby universes, holography, and the swampland, arXiv:2004.06738.

[100] D. Marolf and H. Maxfield, The Page curve and baby universes, arXiv:2105.12211.

[101] T. Anegawa, N. Iizuka, and D. Kabat, A comment on a fine-grained description of evaporating black holes with baby universes, J. High Energy Phys. 09 (2022) 158.

[102] C. Peng, J. Tian, and J. Yu, Baby universes, ensemble averages and factorizations with matters, arXiv:2111.14856.

[103] J. J. Heckman, A. P. Turner, and X. Yu, Disorder averaging and its UV discontents, Phys. Rev. D **105**, 086021 (2022).