Topological defects as Lagrangian correspondences

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Topological defects attract much recent interest in high-energy and condensed matter physics, because they encode (noninvertible) symmetries and dualities. We study codimension-1 topological defects from a Hamiltonian point of view, with the defect location playing the role of "time." We show that the Weinstein symplectic category governs topological defects and their fusion: Each defect is a Lagrangian correspondence, and defect fusion is their geometric composition. We illustrate the utility of these ideas by constructing S- and T-duality defects in string theory, including a novel topology-changing non-Abelian T-duality defect.

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I. INTRODUCTION

Quantum field theories are about more than the physics of pointlike particles. Even outside of string and M theory—whose *raison d'être* is the physics of fundamental extended objects—one finds that line, surface, and higher-dimensional objects have much to say, even if they were not originally "put in by hand" in the theory in question. Early examples are Wilson loops in QCD; their vacuum expectation values probe quark confinement.

Extended objects and operators supported on such appear generically as mediators of dualities, global symmetries, and their "higher" or "generalized" counterparts [1]. In this paper, we will be exclusively study codimension-1 extended objects (of dimension $\dim M - 1$ if the theory in question lives on the manifold M) which we will call *defects*. (They are also called "walls" or "interfaces.")

We will, in particular, be interested in *topological defects*—ones which may be freely deformed such that correlation functions are unchanged—due to their close connection to dualities [2–5]. The physical picture is simple: We envision a scenario where the defect locus separates spacetime M into phase 1 (say, inside) and phase 2 (say, outside), where phases 1 and 2 can be two copies of the same theory or possibly two different theories. For, e.g., a spherical defect of radius t, as t goes from 0 to $+\infty$ we observe a transition from phase 2 to phase 1; by inserting operators inside or outside this defect sphere, we thus obtain the action of the associated duality transformation on all local operators. Moreover, two topological defects can be *fused* by deforming them so they lie close to each other; this connects the algebraic aspects of symmetry—(semi) group structure—to the defect picture. Fusion accommodates both conventional symmetries—related to defects that may fuse into the "invisible" defect—as well as more exotic noninvertible symmetries.

Since the presence of one or more defects breaks Lorentz invariance, we might as well study them in a *Hamiltonian formulation*. The Hamiltonian "time" t will be such that $t = \text{const}$ is the defect locus; varying the constant gives a family of defects. We will discuss arbitrary systems in a Hamiltonian path-integral formulation, working in finite dimensions for technical simplicity.

The main takeaway from this work will be that *topological defects and their fusion furnish the Weinstein symplectic category* [6,7] of symplectic manifolds with Lagrangian correspondences—which are related to canonical transformations in Hamiltonian mechanics—as morphisms. We will demonstrate in Sec. V that this gives a practical and efficient way to identify topological defects in diverse physical contexts and to study their fusion. In the rest of this paper, we will explain what this statement means and give a brief but reasonably complete account of why it is true.

We start with textbook Hamiltonian mechanics and canonical transformations. Consider a *phase-space action principle* for, e.g., a particle on $\mathbb{R}$:

$$S[x,p] = \int_{T_\mathrm{I}}^{T_\mathrm{F}} \mathrm{d}t\{p(t)\dot{x}(t) - h(x(t),p(t))\}, \qquad (1)$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$. The Hamiltonian h is a function depending on a point $(x,p) \in \mathbb{R}^2$ of *phase space*; it characterizes

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the system in question. (We assume it is time independent for simplicity.) Time evolution with fixed initial and final boundary conditions $x(T_{\rm I}) = x_{\rm I}$, $x(T_{\rm F}) = x_{\rm F}$ is determined by extremizing S, yielding Hamilton's equations.

A *canonical transformation* is supposed to express x and p in terms of new variables x_2 and p_2 such that the equations of motion arise from an action $S[x_2, p_2]$ of the form (1), possibly with different Hamiltonian $h \to h_2$. This will be the case if there exists a function F with

$$p\dot{x} - h = p_2\dot{x}_2 - h_2 + {\rm d}F/{\rm d}t. \tag{2}$$

F is called the *generating function*.

With some assumptions, F can determine the transformation. For example, assume the map $(x, p) \to (x_2, p_2)$ is such that we can unambiguously specify any point in phase space by fixing original and new position variables (x, x_2) and also assume F depends on (x, x_2). (This is a "type 1 generating function.") The velocities $\dot{x}$ and $\dot{x}_2$ are linearly independent of each other and of (x, x_2), yielding

$$p = \partial F/\partial x, \qquad p_2 = -\partial F/\partial x_2, \qquad h_2 = h; \tag{3}$$

this can be solved to produce a map $(x, p) \to (x_2, p_2)$ under favorable conditions. (We return to this later.)

II. END-POINT CONTRIBUTIONS FOR MECHANICS ON MANIFOLDS

Let us review a subtlety in Hamiltonian dynamics on a symplectic manifold M, relevant whenever the symplectic form ω does not admit a symplectic potential 1-form θ with $\omega = {\rm d}\theta$ that is nonsingular everywhere (i.e., ω is not exact—here and henceforth, we employ the language of differential forms; d is the de Rham differential). We assume ω obeys a Dirac quantization condition:

$$\int_C \omega = 2\pi\mathbb{Z} \tag{4}$$

for C any closed two-dimensional cycle inside M.

The previous section describes the case $M = T^\star\mathbb{R}$ with symplectic form $\omega = {\rm d}(p{\rm d}x) = {\rm d}p \wedge {\rm d}x$, and $\theta = p{\rm d}x$. The problem of generalizing the action (1) to a symplectic manifold where ω is not exact is mathematically the same as the problem of coupling a charged particle to the electromagnetic field generated by a magnetic monopole. The subtlety in the monopole case is also that the electromagnetic potential $\mathcal{A}$ is not well defined everywhere, so that the usual coupling $\int {\rm d}t \mathcal{A}_\mu(x(t))\dot{x}^\mu$ to the particle worldline needs a careful prescription.

The correct prescription was given long ago by Wu and Yang [8]. First, we cover M by open subsets U_α such that each U_α and all intersections $U_\alpha \cap U_\beta$ are contractible. Then, for each α there exists a locally defined symplectic potential 1-form θ_α such that ${\rm d}\theta_\alpha = \omega|_{U_\alpha}$ due to the Poincaré lemma. ($\omega|_{U_\alpha}$ is the restriction.) On overlaps, using the Poincaré lemma again, we find scalar functions $g_{\alpha\beta}$ describing a "gauge transformation" from θ_α to θ_β:

$$\theta_\beta - \theta_\alpha = g_{\alpha\beta}. \tag{5}$$

Given any curve $\gamma\colon [T_{\rm I}, T_{\rm F}] \to M$, we write the analog of the action (1) by splitting the curve at arbitrary points within any overlap $U_\alpha \cap U_\beta$. If, e.g., the curve happens to lie entirely within two contractible patches, $U_\alpha \ni \gamma(T_{\rm I})$ and $U_\beta \ni \gamma(T_{\rm F})$, with $\gamma(t') \in U_\alpha \cap U_\beta$, then we write

$$\int_{T_{\rm I}}^{t'} \gamma^\star\theta_\alpha + \int_{t'}^{T_{\rm F}} \gamma^\star\theta_\beta + g_{\alpha\beta}(\gamma(t')). \tag{6}$$

The $g_{\alpha\beta}$ term ensures independence from the arbitrary choice of intermediate t'. Given the quantization condition (4), the construction is independent of choices [9].

The upshot is that we may make arbitrary choices of local symplectic potential, *at the price of generating contributions that depend on the end points* $\gamma(T_{\rm I})$ *and* $\gamma(T_{\rm F})$ (and possibly on the intermediate points within overlaps). We thus write the action for Hamiltonian mechanics on a symplectic manifold as [where we reinstated the Hamiltonian $h \in C^\infty(M)$ and where $\gamma(T_{\rm I}) \equiv p_{\rm I}$ $\gamma(T_{\rm F}) \equiv p_{\rm F}$]

$$S[\gamma] = S_{\rm I}(p_{\rm I}) + S_{\rm F}(p_{\rm F}) + \int_{T_{\rm I}}^{T_{\rm F}} \gamma^\star\theta - \gamma^\star(h){\rm d}t. \tag{7}$$

The "end-point contribution" functions $S_{\rm I}$ and $S_{\rm F}$ compensate for the ambiguity in the choice of symplectic potential near the end points $p_{\rm I}$ and $p_{\rm F}$, while $\int_{T_{\rm I}}^{T_{\rm F}} \gamma^\star\theta$ is interpreted via the Wu-Yang prescription [as in Eq. (6)]. In particular, we can set $S_{\rm I} = S_{\rm F} = 0$ at the price of fixing the symplectic potentials near each end point.

III. LAGRANGIAN CORRESPONDENCES AND THEIR COMPOSITION

A *Lagrangian submanifold*, or just *Lagrangian* L of a symplectic manifold M, is a maximal submanifold where ω vanishes; explicitly, $\iota_L^\star\omega = 0$ for ι_L the inclusion map $\iota_L\colon L \hookrightarrow M$. [In other words, $\omega(v_1, v_2) = 0$ for all vectors v_1 and v_2 tangent to L.] Maximality in the case $\dim M < \infty$ means $\dim L = \dim M/2$. For the particle on the line (1), both submanifolds $p = 0$ and $x = 0$ are Lagrangian.

In general, a choice of Lagrangian submanifold amounts to a (local) choice of "position" and "momentum" variables on *any* symplectic manifold: This is realized via the *Weinstein Lagrangian neighborhood theorem*, which states that an open neighborhood of L in M is isomorphic to an open neighborhood of the zero section of the cotangent bundle $T^\star L$, and $L \subset M$ maps to the zero section

$L \hookrightarrow T^\star L$. A crucial corollary is that, given a Lagrangian L, we can always find a coordinate system on M with $\dim M/2$ momenta p_a and positions x^a, and an "adapted" local symplectic potential θ_L with

$$\theta_L = p_a \mathrm{d}x^a \tag{8}$$

with $p_a = 0$ for all $a = 1, 2, \ldots (\dim M/2)$ specifying the chosen Lagrangian L.

A. Canonical transformations are also properly understood as Lagrangians

For example, the "type 1 transformation" of Eq. (3) may be described via the Lagrangian submanifold L_0 specified as the locus $p = p_2 = 0$ in the symplectic manifold $T^\star\mathbb{R} \times T^\star\mathbb{R}$, with symplectic form

$$\mathrm{d}p \wedge \mathrm{d}x - \mathrm{d}p_2 \wedge \mathrm{d}x_2. \tag{9}$$

The adapted potential is $\theta_{L_0} = p\mathrm{d}x - p_2\mathrm{d}x_2$. Then, given any function $F \in C^\infty(L_0)$, we find a Lagrangian L_F specified by Eq. (3); geometrically, L_F is the image of L_0 under the Hamiltonian flow generated by F. Conversely, the general apparatus of generating functions amounts to finding Lagrangians specified, in this way, by functions F. (The type is encoded in the choice of L_0.) From this perspective, the invertibility of Eq. (3) is immaterial.

A *Lagrangian correspondence* from symplectic manifold (M_1, ω_1) to symplectic manifold (M_2, ω_2) is a Lagrangian L_{12} inside $(M_1 \times M_2,\ \omega_2 - \omega_1)$. We have just seen how Lagrangian correspondences appear as canonical transformations in mechanics. The concept appears more prominently in mathematics in the context of the *Weinstein symplectic category* [6,7]. Put briefly, the idea is that Lagrangian correspondences should be considered as morphisms between symplectic manifolds. The motivation comes from quantization: We should assign a Hilbert space H_1 to a symplectic manifold M_1 and—perhaps less obviously—a state $|\psi\rangle$ to a Lagrangian $L \subset M$. If $M_1 \times M_2$ is assigned the tensor product $H_1 \otimes H_2$, then a linear map $H_1 \to H_2$ should map to elements of $H_1^\star \otimes H_2$, which should come from Lagrangian correspondences L_{12}.

Therefore, two Lagrangian correspondences L_{12} (from M_1 to M_2) and L_{23} (from M_2 to M_3) ought to be composable into a third one L_{13}. We can define the set

$$L_{13} = \Pi_{13}((L_{12} \times L_{23}) \cap (M_1 \times \Delta_{M_2} \times M_3)), \tag{10}$$

where $\Delta_{M_2}: M_2 \to M_2 \times M_2$ is the diagonal, and $\Pi_{13}: M_1 \times M_2 \times M_2 \times M_3 \to M_1 \times M_3$ is the projection. This set is not a manifold *unless* the intersection is transverse; in that case, L_{13} is an immersed submanifold and L_{13} is a Lagrangian correspondence called the *geometric composition* of L_{12} and L_{23}. We refer to Sec. 2 in Ref. [10] for more on Lagrangian correspondences and to the recent review [11] for context and mathematical applications thereof.

IV. TOPOLOGICAL HAMILTONIAN DEFECTS

We now study a defect on the worldline of a particle at time $t = t_{12}$ via a generalization of the action (7). We write $S = I_{12} + S^{\mathrm{D}}_{12}$, with $S^{\mathrm{D}}_{12}(p_{12})$ a function depending on the defect location $p_{12} \equiv (\gamma_1(t_{12}), \gamma_2(t_{12}))$ and

$$I_{12} = \int_{-\infty}^{t_{12}} \gamma_1^\star\theta_1 - \gamma_1^\star(h_1)\mathrm{d}t + \int_{t_{12}}^{+\infty} \gamma_2^\star\theta_2 - \gamma_2^\star(h_2)\mathrm{d}t. \tag{11}$$

The defect separates "phase 1" ($t < t_{12}$) where the particle moves on the symplectic manifold (M_1, ω_1) with Hamiltonian h_1 along the curve $\gamma_1: (-\infty, t_{12}] \to M_1$ from "phase 2" ($t > t_{12}$). Henceforth, we are not concerned with the end points T_{I}, T_{F}, so we set $T_{\mathrm{I}} = -\infty$ and $T_{\mathrm{F}} = +\infty$. However, the *defect action* S^{D}_{12} will be important.

A topological defect is one that can be moved around at no cost, whence the defining condition is

$$\frac{\mathrm{d}S}{\mathrm{d}t_{12}} = 0. \tag{12}$$

We will derive constraints on p_{12} from this.

To proceed we employ a *folding trick* [12]: Define the "folded" curve Γ_{12} on $M_1 \times M_2$ as

$$\Gamma_{12}(\tau) = (\gamma_1(t_{12} - \tau), \gamma_2(t_{12} + \tau)), \tag{13}$$

and then S takes the form (7) with a single end point at $\tau = 0$:

$$S[\Gamma_{12}] = S^{\mathrm{D}}_{12}(p_{12}) + \int_0^{+\infty} \Gamma_{12}^\star\Theta_{12} - \Gamma_{12}^\star(H_{12})\mathrm{d}\tau. \tag{14}$$

Here, $\Theta_{12} = \theta_2 - \theta_1$ is a (local) symplectic potential for the symplectic form $\omega_2 - \omega_1$ on $M_1 \times M_2$. (There was a sign change, because $t = t_{12} - \tau$ is orientation reversing.) The integral is again defined via the Wu-Yang prescription, which is sensible since $\omega_2 - \omega_1$ satisfies the quantization condition (4) if ω_1 and ω_2 do. The folded Hamiltonian is the sum $H_{12} = h_1 + h_2$.

Consistency of the variational principle (14) requires boundary conditions at $\tau = 0$. We anticipate a future calculation and select the boundary condition $\Gamma_{12}(0) \equiv p_{12} \in L_{12}$ for L_{12} a Lagrangian correspondence from M_1 to M_2, i.e., a Lagrangian for $(M_1 \times M_2,\ \omega_2 - \omega_1)$. We can, thus, employ the Lagrangian neighborhood theorem and produce an adapted (to L_{12}) potential $\Theta_{12} = P_A \wedge \mathrm{d}X^A$ near $\Gamma_{12}(0) = p_{12}$, so that the boundary condition is simply $P_A(0) = 0$ for *all* curves. [The index A takes $(\dim M_1 + \dim M_2)/2$ values.] Since we have made a choice of symplectic potential near the end point, off shell, we can thus set $S^{\mathrm{D}}_{12} = 0$.

If we write $\mathrm{d}/\mathrm{d}t_{12} = \delta$, then using Eq. (13)

$$\delta\Gamma_{12}^{\star}(H_{12}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\Gamma_{12}^{\star}(h_2 - h_1) \tag{15}$$

and (where $\Omega_{12} = \omega_2 - \omega_1$)

$$\delta(\Gamma_{12}^{\star}\Theta_{12}) = \frac{\mathrm{d}}{\mathrm{d}\tau}(P_A \delta X^A)\mathrm{d}\tau + \Omega_{12}(\delta\Gamma_{12}, \dot{\Gamma}_{12})\mathrm{d}\tau. \tag{16}$$

In the last term, $\dot{\Gamma}_{12}$ is the tangent vector to the curve (13), while $\delta\Gamma_{12}$ is the vector obtained by the $\delta = \mathrm{d}/\mathrm{d}t_{12}$ derivative. However, the two are related by $\delta\Gamma_{12} = C\dot{\Gamma}_{12}$ for a certain constant matrix C [due to Eq. (13)]; C satisfies

$$C^2 = 1, \qquad \Omega_{12}(CV, CU) = \Omega_{12}(V, U), \tag{17}$$

whence $\Omega_{12}(\delta\Gamma_{12}, \dot{\Gamma}_{12}) = 0$. (In "factorized" coordinates, C changes the sign of vectors pointing along M_1, from which these identities are now obvious.) Using the boundary condition $P_A = 0$ at $\tau = 0$, the first term in Eq. (16) also fails to contribute, so we obtain

$$\delta S = \Gamma_{12}^{\star}(h_1 - h_2)|_{\tau=0} = 0. \tag{18}$$

Therefore, the topological defect condition (12) holds *if* the defect location p_{12} lies on a Lagrangian correspondence L_{12} *and* the Hamiltonians $h_{1,2}$ match on each side of the defect. This is essentially the matching of stress-energy tensors on topological defects in field theory (see, e.g., [5]).

We now discuss fusion. We take two topological defects at locations $t = t_{12}$ and $t = t_{23}$ that define a segmented worldline along symplectic manifolds M_1 (for $t \leq t_{12}$), M_2 (for $t_{12} \leq t \leq t_{23}$), and M_3 (for $t \geq t_{23}$), with symplectic forms and Hamiltonian functions (ω_1, h_1) on M_1, etc. We have associated Lagrangian correspondences L_{12} (to the t_{12} defect) and L_{23} (to the t_{23} defect) as before.

To write the action, we introduce the worldline 1-forms

$$\alpha_{1,2,3} = \gamma_{1,2,3}^{\star}\theta_{1,2,3} - \gamma_{1,2,3}^{\star}(h_{1,2,3})\mathrm{d}t. \tag{19}$$

For the action, we take simply

$$S = \int_{-\infty}^{t_{12}} \alpha_1 + \int_{t_{12}}^{t_{23}} \alpha_2 + \int_{t_{23}}^{+\infty} \alpha_3, \tag{20}$$

which is again interpreted via the Wu-Yang prescription and where we have made a choice of symplectic potentials close to t_{12} and t_{23} that are adapted to the Lagrangian correspondences so as to set the defect actions to zero.

Since both defects are topological, we can shift them around freely. By *defect fusion*, we mean the defect obtained in the coincidence limit $t_{12} \to t_{23}$. This ought to be a topological defect between M_1 and M_3 and, thus, a Lagrangian correspondence L_{13} from M_1 to M_3.

We will calculate L_{13} from L_{12} and L_{23}. Let us write

$$S = \int_{-\infty}^{t_{12}} \alpha_1 + \int_{t_{12}}^{+\infty} \alpha_2 + \int_{-\infty}^{t_{23}} \alpha_2 + \int_{t_{23}}^{+\infty} \alpha_3 - \int_{-\infty}^{+\infty} \alpha_2. \tag{21}$$

This entails an arbitrary extension of γ_2 to the entire real line; the arbitrariness will drop out shortly. The point of the rewriting is that we may fold the *first four* terms pairwise to obtain terms of the form (14). This yields

$$\int_0^{+\infty} \Gamma_{12}^{\star}\Theta_{12} + \Gamma_{23}^{\star}\Theta_{23} - (\Gamma_{12}^{\star}(H_{12}) + \Gamma_{23}^{\star}(H_{23}))\mathrm{d}\tau. \tag{22}$$

Here, Θ_{12} is a symplectic potential for $\omega_2 - \omega_1$, Θ_{23} is one for $\omega_3 - \omega_2$, $H_{12} = h_1 + h_2$, and $H_{23} = h_2 + h_3$. Γ_{12} is the curve (13), while Γ_{23} is given similarly.

We interpret this via the twice-folded curve $\Gamma_{1223}(\tau) = (\Gamma_{12}(\tau), \Gamma_{23}(\tau))$ on the manifold $M_1 \times M_2 \times M_2 \times M_3$:

$$(\gamma_1(t_{12} - \tau), \gamma_2(t_{12} + \tau), \gamma_2(t_{23} - \tau), \gamma_3(t_{23} + \tau)). \tag{23}$$

For $\tau = 0$, this is (p_{12}, p_{23}), where $p_{12} \in L_{12}$ and $p_{23} \in L_{23}$.

Since both original defects are topological, we can send, e.g., $t_{23} \to 0$; then the coincidence limit is $t_{12} \to 0$, and the twice-folded curve becomes

$$\Gamma_{1223}(\tau) = (\gamma_1(-\tau), \gamma_2(+\tau), \gamma_2(-\tau), \gamma_3(+\tau)). \tag{24}$$

We see γ_2 is traversed twice as τ goes from 0 to $+\infty$; this produces an integral that cancels the last term of Eq. (21), thus eliminating the "intermediate" manifold M_2. Moreover, for $\tau = 0$ we see that Eq. (24) takes the form (p_1, p_2, p_2, p_3) for points $p_{1,2,3} \in M_{1,2,3}$. Since $\Gamma_2(0) \equiv p_{12} = (p_1, p_2)$ and $\Gamma_{23}(0) \equiv p_{23} = (p_2, p_3)$ each lie on L_{12} and L_{23}, we have arrived precisely at the geometric composition of the Lagrangian correspondences (10). The Hamiltonians on each side of the fused defect L_{13} trivially agree with each other, so this completes the argument.

V. DISCUSSION

Since canonical transformations (in the physics sense) give rise to Lagrangian correspondences, we can construct a topological defect for every canonical transformation. We exploit this to construct duality defects. As a first example, take $d = 4$ U(1) Euclidean gauge theory, with complex coupling $\tau = \theta/(2\pi) + 4\pi i/g^2$ (of which g^{-2} is the coupling and θ the theta angle). The phase space for electromagnetism is spanned by the positions $\mathcal{A}_i(\sigma)$ ($i = 1, 2, 3$, $\sigma \in \mathbb{R}^3$) and their conjugate momenta $\Pi^i(\sigma)$; the former are the spatial components of the gauge potential 1-form $\mathcal{A}$. In Ref. [13], there is a canonical transformation that implements $\tau \to -\tau^{-1}$; thus, there exists a topological defect between the theory with coupling τ and the theory with coupling $-\tau^{-1}$, as was found in Refs. [4,5] entirely differently. Explicitly, we have the type-1 generating function

$$F = \int_{M_4} \mathcal{F} \wedge \tilde{\mathcal{F}} \tag{25}$$

in terms of the original and dual field strengths $\mathcal{F} = \mathrm{d}\mathcal{A}$ and $\tilde{\mathcal{F}} = \mathrm{d}\tilde{\mathcal{A}}$. (For this to work, we need the *reduced* phase space for electromagnetism, obtained by symplectic reduction modulo the Gauss law constraint $\partial_i \Pi^i = 0$.) We can similarly obtain S-duality defects on the world volume of the D3 brane in type-IIB string theory [14] (including fermions [15]), as well as between a IIB superstring and a D1-brane [16].

We may also easily recover world sheet T-duality defects from the T-duality canonical transformation given in Ref. [17]. This extends to other flavors of T-duality, including Poisson-Lie (via Refs. [18,19]) and even fermionic T-duality (via Ref. [20]); for the latter case, topological defects were also found in Ref. [21] with different methods. In the Poisson-Lie case, we have thus made contact with the recent "Poisson-Lie defects" of Ref. [22].

We can say more about Poisson-Lie T-duality defects, however. In Ref. [23], we recently cointroduced a joint generalization of topological [24,25] and Poisson-Lie T-dualities in the form of Lagrangian correspondences between the phase spaces for string propagation on a principal *bibundle* $G \hookrightarrow M \dashrightarrow B$ and its dual bibundle $\tilde{G} \hookrightarrow \tilde{M} \dashrightarrow B$, whose fibers G and $\tilde{G}$ are Poisson-Lie dual groups; this "bibundle duality" can realize topology changes in the global fibration structure of target space akin to those of topological (Abelian) T-duality, which is a special case. With the result of the current paper, we can thus realize bibundle duality via a topological worldsheet defect. (This way, the Drinfeld double bibundles of Ref. [23] give novel "bibranes" in the terminology of Ref. [2].)

There are also examples of dualities that may be understood this way outside of string theory (which is admittedly overrepresented above on account of the author's preferences and expertise). For instance, bosonization in two-dimensional field theory has been described in terms of canonical transformations [26,27] and, thus, may be realized via topological defects, like S-duality in $d = 4$ gauge theory was above.

There are a few notable omissions from our brief treatment. The first is the introduction of time dependence (in the Hamiltonians h_1 and h_2) which *a priori* motivates introducing time dependence in the defect action $S^{\mathrm{D}}_{12}(p_{12}, t_{12})$. This changes little: Choosing $\Gamma_{12}(0) \in L_{12}$ as a boundary condition (for time-independent L_{12}) along with appropriate end-point contributions eventually forces $S^{\mathrm{D}}_{12} = S^{\mathrm{D}}_{12}(t_{12})$ without loss of generality. The effect is to introduce a discontinuity in the Hamiltonians, so $h_2 = h_1 + \partial S^{\mathrm{D}}_{12}/\partial t_{12}$ on the defect.

Another omission has to do with the transversality issue discussed below Eq. (10): The fusion of two topological defects may fail to be a defect, in the sense that the corresponding Lagrangian may be singular (as a manifold). (For this reason, Weinstein himself called it the *symplectic 'category'*.) Fortunately, it was shown in Proposition 5.2.1 in Ref. [10] and Theorem 2.3 in Ref. [28] that the intersection of Eq. (10) is rendered transverse if the Lagrangians are perturbed appropriately; this could be realized in our picture by switching on small defect actions (S^{D}_{12} and S^{D}_{23}). The physical implications of this (if any) are left for the future.

Finally, we will also be leaving the treatment of gauge theories, i.e., Hamiltonian systems with first-class constraints, for the future. That case includes temporal reparametrization invariance, which introduces many subtleties; however, we note that composition of correspondences already allows us to construct topological defects in the *reduced (gauged) theory* from topological defects in the original theory: The key point is that coisotropic reduction gives a Lagrangian correspondence between the original and reduced symplectic manifolds (see Sec. 3 in Ref. [29]). In this context, we also expect that introducing degrees of freedom lying on the defect might be necessary, as is common when discussing topological defects in field theory.

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