Holographic renormalization in the Hamilton-Jacobi formulation with exact ansatz generation

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In AdS/CFT corresponding, the UV divergence of generating functional on the field theory can be removed as the IR divergence in the gravity. This geometric process is well known as holographic renormalization. The standard method of holographic renormalization is based on the Fefferman-Graham expansion, which is strict and universal but technically cumbersome. To improve the technique, different methods have been proposed. Here we develop an alternative approach to holographic renormalization based on the Hamilton-Jacobi formulation of gravity. Compared to previous approaches, its distinguishing feature is the generation of exact ansatz of counterterms. We apply this approach to several typical holographic models, which consistently performs well.

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I. INTRODUCTION

Anti-de Sitter/conformal field theory (AdS/CFT) corresponding not only provides a gravitational lens for the strongly coupled quantum many-body system but also inspires the theory of quantum gravity [1]. Holographic renormalization (HR), which removes the UV divergence on the boundary field theory by isolating the IR divergence in the bulk gravity, is one of the essential components in the AdS/CFT [2,3].

There are various ways to perform the HR [4–20]. Among others, the standard method is strict, universal, and conceptually simple [4,7,8]. However, its core component, the Fefferman-Graham (FG) expansion (especially its inversion), is technically cumbersome [21], which can be partially attributable to the breaking of covariance in the procedure. The problem of covariance is absent in a class of approaches based on the Hamilton-Jacobi (HJ) formulation of gravity, where the radial coordinate plays the role as time. This class of approach was first proposed by de Boer, Verlinde, and Verlinde (dBVV) [9,10], who derive the counterterms in the derivative expansion by iteratively solving the radial HJ equation. Note that here the HJ equation is reduced to the Hamiltonian constraint which ensures the invariance under the radial diffeomorphism. The dBVV's approach is considerably improved by Kalkkinen, Martelli, and Muck [11,12]. In particular, the logarithmic counterterms that have not been explicitly obtained in [9,10] are isolated by relating them to the breakdown of the recursive equation. The main flaw common to these works is the requirement to postulate an ansatz consisting of all potential divergent terms. Since usually the ansatz is constrained only to be local and covariant, it is likely to contain plenty of redundancy while sometimes the sufficient ansatz is difficult to figure out. In Refs. [13,14], Papadimitriou and Skenderis put forward a systematic method without relying on the ansatz. The crucial difference is that the covariant expansion is organized according to the eigenfunctions of the dilatation operator comprised of induced metric and scalar fields. However, as pointed out in [15], the eigenfunctions of the dilatation operator for an arbitrary scalar potential would not serve as a practical basis for the expansion since they would be highly nontrivial.[1] This problem has been addressed in [15], where the dilatation operator is replaced by the operator relevant only to the induced metric and usually the resulting recursive equations are the functional differential equations.

In 2016, Evang and Hadjiantonis proposed a practical approach to the HJ formulation of HR, which returns to the

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[1]The dilatation operator can be understood as the asymptotic form of the functional representation of the radial derivative. In Ref. [13], it has been noted that the functional representation must be modified when the leading asymptotics of bulk fields are of the form $r\exp(-dr/2)$. This happens when the scaling dimension of dual operators is $\Delta = d/2$. The well-known Freedman-Gubser-Pilch-Warner (FGPW) model is exactly this case [22].

derivative expansion and the postulation of ansatz [19]. In contrast to previous approaches for solving the Hamiltonian constraint, this one preserves the radial partial-derivative term of the HJ equation.[2] We hence refer to the former as Hamiltonian approaches and the latter HJ approach. The HJ approach has been illustrated in several Einstein-scalar theories, including the FGPW model, the dilaton-axion system with constant potential and others. Interestingly, it handles power and logarithmic divergences in completely the same way, that is, there is no difference between their derivations such as the breakdown of recursive equations. In Ref. [20], it is clarified that the HJ approach is not conflicted with the Hamiltonian constraint, because there only a part of the HJ equation is actually used to perform HR while the Hamiltonian constraint is not used. Thus, the HJ approach is strict and applicable to the theories with or without the diffeomorphism symmetry.

In this paper, we will develop a new version of the HJ approach to HR. The key improvement we will make is a systematic method to generate the exact ansatz. By "exact," it means no omission and no redundancy. Moreover, it will be illustrated that a general solution to the coefficients in the ansatz can be derived. In the Appendix, we will provide some technical details and apply the new approach to some typical holographic models. In particular, we will study a generalized holographic axion model, which has interesting applications in the duality between gravity and quantum matter [24–27]. Since its action allows for the derivative term with nonintegral powers, this model might be a challenge for previous HR approaches that require a complete list of all possible ansatz or similar information. In fact, this was the original motivation for developing our new approach.

II. BENCHMARK

We will develop the benchmark of the approach in terms of the Einstein gravity coupled to scalars fields in the $d+1$-dimensional asymptotic AdS spacetime. The bulk action of the theory is

$$S = -\int_M d^{d+1}x\sqrt{g}(\mathcal{R} - g^{\mu\nu}G_{IJ}\partial_\mu\Phi^I\partial_\nu\Phi^J - V), \quad (1)$$

where V is a potential of scalar fields and G_{IJ} denote their symmetric couplings.

A. HJ formulation

Let us introduce the HJ equation of this gravity system. Using Eq. (1) and the Arnowitt-Deser-Misner (ADM) metric with FG gauge [28]

$$ds^2 = dr^2 + \gamma_{ij}dx^i dx^j, \quad (2)$$

one can obtain the radial Hamiltonian [19,20]

$$H = \int_{\partial M} d^d x\left[\frac{1}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2 + \frac{1}{4}G^{IJ}\pi_I\pi_J\right) + \sqrt{\gamma}L_S\right], \quad (3)$$

where π_{ij} and π_I are the canonical momenta conjugate to the induced metric γ^{ij} and the scalar fields Φ^I. We would like to refer L_S as the counterterm seed due to its role played in the HR, which is given by

$$L_S = R - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi). \quad (4)$$

In classical mechanics, the canonical momenta can be expressed as the variation of the on-shell action $S_{\rm os}$ [29]. As a result, the HJ equation can be written as[3]

$$H\left(\gamma_{ij}, \Phi^I; \frac{\delta S_{\rm os}}{\delta\gamma_{ij}}, \frac{\delta S_{\rm os}}{\delta\Phi^I}\right) + \frac{\partial S_{\rm os}}{\partial r} = 0. \quad (5)$$

In section A of the Appendix, we have reviewed that the HJ equation can be divided into two parts. We focus on the counterterm part

$$H_{\rm ct} + \frac{\partial S_{\rm ct}}{\partial r} = 0, \quad (6)$$

where

$$H_{\rm ct} = -\int_{\partial M} d^d x[\{S_{\rm ct}, S_{\rm ct}\} + \sqrt{\gamma}L_S], \quad (7)$$

$$\{S_{\rm ct}, S_{\rm ct}\} = \frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_{\rm ct}}{\delta\gamma_{ij}}\frac{\delta S_{\rm ct}}{\delta\gamma_{kl}}\gamma_{ijkl} + \frac{1}{4}G^{IJ}\frac{\delta S_{\rm ct}}{\delta\Phi^I}\frac{\delta S_{\rm ct}}{\delta\Phi^J}\right), \quad (8)$$

$$\gamma_{ijkl} = \gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}. \quad (9)$$

For later use, we rewrite the counterterm as

$$S_{\rm ct} = -2\int_{\partial M} d^d x\sqrt{\gamma}U(\gamma^{ij}, \Phi^I, r). \quad (10)$$

For the sake of brevity, we define

[2]This idea is partially inspired by Ref. [23], where the HJ equation is used to isolate the infrared divergence of the scalar field in a fixed de-Sitter background.

[3]For the theory with diffeomorphism symmetry, the Hamiltonian constraint $H = 0$ should be respected. Thus, one might wonder why the partial derivative $\partial S_{\rm os}/\partial r$ is kept in general. This issue has been explained carefully in section 2 of Ref. [20]. Here we emphasize that Eq. (5) always holds no matter if the theory has the diffeomorphism symmetry or not.

$$K = 4Y_{ij}Y^{ij} - \frac{1}{d-1}(U-2Y)^2 - U^2, \tag{11}$$

$$Y_{ij} = \int \frac{\delta U}{\delta \gamma^{ij}}, \qquad Y^{ij} = -\int \frac{\delta U}{\delta \gamma_{ij}}, \qquad Y = \gamma^{ij}\int \frac{\delta U}{\delta \gamma^{ij}},$$
$$P_I = \int \frac{\delta U}{\delta \Phi^I}, \tag{12}$$

$$\int \delta U = \frac{1}{\sqrt{\gamma}}\int_{\partial M} d^d x \sqrt{\gamma}\delta U. \tag{13}$$

Then Eq. (6) can be reshaped as

$$2\frac{\partial U}{\partial r} + L_S + K + G^{IJ}P_I P_J = 0, \tag{14}$$

which holds up to total derivatives, since it should be understood as an integral equation.

B. Generation of ansatz

One of the key points in various HR approaches is how to organize the counterterms. Considering that the counterterms are built from some boundary invariants with different degrees of divergence, we choose to expand them according to these degrees of divergence, which can be formally written as

$$U = U_{(k_0)} + U_{(k_1)} + U_{(k_2)} + \cdots. \tag{15}$$

Some remarks on the expansion are in order. First, we suppose that $U_{(k_0)}$ is independent with boundary fields. This is true at least for asymptotic AdS spacetimes. Second, the divergence degree k_i is defined by the asymptotic behavior of $U_{(k_i)}$, which can be expressed as $U_{(k_i)} \sim e^{-k_i r}$. Although k_i is often a number, it is not necessary.[4] However, one should be careful that the number of terms in the expansion (15) should be finite.[5] Third, we do not need to postulate the concrete ansatz of each $U_{(k_i)}$, nor do we specify its divergence degree k_i in advance. Both of them will be emergent. Fourth, we assume a variation identity[6]

$$Y_{(k_i)} = \bar{k}_i U_{(k_i)} + \mathrm{TD} \quad \text{with} \quad k_i \geq 2\bar{k}_i, \tag{16}$$

[4]Whether k_i is a number or not, we can use $U_{(k_i)} \sim e^{-k_i r}$ to sort the counterterms. In Appendix D 2, we will deal with the FGPW model where the divergence degree related to the scalar with $\Delta = d/2$ is not a number.

[5]In view of this, the expansion according to divergence degrees is not suitable for the improved holographic QCD model presented in [30], because it has been revealed that there are divergences from any power of the dilaton [15].

[6]This identity has been found before in different systems [19,20]. A similar identity has been used in [15] and it can be traced back to [6] for pure gravity.

where $\bar{k}_i$ is the number of inverse metrics in $U_{(k_i)}$ and TD means total derivatives. In section B of the Appendix, we will prove that Eq. (16) holds very generally.

Now let us expand Eq. (14). Substituting Eq. (15) into Eq. (14), we have

$$2\frac{\partial}{\partial r}\sum_i U_{(k_i)} + \sum_i L_{S(k_i)} + \sum_{m,n} H_{(k_m,k_n)} = 0, \tag{17}$$

where

$$\begin{aligned} H_{(k_m,k_n)} = {} & 4Y_{(k_m)ij}Y^{ij}_{(k_n)} + G^{IJ}P_{I(k_m)}P_{J(k_n)} \\ & - \frac{1}{d-1}(U_{(k_m)} - 2Y_{(k_m)})(U_{(k_n)} - 2Y_{(k_n)}) \\ & - U_{(k_m)}U_{(k_n)}. \end{aligned} \tag{18}$$

Since the variation of $U_{(k_0)}$ vanishes, the leading order of Eq. (17) is reduced to:

$$\begin{aligned} & 2\frac{\partial U_{(k_0)}}{\partial r} - \frac{1}{d-1}(U_{(k_0)} - 2Y_{(k_0)})^2 \\ & - U^2_{(k_0)} + G^{IJ}P_{I(k_0)}P_{J(k_0)} + L_{S(k_0)} \\ & = 2\frac{\partial U_{(k_0)}}{\partial r} - \frac{d}{d-1}U^2_{(k_0)} + d(d-1) = 0, \end{aligned} \tag{19}$$

which has the solution $U_{(k_0)} = 1 - d + \mathcal{O}(e^{-dr})$. Note that the higher order $\mathcal{O}(e^{-dr})$ does not correspond to a real divergent term since $\sqrt{\gamma}e^{-dr} \sim \mathcal{O}(1)$. With $U_{(k_0)}$ in hand, one can rewrite any other order of Eq. (17) as

$$2\frac{\partial}{\partial r}U_{(k_i)} + 2(d - 2\bar{k}_i)U_{(k_i)} + L_{S(k_i)} + \sum_{m,n\to i} H_{(k_m,k_n)} = 0, \tag{20}$$

where $m, n \to i$ denotes that the choice of m and n in the range $(0, i]$ such that $H_{(k_m,k_n)}$ has the divergence degree k_i. Note that in derivation of Eq. (20), we have used the variation identity (16).

To proceed, we set

$$U_{(k_i)} = \sum_a C^{(a)}_{(k_i)}(r)\bar{U}^{(a)}_{(k_i)}, \tag{21}$$

where $\bar{U}^{(a)}_{(k_i)}$ are some independent scalars made of boundary fields and their derivatives. The coefficients $C^{(a)}_{(k_i)}(r)$ are assumed to depend on r in general, following Refs. [19,20]. Without loss of generality, we will focus on one term $C_{(k_i)}(r)\bar{U}_{(k_i)}$, where we have suppressed the index (a) for brevity. Then Eq. (20) indicates

$$2\frac{\partial}{\partial r}C_{(k_i)}\bar{U}_{(k_i)} + 2(d-2\bar{k}_i)C_{(k_i)}\bar{U}_{(k_i)} + L_{A(k_i)} + L_{P(k_i)} = 0, \tag{22}$$

where

$$L_{A(k_i)} = L_{S(k_i)} + \sum_{\substack{m,n\to i\\ m,n\neq i}} H_{(k_m,k_n)}, \tag{23}$$

$$L_{P(k_i)} = \sum_{\substack{m,n\to i\\ m=i \text{ or } n=i}} G^{IJ}P_{I(k_m)}P_{J(k_n)} + \sum_{\substack{m,n\to i\\ m=i \text{ and } n=i}} G^{IJ}P_{I(k_m)}P_{J(k_n)} \tag{24}$$

$$= C_{(k_i)}\bar{U}_{(k_i)}b_{1(k_i)} + C^2_{(k_i)}\bar{U}_{(k_i)}b_{2(k_i)}. \tag{25}$$

In section C of the Appendix, we will analyze two constants $b_{1(k_i)}$ and $b_{2(k_i)}$ in detail. One can find that either $b_{1(k_i)}$ or $b_{2(k_i)}$ must be zero. We will refer them as mass parameters since usually they are related to the mass of scalar fields.

Observing Eqs. (22)–(25), one may notice an important fact: the ith order counterterms are absent if $L_{A(k_i)} = 0$. This can be seen by solving Eq. (22) without $L_{A(k_i)}$:

$$2\frac{\partial}{\partial r}C_{(k_i)} + 2(d-2\bar{k}_i)C_{(k_i)} + b_{1(k_i)}C_{(k_i)} + b_{2(k_i)}C^2_{(k_i)} = 0. \tag{26}$$

Its solutions can be divided into three classes.

(1) For $b_{1(k_i)} = b_{2(k_i)} = 0$, the solution is

$$C_{(k_i)} = \alpha e^{-(d-2\bar{k}_i)r} + \cdots. \tag{27}$$

(2) For $b_{1(k_i)} \neq 0$ and $b_{2(k_i)} = 0$, the solution is

$$C_{(k_i)} = \alpha e^{-(d-2\bar{k}_i+b_{1(k_i)}/2)r} + \cdots. \tag{28}$$

(3) For $b_{1(k_i)} = 0$ and $b_{2(k_i)} \neq 0$, the solution is

$$C_{(k_i)} = \begin{cases} \alpha e^{-(d-2\bar{k}_i)r} + \cdots, & d-2\bar{k}_i \neq 0 \\ \frac{2}{b_{2(k_i)}r} + \cdots, & d-2\bar{k}_i = 0 \end{cases}. \tag{29}$$

Here α is an arbitrary integral constant. Immediately, this implies that the solutions with α are not relevant to any real divergent terms, because the divergent part of counterterms should be unique.[7] Moreover, the second line of Eq. (29) does not yield real divergent terms either, since

$$\sqrt{\gamma}C_{(k_i)}\bar{U}_{(k_i)} \sim e^{dr}\cdot\frac{1}{r}\cdot e^{-k_i r} = \frac{e^{-(k_i-2\bar{k}_i)r}}{r} < \mathcal{O}(1), \tag{30}$$

where we have used $k_i \geq 2\bar{k}_i$.

As a result, one can see that the ansatz $U_{(k_i)}$ at order i should be emergent in $L_{A(k_i)}$. Put differently, the divergence degree and the ansatz at each order can be iteratively generated. To be clearer, we define the ansatz generator

$$L_A = L_S + \sum_{m,n} H_{(k_m,k_n)}. \tag{31}$$

One should input the counterterm seed of order greater than $i-1$ and the ansatz of order in the range $(0, i-1]$. The $L_{A(k_i)}$ defined by Eq. (23) can be identified as the term in L_A with k_i closest to k_{i-1} but greater than it.

C. Solution of coefficients

We will specify the coefficients in the ansatz by solving Eq. (22). To save symbols, let us still denote any term in $L_{A(k_i)}$ as $\bar{U}_{(k_i)}$ and the ansatz as $C_{(k_i)}\bar{U}_{(k_i)}$. Thus, Eq. (22) can be reduced to

$$2\frac{\partial}{\partial r}C_{(k_i)} + 2(d-2\bar{k}_i)C_{(k_i)} + 1 + b_{1(k_i)}C_{(k_i)} + b_{2(k_i)}C^2_{(k_i)} = 0. \tag{32}$$

It also has three classes of solutions, which are listed below.

(1) For $b_{1(k_i)} = b_{2(k_i)} = 0$,

$$C_{(k_i)} = \begin{cases} -\frac{1}{2(d-2\bar{k}_i)} + \mathcal{O}(e^{-(d-2\bar{k}_i)r}), & d-2\bar{k}_i \neq 0 \\ -\frac{r}{2} + \mathcal{O}(1), & d-2\bar{k}_i = 0 \end{cases}. \tag{33}$$

(2) For $b_{1(k_i)} \neq 0$, $b_{2(k_i)} = 0$,

$$C_{(k_i)} = \begin{cases} -\frac{1}{2(d-2\bar{k}_i)+b_{1(k_i)}} + \mathcal{O}(e^{-(d-2\bar{k}_i+b_{1(k_i)}/2)r}), & d-2\bar{k}_i \neq \frac{-b_{1(k_i)}}{2} \\ -\frac{r}{2} + \mathcal{O}(1), & d-2\bar{k}_i = \frac{-b_{1(k_i)}}{2} \end{cases}. \tag{34}$$

[7] In this work, we are not concerned with finite counterterms. They depend on the choice of renormalization group (RG) scheme.

(3) For $b_{1(k_i)} = 0$ and $b_{2(k_i)} \neq 0$,

$$C_{(k_i)} = \begin{cases} \frac{-1}{(d-2\bar{k}_i)+\sqrt{(d-2\bar{k}_i)^2-b_{2(k_i)}}} + \mathcal{O}(e^{-\sqrt{(d-2\bar{k}_i)^2-b_{2(k_i)}}r}), & d-2\bar{k}_i \neq \sqrt{b_{2(k_i)}} \\ -\frac{1}{d-2\bar{k}_i} + \frac{2}{(d-2\bar{k}_i)^2}\frac{1}{r} + \mathcal{O}(r^{-2}), & d-2\bar{k}_i = \sqrt{b_{2(k_i)}} \end{cases}. \tag{35}$$

Here we keep the coefficients up to the presence of integral constants. Thus, the higher orders can be dropped since they should not be relevant to any real divergent terms. As a result, we have obtained the coefficients at the ith order. Each coefficient is determined by four constants $d, \bar{k}_i, b_{1(k_i)}, b_{2(k_i)}$. It is worth pointing out that there are two important special cases. One is relevant to the theories with constant potential. Since $b_{1(k_i)} = b_{2(k_i)} = 0$ and $k_i = 2\bar{k}_i$ therein, the coefficients $C_{(k_i)}$ are universal for all counterterms at any given (d, k_i), see section C in the Appendix. The other is relevant to any logarithm divergence, which appears at $d - 2\bar{k}_i = 0$ or $-b_{1(k_i)}/2$ and is associated with the universal coefficient $-r/2$.

III. GENERAL ALGORITHM

We will extract the spirit of the above benchmark and promote it to be a more general algorithm, which can be stated as follows.

(1) Derive the HJ equation and separate the counterterm part from it.

(2) Take the formal expansion according to the divergence degree. Express any order of counterterms as $U_{(k_i)} = C_{(k_i)}(r)\bar{U}_{(k_i)}$ and build up the recursive equation

$$2\frac{\partial}{\partial r}C_{(k_i)}\bar{U}_{(k_i)} + F(C_{(k_i)})\bar{U}_{(k_i)} + L_{A(k_i)} = 0. \tag{36}$$

Here $F(C_{(k_i)})$ is a function of the coefficient[8] but $L_{A(k_i)}$ is independent with the coefficient.

(3) Generate the exact ansatz of the counterterms iteratively from $L_{A(k_i)}$.

(4) Set $L_{A(k_i)} = \bar{U}_{(k_i)} = 1$ in Eq. (36) to obtain an ODE of the coefficient. Solve the ODE for a general solution.

Note that we have assumed that the solution of Eq. (36) without $L_{A(k_i)}$ is not relevant to divergent counterterms.[9]

[8]For all the models in this paper, the function is the sum of linear and quadratic terms. But it would be changed for a more general model.

[9]The counterexamples may be very rare, even if they would exist.

IV. EXAMPLES AND ASSESSMENTS

In section D of the Appendix, we will use the above approach to work out the counterterms explicitly in some typical holographic models. Keeping these examples in mind, we will assess the approach from four aspects.

A. Simplicity

The main calculation we need is to take variation and solve ODE. The difficulty of variation (with respect to the tensor) is greatly reduced by some symbolic computing programs (such as the Mathematica package xAct [31]). As for ODE, they are first order and usually simple, at least for the holographic models we encounter in this paper. In particular, when the general solution is found and the variation is prepared, the counterterms can be obtained only by algebra calculation.

B. Universality

We have taken the model with the action (1) as the benchmark. This is a rather general model. In fact, most models in the Appendix are its special cases. They include the theory of gravity coupled to a dilaton with a general potential [3], the FGPW model where one of two scalars has $\Delta = d/2$ [19,22], the axion-dilaton model with constant potential,[10] and the axion-dilaton model with asymptotic polynomial potential [32]. As the illustration of more general algorithm, we also study two models that cannot be described by the action (1). They are the massive gravity that breaks the diffeomorphism symmetry [33–37] and the generalized holographic axion model that allows for the derivative term with nonintegral powers [24–27].

C. Exactness

The HJ approach to the HR of the axion-dilaton model has been studied in [19], where the first step is to postulate the ansatz for each counterterm. For the highest order at $d = 4$, their ansatz has 28 terms. Note that the terms up to total derivatives have already been omitted. Nevertheless, the redundancy is considerable, since the number of real counterterms is only 16. The situation is more serious for

[10]In this model, the scalar fields are dual to marginal operators and they do not enjoy the same suppression as the scalar fields that are dual to the relevant operators. This issue is handled in [19] by allowing the coefficients in the ansatz as the functions of marginal scalars. In our work, it is not necessary. Different scalar fields are treated in the same way.

the nonlinear holographic axion model: it is difficult to postulate the sufficient ansatz due to the noninteger powers. As a comparison, we generate the exact ansatz in the Appendix for all the models including these two.

D. Fluency

As an advantage inherited from the HJ approach [19,20], the logarithmic divergence does not cause any more trouble than the power divergence. On the contrary, here it is somewhat simpler to deal with the logarithmic divergence. This can be understood as follows. After generating the exact ansatz at certain order with the logarithmic divergence, we immediately know that all coefficients are $-r/2$, which has been pointed out as the second special case below Eq. (35). The logarithmic divergence is present in general at the highest order when d is even. It also appears in massive gravity and holographic axion model when d is odd. Since the highest order has the most independent counterterms, the universal coefficient brings considerable simplification.

V. SUMMARY

In this work, we developed an alternative approach to holographic renormalization based on the HJ formulation of gravity. Its distinguishing feature is the generation of exact ansatz of counterterms. Although our primary focus is on the technical aspects, this approach brings new understandings of how counterterms are organized and generated. As one can see, the ith counterterm is determined by the exact ansatz $L_{A(k_i)}$ and the coefficient function $F(C_{(k_i)})$ in Eq. (36). Meanwhile, the existence of general solutions to the coefficients informs us of all types of counterterms in a certain class of theory. In particular, the benchmark model allows three types of counterterms, which are power law, logarithmic, and inverse logarithmic. Furthermore, the general conditions under which these types appear may be helpful in designing a special UV which is required by some bottom-up holographic models. For example, if we expect a conformal anomaly when d is odd, one way is to design a model such that the counterterm seed has a term with half-integer inverse metrics. This is exactly the fact that massive gravity exhibits. In the future, it would be interesting to explore whether the current approach can be extended beyond the standard AdS/CFT, which may break the conformal symmetry [38–42], keep the finite coupling [43–45], and even deviate from the large N limit [46,47].

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APPENDIX A: DECOMPOSE HJ EQUATION

Let us start from the HJ equation:

$$H\left(\gamma_{ij}, \Phi^I; \frac{\delta S_{\mathrm{os}}}{\delta\gamma_{ij}}, \frac{\delta S_{\mathrm{os}}}{\delta\Phi^I}\right) + \frac{\partial S_{\mathrm{os}}}{\partial r} = 0. \tag{A1}$$

Suppose that S_{ren} and S_{ct} are the renormalized part and the counterterm part of S_{os}, respectively. Then Eq. (A1) can be decomposed into

$$H_{\mathrm{ren}} + \frac{\partial S_{\mathrm{ren}}}{\partial r} - H_{\mathrm{ct}} - \frac{\partial S_{\mathrm{ct}}}{\partial r} = 0. \tag{A2}$$

Here $H_{\mathrm{ct}} \equiv -(H - H_{\mathrm{ren}})$ and we define H_{ren} as the part of H relevant to S_{ren}. Using the expression of Hamiltonian (3), we read

$$H_{\mathrm{ren}} = \int_{\partial M} d^d x [2\{-S_{\mathrm{ct}}, S_{\mathrm{ren}}\} + \{S_{\mathrm{ren}}, S_{\mathrm{ren}}\}], \tag{A3}$$

$$H_{\mathrm{ct}} = -\int_{\partial M} d^d x [\{S_{\mathrm{ct}}, S_{\mathrm{ct}}\} + \sqrt{\gamma} L_S], \tag{A4}$$

where L_S is the counterterm seed and the bracket $\{S_{\mathrm{a}}, S_{\mathrm{b}}\}$ is defined through

$$\{S_{\mathrm{a}}, S_{\mathrm{b}}\} = \frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_{\mathrm{a}}}{\delta\gamma_{ij}}\frac{\delta S_{\mathrm{b}}}{\delta\gamma_{kl}}\gamma_{ijkl} + \frac{1}{4}G^{IJ}\frac{\delta S_{\mathrm{a}}}{\delta\Phi^I}\frac{\delta S_{\mathrm{b}}}{\delta\Phi^J}\right). \tag{A5}$$

We can change Eq. (A3) a little as

$$H_{\mathrm{ren}} = \int_{\partial M} d^d x [2\{S_{\mathrm{os}}, S_{\mathrm{ren}}\} - \{S_{\mathrm{ren}}, S_{\mathrm{ren}}\}]. \tag{A6}$$

Note that the second term in Eq. (A6) is vanishing near the boundary. This is because it is much smaller than the first term by definition and we will show below that the first term is not greater than $\mathcal{O}(1)$, see Eq. (A9).

To calculate the variations of S_{os}, we can equate two forms of the momentum

$$\frac{\delta S_{\mathrm{os}}}{\delta\gamma_{ij}} = \frac{\partial L}{\partial\dot{\gamma}_{ij}} = \sqrt{\gamma}(\mathcal{K}^{ij} - \mathcal{K}\gamma^{ij}), \tag{A7}$$

$$\frac{\delta S_{\mathrm{os}}}{\delta\Phi^I} = \frac{\partial L}{\partial\dot{\Phi}^I} = 2\sqrt{\gamma}G_{IJ}\dot{\Phi}^J, \tag{A8}$$

where the extrinsic curvature tensor $\mathcal{K}^i{}_j = \gamma^{ik}\dot{\gamma}_{kj}/2$. Inserting Eqs. (A7)–(A8) into Eq. (A6), we find

$$H_{\mathrm{ren}} = \int_{\partial M} d^d x \left(\frac{\delta S_{\mathrm{ren}}}{\delta\gamma_{ij}}\dot{\gamma}_{ij} + \frac{\delta S_{\mathrm{ren}}}{\delta\Phi^I}\dot{\Phi}^I\right). \tag{A9}$$

Furthermore, we consider that S_{ren} can be taken as the functional of $(\bar{\gamma}_{ij}, \bar{\Phi}^I)$ or (γ_{ij}, Φ^I, r). This is viewed from the field theory and its gravity dual, respectively. Keeping this in mind, we can derive

$$\frac{\partial S_{\rm ren}}{\partial r}+H_{\rm ren}=\frac{\partial S_{\rm ren}(\gamma_{ij},\Phi^I,r)}{\partial r}+\int_{\partial M}d^dx\left(\frac{\delta S_{\rm ren}}{\delta\gamma_{ij}}\dot{\gamma}_{ij}+\frac{\delta S_{\rm ren}}{\delta\Phi^I}\dot{\Phi}^I\right) \tag{A10}$$

$$=\frac{dS_{\rm ren}(\bar{\gamma}_{ij},\bar{\Phi}^I)}{dr}=0. \tag{A11}$$

Combining it with Eq. (A2), we can obtain the counterterm part of the HJ equation[11]

$$H_{\rm ct}+\frac{\partial S_{\rm ct}}{\partial r}=0. \tag{A12}$$

APPENDIX B: VARIATION IDENTITIES

We will study two variation identities. Suppose that a boundary invariant with divergence degree k can be formally written as

$$U_{(k)}=(\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_aY_{b\dots}\cdots). \tag{B1}$$

Here $(X\cdots)$ denotes the product of (one or more) X and $Y_{b\dots}$ denotes any tensor which is made by the product of Φ^I and $R^a{}_{bcd}$ as well as their covariant derivatives.

For convenience, we define an operator

$$\int X\bar{\delta}\equiv\gamma^{ij}\frac{1}{\sqrt{\gamma}}\int d^dx\sqrt{\gamma}X\frac{\delta}{\delta\gamma^{ij}}, \tag{B2}$$

where X can denote any tensor. Using the chain rule, we have

$$\begin{aligned}\int\bar{\delta}U_{(k)}=&\int\bar{\delta}(\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_aY_{b\dots}\cdots)\\&+\int(\gamma^{ab}\cdots)\bar{\delta}(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_aY_{b\dots}\cdots)\\&+\int(\gamma^{ab}\cdots)(\Phi^I\cdots)\bar{\delta}(R^a{}_{bcd}\cdots)(\nabla_aY_{b\dots}\cdots)\\&+\int(\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)\bar{\delta}(\nabla_aY_{b\dots}\cdots).\end{aligned} \tag{B3}$$

We will calculate the four terms respectively:

$$\int\bar{\delta}(\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_aY_{b\dots}\cdots)=\bar{k}U_{(k)}, \tag{B4}$$

$$\int(\gamma^{ab}\cdots)\bar{\delta}(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_aY_{b\dots}\cdots)=0, \tag{B5}$$

$$\int(\gamma^{ab}\cdots)(\Phi^I\cdots)\bar{\delta}(R^a{}_{bcd}\cdots)(\nabla_aY_{b\dots}\cdots)=\mathrm{TD}, \tag{B6}$$

$$\int(\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)\bar{\delta}(\nabla_aY_{b\dots}\cdots)=\mathrm{TD}. \tag{B7}$$

The former two equations are obvious. Note that $\bar{k}$ is the number of inverse metric in $(\gamma^{ab}\cdots)$. The latter two equations are derived as follows:

$$\begin{aligned}\int X\bar{\delta}R^a{}_{bcd}&=\int X(\nabla_c\bar{\delta}\Gamma^a_{bd}-\nabla_d\bar{\delta}\Gamma^a_{bc})\\&=\int(-\nabla_cX\bar{\delta}\Gamma^a_{bd}+\nabla_dX\bar{\delta}\Gamma^a_{bc})\\&=\int-\nabla_cX\frac{1}{2}\gamma^{ae}(\nabla_b\bar{\delta}\gamma_{ed}+\nabla_d\bar{\delta}\gamma_{be}-\nabla_e\bar{\delta}\gamma_{bd})-(c\to d)\\&=\int\frac{1}{2}\gamma^{ae}(\nabla_b\nabla_cX\bar{\delta}\gamma_{ed}+\nabla_d\nabla_cX\bar{\delta}\gamma_{be}-\nabla_e\nabla_cX\bar{\delta}\gamma_{bd})-(c\to d)\\&=-\frac{1}{2}(\nabla_b\nabla_cX\delta^a_d+\nabla_d\nabla_cX\delta^a_b-\nabla^a\nabla_cX\gamma_{bd})-(c\to d),\end{aligned} \tag{B8}$$

[11]This equation has been derived before in [20]. Here the procedure is more general since we have not invoked the explicit asymptotic behavior of the metric and scalar fields. Moreover, Eq. (A12) is similar to Eq. (27) in [3], and their relation has been explained in [20].

$$\begin{aligned}\int X\bar{\delta}(\nabla_i Y_{j\ldots}) &= \int X\bar{\delta}(\partial_i Y_{j\ldots} - \Gamma^k_{ij} Y_{k\ldots} + \cdots) \\ &= \int X[(-\bar{\delta}\Gamma^k_{ij} Y_{k\ldots} + \cdots) + \nabla_i \bar{\delta} Y_{j\ldots}] \\ &= \int \left[-X\frac{1}{2}\gamma^{ke}(\nabla_i\bar{\delta}\gamma_{ej} + \nabla_j\bar{\delta}\gamma_{ie} - \nabla_e\bar{\delta}\gamma_{ij})Y_{k\ldots} + \cdots + X\nabla_i\bar{\delta}Y_{j\ldots}\right] \\ &= \int \left\{\frac{1}{2}[\nabla_i(XY^e_{\ldots})\bar{\delta}\gamma_{ej} + \nabla_j(XY^e_{\ldots})\bar{\delta}\gamma_{ie} - \nabla_e(XY^e_{\ldots})\bar{\delta}\gamma_{ij}] + \cdots - \nabla_i X\bar{\delta}Y_{j\ldots}\right\} \\ &= \frac{1}{2}[-\nabla_i(XY_{j\ldots}) - \nabla_j(XY_{i\ldots}) + \nabla_k(XY^k_{\ldots})\gamma_{ij}] + \cdots - \int \nabla_i X\bar{\delta}Y_{j\ldots}\end{aligned} \tag{B9}$$

Here $\cdots$ in Eq. (B9) denote the contributions according to the suppressed index … in the tensor $Y_{j\ldots}$. Moreover, we note that the last term above is a total derivative, which can be seen by iteratively using Eq. (B5), Eq. (B6), and Eq. (B9). Combining Eqs. (B4)–(B7), we have proved[12]

$$\int \bar{\delta}U_{(k)} = \bar{k}U + \mathrm{TD}. \tag{B10}$$

From the form of Eq. (B1), it is obvious that there is a constraint $2\bar{k} \le k$.

Next, we turn to the operator

$$\int X\tilde{\delta} \equiv \Phi^I \frac{1}{\sqrt{\gamma}} \int d^d x \sqrt{\gamma} X \frac{\delta}{\delta\Phi^I}. \tag{B11}$$

Acting it on Eq. (B1), we read

$$\begin{aligned}\int \tilde{\delta}U_{(k)} &= \int \tilde{\delta}(\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_a Y_{b\cdots}\cdots) \\ &+ \int (\gamma^{ab}\cdots)\tilde{\delta}(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_a Y_{b\cdots}\cdots) \\ &+ \int (\gamma^{ab}\cdots)(\Phi^I\cdots)\tilde{\delta}(R^a{}_{bcd}\cdots)(\nabla_a Y_{b\cdots}\cdots) \\ &+ \int (\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)\tilde{\delta}(\nabla_a Y_{b\cdots}\cdots).\end{aligned} \tag{B12}$$

Each line yields

$$\int \tilde{\delta}(\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_a Y_{b\cdots}\cdots) = 0, \tag{B13}$$

[12]This identity may still hold even if the counterterm is more general than Eq. (B1). For examples, see the massive gravity in section D and the holographic axion model in section E.

$$\int (\gamma^{ab}\cdots)\tilde{\delta}(\Phi^I\cdots)(R^a{}_{bcd}\cdots)(\nabla_a Y_{b\cdots}\cdots) = \tilde{k}_1 U_{(k)}, \tag{B14}$$

$$\int (\gamma^{ab}\cdots)(\Phi^I\cdots)\tilde{\delta}(R^a{}_{bcd}\cdots)(\nabla_a Y_{b\cdots}\cdots) = 0, \tag{B15}$$

$$\int (\gamma^{ab}\cdots)(\Phi^I\cdots)(R^a{}_{bcd}\cdots)\tilde{\delta}(\nabla_a Y_{b\cdots}\cdots) = \tilde{k}_2 U_{(k)}, \tag{B16}$$

where $\tilde{k}_1$ and $\tilde{k}_2$ denote the numbers of Φ^I in $(\Phi^I\cdots)$ and $(\nabla_a Y_{b\cdots}\cdots)$, respectively. Note that one can use

$$\int X\tilde{\delta}(\nabla_a Y_{b\cdots}) = \int -\nabla_a X\tilde{\delta}Y_{b\cdots} \tag{B17}$$

and Eqs. (B14)–(B15) iteratively to prove Eq. (B16). Thus, we have

$$\int \tilde{\delta}U_{(k)} = \tilde{k}U_{(k)}, \tag{B18}$$

where $\tilde{k} = \tilde{k}_1 + \tilde{k}_2$ means the number of Φ^I in $U_{(k)}$. Note that there is also a constraint $2\bar{k} + \Delta_-\tilde{k} \le k$ associated with Eq. (B18). Here we have set $\Phi^I \sim e^{-\Delta_- r}$.

APPENDIX C: MASS PARAMETERS

The two parameters $b_{(1)}$ and $b_{(2)}$ are defined by

$$L_{P(k_i)} = \sum_{\substack{m,n\to i \\ m=i \text{ or } n=i}} G^{IJ} P_{I(k_m)} P_{J(k_n)} + \sum_{\substack{m,n\to i \\ m=i \text{ and } n=i}} G^{IJ} P_{I(k_m)} P_{J(k_n)} \tag{C1}$$

$$= C_{(k_i)}\bar{U}_{(k_i)} b_{1(k_i)} + C^2_{(k_i)}\bar{U}_{(k_i)} b_{2(k_i)}, \tag{C2}$$

where

$$P_{I(k_m)} = \int \frac{\delta U_{(k_m)}}{\delta \Phi^I}, \qquad P_{J(k_n)} = \int \frac{\delta U_{(k_n)}}{\delta \Phi^J}. \tag{C3}$$

Now let us calculate the divergence degrees of two terms in Eq. (C1). Without loss of generality, we have

$$\begin{aligned}[G^{IJ}P_{I(k_i)}P_{J(k_n)}] &= [G^{IJ}] + [P_{I(k_i)}] + [P_{J(k_n)}] \\ &= [G^{IJ}] + k_i + k_n - [\Phi^I] - [\Phi^J] = k_i,\end{aligned} \tag{C4}$$

$$\begin{aligned}[G^{IJ}P_{I(k_i)}P_{J(k_i)}] &= [G^{IJ}] + [P_{I(k_i)}] + [P_{J(k_i)}] \\ &= [G^{IJ}] + 2k_i - [\Phi^I] - [\Phi^J] = k_i,\end{aligned} \tag{C5}$$

where $[\cdots]$ denotes the divergence degree of $\cdots$. We will analyze the parameters $b_{(1)}$ and $b_{(2)}$ in terms of Eq. (C4) and Eq. (C5). We will focus on the situations related to holographic models studied in the next section.

(1) Since $k_n \neq k_i$ in Eq. (C4), the two equations cannot hold at the same time. This means either $b_{1(k_i)} = 0$ or $b_{2(k_i)} = 0$.

(2) Consider the benchmark model with constant potential, where $[\Phi^I] = [\Phi^J] = 0$ and $[G^{IJ}] = 0$. Then neither Eq. (C4) nor Eq. (C5) holds and we have $b_{1(k_i)} = b_{2(k_i)} = 0$ and $k_i = 2\bar{k}_i$.

(3) Consider the benchmark model with massive scalar fields and suppose G^{IJ} as a diagonal constant matrix. The two equations have the solutions $k_n = 2[\Phi^I]$ and $k_i = 2[\Phi^I]$. Keeping this in mind, one can find

$$\begin{aligned} b_{1(k_i)} &= \frac{4G^{II}V_{II}\tilde{k}_i}{d + \sqrt{d^2 - b_{2(2[\Phi^I])}}}, \\ b_{2(2[\Phi^I])} &= -4G^{II}V_{II}, \end{aligned} \tag{C6}$$

where $\tilde{k}_i$ is the number of scalar field Φ^I in $U_{(k_i)}$, V_{II} is the constant before the term $\Phi^I\Phi^I$ in the potential, G^{II} is the inverse of scalar couplings, and the same index I is not summed up. Note that we have used Eq. (B18) to derive Eq. (C6). Since $G^{II}V_{II} = m^2$ and $\Delta_- = \frac{d}{2} - \sqrt{(\frac{d}{2})^2 + m^2}$, where m is the mass of Φ^I, Eq. (C6) can be reduced to

$$b_{1(k_i)} = -2\tilde{k}_i\Delta_-, \qquad b_{2(2\Delta_-)} = -4m^2. \tag{C7}$$

We therefore refer $b_{(1)}$ and $b_{(2)}$ as the mass parameters.

(4) In section D. 5, we will study the holographic axion model, where the divergence degree $[\Phi^I] = 0$ but the effective couplings $[\bar{G}^{IJ}] \neq 0$. Apparently, two equations have the solutions $k_n = -[\bar{G}^{IJ}]$ and $k_i = -[\bar{G}^{IJ}]$. However, they are not real solutions since $k_i > -[\bar{G}^{IJ}]$ therein. So we still have $b_{1(k_i)} = b_{2(k_i)} = 0$.

APPENDIX D: APPLICATIONS

We will apply the HJ approach developed in the main text to six typical models. The HR of the former five models has been studied before by different approaches [3,15,19,20,32]. The counterterms of the fifth model have been used in [32] but are not given explicitly. The two special cases of the last model have also been studied in [48] and [24]. One can see that our results are consistent with theirs.

1. Dilaton model: General potential

Suppose that the boundary dimension is $d = 4$ and there is only one scalar field ϕ. From the HJ equation, we read the (inverse) coupling $G^{11} = 2$ and the counterterm seed

$$L_S = R - \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi - V(\phi), \tag{D1}$$

where the scalar potential is

$$V(\phi) = V_0 + V_1\phi + V_2\phi^2 + V_3\phi^3 + V_4\phi^4 + \cdots, \tag{D2}$$

with $V_0 = 12$, $V_1 = 0$, and $V_2 = -3/2$. The mass square is $m^2 = -3$ and the scaling dimension is $\Delta = 3$.

We will derive the counterterms by three steps.

(1) Generate ansatz

With $U_{(k_0)} = -3$ and L_S in hand, we can calculate Eq. (31) iteratively and generate the ansatz at all order with $k_i \leq d$:

$$L_{A(2)} = R - V_2\phi^2, \qquad L_{A(3)} = -V_3\phi^3, \tag{D3}$$

$$\begin{aligned} L_{A(4)} = &-\frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi - V_4\phi^4 + 4Y_{(2)ij}Y^{ij}_{(2)} \\ &+ 2P^2_{(3)} - U^2_{(2)} - \frac{1}{3}(U_{(2)} - 2Y_{(2)})^2. \end{aligned} \tag{D4}$$

(2) Take variation

What we need is

$$Y_{(2)ij} = C^{(1)}_{(2)}R_{ij}, \qquad P_{(3)} = -3V_3\phi^2C_{(3)}. \tag{D5}$$

Note that the upper index (1) in $C^{(1)}_{(2)}$ is used to indicate that $C^{(1)}_{(2)}$ is associated with the first term in $L_{A(2)}$.

(3) Specify coefficients

Using Eq. (C7), one can read the mass parameters

$$b^{(2)}_{2(2)} = 12, \qquad b_{1(3)} = -6. \tag{D6}$$

The coefficients can be obtained by Eqs. (33), (34), and (35). In particular, keep in mind that they are universal at the highest order: $C_{(d)} = -\frac{r}{2}$. Then we have

$$C_{(2)}^{(1)} = -\frac{1}{4}, \qquad C_{(2)}^{(2)} = -\frac{1}{6}, \qquad C_{(3)} = -\frac{1}{2}, \\ C_{(4)} = -\frac{r}{2}. \tag{D7}$$

We collect $U_{(k_i)}$ at all orders (except $U_{(k_0)} = 1 - d$):

$$U_{(2)} = -\frac{1}{4}(R + \phi^2), \qquad U_{(3)} = \frac{1}{2}V_3\phi^3, \\ U_{(4)} = -\frac{r}{2}L_{A(4)}. \tag{D8}$$

Putting Eq. (D4), Eq. (D5), Eq. (D7), and Eq. (D8) together, it yields

$$U_{(4)} = -\frac{r}{2}\left[\frac{1}{4}R_{ij}R^{ij} - \frac{1}{12}R^2 - \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi - \frac{1}{12}R\phi^2 + \left(\frac{9}{2}V_3^2 - V_4 - \frac{1}{12}\right)\phi^4\right]. \tag{D9}$$

2. FGPW model: Dimension $\Delta = d/2$

Suppose that the boundary dimension is $d = 4$ and there are two scalar fields ψ and ϕ. The couplings are the identity matrix $G^{IJ} = \mathrm{diag}(1, 1)$ and the counterterm seed is

$$L_S = R - \gamma^{ij}\partial_i\psi\partial_j\psi - \gamma^{ij}\partial_i\phi\partial_j\phi - V(\psi, \phi), \tag{D10}$$

where the scalar potential is

$$V(\psi, \phi) = -12 - 4\phi^2 - 3\psi^2 + \psi^4 + \cdots. \tag{D11}$$

The mass squares are $m^2 = -3$ and -4 for two scalars fields, respectively. Accordingly, the scaling dimensions are $\Delta = 3$ and 2. The latter indicates a nontrivial asymptotic behavior $\phi \sim re^{-2r}$. In order to sort the counterterms that involve ϕ, we set $\phi \sim e^{-\bar{\Delta}r}$ where $\bar{\Delta} = 2 - \ln r/r$ is not a number. However, it is convenient to understand $\bar{\Delta}$ as an effective dimension with $1 < \bar{\Delta} < 2$.

In the following, we will derive the counterterms. Since the procedure is similar to the above example, we will omit unnecessary text descriptions for brevity.

(1) Generate ansatz

$$L_{A(2)} = R + 3\psi^2, \qquad L_{A(2\bar{\Delta})} = 4\phi^2, \tag{D12}$$

$$L_{A(4)} = -\gamma^{ij}\partial_i\psi\partial_j\psi - \psi^4 + 4Y_{(2)ij}Y_{(2)}^{ij} - U_{(2)}^2 - \frac{1}{3}(U_{(2)} - 2Y_{(2)})^2. \tag{D13}$$

Note that the divergence degree of $L_{A(2\bar{\Delta})}$ is $2\bar{\Delta}$, which is not a number. But this did not cause any problems with our approach.

(2) Take variation

$$Y_{(2)ij} = C_{(2)}^{(1)}R_{ij}. \tag{D14}$$

(3) Specify coefficients

$$b_{2(2)}^{(2)} = 12, \qquad b_{2(2\bar{\Delta})} = 16. \tag{D15}$$

$$C_{(2)}^{(1)} = -\frac{1}{4}, \qquad C_{(2)}^{(2)} = -\frac{1}{6}, \\ C_{(2\bar{\Delta})} = -\frac{1}{4} + \frac{1}{8r}, \qquad C_{(4)} = -\frac{r}{2}. \tag{D16}$$

We collect $U_{(k_i)}$ at all orders:

$$U_{(2)} = -\frac{1}{4}(R + 2\psi^2), \qquad U_{(2\bar{\Delta})} = \left(-1 + \frac{1}{2r}\right)\phi^2, \\ U_{(4)} = -\frac{r}{2}L_{A(4)}. \tag{D17}$$

One can find that $U_{(4)}$ can be rewritten as

$$U_{(4)} = -\frac{r}{2}\left(\frac{1}{4}R_{ij}R^{ij} - \frac{1}{12}R^2 - \gamma^{ij}\partial_i\psi\partial_j\psi - \frac{1}{6}R\psi^2 - \frac{4}{3}\psi^4\right). \tag{D18}$$

3. Axion-dilaton model: Constant potential

Suppose that the boundary dimension is $d = 4$ and there are two massless scalar fields: the dilaton ϕ and the axion χ. The couplings are $G^{IJ} = \mathrm{diag}(1, 1/Z)$. Here Z is a function of ϕ and $I = 1, 2$ refer ϕ and χ, respectively. The counterterm seed is

$$L_S = R - \gamma^{ij}\nabla_i\phi\nabla_j\phi - Z\gamma^{ij}\nabla_i\chi\nabla_j\chi - V. \tag{D19}$$

Unlike the previous two examples, now the potential is constant $V = -12$.

(1) Generate ansatz

$$L_{A(2)} = R - \gamma^{ij}\nabla_i\phi\nabla_j\phi - Z\gamma^{ij}\nabla_i\chi\nabla_j\chi, \tag{D20}$$

$$L_{A(4)} = P_{1(2)}^2 + \frac{1}{Z}P_{2(2)}^2 + 4Y_{(2)ij}Y_{(2)}^{ij} - \frac{4}{3}U_{(2)}^2. \tag{D21}$$

(2) Take variation

$$Y_{(2)ij} = C_{(2)}(R_{ij} - \nabla_i\phi\nabla_j\phi - Z\nabla_i\chi\nabla_j\chi), \tag{D22}$$

$$P_{1(2)} = C_{(2)}(2\nabla^2\phi - Z'\gamma^{ij}\nabla_i\chi\nabla_j\chi), \\ P_{2(2)} = 2C_{(2)}\gamma^{ij}\nabla_i(Z\nabla_j\chi). \tag{D23}$$

(3) Specify coefficients

Since the potential is constant, we have $b_{1(k_i)} = b_{2(k_i)} = 0$ and $2\bar{k}_i = k_i$. This indicates $C_{(k_i)} = -\frac{1}{2(d-k_i)}$ for $d \neq k_i$ or $-\frac{r}{2}$ for $d = k_i$. Concretely, we have

$$C_{(2)} = -\frac{1}{4}, \qquad C_{(4)} = -\frac{r}{2}. \tag{D24}$$

We collect $U_{(k_i)}$ at all orders[13]:

$$U_{(2)} = -\frac{1}{4}L_{A(2)}, \qquad U_{(4)} = -\frac{r}{2}L_{A(4)}. \tag{D25}$$

One can see that $U_{(4)}$ can be expanded as 16 independent terms [19].

4. Axion-dilaton model: Asymptotic polynomial potential

Suppose that the boundary dimension is $d = 4$ and there are two scalar fields: one is a massless axion and the other is a dilaton with nontrivial potential [32]. The counterterm seed can be written as

$$L_S = R - \frac{1}{2}\gamma^{ij}\nabla_i\phi\nabla_j\phi - \frac{1}{2}Z\gamma^{ij}\nabla_i\chi\nabla_j\chi - V, \tag{D26}$$

where the coupling and potential are given by

$$Z = e^{2\gamma\phi} = 1 + 2\gamma\phi + 2\gamma^2\phi^2 \cdots \tag{D27}$$

$$\begin{aligned} V &= -12\cosh(\sigma\phi) - b\phi^2 \\ &= -12 - (b + 6\sigma^2)\phi^2 - \frac{\sigma^4\phi^4}{2} + \cdots. \end{aligned} \tag{D28}$$

Note that they have been expanded as polynomials near the boundary and are characterized by three parameters γ, σ, and b. The scaling dimension Δ of the scalar operator dual to the dilaton is related to σ and b by

$$b + 6\sigma^2 = \frac{\Delta(4-\Delta)}{2}. \tag{D29}$$

We will set $\Delta = 3$ that was chosen to study thermodynamics in [32].

(1) Generate ansatz

$$\begin{aligned} L_{A(2)} &= R - \frac{1}{2}\gamma^{ij}\nabla_i\chi\nabla_j\chi + (b + 6\sigma^2)\phi^2, \\ L_{A(3)} &= -\gamma\phi\gamma^{ij}\nabla_i\chi\nabla_j\chi, \end{aligned} \tag{D30}$$

$$\begin{aligned} L_{A(4)} = &-\frac{1}{2}\gamma^{ij}\nabla_i\phi\nabla_j\phi - \gamma^2\phi^2\gamma^{ij}\nabla_i\chi\nabla_j\chi \\ &+ \frac{\sigma^4\phi^4}{2} + 2P_{2(2)}^2 + 2P_{(3)}^2 + 4Y_{(2)ij}Y_{(2)}^{ij} \\ &- U_{(2)}^2 - \frac{1}{3}(U_{(2)} - 2Y_{(2)})^2. \end{aligned} \tag{D31}$$

(2) Take variation

$$Y_{(2)ij} = C_{(2)}^{(1)}R_{ij} - C_{(2)}^{(2)}\frac{1}{2}\nabla_i\chi\nabla_j\chi, \tag{D32}$$

$$P_{2(2)} = C_{(2)}^{(2)}\gamma^{ij}\nabla_i\nabla_j\chi, \qquad P_{(3)} = -C_{(3)}\gamma\gamma^{ij}\nabla_i\chi\nabla_j\chi. \tag{D33}$$

(3) Specify coefficients

$$b_{2(2)}^{(3)} = 12, \qquad b_{1(3)} = -2. \tag{D34}$$

$$\begin{aligned} &C_{(2)}^{(1)} = -\frac{1}{4}, \qquad C_{(2)}^{(2)} = -\frac{1}{4}, \qquad C_{(2)}^{(3)} = -\frac{1}{6}, \\ &C_{(3)} = -\frac{1}{2}, \qquad C_{(4)} = -\frac{r}{2}. \end{aligned} \tag{D35}$$

We collect $U_{(k_i)}$ at all orders:

$$U_{(2)} = -\frac{1}{4}R + \frac{1}{8}\gamma^{ij}\nabla_i\chi\nabla_j\chi - \frac{1}{6}(b + 6\sigma^2)\phi^2, \tag{D36}$$

$$U_{(3)} = \frac{1}{2}\gamma\phi\gamma^{ij}\nabla_i\chi\nabla_j\chi, \qquad U_{(4)} = -\frac{r}{2}L_{A(4)}. \tag{D37}$$

5. Massive gravity: Breaking diffeomorphism symmetry

Massive gravity cannot be described by the benchmark action in the main text. Particularly, it breaks the diffeomorphism symmetry, which indicates that the ADM metric cannot be gauged as usual. Nevertheless, it is found that for the popular holographic model [33–37], the lapse still can be normalized and sometimes the shift vector is falling off fast enough near the boundary. Thus, the HJ equation and its counterterm part still have the same form as before, except that the counterterm seed is changed as [20]

$$L_S = R + d(d-1) + \sum_{n=1}^{4}\alpha_n e_n(X), \tag{D38}$$

[13]By matching the scalar fields and specifying the coupling, these counterterms can be reduced to obtain the counterterms of some important but simpler five-dimensional axion-dilaton models, see [49,50] for two examples.

where $e_n(X)$ are the symmetric polynomials of the eigenvalues of the $d\times d$ matrix $X^i{}_j$,

$$e_1=[X],\qquad e_2=[X]^2-[X^2],\\ e_3=[X]^3-3[X][X^2]+2[X^3], \tag{D39}$$

$$e_4=[X]^4-6[X]^2[X]+8[X^3][X]+3[X^2]^2-6[X^4]. \tag{D40}$$

Note that we have denoted $[X]=X^i{}_i$ and will set $d=4$. Obviously, the matrix $X^i{}_j$ is the key object, which is defined as the square root of $\gamma^{ij}f_{ij}$. Here f_{ij} are the reference metric on the boundary.

The HR of massive gravity can be performed by the general algorithm in main text. One can expect that it is similar to the last model. This is because their coefficient functions $F(C_{(k_i)})$ in the recursive equations (36) are equal to the same $2(d-2\bar{k}_i)$. Note that $e_n(X)$ in the counterterm seed and other ansatz generated below do not take the form as Eq. (B1) but the variation identity (B10) still holds, where the number $\bar{k}$ is counted by treating $X^i{}_j$ as the object with $1/2$ inverse metric [20].

(1) Generate ansatz

$$L_{A(1)}=\alpha_1e_1, \tag{D41}$$

$$L_{A(2)}=\alpha_2e_2+R+4Y_{(1)ij}Y^{ij}_{(1)}-U^2_{(1)}, \tag{D42}$$

$$L_{A(3)}=\alpha_3e_3+8Y_{(1)ij}Y^{ij}_{(2)}-2U_{(1)}U_{(2)}, \tag{D43}$$

$$L_{A(4)}=\alpha_4e_4+4Y_{(2)ij}Y^{ij}_{(2)}+8Y_{(1)ij}Y^{ij}_{(3)}\\ -\frac{4}{3}U^2_{(2)}-2U_{(1)}U_{(3)}. \tag{D44}$$

(2) Take variation

$$Y_{(1)ij}=C_{(1)}\alpha_1\frac{1}{2}X_{ij}, \tag{D45}$$

$$Y_{(2)ij}=C_{(2)}[R_{ij}+(\alpha_2-C^2_{(1)}\alpha_1^2)(2[X]X_{ij}-[X^2]_{ij})], \tag{D46}$$

$$Y_{(3)ij}=C_{(3)}\left\{\left(\frac{\alpha_1^3}{144}-\frac{\alpha_1\alpha_2}{4}+3\alpha_3\right)\left(\frac{1}{2}[X]^2X_{ij}-\frac{1}{2}X_{ij}[X^2]-[X][X^2]_{ij}+[X^3]_{ij}\right)\right.\\ +\frac{\alpha_1}{12}((2\nabla^k\nabla_{(j}X_{i)k}-\nabla^k\nabla_kX_{ij}-\nabla_i\nabla_j[X])-3R_{k(i}X^k_{j)})\\ \left.+\frac{\alpha_1}{12}\left([X]R_{ij}+\frac{1}{2}RX_{ij}+\gamma_{ij}\nabla^k\nabla_kX-\nabla_i\nabla_j[X]\right)\right\}. \tag{D47}$$

Here we have entered $C_{(1)}$ and $C_{(2)}$ given below to simplify $Y_{(3)ij}$.

(3) Specify coefficients

$$C_{(1)}=-\frac{1}{6},\qquad C_{(2)}=-\frac{1}{4},\\ C_{(3)}=-\frac{1}{2},\qquad C_{(4)}=-\frac{r}{2}. \tag{D48}$$

We collect $U_{(k_i)}$ at all orders:

$$U_{(1)}=-\frac{1}{6}L_{A(1)},\qquad U_{(2)}=-\frac{1}{4}L_{A(2)},\\ U_{(3)}=-\frac{1}{2}L_{A(3)},\qquad U_{(4)}=-\frac{r}{2}L_{A(4)}. \tag{D49}$$

6. Generalized axion model: Noninteger power

We will study an interesting model in AdS/CMT: the generalized holographic axion model [27]. We write down its action

$$S=-\int_M d^4x\sqrt{g}(\mathcal{R}+6-V(X)). \tag{D50}$$

It has the derivative term $V=X^n$, where $X=g^{\mu\nu}X_{\mu\nu}$ and $X_{\mu\nu}=G_{IJ}\partial_\mu\chi^I\partial_\nu\chi^J$ with $G_{IJ}=\frac{1}{2}\mathrm{diag}(1,1)$. Note that χ^I with $I=1,2$ denote two massless scalar fields.

One obvious feature of this model is that the power n can be a decimal. In fact, it allows for explicit breaking of translational symmetry when $1/2\le n<5/2$ and spontaneous breaking for $n>5/2$ [51]. The HR of this model has not been systematically studied before except for the cases with $n=1$ [48] and $1/2$ [24].

Let us build up the HJ formulation of this model. Using the metric

$$ds^2=dr^2+\gamma_{ij}dx^idx^j, \tag{D51}$$

one can obtain the Lagrangian from the action

$$L=-\int_{\partial M}d^3x\sqrt{\gamma}[R+6+\mathcal{K}^2-\mathcal{K}_{ij}\mathcal{K}^{ij}\\ -V(G_{IJ}\dot{\chi}^I\dot{\chi}^J+\gamma^{ij}\chi_{ij})]. \tag{D52}$$

It yields the canonical momenta

$$\pi^{ij} \equiv \frac{\partial L}{\partial \dot{\gamma}_{ij}} = \sqrt{\gamma}(\mathcal{K}^{ij} - \mathcal{K}\gamma^{ij}), \tag{D53}$$

$$\pi_I \equiv \frac{\partial L}{\partial \dot{\chi}^I} = 2\sqrt{\gamma}V'(X)G_{IJ}\dot{\chi}^J. \tag{D54}$$

The Hamiltonian can be defined by a Legendre transformation of Lagrangian

$$H \equiv \int_{\partial M} d^3x(\pi^{ij}\dot{\gamma}_{ij} + \pi_I\dot{\chi}^I) - L. \tag{D55}$$

We need to solve $\dot{\chi}^I$ from the nonlinear equation (D54). This is difficult for a decimal n. Fortunately, in the holographic application of this model, it is usually assumed implicitly

$$V(G_{IJ}\dot{\chi}^I\dot{\chi}^J + \gamma^{ij}\chi_{ij}) = V(\chi) + V'(\chi)G_{IJ}\dot{\chi}^I\dot{\chi}^J + \cdots, \tag{D56}$$

where $\chi = \gamma^{ij}\chi_{ij}$ and $\chi_{ij} = G_{IJ}\partial_i\chi^I\partial_j\chi^J$. Suppose that the asymptotic behavior of the axions

$$\chi^I = \chi^I_{(0)}(x^i) + e^{-\bar{\Delta}r}\chi^I_{(1)}(x^i) + \cdots. \tag{D57}$$

The expansion (D56) implies

$$\bar{\Delta} > 1. \tag{D58}$$

Furthermore, one can find that the higher order terms in Eq. (D56) do not contribute to the counterterms, provided that

$$2(n-1) + 2\bar{\Delta} \geqslant 3. \tag{D59}$$

Here we will focus on the situation satisfied with Eq. (D58) and Eq. (D59). More complete analysis will be given in [52].

Now let us go back to Eq. (D54). With Eq. (D56) in mind, it can be solved as

$$\dot{\chi}^I = \frac{G^{IJ}\pi_J}{2\sqrt{\gamma}V'(\chi)}, \tag{D60}$$

and the Hamiltonian has the form

$$H = \int_{\partial M} d^3x\left[\frac{1}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2 + \frac{1}{4}\bar{G}^{IJ}\pi_I\pi_J\right) + \sqrt{\gamma}L_S\right], \tag{D61}$$

where the effective coupling and the counterterm seed are

$$\bar{G}^{IJ} = \frac{1}{V'(\chi)}G^{IJ}, \qquad L_S = R + 6 - V(\chi). \tag{D62}$$

With the Hamiltonian in hand, we can derive the counterterm part of the HJ equation. The procedure is almost the same as in section A. The only thing worth mentioning is that despite the appearance of nontrivial coupling $\bar{G}^{IJ}$ in the Hamiltonian, it disappears in its renormalized counterpart, since here we have

$$\frac{\delta S_{\mathrm{os}}}{\delta \Phi^I} = \frac{\partial L}{\partial \dot{\Phi}^I} = 2\sqrt{\gamma}G_{IJ}V'(\chi)\dot{\Phi}^J. \tag{D63}$$

Furthermore, one can see that $\bar{G}^{IJ}$ and $V(\chi)$ are not polynomials in general, which makes the ansatz generated below go beyond the form (B1). Nevertheless, the variation identity (B10) still holds while $\bar{k}$ may be a decimal. Collecting these facts, one can find that the general algorithm is applicable to this model. In fact, it is similar to the previous two models since their coefficient functions $F(C_{(k_i)})$ in the recursive equations are the same.

(1) Generate ansatz

$$L_{A(2n)} = -V, \qquad L_{A(2)} = R, \tag{D64}$$

$$L_{A(4n)} = 4Y_{(2n)ij}Y^{ij}_{(2n)} + \frac{d+4n(n-1)}{1-d}U^2_{(2n)}, \tag{D65}$$

$$L_{A(2n+2)} = 8Y_{(2n)ij}Y^{ij}_{(2)} + \bar{G}^{IJ}P_{I(2n)}P_{J(2n)} + \frac{2(d-2+2n)}{1-d}U_{(2n)}U_{(2)}, \tag{D66}$$

$$L_{A(6n)} = 8Y_{(2n)ij}Y^{ij}_{(4n)} + \frac{d+2n(4n-3)}{1-d}2U_{(2n)}U_{(4n)}. \tag{D67}$$

(2) Take variation

$$Y_{(2n)ij} = -C_{(2n)}V'\chi_{ij}, \quad P_{I(2n)} = 2C_{(2n)}\nabla_i(V'\nabla^i\chi_I), \quad Y_{(2)ij} = C_{(2)}R_{ij}, \tag{D68}$$

$$Y_{(4n)ij} = C_{(4n)}C^2_{(2n)}\left\{\frac{d+4n(n-1)}{1-d}2n\chi^{2n-1}\chi_{ij} + 8n^2\chi^{2n-3}[(n-1)\chi_{kl}\chi^{kl}\chi_{ij} + \chi\chi^k_i\chi_{kj}]\right\}. \tag{D69}$$

(3) Specify coefficients

We have $C_{(k_i)} = -\frac{1}{2(d-k_i)}$ for $d \neq k_i$ or $-\frac{r}{2}$ for $d = k_i$. We have not written the concrete expression at each order since n has not been specified.

We collect $U_{(k_i)}$ at all orders:

$$U_{(2n)} = C_{(2n)} L_{A(2n)}, \qquad U_{(2)} = C_{(2)} L_{A(2)}, \tag{D70}$$

$$U_{(4n)} = C_{(4n)} L_{A(4n)}, \qquad U_{(2n+2)} = C_{(2n+2)} L_{A(2n+2)},$$
$$U_{(6n)} = C_{(6n)} L_{A(6n)}. \tag{D71}$$

Since here $d = 3$, the logarithmic divergence appears at $n = \frac{3}{2}, \frac{3}{4}$, and $\frac{1}{2}$. Note that we have sorted the orders based on $\frac{1}{2} < n < 1$. For $n = \frac{1}{2}$, $U_{(2)}$ and $U_{(4n)}$ appear at the same order, and so do $U_{(2n+2)}$ and $U_{(6n)}$.[14] For $3/2 \geq n \geq 1$, one can see that only $U_{(2)}$ and $U_{(2n)}$ in Eq. (D70) and Eq. (D71) are relevant to real counterterms. For $n > 3/2$, only $U_{(2)}$ remains to be relevant.

[14]In Ref. [24], only $U_{(2n)}$ has been obtained. Using the concrete black hole solution given in [25], we have double checked that other counterterms are necessary for the finiteness of the renormalized action [52].

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