

Causality bounds chaos in geodesic motion

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Predictability is grounded by causality while may be practically restricted by the occurrence of chaos. To reveal the relation between these two popular notions, we study chaos in geodesic motion in generic curved spacetimes with external potentials, where causality is controlled by a scalar potential. We develop a reparametrization-independent method to analytically estimate the Lyapunov exponent λ of particle motion. We show that causality gives the universal upper bound $\lambda \propto E$ ($E \rightarrow \infty$), which coincides with the chaos energy bound proposed by Murata, Tanahashi, Watanabe, and one of the authors (K. H.). We also find that the chaos bound discovered by Maldacena, Shenker, and Stanford can be violated in particular potentials, even with causality. Our estimates, although waiting for numerical confirmation, reveal the hidden nature of physical theories: causality bounds chaos.

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I. INTRODUCTION

Predictability is an indispensable property of physics since it enables us to justify theories through experiments. The reason why causality is considered one of the most fundamental principles is that it is the foundation for predictability. However, even in completely causal (or deterministic) theories, the practical predictability of dynamical systems can be lost in time evolution. This is chaos, in which the infinitesimal difference of initial values can grow exponentially in time. What is the relation between these two popular notions—causality and chaos?

Causality is formulated in several contexts, but it is better defined in relativistic theories. This “relativistic causality” plays a key role in general relativity.¹ The peculiarity is enlightened when curved spacetimes have event horizons, in the initial value problems using Cauchy surfaces. Causality principle prohibits some spacetimes; for example, the Gödel solution [1] in general relativity is regarded as unphysical because of the presence of closed timelike curves, which leads to pathological occurrences such as the grandfather paradox. Hence, causality principle has been

used in confirming the relativistic consistencies, for instance, of modified gravity theories [2–6].

Chaos, remember, was born in Poincaré’s analysis of three-body problems, regarding gravitational motion of stars in spacetimes. Consequently, the chaos of geodesic motion in curved spacetime has been investigated for decades and applied to the explications of the properties of spacetimes [7–10]. For example, the Lyapunov exponent of null rays was found to relate to quasinormal mode frequencies of black holes [11]. Chaos of particle motion around black holes has been studied extensively [12,13], and it was revealed that the Lyapunov exponent λ for the motion near the horizon becomes $\lambda = \kappa$ [14], where κ is the surface gravity of the black hole.

Therefore, for the investigation of possible relations between causality and chaos, geodesic motion of a particle in curved spacetimes is a suitable ground. In classical relativistic systems, causality bounds particle motion in/on light cones, which should restrict the complexity of motion. This leads us to the expectation that causality generically gives some fundamental bound on the complicated time evolution of degrees of freedom, chaos. However, little literature has studied quantitatively the relation between chaos and causality.

In this respect, we prepare two proposed upper bounds on the Lyapunov exponents of chaos. The first one is the renowned chaos bound proposed by Maldacena, Shenker, and Stanford (MSS) [15]. It asserts that the Lyapunov exponent of out-of-time-ordered correlators (OTOCs) [16,17] defined in large N quantum field theory is bounded as

$$\lambda \leq \frac{2\pi T}{\hbar}, \quad (1)$$

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¹The importance of (relativistic) causality is hidden in Newtonian theories. They are understood as the light speed $c \rightarrow \infty$ limit of the relativistic ones.

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where T is the temperature of the system. Originally, this bound was discovered with AdS/CFT correspondence [18] through thought experiments of shock waves near black hole horizons [17,19], whose surface gravity is expressed as $\kappa = 2\pi T/\hbar$. As the horizon is defined by causality, the MSS chaos bound could be a manifestation of the causality in the bulk curved spacetime, once the holographic principle is assumed.

Another proposal of a universal bound on the Lyapunov exponent is the chaos energy bound [20]. This conjectures an upper bound of the energy dependence of the Lyapunov exponent λ ,

$$\lambda(E) \stackrel{E \rightarrow \infty}{\propto} E^c, \quad c \leq 1, \quad (2)$$

using the energy E of the system, which is consistent in the large N limit with the MSS bound. This bound was motivated by the thermodynamic consistency of OTOCs and is applicable to any classical/quantum Hamiltonian mechanics and field theories, on some reasonable assumptions such as locality. It remains unclear whether this energy bound can be derived from more fundamental principles of physics such as causality.

The purpose of this paper is to find any relation between chaos and causality, whose significance has tended to be overlooked since causality is assumed for granted. We especially focus on chaos in relativistic systems and relativistic causality. Specifically, we study how chaos is affected by a local expression of causality that physical particles cannot be spacelike. We estimate the Lyapunov exponent of the geodesic motion and elucidate that causality gives the upper bound of its energy dependence, which coincides with the proposed chaos energy bound (2).

We consider generic situations in which a particle is moving in a general stationary and spherically symmetric spacetime and is coupled to general vector/scalar potentials. There, the causality of the particle motion is controlled by the scalar potential V with $p^2 = -(m^2 + V)$, where p and m are the four-momentum and the mass of the particle, respectively. Requiring that the physical particle should never go back to the past imposes the restriction $m^2 + V \geq 0$ on allowed scalar potentials. We find that, owing to this restriction stemmed by causality, *the energy bound* (2) is always satisfied. This is a partial derivation of the universal bound on the energy dependence of chaos from the causality principle. On the other hand, we also show that *the MSS bound* can be violated under particular potentials, even with the causality restriction. With these results, we claim that the causality actually provides a bound on chaos, taking the form of the chaos energy bound (2), while the MSS bound (1) is a stronger one, possibly dictated by holography.

Note that our analytic study merely estimates, not calculates exactly, the upper bound of the Lyapunov exponent. It points out the possibility of the violation of the MSS bound; however, whether the bound is violated in actual chaotic

motion is another issue which we do not address because the analytic calculation of Lyapunov exponents in our generic situations is quite difficult. More numerical works are needed for verifying the satisfaction/violation of the bounds in each concrete model. In particular, the violation of the MSS bound suggests that, beyond causality, some fundamental principle that rules out those potentials which violate the bound is missing for holographic descriptions of gravity.

This paper is organized as follows: First, in Sec. II, we provide a method to analytically estimate the maximal Lyapunov exponent λ of geodesic motion in generic spacetimes with vector/scalar potentials. The method is applicable in any parametrization of the geodesic with an einbein e and is used to study two chaos bounds, the energy bound and the MSS bound. In Sec. III, we evaluate the Lyapunov exponent in the $e = 1$ gauge and investigate the energy bound (2). We show that this bound can be violated by the scalar potential; however, causality restricts its asymptotic behavior in the $E \rightarrow \infty$ limit and protects the bound. In Sec. IV, we convert the Lyapunov exponents in the $e = 1$ gauge to those in the static gauge, using the method developed in Sec. II, and investigate the MSS bound (1). There we show that for particular potentials the Lyapunov exponent λ can diverge in the $E \rightarrow \infty$ limit and trivially violate the MSS bound. The final section is for our conclusion and discussion.

II. REPARAMETRIZATION-INDEPENDENT ESTIMATION OF THE LYAPUNOV EXPONENT

In this section, we provide a method to analytically estimate the maximal Lyapunov exponent λ of geodesic motion in generic spacetimes. First, in Sec. II A, we illustrate how to estimate λ in the simplest case of the four-dimensional Schwarzschild spacetime. Then, in Sec. II B, we describe our method for generic situations in which a particle is moving in a general stationary and spherically symmetric spacetime and is coupled to general vector/scalar potentials. The method is applicable in any parametrization of the geodesic and will be used to study the chaos bounds in the later sections.

A. Example: Schwarzschild spacetime

In generic situations, the Lyapunov exponent in chaotic motion is measured numerically, while our interest in the chaos bounds needs an analytic expression for the exponent. Hence, in this paper, we rely on an intuitive estimation method for the Lyapunov exponent focusing on the accelerated particle motion at separatrices in the phase space. This amounts to the estimation of a curvature of the effective potential at its local maxima. The method was used in [14] and subsequent papers in the context of chaotic motion in spacetimes and the AdS/CFT correspondence.

In this subsection, as an introductory example, let us consider the radial component of motion of a free

particle in the Schwarzschild spacetime with four dimensions. The illustration is for two purposes: a demonstration in the simplest and well-known example, and a caution for an unreasoning use of popular conservation laws without looking at the worldline gauge. In fact, we will find in Sec. II B that the calculation in this subsection depends on a special choice of the parametrization of the geodesic.

The metric in the four-dimensional Schwarzschild spacetime is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\varphi^2, \quad (3)$$

where we put $\theta = \pi/2$ from the spherical symmetry. We have the conserved quantities

$$E = \left(1 - \frac{2M}{r}\right)p^t, \quad (4)$$

$$L = r^2 p^\varphi, \quad (5)$$

where they are the Killing energy and the Killing angular momentum, respectively, and p^μ is the four-momentum of the particle. The standard normalization of four-momentum

$$-m^2 = p^2 \quad (6)$$

leads to

$$-m^2 = -\frac{E^2}{1 - 2M/r} + \frac{\dot{r}^2}{1 - 2M/r} + \frac{L^2}{r^2}, \quad (7)$$

which is rephrased as

$$E^2 = \dot{r}^2 + \left(m^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right). \quad (8)$$

We define the effective potential as

$$V_{\text{eff}}(r) := \left(m^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right), \quad (9)$$

then the energy integral (8) describes a particle moving in the potential V_{eff} .

To estimate the Lyapunov exponent, we closely look at the motion around a stationary point $r = r_0$ of the potential $V_{\text{eff}}(r)$. We expand the radial coordinate as $r = r_0 + \varepsilon$, then the energy integral (8) can be cast into the form

$$E^2 - V_{\text{eff}}(r_0) = \dot{\varepsilon}^2 + \frac{1}{2}V''_{\text{eff}}(r_0)\varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad (10)$$

When $V''_{\text{eff}}(r_0)$ is negative, r_0 is a separatrix where circular orbits become unstable, and the system is described by an inverted harmonic oscillator, which has solutions of the form $\varepsilon = Ae^{\lambda\tau}$, where $\lambda := \sqrt{-V''_{\text{eff}}(r_0)/2}$. This λ is interpreted as the estimation of the Lyapunov exponent of the geodesic motion, which is given in the case of the Schwarzschild spacetime as

$$\lambda = \sqrt{-\frac{L^2}{r^5}\left(-\frac{2Mm^2}{L^2}r^2 + 3r - 12M\right)}. \quad (11)$$

B. Generic spacetimes

In the analysis in Sec. II, we borrowed the popular conserved quantities and the four-momentum (6). However, the gauge dependence of the worldline time τ was hidden in such a formulation. Since the Lyapunov exponent λ is defined through the form $r \sim e^{\lambda\tau}$, it should depend on the worldline parametrization of the geodesic. In this subsection, we describe a reparametrization-independent estimation of the Lyapunov exponent applicable for more general situations: We formulate geodesic motion without fixing the gauge of the worldline time τ and apply the inverted-harmonic-oscillator approximation to estimate the Lyapunov exponent.

Let us consider a relativistic particle with mass m and charge q in a D -dimensional stationary spherically symmetric spacetime with the metric given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\varphi^2, \quad (12)$$

where we put the $D - 3$ angular coordinates to take the value $\pi/2$ from the spherical symmetry, and name the remainder φ . Note that we do not specify the gravity theory.

The Lagrangian of the particle coupled to an electromagnetic potential $A_\mu = A_\mu(r)$ and a scalar potential $V = V(r)$ is given by

$$\mathcal{L} = \frac{1}{2e}g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - \frac{em^2}{2} + qA_\mu\dot{X}^\mu - \frac{1}{2}eV, \quad (13)$$

where dots denote the derivative with respect to the geodesic parameter τ and $e = e(\tau)$ is an einbein, which is the worldline metric. The equations of motion derived from this Lagrangian are

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\rho}\dot{X}^\nu\dot{X}^\rho - \frac{\dot{e}}{e}\dot{X}^\mu = eqF^\mu_{\nu}\dot{X}^\nu - \frac{1}{2}e^2V^{;\mu}, \quad (14)$$

$$-g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = e^2(m^2 + V). \quad (15)$$

Here (14) is the geodesic equation,² and Eq. (15) is the one corresponding to (6) in Sec. II A. We can also find conserved quantities

$$E = \frac{f}{e} \dot{t} - qA_t, \quad (16)$$

$$L = \frac{r^2}{e} \dot{\varphi} + qA_\varphi, \quad (17)$$

whose conservation follows from the geodesic equations. Note that these are invariant under reparametrization.

$$0 = \left[\frac{f'}{2f^2} (E + qA_t)^2 - \frac{1}{r^3} (L - qA_\varphi)^2 - q \frac{A'_t}{f} (E + qA_t) - q \frac{A'_\varphi}{r^2} (L - qA_\varphi) + \frac{1}{2} V' \right]_{r=r_0}, \quad (18)$$

$$\begin{aligned} \frac{\lambda^2}{e(r_0)^2} = & \left[- \left(\frac{f''g}{2f^2} - \frac{f'^2g}{f^3} \right) (E + qA_t)^2 - \frac{3g}{r^4} (L - qA_\varphi)^2 - \frac{1}{2} gV'' \right. \\ & \left. + qg \left(\left(\frac{A''_t}{f} - \frac{2A'_t f'}{f^2} \right) (E + qA_t) + \frac{qA'^2_t}{f} + \left(\frac{A''_\varphi}{r^2} - \frac{4A'_\varphi}{r^3} \right) (L - qA_\varphi) - \frac{qA'^2_\varphi}{r^2} \right) \right]_{r=r_0}, \quad (19) \end{aligned}$$

$$[(L - qA_\varphi)^2]_{r=r_0} \simeq \left[r^2 \left(\frac{(E + qA_t)^2}{f} - (m^2 + V) \right) \right]_{r=r_0} \quad (20)$$

in the $A \ll r_0$ limit, where we have dropped the kinetic term proportional to \dot{i}^2 in (20). This is the case when the particle starts to “roll off” the extremum of the effective potential and is still moving in the vicinity of it. Here, r_0 determined by $\mathcal{O}(A^0)$ of the geodesic equation (18) is identified as the position of the extrema of the effective potential, and λ by $\mathcal{O}(A^1)$ of the geodesic equation (19) is the Lyapunov exponent. The energy integral (20) relates L with E at $r = r_0$, which will be used in later sections. As a consistency check, we note that (18) does not depend on $e(\tau)$, which should be the case because the location of the extrema should be independent of the intrinsic parameter τ .

In this manner, we have obtained the analytic estimation (19) of the Lyapunov exponent for the geodesic motion in generic spacetimes with the external fields, without fixing the gauge of the worldline parameter. As a by-product, we find that the formulation used in Sec. II A implicitly assumed the $e = 1$ gauge, as is seen in the comparison between (6) and (15).

Our result shows that the einbein dependence of λ appears, remarkably, only through the form of λ/e .

²Now, strictly speaking, the particle does not move on geodesics. It does not free-fall, but feels the external Lorentz and scalar forces. However, since its motion obeys the geodesic equation (14), we just call its trajectory “geodesic” below.

The most important remark is that the norm of the four-momentum of the particle (15) is no longer conserved due to the scalar potential $V(r)$. This external potential allows a timelike (or null) particle to change to be spacelike depending on its radial coordinate. In this sense, the causality of the particle is controlled by V .

Let us assume the solution of the form $r = r_0 + Ae^{\lambda\tau}$ and $e = e(r_0) + e'(r_0)Ae^{\lambda\tau}$ expected in the inverted-harmonic-oscillator approximation and determine r_0 and λ from the equations of motion (14) and (15). Substituting these solutions gives

This characteristic dependence will be used to convert λ between different gauge choices in Sec. IV.

III. CAUSALITY BOUNDS THE ENERGY DEPENDENCE OF CHAOS

In this section, we elucidate that causality gives the chaos energy bound (2) on the energy dependence of the Lyapunov exponent λ . We choose a natural gauge $e = 1$ in which the gauge choice does not refer to the conserved quantities such as energy. Although the expression (19) appears complicated, we find that the high energy limit can simplify the argument. We begin in Sec. III A with the Schwarzschild case again and show that λ is proportional to E in the $E \rightarrow \infty$ limit, namely, the chaos energy bound (2) is saturated. Then, in Sec. III B, we investigate the same limit in our generic situations formulated in Sec. II B. The important point is that the scalar potential V controls the causality of the particle via (15). To reconcile the potential with the causality, we impose

$$m^2 + V(r) \geq 0$$

and reveal that this restriction gives an upper bound to the asymptotic order of V , which results in the chaos energy bound (2).

A. Saturation of the chaos energy bound in the Schwarzschild spacetime

We have analytically estimated the Lyapunov exponent λ (11) for the geodesic motion in the Schwarzschild

spacetime in Sec. II A. Here, we prove that it saturates the chaos energy bound (2), i.e., it realizes maximum energy dependence on the bound. We find that the $E \rightarrow \infty$ limit resolves the complicated energy dependence of λ and extracts its universal behavior.

The extremum of the effective potential at which λ is estimated is determined by the zero of the potential slope

$$V'_{\text{eff}}(r) = -\frac{2L^2}{r^4} \left(-\frac{Mm^2}{L^2} r^2 + r - 3M \right). \quad (21)$$

Let us start with the massless particle case for which this equation is simplified. In fact, V_{eff} has the local maximum at $r_0 = 3M$. The relation (8) reduces to $E^2 \simeq L^2/27M^2$ at $r = r_0$ when we drop the kinetic term, then we find

$$\lambda^2 = \frac{E^2}{3M^2}. \quad (22)$$

In particular, this result shows $\lambda \propto E^1$, and the chaos energy bound (2) is saturated. We note that we have not taken the $E \rightarrow \infty$ limit.

For the massive particle, V_{eff} has the local maximum at the inner one of two stationary points

$$r_0 = \frac{L^2}{2Mm^2} \left(1 - \sqrt{1 - \frac{12M^2m^2}{L^2}} \right). \quad (23)$$

We note $r_0 > 2M$ [in the case of $|L| \geq 2\sqrt{3}M$ in which $V'_{\text{eff}}(r) = 0$ has real solutions]. Then, we find

$$\begin{aligned} E^2 &\simeq \left(m^2 + \frac{L^2}{r^2} \right) \left(1 - \frac{2M}{r} \right) \\ &= m^2 \left(1 - \frac{(1+u)(1-2u)}{9(1-u)} \right) \end{aligned} \quad (24)$$

at $r = r_0$, where $u := \sqrt{1 - 12M^2m^2/L^2}$. Solving this about u offers

$$\begin{aligned} u &= \frac{9(E/m)^2 - 8}{4} \left(\sqrt{1 + \frac{8}{9(E/m)^2 - 8}} - 1 \right) \\ &= 1 - \frac{2}{9} \left(\frac{m}{E} \right)^2 - \frac{16}{27} \left(\frac{m}{E} \right)^4 + \mathcal{O} \left(\frac{m}{E} \right)^6 \end{aligned} \quad (25)$$

in the expansion for $E^2 \gg m^2$. Now we have $r_0 = 6M/(1+u)$, thus λ is calculated as

$$\begin{aligned} \lambda^2 &= \frac{m^2}{108M^2} \frac{u(1+u)^3}{1-u} \\ &= m^2 \left(\frac{1}{3M^2} \left(\frac{E}{m} \right)^2 - \frac{29}{27M^2} + \mathcal{O} \left(\frac{m}{E} \right)^2 \right) \\ &\xrightarrow{E \rightarrow \infty} \frac{E^2}{3M^2}. \end{aligned} \quad (26)$$

This asymptotic value coincides with the massless case (22), and the energy bound is saturated again.

Here, we have shown that the chaos energy bound is saturated irrespective of the particle mass, even though the L dependence of r_0 for the massive particle makes the E dependence of λ complicated.³ This simplification suggests that the $E \rightarrow \infty$ limit unveils the existence of the universal energy bound of chaos (2).

B. Causality ensures the chaos energy bound in generic spacetimes

We investigate the $E \rightarrow \infty$ limit of λ for generic situations (19) in the $e(\tau) \equiv 1$ gauge. This gauge choice is natural for two reasons: First, it is equivalent to taking the geodesic parameter as $\tau = (\text{proper time})/m$ (or its appropriate massless limit), and second, it does not refer to the integral constants (such as energy) which do not appear in the Lagrangian. We show that the scalar potential can generically violate the chaos energy bound (2); however, causality restricts its asymptotic behavior in the $E \rightarrow \infty$ limit and protects the universal bound. We also find the bound is always saturated in the potentials which vanish asymptotically. Another gauge choice will be studied in the next section.

In general, r_0 determined by (18) must depend on E complicatedly. Thus, as in (26), we adopt the $E \rightarrow \infty$ limit to extract the universal asymptotic behavior of the Lyapunov exponent λ (19). We analyze the high energy behavior based on the ansatz that the energy dependence of the quantities showing up in the estimation of the Lyapunov exponent can be expressed by a power. For r_0 , we assume⁴

$$r_0 \xrightarrow{E \rightarrow \infty} \mathcal{O}(E^n) \quad (n \in \mathbb{R}). \quad (27)$$

At high energy, while for $n = 0$ the functions of r such as f , g , A , and V are $\mathcal{O}(E^0)$, for the case of $n \neq 0$ they can have nontrivial E dependence depending on their asymptotic form. We need to determine the form of these functions.

³We can prove that, in any stationary spherical symmetric spacetime, r_0 for the free massless particle cannot depend on L (so λ is proportional to E), while for the massive one it can.

⁴Note that many physical solutions hold our power ansatz; exponential divergences correspond to the $n \rightarrow \infty$ case; logarithmic divergences or oscillations correspond to $n = 0$.

First, from now on, let us assume

$$ln \leq 2. \quad (34)$$

$$f(r), g(r) \xrightarrow{r \rightarrow \infty} 1 - \Lambda r^2 + \mathcal{O}\left(\frac{1}{r}\right) \quad (28)$$

where Λ is a coefficient parameter corresponding to the cosmological constant in pure Einstein gravity.⁵ The right-hand side means that the geometry asymptotes to maximally symmetric spacetimes (Minkowski, de Sitter, or anti-de Sitter). In addition, we consider only $n \geq 0$; since otherwise, in the high energy limit, the orbit may fall into the event horizon for the case of black hole spacetimes. Next, let us consider vector potential A_μ . If we assume

$$A_\mu \xrightarrow{r \rightarrow \infty} \mathcal{O}(r^k) \quad (k \in \mathbb{R}) \quad (29)$$

$$\xrightarrow{E \rightarrow \infty} \mathcal{O}(E^{kn}), \quad (30)$$

then we get

$$kn \leq 1; \quad (31)$$

otherwise, E is negligible compared to A_t , so (18), which determines r_0 , becomes independent of E and contradicts $n > 0$. Hence, we find $E + qA_t \rightarrow \mathcal{O}(E^1)$. Finally, for the scalar potential V , which is the most important for our causality analysis, we assume

$$V(r) \xrightarrow{r \rightarrow \infty} \mathcal{O}(r^l) \quad (l \in \mathbb{R}) \\ \xrightarrow{E \rightarrow \infty} \mathcal{O}(E^{ln}). \quad (32)$$

Then we find that (18) puts no restriction on l ; however, the causality with (15) does bound l ; demanding physical particles never to be spacelike, that is, $-g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu \geq 0$ for $\forall r$ along with the geodesic, gives the restriction

$$m^2 + V(r) \geq 0 \quad (33)$$

on V . This is our causality constraint. Then, since $(L - qA_\phi)^2$ in (20) is positive, ln is bounded by the asymptotic energy exponent of $(E + qA_t)^2/f$. This bound depends on the behavior of $f(r)$ in the $E \rightarrow \infty$ limit or the asymptotic geometry of spacetimes; thus, let us look at it carefully in order.

For the asymptotically flat case, we have $\Lambda = 0$ and $f, g \rightarrow 1$. With $n > 0$, we have $E^2/f \rightarrow \mathcal{O}(E^2)$ and (20) with causality (33) bounds V as

⁵Here, we only specify the asymptotic behavior of the spacetime but not the gravity theory or its matter contents. Therefore, our argument can apply to any stationary spherically symmetric spacetimes with at least one unstable circular orbit in any gravity theory.

Then the energy dependence of the terms in the right-hand side of (18) is found as follows:

$$(\text{first term}) = \mathcal{O}(E^{2(1-n)}), \quad (35)$$

$$(\text{second term}) = \mathcal{O}(E^{2-n}), \quad (36)$$

$$(\text{third term}) = \mathcal{O}(E^{1+kn-n}) \leq \mathcal{O}(E^{2-n}), \quad (37)$$

$$(\text{fourth term}) = \mathcal{O}(E^{1+kn-2n}) \leq \mathcal{O}(E^{2-2n}), \quad (38)$$

$$(\text{fifth term}) = \mathcal{O}(E^{(l-1)n}) \leq \mathcal{O}(E^{2-n}). \quad (39)$$

If $kn < 1$ and $ln < 2$, we find $n = 0$; otherwise, in the $E \rightarrow \infty$ limit, only the second term is dominant in (18), so r_0 becomes independent of E and contradicts $n > 0$. In the $n = 0$ case, only the first and second terms are dominant in (19), and $\lambda \rightarrow \mathcal{O}(E^1)$ is shown. Meanwhile, if $kn = 1$ ($ln = 2$), the third (fifth) term in (18) is of the same order as the second, and r_0 can depend on E nontrivially. Then, we find $\lambda \rightarrow \mathcal{O}(E^{1-n})$ and the energy bound is satisfied (but not saturated in any cases).

For the asymptotically (anti-)de Sitter case, we have $\Lambda \neq 0$ and $f, g \rightarrow \mathcal{O}(E^{2n})$. With $n > 0$, we have $E^2/f \rightarrow \mathcal{O}(E^{2(1-n)})$ and (20) with causality (33) bounds V as

$$(l+2)n \leq 2, \quad n \leq 1. \quad (40)$$

Then the energy dependence of the terms in (18) is as follows:

$$(\text{first term}) = \mathcal{O}(E^{2-n}), \quad (41)$$

$$(\text{second term}) = \mathcal{O}(E^{2-n}), \quad (42)$$

$$(\text{third term}) = \mathcal{O}(E^{1+kn-n}) \leq \mathcal{O}(E^{2-n}), \quad (43)$$

$$(\text{fourth term}) = \mathcal{O}(E^{1+kn-n}) \leq \mathcal{O}(E^{2-n}), \quad (44)$$

$$(\text{fifth term}) = \mathcal{O}(E^{(l+1)n}) \leq \mathcal{O}(E^{2-n}). \quad (45)$$

If $kn < 1$ and $(l+2)n < 2$, we find $n = 0$; otherwise, in the $E \rightarrow \infty$ limit, only the first and second terms are dominant in (18), so r_0 becomes independent of E since $(E + qA_t)^2$ is eliminated as an overall factor and contradicts $n > 0$. Then, $\lambda \rightarrow \mathcal{O}(E^1)$ is shown as the flat case. Meanwhile, if $kn = 1$ ($(l+2)n = 2$), the third and fourth (only fifth) terms are of the same order as the first and second, and r_0 can depend on E nontrivially. Then we find $\lambda \rightarrow \mathcal{O}(E^{1-n})$ and the energy bound is satisfied in any cases again.

We have evaluated the energy dependence of the Lyapunov exponent λ in the $E \rightarrow \infty$ limit in both the asymptotically flat/(A)dS cases, and we obtained

$$\lambda_{e=1} \rightarrow \begin{cases} \mathcal{O}(E^1) & (kn < 1 \text{ and } ln < 2, \text{ hence } n = 0) \\ \mathcal{O}(E^{1-n}) & (kn = 1 \text{ or } ln = 2, n > 0) \end{cases} \quad (\Lambda = 0) \quad (46)$$

$$\lambda_{e=1} \rightarrow \begin{cases} \mathcal{O}(E^1) & (kn < 1 \text{ and } (l+2)n < 2, \text{ hence } n = 0) \\ \mathcal{O}(E^{1-n}) & (kn = 1 \text{ or } (l+2)n = 2, 0 < n \leq 1) \end{cases} \quad (\Lambda \neq 0). \quad (47)$$

Without the causality constraint, the asymptotic energy exponent l of $V(r)$ can be arbitrarily large, and the chaos energy bound is violated. On the other hand, we have discovered that causality gives the upper bound on l and protects the chaos energy bound.⁶ Since causality is one of the most fundamental principles of physics, our result suggests that causality ensures the universal bound (2) for any chaotic system.

IV. CAUSALITY ALONE CANNOT AVOID VIOLATION OF THE MSS BOUND

We have evaluated above the Lyapunov exponent λ in the $e = 1$ gauge and revealed that the causality of the particle ensures the universal bound (2) on the energy dependence of chaos. In this section, in contrast, we study another chaos bound: the MSS bound (1). Since this bound is motivated by the AdS/CFT correspondence, the Lyapunov exponent needs to be measured in the static gauge. We make a gauge transformation upon the result given in the previous section and show that, at high energy, the MSS bound can be violated.

We first remind the reader that the MSS bound is for the boundary theory of the AdS/CFT correspondence, thus the

Lyapunov exponent λ in the expression of the MSS bound needs to be measured with the boundary time t . This corresponds to the Killing time of the bulk spacetime on the gravity side. The boundary time corresponds to the target space time on the gravity side. Thus, we need to convert λ in the $e = 1$ gauge to that in the static gauge using the reparametrization-independent representation (19).

Under this reparametrization, r_0 is invariant, and the conditions on n , k , and l are the same as those in Sec. III B. The transformation law is simply

$$\lambda_{\text{static}} = e(\tau = t(\tau))|_{r_0} \lambda_{e=1}. \quad (48)$$

Because, in the static gauge, we obtain

$$e(\tau = t(\tau)) = \frac{f(r)}{E + qA_t} \quad (49)$$

from (16), we find that $e(\tau = t)|_{r_0}$ is $\mathcal{O}(E^{-1})$ in the asymptotically flat case and $\mathcal{O}(E^{2n-1})$ in the asymptotically (A)dS case. Hence, the energy dependence of λ becomes

$$\lambda_{\text{static}} \rightarrow \begin{cases} \mathcal{O}(E^0) & (kn < 1 \text{ and } ln < 2, \text{ hence } n = 0) \\ \mathcal{O}(E^{-n}) & (kn = 1 \text{ or } ln = 2, n > 0) \end{cases} \quad (\Lambda = 0), \quad (50)$$

$$\lambda_{\text{static}} \rightarrow \begin{cases} \mathcal{O}(E^0) & (kn < 1 \text{ and } (l+2)n < 2, \text{ hence } n = 0) \\ \mathcal{O}(E^n) & (kn = 1 \text{ or } (l+2)n = 2, 0 < n \leq 1) \end{cases} \quad (\Lambda \neq 0). \quad (51)$$

Among the various cases, we find that in the last case, for $\Lambda \neq 0$ with $kn = 1$ or $(l+2)n = 2$, the exponent λ diverges in the $E \rightarrow \infty$ limit and violates the MSS bound: Since the upper bound $2\pi T/\hbar$ in the MSS bound is described only by the quantities of spacetimes, this divergence obviously violates the bound.

⁶In particular, popular causal setups of the gravity-matter actions may provide potentials which vanish in $r \rightarrow \infty$, i.e., $k, l < 0$. Our analysis in this section includes those cases and leads to the saturation of the chaos energy bound.

The violation condition we have found here suggests that spacetimes with external potentials having such values of k or l , which are still causal, may not be realized holographically.⁷ This is a natural result from the viewpoint of the expectation that the holographic duality works for not all possible gravitational backgrounds which are causal.

⁷It is amusing to note that this divergence of λ can occur particularly in the asymptotically AdS ($\Lambda < 0$) case, where the MSS bound is motivated by the AdS/CFT correspondence.

V. CONCLUSION AND DISCUSSION

We have considered the geodesic motion in generic situations in which a particle is moving in a general stationary and spherically symmetric spacetime with the metric (12) and is coupled to general vector/scalar potentials (13), and we have investigated the bound on the Lyapunov exponent λ imposed by causality. In Sec. II, we estimated λ as (19) without fixing the gauge of the worldline time. In Sec. III, we described our main results that causality of the particle motion gives a universal upper bound on the energy dependence of λ , which coincides with the proposed chaos energy bound (2). It was important there that causality was reflected in (15) and gave the restriction (34) and (40) on the asymptotic form of the scalar potential. In contrast, in Sec. IV, we pointed out that the celebrated MSS bound could be violated in the particular potentials (51), even with such a causal restriction. This suggests that, beyond causality, some fundamental principle that rules out those potentials which violate the bound is needed for holographic descriptions of gravity. Causality is not strong enough to ensure the holographic principle.

Let us make a brief comment on the gauges we chose in this paper. For that purpose, we remind the reader of another perspective in the study of chaos in gravity theories: chaos in the time evolution of the gravitational field itself, which plays an important role in cosmology [21,22]. There, subtleties in defining chaos, which is characterized by a positive Lyapunov exponent, are led by the noninvariance of the exponent of the gravitational field under general coordinate transformations [23–29]. One may notice that this subtlety applies to our study, too, because of the noninvariance under the reparametrizations of the worldline time τ . In fact, the energy dependence of the Lyapunov exponent in the $e = 1$ gauge [(46) and (47)] has changed as in (50) and (51) in the static gauge. However, we argue for several reasons that this issue is resolved in our case and the most important energy dependence is the one in the $e = 1$ gauge: First, the $e = 1$ gauge is equivalent to choosing the gauge of the worldline time for a massive particle as $\tau = (\text{proper time})/m$, which is the intrinsic time observed in the comoving frame of the particle. Second, an einbein e is unnatural if it depends on the energy E , whose conservation follows from the equations of motion, since the gauge of the worldline time must be fixed before integrating the equations. Then the energy dependence of λ should not change from (46) and (47), and it should satisfy the chaos energy bound (2) in the natural gauges.

Several discussions are in order. Below we make comments on the need for (i) numerical simulations, (ii) analysis in less-symmetric cases, and (iii) study in relation to AdS/CFT correspondence. First, our analytic estimate uses the separatrices of the effective potential which the moving particle feels. Although the curvature of the potential produces an exponential behavior of the particle motion, it is merely an estimate of the Lyapunov exponent of the

whole motion of the particle. To determine the actual Lyapunov exponent, one needs numerical studies for each chosen setup of curved spacetimes and potentials. Our study focused on universal properties and thus used the analytic estimate, and it is obvious that more numerical studies are necessary to check the violation of the MSS bound. We emphasize that we have not specified the gravity theory and our results apply to generic spacetimes not only in general relativity but also in its extensions like various modified gravity theories⁸, and numerical/analytic calculation of the Lyapunov exponents in such theories (see [30–37] for relevant studies) may lead to further progress in chaos study.

Second, we should add a caveat about the symmetry of our systems and chaoticity. We have focused on geodesic motion in general stationary and spherically symmetric spacetimes. As we have seen above, this reduces the problem to just a one-dimensional one by extracting the radial component of the particle motion, thus according to the Poincaré-Bendixson theorem, there should be no chaos. What we assume is that the separatrices we used would serve as the nest of chaos in generic less-symmetric cases in which chaos is present. The reason why we were forced to use the symmetric spacetimes is that, for the analytic evaluation of the Lyapunov exponent, we needed the energy exponents of the conserved quantities coming from the spacetime symmetry. In general, chaos emerges in nonintegrable systems which have not enough number of conserved quantities, thus it is important to extend our reparametrization-independent estimation of the Lyapunov exponent (19) to less symmetric spacetimes.

Third, what is “some fundamental principle for holographic description” suggested by the violation of the MSS bound? The most conservative candidate is the causality of the boundary theory, not of the bulk gravity theory which we studied. In AdS/CFT correspondence, although a particle in the boundary is described by a local operator, one falling into the bulk is by a composite and nonlocal operator. Hence, the causality constraint of the boundary theory may be nontrivial and more constraining than the bulk one.⁹ There is the possibility that this hidden constraint rules out the potentials which violate the MSS bound. It should be important to verify this conjecture.

Finally, we remark on a general perspective of the antithetical nature of causality and chaos. The chaos energy

⁸It is expected that the asymptotic behavior of the metric remains unchanged even if higher curvature terms are added to the Einstein-Hilbert action because those are considered sub-leading corrections in the effective field theory description. Therefore, our assumption on the metric (28) would be consistent with solutions in various gravity theories as well as in general relativity.

⁹The gravity side could be a string theory, in which case the chaos in the bulk is also nonlocal; see [38] for an early study of string chaos in the AdS/CFT correspondence.

bound (2) was motivated by the thermodynamic consistency of the definition of the Lyapunov exponent. Is there anything to do with the relativistic causality which was important for deriving (2) in this paper? This question reminds us of Hawking [39] having discovered that causality division surfaces, i.e., event horizons, are necessarily thermal. In other words, consistency of thermal ensembles in quantum mechanics is required by the singular causal structure. In this respect, what we see in this paper may be just a tiny part of the physical consistency conditions in the whole theory space unseen by us. It is

truly amusing to explore more on yet-to-be-discovered relations between the antithetical notions in physics: causality and chaos.

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