Fermion scattering on topological solitons in the $\mathbb{CP}^{N-1}$ model

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The scattering of Dirac fermions in the background fields of topological solitons of the $(2+1)$-dimensional $\mathbb{CP}^{N-1}$ model is studied using analytical and numerical methods. It is shown that the exact solutions for fermionic wave functions can be expressed in terms of the confluent Heun functions. The question of the existence of bound states for the fermion-soliton system is then investigated. General formulas describing fermion scattering are obtained, and a symmetry property for the partial phase shifts is derived. The amplitudes and cross sections of the fermion-soliton scattering are obtained in an analytical form within the framework of the Born approximation, and the symmetry properties and asymptotic forms of the Born amplitudes are investigated. The dependences of the first few partial phase shifts on the fermion momentum are obtained by numerical methods, and some of their properties are investigated and discussed.

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I. INTRODUCTION

A number of $(2+1)$-dimensional field models admit the existence of planar topological solitons [1–3], which play an important role in field theory, high-energy physics, condensed matter physics, cosmology, and hydrodynamics. The vortices of the effective theory of superconductivity [4] and of the $(2+1)$-dimensional Abelian Higgs model [5] are probably the most important topological solitons of this type. The next most important is the topological soliton of the $(2+1)$-dimensional nonlinear $O(3)$ sigma model [6]. One feature of the soliton solutions of the nonlinear $O(3)$ sigma model is the presence of an arbitrary parameter determining their spatial size. This is because the static energy functional of the $(2+1)$-dimensional $O(3)$ sigma model is invariant under scale transformations.

Nonlinear sigma models can also be formulated for orthogonal groups $O(N)$ with $N \geq 4$, but unlike the $O(3)$ sigma model, these models have no soliton solutions. However, there is another family of nonlinear scalar field models whose properties are similar to those of the nonlinear $O(N)$ sigma models in many respects, but which have topological soliton solutions for an arbitrary number of fields. These are the so-called $\mathbb{CP}^{N-1}$ models [7–10]. For $N = 2$, the $\mathbb{CP}^{N-1}$ model is reduced to the $O(3)$ sigma model, but for $N \geq 3$, the $\mathbb{CP}^{N-1}$ model is a better generalization than the $O(N+1)$ sigma model, as it continues to have soliton solutions [11,12] even in this case.

Since their appearance in the late 1970s, the $\mathbb{CP}^{N-1}$ models have consistently attracted interest, primarily based on the fact that the two-dimensional $\mathbb{CP}^{N-1}$ models are an useful instrument for studying nonperturbative effects in the four-dimensional Yang-Mills models. The two-dimensional $\mathbb{CP}^{N-1}$ models share many common properties with four-dimensional Yang-Mills models, including conformal invariance at the classical level, asymptotic freedom in the ultraviolet regime [13], strong coupling in the infrared regime, and the existence of a topological term and instantons [11,12] resulting in a complex structure of the vacuum at the quantum level. The lower dimensionality of the $\mathbb{CP}^{N-1}$ models facilitates the analysis of nonperturbative effects in the strong coupling regime, compared to the more complex four-dimensional Yang-Mills models. In addition, two-dimensional $\mathbb{CP}^{N-1}$ models can be considered as effective field theories describing low-energy dynamics on the world sheet of non-Abelian vortex strings in a class of four-dimensional gauge theories [14–19]. The $\mathbb{CP}^{N-1}$ models also have interesting applications in the field of condensed matter physics [20], and particularly in (anti)ferromagnetism, the Hall effect, and the Kondo effect. They also find application in the study of the sphaleron-induced fermion number violation at high temperature [21].

The $\mathbb{CP}^{N-1}$ model can be extended to include fermionic matter fields. This can be achieved either by a supersymmetric extension of the $\mathbb{CP}^{N-1}$ model or by minimal

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coupling between fermionic fields and a composite gauge field of the $\mathbb{CP}^{N-1}$ model (the so-called minimal model [22]). The supersymmetric extension of the $\mathbb{CP}^{N-1}$ model involves Majorana fermion fields that satisfy nontrivial constraints, whereas the minimal model deals with unconstrained Dirac fermion fields. In the present paper, we investigate a fermion-soliton system in the minimal model within the background field approximation. In particular, we find that the fermionic wave functions are expressed in terms of the confluent Heun functions, and that the fermion-soliton system has no bound states. The results obtained here can be used to describe the interaction of fermions with two-dimensional or threadlike three-dimensional topological defects in condensed matter physics. We note that it was stated in Ref. [23] that the wave functions of a Dirac fermion minimally coupled to the two-dimensional $\mathbb{CP}^1$ model can be expressed in terms of the confluent Heun functions. Furthermore, it was shown in Refs. [24,25] that the fermion scattering on a one-dimensional kink or Q-ball can also be described in terms of Heun-type functions.

This paper is structured as follows. In Sec. II, we describe briefly the Lagrangian, symmetries, field equations, and topological solitons of the $\mathbb{CP}^{N-1}$ model. In Sec. III, we study the fermion-soliton scattering in the background field approximation. We show that for $\mathbb{CP}^{N-1}$ solitons with winding numbers $n = \pm 1$, the fermionic wave functions can be expressed in terms of the confluent Heun functions. We also consider the question of the existence of bound fermionic states for these solitons, and establish a symmetry property for partial phase shifts. In Sec. IV, we give an analytical description of the fermion scattering within the framework of the Born approximation. In Sec. V, we present numerical results for the first few partial phase shifts, and compare the exact results with those obtained in the Born approximation. In the final section, we briefly summarize the results obtained in the present work.

Throughout the paper, the natural units $c = 1$ and $\hbar = 1$ are used.

II. LAGRANGIAN, FIELD EQUATIONS, AND TOPOLOGICAL SOLITONS OF THE MODEL

The Lagrangian density of the $\mathbb{CP}^{N-1}$ model minimally interacting with fermionic fields has the form

$$\mathcal{L} = g^{-1}(D_\mu n_a)^* D^\mu n_a + i\bar{\psi}_a \gamma^\mu D_\mu \psi_a - M\bar{\psi}_a \psi_a, \quad (1)$$

where n_a are complex scalar fields, ψ_a are fermionic fields, g is a coupling constant, and the index a runs from one to N. The complex scalar fields n_a satisfy the normalization condition $n_a^* n_a = 1$, where summation over repeated indices is implied. In Eq. (1), the covariant derivatives of fields are

$$D_\mu n_a = \partial_\mu n_a + iA_\mu n_a, \quad (2a)$$

$$D_\mu \psi_a = \partial_\mu \psi_a + iA_\mu \psi_a, \quad (2b)$$

where A_μ is a vector gauge field.

By varying the action $S = \int \mathcal{L} d^2x dt$ in the fields n_a, $\bar{\psi}_a$, and A_μ, and taking into account the constraint $n_a^* n_a = 1$ by means of the Lagrange multiplier method, we obtain the field equations for the minimal $\mathbb{CP}^{N-1}$ model:

$$D_\mu D^\mu n_a - (n_b^* D_\mu D^\mu n_b) n_a = 0, \quad (3)$$

$$(i\gamma^\mu D_\mu - M)\psi_a = 0, \quad (4)$$

$$A_\mu - i n_a^* \partial_\mu n_a - \frac{g}{2}\bar{\psi}_a \gamma_\mu \psi_a = 0. \quad (5)$$

It follows from Eq. (5) that the gauge field A_μ is not a dynamic one, and is expressed in terms of the fields n_a and ψ_a.

The minimal $\mathbb{CP}^{N-1}$ model possesses a number of symmetries. The invariance of the model (1) under the global $U(N)$ transformations $n_a \to U_{ab} n_b$, $\psi_a \to U_{ab}\psi_b$ results in the existence of the Noether current, which is a vector field with values in complex anti-Hermitian $(N \times N)$-matrices, with matrix entries

$$(j_\mu)_{ab} = g^{-1}[n_a (D_\mu n_b)^* - (D_\mu n_a) n_b^*] + i\bar{\psi}_a \gamma_\mu \psi_b. \quad (6)$$

In addition, the model is also invariant under the local $U(1)$ transformations

$$n_a(x) \to e^{i\Lambda(x)} n_a(x), \quad (7a)$$

$$\psi_a(x) \to e^{i\Lambda(x)} \psi_a(x), \quad (7b)$$

$$A_\mu(x) \to A_\mu(x) + i e^{-i\Lambda(x)} \partial_\mu e^{i\Lambda(x)}. \quad (7c)$$

The corresponding gauge current is the trace of the matrix-valued current in Eq. (6).

A characteristic feature of the $\mathbb{CP}^{N-1}$ models is that they possess localized solutions [11,12], which can be interpreted either as instantons in the two-dimensional Euclidean case or as static topological solitons in the $(2+1)$-dimensional case. All of these solutions can be obtained in analytical form. In particular, the $\mathbb{Z}_{|n|}$ symmetric soliton solution of the $(2+1)$-dimensional $\mathbb{CP}^{N-1}$ model can be written as

$$\mathbf{n}(\rho, \theta) = \frac{\lambda^{|n|}\mathbf{u} + \rho^{|n|} e^{in\theta}\mathbf{v}}{(\lambda^{2|n|} + \rho^{2|n|})^{1/2}}, \quad (8)$$

where n is a nonzero integer, ρ and θ are polar coordinates, $\mathbf{u} = (1, 0, \ldots, 0)$ and $\mathbf{v} = (0, 0, \ldots, 1)$ are orthonormal N-dimensional vectors, and λ is a scaling parameter, which determines the effective size of the soliton. The gauge field A_μ that corresponds to the solution (8) is

$$A_\mu = n\frac{\rho^{2(|n|-1)}}{\rho^{2|n|}+\lambda^{2|n|}}\quad(0, y, -x), \tag{9}$$

where we factor out the common factor of the covariant components of the gauge field.

Equation (8) tells us that $\mathbf{n}\to e^{in\theta}\mathbf{v}$ as $\rho\to\infty$, and, consequently, the solution (8) tends to the same element of the complex projective space $\mathbb{CP}^{N-1}$ in this limit. It follows that the field configurations in Eq. (8) map from the compactified plane (which is topologically equivalent to the two-sphere S^2) to $\mathbb{CP}^{N-1}$. Since the second homotopic group $\pi_2(\mathbb{CP}^{N-1})=\mathbb{Z}$, the field configurations in Eq. (8) can be labeled by an integer Q called the winding number, as explicitly given in Ref. [11]:

$$Q = -\frac{1}{2\pi}\int d^2x\,\epsilon_{ij}\partial_i A_j = -\frac{1}{2\pi}\int_{S^1} A_i dx^i, \tag{10}$$

where ϵ_{ij} is the two-dimensional antisymmetric tensor and $\epsilon_{12}=1$. It can easily be shown that for the soliton solutions (8), the winding number $Q=n$, meaning that they are stable to transition into field configurations belonging to the topologically trivial sector with $n=0$.

In contrast to the usual solutions satisfying second-order field equations, the solution (8) also satisfies the first-order equations,

$$D_i\mathbf{n}\pm i\epsilon_{ij}D_j\mathbf{n}=0. \tag{11}$$

Depending on the sign, this is called the self-duality or antiself-duality condition. It can be shown [11] that the energy of any solution in the topological sector with $Q=n$ satisfies the inequality

$$E\geq 2\pi|n|g^{-1}, \tag{12}$$

and that saturation of this inequality is possible only for (anti-)self-dual solutions satisfying Eq. (11). Note that for $N>2$, the $\mathbb{CP}^{N-1}$ models also have nonself-dual solutions [26–32] that do not satisfy Eq. (11) and are only unstable saddle points of the energy functional.

The energy density of the soliton solution (8) is

$$\begin{aligned}\mathcal{E} &= g^{-1}[(D_0n_a)^* D_0 n_a + (D_i n_a)^* D_i n_a]\\ &= 2n^2\lambda^2 g^{-1}\frac{(\lambda\rho)^{2(|n|-1)}}{(\rho^{2|n|}+\lambda^{2|n|})^2},\end{aligned} \tag{13}$$

and the soliton energy

$$E_{\mathrm{s}} = 2\pi\int_0^\infty \mathcal{E}(\rho)d\rho = 2\pi|n|g^{-1} = 2\pi|Q|g^{-1}. \tag{14}$$

We see that the soliton energy (14) saturates the inequality (12), and therefore is the absolute minimum in the topological sector with a given Q. The soliton energy does not depend on the scaling parameter λ since the bosonic part of the action of the model (1) is invariant under scale transformations $\mathbf{x}\to a\mathbf{x}$.

III. FERMIONS IN THE BACKGROUND FIELD OF A $\mathbb{CP}^{N-1}$ SOLITON

We consider fermion scattering on the topological $\mathbb{CP}^{N-1}$ soliton within the background field approximation, i.e., we neglect the fermion backreaction on the soliton field configuration (8). Equation (5) tells us that the gauge field $A_\mu = in_a^*\partial_\mu n_a + 2^{-1}g\bar\psi_a\gamma_\mu\psi_a$. The background field approximation involves neglecting the fermionic term $2^{-1}g\bar\psi_a\gamma_\mu\psi_a$ in comparison with the bosonic term $in_a^*\partial_\mu n_a$. In this case, the gauge field $A_\mu = in_a^*\partial_\mu n_a$, and it follows from Eqs. (2) and (3) that there is no fermion backreaction on the soliton field. An analysis shows that this approximation is possible under the condition

$$g\ll |n|\lambda^{-1}\varrho_F^{-1}, \tag{15}$$

where ϱ_F is the two-dimensional density of incident fermions.

The presence of the fermionic part in the gauge field A_μ causes the Dirac equation (4) to be nonlinear (cubic) in fermionic fields, which does not allow us to obtain an analytical solution. In order to avoid this, we must neglect this nonlinear cubic term compared to the linear Dirac mass term in Eq. (4). This is possible if the condition

$$g\ll M\varrho_F^{-1} \tag{16}$$

is satisfied, where M is the fermion mass.

We see that the conditions (15) and (16) can always be fulfilled if the density of incident fermions is sufficiently low. From the viewpoint of QFT, however, we are talking about the scattering of a fermion of mass M on a $\mathbb{CP}^{N-1}$ soliton of mass $M_{\mathrm{s}}=2\pi|n|g^{-1}$. To allow us to neglect the recoil of the $\mathbb{CP}^{N-1}$ soliton in fermion scattering, the mass M_{s} must be much larger than the energy ε of the incident fermion, which leads to the condition

$$g\ll 2\pi|n|\varepsilon^{-1} < 2\pi|n|M^{-1}. \tag{17}$$

The conditions (15), (16), and (17) do not contradict each other, and can always be satisfied if the coupling constant g is sufficiently small. In this case, the Dirac equation (4) can be written in the Hamiltonian form

$$i\partial_t\psi_a = H\psi_a, \tag{18}$$

where the Hamiltonian

$$H = -i\alpha^k[\partial_k - (n_b^*\partial_k n_b)] + \beta M, \tag{19}$$

the matrices $\alpha^i = \gamma^0\gamma^i$ and $\beta = \gamma^0$, and the Dirac matrices

$$\gamma^0 = \sigma_3, \qquad \gamma^1 = -i\sigma_1, \qquad \gamma^2 = -i\sigma_2. \tag{20}$$

Note that all components ψ_a of the fermionic multiplet $(\psi_1, \ldots, \psi_N)$ satisfy the same equation (18).

We now discuss the symmetry properties of the Dirac equation (18) under discrete transformations. Let $\psi(t, \mathbf{x})$ be a solution to the Dirac equation (18) in the background field of $\mathbb{CP}^{N-1}$ soliton (8). It can easily be shown that in this case,

$$\psi^C(t, \mathbf{x}) = \sigma_1\psi^*(t, \mathbf{x}), \tag{21}$$

$$\psi^P(t, \mathbf{x}) = \sigma_3\psi(t, -\mathbf{x}), \tag{22}$$

and

$$\psi^{\Pi_2 T}(t, x, y) = \psi^*(-t, x, -y) \tag{23}$$

are also solutions to this equation. The solutions (21)–(23) are obtained from the original solution $\psi(t, \mathbf{x})$ by means of the C, P, and combined $\Pi_2 T$ transformations, respectively, where the symbol Π_2 denotes the operation of coordinate reflection about the Ox_1 axis.

A. Exact fermionic wave functions

It can easily be shown that the Hamiltonian (19) commutes with the angular momentum operator

$$J_3 = -i\partial_\theta + \sigma_3/2. \tag{24}$$

The presence of the conserved angular momentum J_3 is due to the fact that according to Eq. (9), the vector field $A_\mu = in_a^*\partial_\mu n_a$ in the Hamiltonian (19) is invariant under in-plane rotations. The common eigenfunctions of the operators H and J_3 have the form

$$\psi_m = \begin{pmatrix} e^{i(m-1/2)\theta} f(\rho) \\ e^{i(m+1/2)\theta} g(\rho) \end{pmatrix} e^{-i\varepsilon t}, \tag{25}$$

where ε and m are the eigenvalues of H and J_3, respectively.

By substituting Eq. (25) into Eq. (18), we obtain a system of first-order differential equations for the radial functions $f(\rho)$ and $g(\rho)$

$$f'(\rho) = \rho^{-1}(A_{mn}(\rho) - 1/2)f(\rho) + (M + \varepsilon)g(\rho), \tag{26}$$

$$g'(\rho) = (M - \varepsilon)f(\rho) - \rho^{-1}(A_{mn}(\rho) + 1/2)g(\rho), \tag{27}$$

where

$$A_{mn}(\rho) = m - n\frac{\rho^{2|n|}}{\lambda^{2|n|} + \rho^{2|n|}}. \tag{28}$$

The system of differential equations (26) and (27) is equivalent to the second-order differential equation

$$f''(\rho) + \rho^{-1}f'(\rho) + [k^2 - \rho^{-2}(1/2 - A_{mn}(\rho))^2 - \rho^{-1}A'_{mn}(\rho)]f(\rho) = 0, \tag{29}$$

where $k^2 = \varepsilon^2 - M^2$, taken together with the differential relation

$$g(\rho) = [\rho^{-1}(1/2 - A_{mn}(\rho))f(\rho) + f'(\rho)](M + \varepsilon)^{-1}. \tag{30}$$

The substitutions $f \to g$, $g \to f$, $A_{mn} \to -A_{mn}$, $\varepsilon \to -\varepsilon$ in Eqs. (29) and (30) lead to the second-order differential equation and differential relation for the radial functions $g(\rho)$ and $f(\rho)$, respectively.

From Eq. (28), it follows that the functions A_{mn} depend only on the dimensionless combination $\tau = -\rho^2/\lambda^2$, which therefore plays the role of a natural independent variable. In terms of this new variable τ, Eq. (29) takes the form

$$f''(\tau) + \tau^{-1}f'(\tau) - 2^{-2}\tau^{-1}[k^2\lambda^2 + \tau^{-1} \times (1/2 - A_{mn}(\tau))^2 + 2A'_{mn}(\tau)]f(\tau) = 0, \tag{31}$$

where

$$A_{mn}(\tau) = m - n\frac{\tau^{|n|}}{(-1)^{|n|} + \tau^{|n|}}. \tag{32}$$

It follows from Eq. (32) that $A_{mn}(\tau)$ has first-order poles at the points

$$\tau_k = e^{i\frac{\pi}{|n|}(2k+|n|+1)}, \quad k = 0, \ldots, |n| - 1. \tag{33}$$

The point $\tau = 0$ and the $|n|$ points of Eq. (33) are the regular singular points of the differential equation (31), whereas the point $\tau = \infty$ is the irregular singular point. At present, analytical solutions to such differential equations are known only when the number of regular singular points does not exceed two [33]. In our case, this means that analytical fermionic wave functions can be found only for the winding numbers $n = \pm 1$, which correspond to the elementary soliton ($n = 1$) or antisoliton ($n = -1$) of the model (1).

We will therefore consider the case where $n = \pm 1$. As $\tau \to 0$, the radial wave function $f(\tau) \sim \tau^{l/2}$, where $l = |m - 1/2|$. By the substitution $f(\tau) = \tau^{l/2}(1 - \tau)^{-n/2}F(\tau)$, the differential equation (31) is reduced to the confluent Heun differential equation [33–35]

$$F''(\tau)+\left[\frac{\gamma}{\tau}+\frac{\delta}{\tau-1}+\epsilon\right]F'(\tau)+\frac{\alpha\tau-q}{\tau(\tau-1)}F(\tau)=0, \quad (34)$$

where the parameters are

$$\alpha=-\frac{1}{4}k^2\lambda^2, \quad (35a)$$

$$\gamma=l+1, \quad (35b)$$

$$\delta=-n, \quad (35c)$$

$$\epsilon=0, \quad (35d)$$

$$q=-\frac{1}{4}k^2\lambda^2+\frac{n}{2}\left(l-\left(m-\frac{1}{2}\right)\right). \quad (35e)$$

Equation (34) has two independent local solutions in the neighborhood of the point $\tau=0$: the first is regular, while the other is irregular and diverges as τ^{-l} for $l>0$ or as $\ln(\tau)$ for $l=0$. To obtain the regular radial wave function, we must choose the regular solution

$$F(\tau)=H_C[q,\alpha,\gamma,\delta,\epsilon,\tau], \quad (36)$$

which is called the confluent Heun function [33–35]. In the same way, we can find a solution for the other radial wave function in the form $g(\tau)=\tau^{l'/2}(1-\tau)^{n/2}G(\tau)$, where $l'=|m+1/2|$. The function $G(\tau)$ is also expressed in terms of the confluent Heun function

$$G(\tau)=H_C[q',\alpha',\gamma',\delta',\epsilon',\tau], \quad (37)$$

where the parameters are

$$\alpha'=\alpha=-\frac{1}{4}k^2\lambda^2, \quad (38a)$$

$$\gamma'=l'+1, \quad (38b)$$

$$\delta'=n, \quad (38c)$$

$$\epsilon'=\epsilon=0, \quad (38d)$$

$$q'=-\frac{1}{4}k^2\lambda^2-\frac{n}{2}\left(l'+\left(m+\frac{1}{2}\right)\right). \quad (38e)$$

The confluent Heun function satisfies the condition $H_C[q,\alpha,\gamma,\delta,\epsilon,0]=1$. In the region $|\tau|<1$, it can be expanded into a uniformly convergent series. Furthermore, it can be analytically extended to the entire complex plane with a branch cut running from 1 to ∞.

Solutions (36) and (37) are defined up to arbitrary multipliers κ_1 and κ_2, respectively, and their ratio $\kappa=\kappa_2/\kappa_1$ can be determined using the differential relation (30) and the series expansion [33,34] of the confluent Heun function at the origin, as follows:

$$\kappa=\begin{cases}-\frac{\lambda}{2}\frac{\varepsilon-M}{m+1/2}, & m>0\\ \frac{2}{\lambda}\frac{1/2-m}{\varepsilon+M}, & m<0\end{cases} \quad (39)$$

Using the results obtained, we can write an analytical expression for the total fermionic wave function in terms of the radial variable ρ:

$$\psi_m=\mathcal{N}\begin{pmatrix}\kappa^{-1/2}(\rho/\lambda)^{l}(1+(\rho/\lambda)^2)^{-n/2}H_C[q,\alpha,\gamma,\delta,\epsilon,-\rho^2/\lambda^2]e^{i(m-1/2)\theta}\\ \kappa^{1/2}(\rho/\lambda)^{l'}(1+(\rho/\lambda)^2)^{n/2}H_C[q',\alpha',\gamma',\delta',\epsilon',-\rho^2/\lambda^2]e^{i(m+1/2)\theta}\end{pmatrix}e^{-i\varepsilon t}, \quad (40)$$

where $\mathcal{N}$ is a normalization factor and the winding number n of the $\mathbb{CP}^{N-1}$ soliton can take the values ±1.

We now find the symmetry properties of the wave function (40) with respect to the discrete transformations (21)–(23). It is easy to see that $\psi_{\varepsilon mn}$ is an eigenfunction of the operators P and $\Pi_2 T$:

$$[\psi_{\varepsilon mn}(t,\mathbf{x})]^P=(-1)^{m-1/2}\psi_{\varepsilon mn}(t,\mathbf{x}), \quad (41)$$

$$[\psi_{\varepsilon mn}(t,\mathbf{x})]^{\Pi_2 T}=(-1)^{(1+m/|m|)/2}\psi_{\varepsilon mn}(t,\mathbf{x}), \quad (42)$$

where the eigenvalues of the operators H and J_3 and the winding number of the $\mathbb{CP}^{N-1}$ soliton are indicated. At the same time, the action of the charge conjugation operator C transforms the wave function $\psi_{\varepsilon mn}$ into the wave function $\psi_{-\varepsilon-m-n}$ corresponding to a negative energy state, with opposite values of the quantum numbers m and n:

$$[\psi_{\varepsilon mn}(t,\mathbf{x})]^C=\psi_{-\varepsilon-m-n}(t,\mathbf{x}), \quad (43)$$

where

$$\psi_{-\varepsilon-m-n}(t,\mathbf{x})=\begin{pmatrix}e^{i(-m-1/2)\theta}g(\rho)\\ e^{i(-m+1/2)\theta}f(\rho)\end{pmatrix}e^{i\varepsilon t}. \quad (44)$$

Note that the permutation of the radial wave functions in Eq. (44) compared to Eq. (25) is equivalent to the replacements $\varepsilon \to -\varepsilon$, $m \to -m$, $n \to -n$, as follows from Eqs. (35)–(40). Equation (43) tells us that in the study of fermion-soliton systems, it is sufficient to restrict ourselves to fermionic ($\propto e^{-i\varepsilon t}$) solutions, since antifermionic ($\propto e^{i\varepsilon t}$) solutions are obtained from the fermionic ones via charge conjugation.

B. Existence of fermionic bound states

Consider the question of the existence of fermionic bound states in the background field of a $\mathbb{CP}^{N-1}$ soliton with winding number $n = \pm 1$. It is convenient to perform the substitution $f(\rho) = \tilde{\rho}^{-1/2} u(\tilde{\rho})$, $g(\rho) = \tilde{\rho}^{-1/2} v(\tilde{\rho})$ to give differential equations for the new radial functions as follows:

$$u''(\tilde{\rho}) - [\tilde{\varkappa}^2 + U(\tilde{\rho}, m, n)] u(\tilde{\rho}) = 0, \tag{45}$$

$$v''(\tilde{\rho}) - [\tilde{\varkappa}^2 + V(\tilde{\rho}, m, n)] v(\tilde{\rho}) = 0, \tag{46}$$

where $\tilde{\rho} = \rho/\lambda$, $\tilde{\varkappa}^2 = \lambda^2 (M^2 - \varepsilon^2)$, and the potentials

$$U(\tilde{\rho}, m, n) = \frac{m(m-1)}{\tilde{\rho}^2} + \frac{n(n-2m+1)}{1+\tilde{\rho}^2} - \frac{n(2+n)}{(1+\tilde{\rho}^2)^2}, \tag{47}$$

$$V(\tilde{\rho}, m, n) = \frac{m(m+1)}{\tilde{\rho}^2} + \frac{n(n-2m-1)}{1+\tilde{\rho}^2} + \frac{n(2-n)}{(1+\tilde{\rho}^2)^2} \tag{48}$$

do not depend on the scale parameter λ. Equations (45) and (46) have the form of a one-dimensional Schrödinger equation with the potentials (47) and (48), respectively. The quantity $-\tilde{\varkappa}^2$ plays the role of energy and must be negative for fermionic bound states. At the same time, the potentials U and V have second-order poles at $\tilde{\rho} = 0$, and tend to zero as $\tilde{\rho} \to \infty$. From the general properties [36] of the Schrödinger equation, it follows that for bound states to exist, U and V must take negative values. An analysis shows that both U and V have areas of negative values only for $m = 1/2$, $n = 1$ and $m = -1/2$, $n = -1$. For other values of m and n, at least one of U and V turns out to be positive for all $\tilde{\rho} \in (0, \infty)$, which makes the existence of bound fermionic states impossible.

Consider one of the possible cases, say $m = 1/2$, $n = 1$. Another possible case, $m = -1/2$, $n = -1$, is reduced to the previous one through the relation $U(\tilde{\rho}, m, n) = V(\tilde{\rho}, -m, -n)$. It follows from Eq. (47) that the potential $U(\tilde{\rho}, 1/2, 1) = -(2\tilde{\rho})^{-2} - 3(1+\tilde{\rho}^2)^{-2} + (1+\tilde{\rho}^2)^{-1}$. Furthermore, Eq. (45) admits a mechanical analogy; it describes the one-dimensional motion of a unit mass particle along the u-axis in time $\tilde{\rho}$. The motion of the particle occurs under the action of the time-dependent linear force $F(\tilde{\rho}) = (\tilde{\varkappa}^2 + U(\tilde{\rho}, 1/2, 1)) u(\tilde{\rho})$. Since for $m = 1/2$ the solution $u(\tilde{\rho}) \sim \tilde{\rho}^{1/2}$ as $\tilde{\rho} \to 0$, the particle has the coordinate $u = 0$ and possesses an infinite speed at the initial time $\tilde{\rho} = 0$. This infinite speed, however, is compensated by the action of the force $F(\tilde{\rho})$, which also tends to infinity as $\tilde{\rho} \to 0$.

We now consider the limiting case $\tilde{\varkappa}^2 = 0$, which corresponds to $\varepsilon = \pm M$. It is easy to see that in this case, the system of first-order differential equations in Eqs. (26) and (27) splits, and its solutions can therefore be expressed in terms of the elementary functions

$$\psi_{M\frac{1}{2}1} = \begin{pmatrix} (\lambda^2 + \rho^2)^{-1/2} \\ 0 \end{pmatrix} e^{-iMt}, \tag{49a}$$

$$\psi_{-M-\frac{1}{2}-1} = \begin{pmatrix} 0 \\ (\lambda^2 + \rho^2)^{-1/2} \end{pmatrix} e^{iMt}. \tag{49b}$$

Equations (49a) and (49b) are special cases of Eq. (40). It follows from Eq. (39) that the multiplier κ tends to zero (infinity) when $\varepsilon \to M$ and $m = 1/2$ ($\varepsilon \to -M$ and $m = -1/2$). The infinity arising in the upper (lower) component of fermionic wave function (40) is compensated, since the normalization factor $\mathcal{N}$ is proportional to $\kappa^{1/2}$ ($\kappa^{-1/2}$). As a result, the lower (upper) component of fermionic wave function (40) vanishes, and we arrive at Eq. (49a) [Eq. (49b)]. In this case, the confluent Heun function that corresponds to the nonzero component of the fermionic wave function (40) degenerates to a constant.

It follows from Eqs. (49a) and (49b) that at large distances from the soliton, the solutions $\psi_{\pm M \pm 1/2 \pm 1} \propto \rho^{-1}$. Hence, the solutions $\psi_{\pm M \pm 1/2 \pm 1}$ cannot be normalized, and therefore cannot be regarded as a part of the discrete spectrum of the Hamiltonian (19). From Eq. (49a), we obtain the solution

$$u_{M\frac{1}{2}1}(\tilde{\rho}) = \tilde{\rho}^{1/2} (1 + \tilde{\rho}^2)^{-1/2} \tag{50}$$

to Eq. (45). It follows from Eq. (50) that the solution $u_{M\frac{1}{2}1}(\tilde{\rho})$ increases monotonically from zero to $2^{-1/2}$ on the interval (0, 1) and then decreases monotonically to zero on the interval $(1, \infty)$. Note that the solution $u_{M\frac{1}{2}1}(\tilde{\rho}) \sim \tilde{\rho}^{-1/2}$ as $\tilde{\rho} \to \infty$.

Next, we define the effective potential $U_{\text{eff}}(\tilde{\rho}, \tilde{\varkappa}) = \tilde{\varkappa}^2 + U(\tilde{\rho}, 1/2, 1)$, where $\tilde{\varkappa}^2 = \lambda^2 (M^2 - \varepsilon^2)$ must be positive for fermionic bound states. The effective potential $U_{\text{eff}}(\tilde{\rho}, \tilde{\varkappa})$ increases monotonically from $-\infty$ to $0.0428454 + \tilde{\varkappa}^2$ on the interval (0, 2.79921), and then decreases monotonically to $\tilde{\varkappa}^2$ on the interval $(2.79921, \infty)$. In addition, $U_{\text{eff}}(\tilde{\rho}, \tilde{\varkappa})$ vanishes at $\tilde{\rho} = \tilde{\rho}_0(\tilde{\varkappa})$, where $\tilde{\rho}_0(0) \approx 1.85216$ and $\tilde{\rho}_0(\tilde{\varkappa})$ decreases monotonically with an increase in $\tilde{\varkappa}$. It follows that the force $F(\tilde{\rho}) = U_{\text{eff}}(\tilde{\rho}, \tilde{\varkappa}) u(\tilde{\rho})$ is attractive when $\tilde{\rho} \in (0, \tilde{\rho}_0(\tilde{\varkappa}))$, and is repulsive when $\rho \in (\tilde{\rho}_0(\tilde{\varkappa}), \infty)$. This means that in order to correspond to a ground state of energy ε_0, the trajectory $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho})$ of the particle must reach a maximum at some point $\tilde{\rho}_{\text{max}} < \tilde{\rho}_0(\tilde{\varkappa}_0)$ in the region of attraction and then decrease monotonically, tending to zero as $\tilde{\rho} \to \infty$. The monotonic decrease of $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho})$ is due to the fact that the radial wave function of the ground state has no nodes. In addition, Eq. (45) tells us that $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho}) \sim \exp(-\tilde{\varkappa}_0 \tilde{\rho})$ as $\tilde{\rho} \to \infty$, where $\tilde{\varkappa}_0^2 = \lambda^2 (M^2 - \varepsilon_0^2)$.

As $\tilde{\varkappa}^2$ increases, the region of attraction $(0, \tilde{\rho}_0(\tilde{\varkappa}))$ of $U_{\text{eff}}(\tilde{\rho}, \tilde{\varkappa})$ decreases while the region of repulsion $(\tilde{\rho}_0(\tilde{\varkappa}), \infty)$ increases. Since the force $F(\tilde{\rho}) = U_{\text{eff}}(\tilde{\rho}, \tilde{\varkappa}) u(\tilde{\rho}) = (\tilde{\varkappa}^2 + U(\tilde{\rho}, 1/2, 1)) u(\tilde{\rho})$, the attraction force decreases and the repulsion force increases with the growth of $\tilde{\varkappa}^2$. We can normalize the wave function $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho})$ of the assumed bound state by the condition $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho}) / u_{M \frac{1}{2} 1}(\tilde{\rho}) \to 1$ as $\tilde{\rho} \to 0$. It then follows from the above that the trajectories $u_{M \frac{1}{2} 1}(\tilde{\rho})$ and $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho})$ must satisfy the inequality $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho}) > u_{M \frac{1}{2} 1}(\tilde{\rho})$. Recall, however, that $u_{M \frac{1}{2} 1}(\tilde{\rho}) \sim \tilde{\rho}^{-1/2}$ and $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho}) \sim \exp(-\tilde{\varkappa}_0 \tilde{\rho})$ as $\tilde{\rho} \to \infty$, and hence the ratio $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho}) / u_{M \frac{1}{2} 1}(\tilde{\rho})$ must tend to zero in this limit, which contradicts the condition $u_{\varepsilon_0 \frac{1}{2} 1}(\tilde{\rho}) > u_{M \frac{1}{2} 1}(\tilde{\rho})$. We can conclude that there are no bound fermionic states with quantum numbers $m = 1/2$, $n = 1$. It follows that there are no bound fermionic states in the background field of a $\mathbb{CP}^{N-1}$ soliton with $n = \pm 1$.

C. General formalism for fermion scattering

We now turn to the description of fermion scattering in the background field of a $\mathbb{CP}^{N-1}$ soliton. For a fermion with initial momentum $\mathbf{k} = (k, 0)$, according to the general principles of the theory of scattering [36,37], the asymptotics of the wave function of the fermionic scattering state has the form

$$\Psi \sim \psi_{\varepsilon,\mathbf{k}} + \frac{1}{\sqrt{2\varepsilon}} u_{\varepsilon,\mathbf{k}'} f(k,\theta) \frac{e^{ik\rho}}{\sqrt{-i\rho}}, \tag{51}$$

where

$$\psi_{\varepsilon,\mathbf{k}} = \frac{1}{\sqrt{2\varepsilon}} \begin{pmatrix} \sqrt{\varepsilon + M} \\ i\sqrt{\varepsilon - M} \end{pmatrix} e^{-ikx} \tag{52}$$

is the wave function of the incoming fermion with momentum $\mathbf{k} = (k, 0)$,

$$u_{\varepsilon,\mathbf{k}'} = \begin{pmatrix} \sqrt{\varepsilon + M} \\ i\sqrt{\varepsilon - M} e^{i\theta} \end{pmatrix} \tag{53}$$

is the spinor amplitude of the wave function of the outgoing fermion with momentum $\mathbf{k}' = (k\cos(\theta), k\sin(\theta))$, and $f(k,\theta)$ is the scattering amplitude.

The scattering amplitude $f(k,\theta)$ can be expanded in terms of the partial scattering amplitudes $f_m(k)$ as

$$f(k,\theta) = \sum_m f_m(k) e^{i(m-1/2)\theta}, \tag{54}$$

where the summation is taken over the half-integer eigenvalues of the angular momentum (24). The partial scattering amplitudes can in turn be written in terms of the partial elements of the S-matrix as

$$f_m(k) = \frac{1}{i\sqrt{2\pi k}} (S_m(k) - 1). \tag{55}$$

Similarly, the wave function (51) can also be decomposed into partial waves as $\Psi = \sum_m \psi_m$. The asymptotic behavior of the partial waves can be expressed in terms of the partial elements of the S-matrix as follows:

$$\psi_m \sim \frac{(-1)^{1/4}}{\sqrt{2\pi k\rho}} \begin{pmatrix} -i\sqrt{\frac{\varepsilon+M}{2\varepsilon}} [i(-1)^{m-1/2} e^{-ik\rho} + S_m e^{ik\rho}] e^{i(m-1/2)\theta} \\ \sqrt{\frac{\varepsilon-M}{2\varepsilon}} [i(-1)^{m+1/2} e^{-ik\rho} + S_m e^{ik\rho}] e^{i(m+1/2)\theta} \end{pmatrix}. \tag{56}$$

Using standard methods from the theory of scattering [36,37], we can write the differential cross section for the elastic fermion scattering in terms of the scattering amplitude $f(k,\theta)$ as

$$d\sigma/d\theta = |f(k,\theta)|^2. \tag{57}$$

In turn, the partial cross sections for the elastic fermion scattering are expressed in terms of the partial scattering amplitudes as

$$\sigma_m = 2\pi |f_m(k)|^2 = k^{-1} |S_m(k) - 1|^2. \tag{58}$$

Note that in $(2+1)$ dimensions, the cross sections $d\sigma/d\theta$ and σ_m have the dimension of length [36]. The unitarity of the S-matrix, $SS^\dagger = S^\dagger S = \mathbb{I}$, results in the unitarity condition for the partial S-matrix elements, $|S_m(k)| = 1$. This condition allows us to express the partial S-matrix elements S_m in terms of the partial phase shifts δ_m as

$$S_m(k) = e^{2i\delta_m(k)}. \tag{59}$$

The scale invariance of the bosonic sector of the model (1) leads to the existence of the parameter λ, which determines the effective size of the soliton solution (8), and hence affects the fermion-soliton scattering. It follows from a dimensional analysis and Eqs. (35)–(40) that the dependence of S_m on the momentum $k = (\varepsilon^2 - M^2)^{1/2}$ and the scale parameter λ enters only through the dimensionless combination $\tilde{k} = k\lambda$.

The phase shifts $\delta_m(\tilde{k})$ are determined only by the arguments of the confluent Heun functions in Eq. (40), and do not depend on the prefactors. These arguments depend on the parameter ε only through the momentum squared $k^2 = \varepsilon^2 - M^2$, i.e., only through ε^2. The charge conjugation in Eq. (21) is reduced to a permutation of the wave function components and to their complex conjugation, which cannot change the phase shifts. This is because all arguments of the confluent Heun functions in Eq. (40) are real, meaning that these functions are also real, and therefore do not change under the complex conjugation. From this and Eq. (43), we come to the conclusion that the phase shifts satisfy the relation

$$\delta_{mn}(\tilde{k}) = \delta_{-m-n}(\tilde{k}), \tag{60}$$

where the dependence of the phase shift on the soliton winding number $n = \pm 1$ is indicated.

IV. FERMION SCATTERING IN THE BORN APPROXIMATION

In Sec. III, we were able to obtain the analytical expression (40) for the fermionic wave functions in the background field of the $\mathbb{CP}^{N-1}$ soliton for winding numbers $n = \pm 1$. The next step would be to obtain an exact analytical expression for the scattering amplitude (54). To do this, according to Eqs. (55) and (59), we need to know exact analytical expressions for the partial phase shifts $\delta_m(k)$. However, unlike the well-studied Bessel functions, exact analytical expressions $\delta_m(k)$ are unknown for the confluent Heun functions appearing in Eq. (40). Hence, we cannot obtain an exact analytical expression for the scattering amplitude. In view of this, it is important to study the fermion scattering in the Born approximation, which gives us a chance to obtain an approximate analytical expression for the scattering amplitude.

It follows from Eqs. (1) and (2) that the fermion-soliton interaction is described by the potential term

$$V_{\text{int}} = \bar{\psi}_a \gamma^\mu A_\mu \psi_a. \tag{61}$$

In the background field approximation, the gauge field A_μ defined by Eq. (9) does not depend on the fermion fields ψ_a. It follows from this and Eq. (61) that all components of the fermionic multiplet $(\psi_1, \ldots, \psi_N)$ interact with the $\mathbb{CP}^{N-1}$ soliton in the same way and independently of each other.

Using Eq. (61), we can write the first-order Born amplitude for the fermion-soliton scattering as follows:

$$f(\mathbf{k}', \mathbf{k}) = -(8\pi k)^{-1/2} \bar{u}_{\varepsilon,\mathbf{k}'} \gamma^\mu A_\mu(\mathbf{q}) u_{\varepsilon,\mathbf{k}}, \tag{62}$$

where

$$A_\mu(\mathbf{q}) = \int A_\mu(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}} d^2x \tag{63}$$

and $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the momentum transfer. The Born amplitude (62) can be expressed in an analytical form. For winding numbers $|n| \geq 2$, the Born amplitude is expressed in terms of the Meijer G-functions [38]; however, for winding numbers $n = \pm 1$, corresponding to the elementary $\mathbb{CP}^{N-1}$ solitons, the Born amplitude can be written in terms of modified Bessel functions of the second kind:

$$f(\mathbf{k}', \mathbf{k}) = \mathrm{i}n\sqrt{2\pi} k^{1/2} \lambda \operatorname{sign}(\vartheta_2 - \vartheta_1) e^{-i(\vartheta_2 - \vartheta_1)/2} \mathrm{K}_1(q\lambda), \tag{64}$$

where the angle ϑ_1 (ϑ_2) defines the direction of motion of the "in" ("out") fermion, $q = 2k \sin(|\vartheta_2 - \vartheta_1|/2)$ is the magnitude of the momentum transfer, and $n = \pm 1$ is the winding number of the $\mathbb{CP}^{N-1}$ soliton. The amplitude of antifermion scattering differs only in terms of its sign from the amplitude of fermion scattering in Eq. (64). Using known criteria [36,37], it can be shown that the Born approximation is applicable under the following conditions

$$k\lambda \gg 1 \quad \text{and} \quad |\vartheta_2 - \vartheta_1| \ll (k\lambda)^{-1/2} \ll 1. \tag{65}$$

It follows from Eq. (65) that the Born approximation is suitable for describing the low-angle scattering of high-energy fermions.

Equation (64) tells us that the amplitude $f(\mathbf{k}', \mathbf{k})$ is Hermitian with respect to the permutation of the fermion momenta

$$f(\mathbf{k}', \mathbf{k}) = f^*(\mathbf{k}, \mathbf{k}'), \tag{66}$$

as it should be in the Born approximation [36,37]. Another symmetry relation

$$f(\mathbf{k}', \mathbf{k}) = f(\tilde{\mathbf{k}}, \tilde{\mathbf{k}}'), \tag{67}$$

where $\tilde{\mathbf{k}} = (k_x, -k_y)$ and $\tilde{\mathbf{k}}' = (k'_x, -k'_y)$, follows from the invariance of the Dirac equation (18) under the $\Pi_2 T$ transformation in Eq. (23). As already mentioned, the scattering amplitude for antifermions is obtained from Eq. (64) by the replacement $n \to -n$. It follows that the scattering of an antifermion on the $\mathbb{CP}^{N-1}$ soliton with winding number $n = \pm 1$ is equivalent to the scattering of a fermion on the $\mathbb{CP}^{N-1}$ soliton with winding number $n = \mp 1$, which corresponds to Eq. (43).

Using Eq. (64) and the known asymptotic forms of the modified Bessel function $\mathrm{K}_1\ (q\lambda)$, we can study the behavior of the Born amplitude for large and small values of the momentum transfer q. For large momentum transfers, we find that the Born amplitude

$$f \sim \mathrm{i}n\pi(\lambda/2)^{1/2}\mathrm{sign}(\vartheta_2-\vartheta_1)e^{-i(\vartheta_2-\vartheta_1)/2} \times e^{-\lambda q}\sin(|\vartheta_2-\vartheta_1|/2)^{-1/2}, \tag{68}$$

where the angles ϑ_1 and ϑ_2 are fixed and $\vartheta_1 \neq \vartheta_2$. It follows from Eq. (68) that the Born amplitude decreases exponentially with an increase in the dimensionless combination λq. We now consider the case of low momentum transfer q and high fixed fermion momentum k, which corresponds to small scattering angles $\Delta\vartheta \equiv |\vartheta_2-\vartheta_1| = 2\arcsin[q/(2k)] \approx q/k$. In this case, the asymptotics of the Born amplitude is

$$f \sim \mathrm{i}n\sqrt{2\pi}k^{1/2}q^{-1} + n\sqrt{\pi/2}k^{-1/2}. \tag{69}$$

We see that in the limit of small q, the Born amplitude diverges as q^{-1}. Furthermore, unlike Eq. (68), the leading asymptotic terms shown in Eq. (69) do not depend on the parameter λ determining the soliton size.

Next, we turn to the study of the partial amplitudes $f_m(k) = (2\pi)^{-1}\int_0^{2\pi} e^{-i(m-1/2)\vartheta}f(k,\vartheta)d\vartheta$ corresponding to the Born amplitude (64). The imaginary part of the integrand diverges as $\vartheta \to 0, 2\pi$ and is odd with respect to $\vartheta = \pi$, and hence the corresponding integral vanishes in the sense of the principal value. The real part of the integrand is finite and even with respect to $\vartheta = \pi$, meaning that the corresponding integral exists and is nonzero. From this result, it is easy to show that the partial Born amplitudes are odd under the replacement $m \to -m$, i.e.,

$$f_m(k) = -f_{-m}(k). \tag{70}$$

The partial Born amplitudes $f_m(k)$ can be expressed in terms of the Meijer G-functions [38]. These expressions, however, can be significantly simplified in the limit of large fermion momenta k as

$$f_m \sim nm\sqrt{\pi/2}\lambda^{-1}k^{-3/2}. \tag{71}$$

From Eqs. (55) and (71), we can obtain asymptotic forms for the partial S-matrix elements and phase shifts in the Born approximation:

$$S_m = e^{2i\delta_m} \sim 1 + i\frac{nm\pi}{k\lambda} \tag{72}$$

and

$$\delta_m \sim \frac{nm\pi}{2k\lambda}. \tag{73}$$

It follows from Eq. (73) that the phase shifts δ_m tend to zero as $k \to \infty$, which is consistent with the basic principles of scattering theory [36,37].

Using Eq. (64), we obtain an expression for the differential cross section of the fermion scattering in the Born approximation

$$d\sigma/d\vartheta = 2\pi k\lambda^2\mathrm{K}_1(2k\lambda\sin(\vartheta/2))^2. \tag{74}$$

We see that $d\sigma/d\vartheta \sim 2\pi k^{-1}\vartheta^{-2}$ as $\vartheta \to 0$. At the same time, $d\sigma/d\vartheta \sim 2\pi k^{-1}(2\pi-\vartheta)^{-2}$ as $\vartheta \to 2\pi$. It follows that the total cross section $\sigma = \int_0^{2\pi}(d\sigma/d\vartheta)d\vartheta$ of the fermion-soliton scattering diverges at the lower and upper limits of the integral. However, the transport cross section $\sigma_{\mathrm{tr}} = \int_0^{2\pi}(1-\cos(\vartheta))(d\sigma/d\vartheta)d\vartheta$ is finite, and can be expressed in terms of the Meijer G-functions, defined according to Ref. [38], as

$$\sigma_{\mathrm{tr}} = 4\pi^2k\lambda^2G^{3,1}_{2,4}\left(4k^2\lambda^2\middle|\begin{matrix}-\frac{1}{2},\frac{1}{2}\\-1,0,1,-1\end{matrix}\right). \tag{75}$$

Using known asymptotic expansion for the Meijer G-function, we obtain the asymptotics of the transport cross section (75) for large fermion momentum k as

$$\sigma_{\mathrm{tr}} \sim \frac{3\pi^3}{16k^2\lambda} + O(\lambda^{-3}k^{-4}). \tag{76}$$

V. NUMERICAL RESULTS

In Sec. III, we found an exact solution for the fermionic wave function in the background field of the $\mathbb{CP}^{N-1}$ soliton with winding number $n = \pm 1$. This exact solution is expressed in terms of the confluent Heun functions [33–35]. We now want to find the partial phase shifts $\delta_m(\tilde{k})$ for a range of values for the dimensionless combination $\tilde{k} = k\lambda$, as these will give the most complete description of the fermion scattering. Since there is no analytic form for the asymptotics of the confluent Heun function in the region of large ρ, we need to use numerical methods to solve this problem.

The exact solution (40) and the general asymptotic form (56) are two-component spinors. Let us define the ratio of the spinor components taken at two successive points ρ and $\rho + \Delta\rho$ as

$$r^i_{\varepsilon mn}(\rho,\Delta\rho) = \psi^i_{\varepsilon mn}(\rho+\Delta\rho)/\psi^i_{\varepsilon mn}(\rho), \tag{77}$$

where the index $i = 1, 2$ numbers the spinor components. For sufficiently large ρ, the exact solution (40) tends to the general asymptotic form (56). It follows that in this case, the ratio $r^i_{\varepsilon mn}$ calculated with Eq. (40) must be close to that calculated with Eq. (56). Equating these two ratios calculated for some $\rho \gg \lambda$ and $\Delta\rho \sim \lambda$, we obtain an

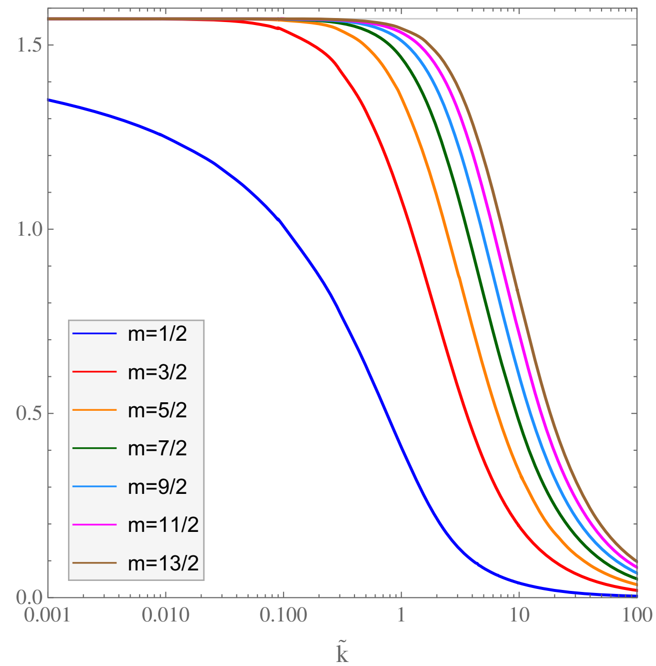

FIG. 1. Dependence of the phase shifts δ_{mn} on the dimensionless combination $\tilde{k} = k\lambda$ for $m = 1/2, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2$, and $n = 1$.

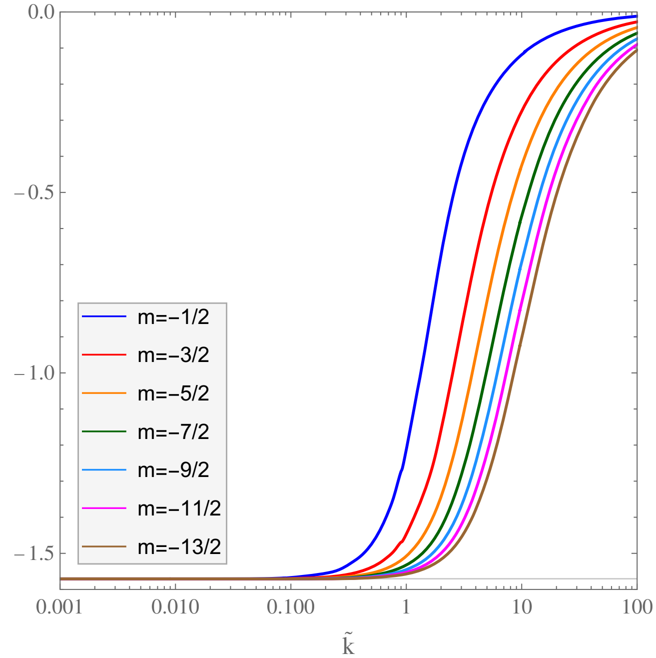

FIG. 2. Dependence of the phase shifts δ_{mn} on the dimensionless combination $\tilde{k} = k\lambda$ for $m = -1/2, -3/2, -5/2, -7/2, -9/2, -11/2, -13/2$, and $n = 1$.

approximate equation to determine the partial S-matrix element $S_m = \exp(2i\delta_m)$ in Eq. (56). Since we can use both $r^1_{\varepsilon mn}$ and $r^2_{\varepsilon mn}$ for this purpose, we have two approximate equations determining S_m. When $\rho \gg \lambda$, $k\rho \gg 1$, and $\Delta\rho \sim \lambda$, the solutions to these two equations become very close to each other, and tend to the same limit as $\rho \to \infty$. We used the arithmetic mean of these two solutions as a numerical value for S_m. To calculate the confluent Heun functions for large values of their arguments, the highly efficient numerical algorithms of the *Mathematica* [39] and MAPLE [40] software packages were used.

Figure 1 shows the dependences of the phase shifts δ_{mn} on the dimensionless combination $\tilde{k} = k\lambda$ for the angular momentum eigenvalues $m = 1/2, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2$, and the soliton winding number $n = 1$. Similarly, Fig. 2 shows the curves $\delta_{mn}(\tilde{k})$ for $m = -1/2, -3/2, -5/2, -7/2, -9/2, -11/2, -13/2$, and $n = 1$. Equation (60) tells us that the curves $\delta_{m-1}(\tilde{k}) = \delta_{-m1}(\tilde{k})$, and these can therefore be obtained from the curves shown in Figs. 1 and 2. We checked Eq. (60) using numerical methods. It follows from Eq. (60) and Figs. 1 and 2 that $\delta_{mn}(0) = \mathrm{sign}(mn)\pi/2$. Note that in the case of short-range forces, the phase shifts vanish if the momentum of a scattered particle tends to zero [36]. In our case, the nonzero value of $\delta_{mn}(0)$ is caused by the long-range ($\propto \rho^{-1}$) character of the gauge field (9).

In the following, we discuss this issue in more detail. Since the phase shifts depend on the dimensionless combination $\tilde{k} = k\lambda$, the regime of small k is equivalent to the regime of small λ. Due to the long-range character of the gauge field A_μ, the function $A_{mn}(\rho)$ included in the system of differential equations (26) and (27) tends to a constant value $m - n$ as $\lambda \to 0$. As a result, the system of differential equations (26) and (27) is simplified, and its solution can be expressed in terms of Bessel functions $J_{m-n\pm1/2}(k\rho)$. The free motion of fermions corresponds to $n = 0$ (the absence of a $\mathbb{CP}^{N-1}$ soliton). In this case, the function $A_{m0}(\rho) = m$, and the solution to the system of differential equations (26) and (27) is expressed in terms of Bessel functions $J_{m\pm1/2}(k\rho)$. Using the well-known asymptotic expansions of the Bessel functions, it is easy to show that the phase shift between $J_{m-n\pm1/2}(k\rho)$ and $J_{m\pm1/2}(k\rho)$ is $\pi n/2$. This can be regarded as the phase shift at zero $\tilde{k}$, and can be written as $\mathrm{sign}(n)\pi/2$ for $n = \pm1$. This expression is compatible with the result $\delta_{mn}(0) = \mathrm{sign}(mn)\pi/2$, as the phase shifts are defined modulo π.

Using analytical and numerical methods, we were able to establish the behavior of the phase shifts δ_{mn} in the region of small $\tilde{k}$ as

$$\delta_{mn}(\tilde{k}) \approx \begin{cases} \frac{\pi}{2} + \frac{\pi}{\ln(\tilde{k}^2)} & \text{if } mn = \frac{1}{2} \\ s_{mn}\frac{\pi}{2} + \alpha_{mn}\tilde{k}^{2\beta_{mn}} & \text{if } mn \neq \frac{1}{2} \end{cases}, \qquad (78)$$

where $\tilde{k} = k\lambda$, $s_{mn} = \mathrm{sign}(mn)$, $\beta_{mn} = |m| + 1/2 - s_{mn}$, and α_{mn} are coefficients satisfying the condition

$\alpha_{mn} = \alpha_{-m-n}$. Based on Eqs. (55), (59), and (78), we can write the corresponding expressions for the partial amplitudes as

$$f_{mn} \approx \begin{cases} \sqrt{\frac{2}{\pi k}}\left[i - \frac{2}{\ln(\tilde{k}^2)}\right] & \text{if } mn = 1/2 \\ \sqrt{\frac{2}{\pi k}}[i - \alpha_{mn}\tilde{k}^{2\beta_{mn}}] & \text{if } mn \neq 1/2 \end{cases} \tag{79}$$

We see that as $\tilde{k} \to 0$, all partial amplitudes tend to the same limiting form $i\sqrt{2/(\pi k)}$. Accordingly, the partial cross sections $\sigma_{mn} = 2\pi|f_{mn}|^2$ tend to $4k^{-1}$, and hence attain the unitary bound in this limit. Note that all partial waves make the same contribution to the fermion scattering when $\tilde{k} = k\lambda \to 0$. This is due to the fact that $|\delta_{mn}(0)| = \pi/2$ for all m.

Let us define $\tilde{k}_{1/2}$ as the value of $\tilde{k}$ at which $|\delta_{mn}|$ takes the value of $\pi/4$, i.e., half of the maximum value $\pi/2$. We have established numerically that the dependence of the parameter $\tilde{k}_{1/2}$ on the eigenvalue m of the angular momentum has the approximate linear form

$$\tilde{k}_{1/2} = k_{1/2}\lambda \approx 1.74|m|. \tag{80}$$

As $|m|$ grows, the main contribution to the angular momentum in Eq. (24) comes from its orbital part. Equation (80) then tells us that the orbital part of the angular momentum is approximately proportional to the fermion momentum $k_{1/2}$ and the linear size λ of the soliton, which is consistent with classical conceptions.

It follows from Figs. 1 and 2 that for fixed $\tilde{k}$, the absolute values of δ_{mn} increase with an increase in $|m|$. This is true for both positive and negative m. Using the formula $\sigma_{mn} = 4k^{-1}\sin^2(\delta_{mn})$, we conclude that the partial cross sections σ_{mn} behave similarly. We see that for all values of k, the contribution of partial waves to the fermion-soliton scattering increases with an increase in $|m|$. This is because the long-range gauge field (9) of the $\mathbb{CP}^{N-1}$ soliton makes a significant contribution to the fermion scattering, even at large distances from the soliton. As $\tilde{k}$ increases, $|\delta_{mn}|$ decreases monotonically and tends to zero as $\tilde{k} \to \infty$. We have found numerically that in this limit, the phase shifts

$$\delta_{mn}(\tilde{k}) \approx \frac{\pi}{2\tilde{k}}\left(nm - \frac{1}{4}\right). \tag{81}$$

These features of the curves $\delta_{mn}(\tilde{k})$ can be understood in the framework of the quasiclassical approximation. Using methods of scattering theory [36], it can be shown that for sufficiently large $|m|$ and $\tilde{k}$, the fermion-soliton scattering is quasiclassical. There is an approximate quasiclassical expression for the phase shifts, which in our case can be written as

$$\delta_{mn}(\tilde{k}) \approx \int\limits_{\tilde{\rho}_0}^{\infty}\left[\tilde{k}^2 - \frac{(m-1/2)^2}{\tilde{\rho}^2} - W(\tilde{\rho}, m, n)\right]^{1/2} d\tilde{\rho} - \int\limits_{\tilde{\rho}_0}^{\infty}\left[\tilde{k}^2 - \frac{(m-1/2)^2}{\tilde{\rho}^2}\right]^{1/2} d\tilde{\rho}, \tag{82}$$

where the potential

$$W(\tilde{\rho}, m, n) = \frac{n(n-2m+1)}{1+\tilde{\rho}^2} - \frac{n(2+n)}{(1+\tilde{\rho}^2)^2} \tag{83}$$

and the lower limit of integration

$$\tilde{\rho}_0 \approx |m - 1/2|\tilde{k}^{-1}. \tag{84}$$

If quasiclassical conditions are fulfilled, then the potential W will be small compared to the term $\tilde{k}^2 - (m-1/2)^2\tilde{\rho}^{-2}$ in the region making the main contribution to the first integral in Eq. (82). Expanding the integrand of the first integral in W and keeping the first expansion term, we can obtain an approximate analytical expression for the phase shifts as

$$\delta_{mn}(\tilde{k}) \approx \frac{\pi n}{2}[(2m-n-1)(2\tilde{k}^2 + (2m-1)^2) + 2\tilde{k}^2(2m+1)][4\tilde{k}^2 + (2m-1)^2]^{-\frac{3}{2}}. \tag{85}$$

From Eq. (85), we can obtain two asymptotic expressions for the phase shifts. The first is valid for $\tilde{k} \to \infty$ and fixed m, and coincides with Eq. (81). The second is valid for $|m| \to \infty$ and fixed $\tilde{k}$, and has the form

$$\delta_{mn}(\tilde{k}) \sim s_{mn}\frac{\pi}{2} - \frac{\pi}{4|m|}, \tag{86}$$

where the factor $s_{mn} = \text{sign}(mn)$. Thus, the asymptotic behavior in Eq. (81) can be obtained within the quasiclassical approximation. Furthermore, Eq. (86) tells us that for fixed $\tilde{k}$, the phase shifts $\delta_{mn}(\tilde{k}) \to s_{mn}\pi/2$ as $|m| \to \infty$, which is consistent with the numerical results. It follows that the partial cross sections $\sigma_{mn} = 4k^{-1}\sin^2(\delta_{mn})$ reach the unitary bound $4k^{-1}$ as $|m| \to \infty$.

As already noted, this behavior of the phase shifts $\delta_{mn}(\tilde{k})$ is due to the slow ($\propto \rho^{-1}$) decrease of the gauge field (9) far from the soliton. This behavior of the gauge field leads to the long-range asymptotics $W \sim n(n-2m-1)\tilde{\rho}^{-2}$ of the quasiclassical potential (83). It is this asymptotic behavior of W that leads to the fact that the phase shifts $\delta_{mn}(\tilde{k}) \to s_{mn}\pi/2$ as $|m| \to \infty$. Indeed, a faster decrease in the gauge field (9) leads to a faster decrease in the quasiclassical potential (83). It can be shown, however, that if the quasiclassical potential W decreases more rapidly than $\tilde{\rho}^{-2}$, the quasiclassical phase shift (82) tends to zero as

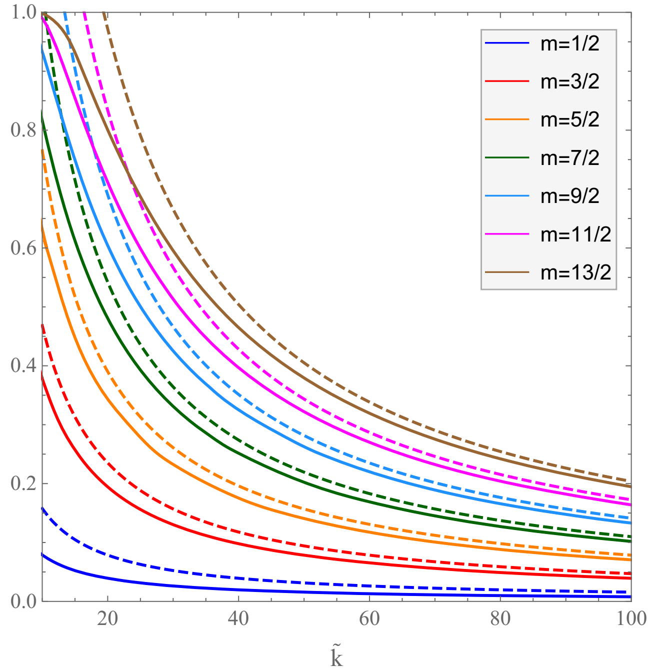

FIG. 3. Dependence of $\sqrt{2\pi k}\mathrm{Re}[f_{mn}]$ on the dimensionless combination $\tilde{k} = k\lambda$ for $m = 1/2, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2$, and $n = 1$. The solid curves correspond to the exact solution (40), and the dashed curves correspond to the Born approximation (64).

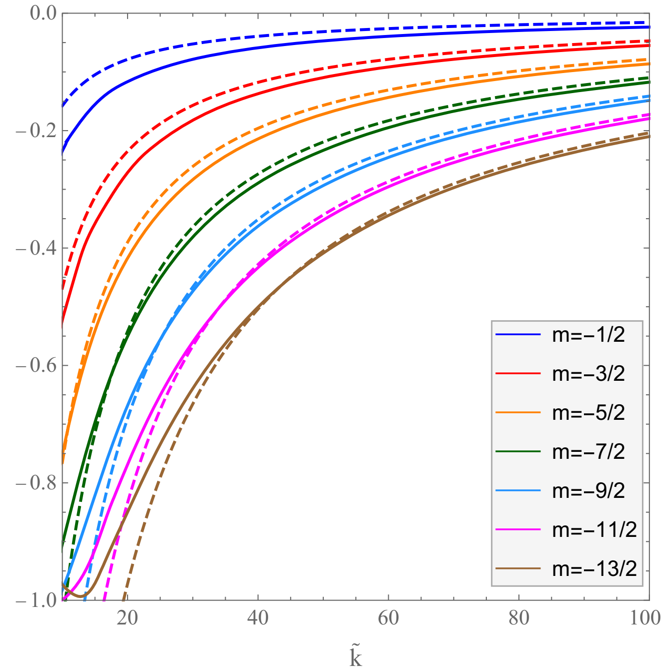

FIG. 4. Dependence of $\sqrt{2\pi k}\mathrm{Re}[f_{mn}]$ on the dimensionless combination $\tilde{k} = k\lambda$ for $m = -1/2, -3/2, -5/2, -7/2, -9/2, -11/2, -13/2$, and $n = 1$. The solid curves correspond to the exact solution (40), and the dashed curves correspond to the Born approximation (64).

$|m| \to \infty$. In this case, the contribution of partial waves with sufficiently large $|m|$ to the fermion scattering becomes negligibly small.

It follows from the results obtained that the difference in the phase shifts is

$$\delta_{mn}(0) - \delta_{mn}(\infty) = s_{mn}\pi/2. \tag{87}$$

This contradicts Levinson's theorem [41], according to which this difference must be equal to π multiplied by the number of bound fermionic states in the partial channel with given values of m and n. Since there are no bound fermionic states in our case, the difference $\delta_{mn}(0) - \delta_{mn}(\infty)$ must be equal to zero, which contradicts Eq. (87). The reason for this is that one of the conditions for the applicability of Levinson's theorem is a rather fast decrease (faster than ρ^{-3}) in the potential term at infinity [37]. In our case, the slow decrease ($\sim\rho^{-2}$) of the potential in Eq. (83) for large ρ makes Levinson's theorem inapplicable.

It follows from the results in Sec. IV that in the Born approximation, the partial amplitudes f_{mn} are real. At the same time, Eqs. (55) and (59) tell us that $\mathrm{Im}[f_{mn}] = \sqrt{\pi k/2}|f_{mn}|^2 > 0$. We see that according to scattering theory [36,37], unitarity is broken in the Born approximation. However, it follows from Eqs. (55), (59), and (81) that $\mathrm{Im}[f_{mn}] \sim \pi^2(1-4mn)^2(32\tilde{k}^2)^{-1}(2\pi k)^{-1/2}$, and hence tends to zero $\propto k^{-5/2}$ as $k \to \infty$. Consequently, the Born approximation becomes applicable in the region of large fermion momenta k.

It was shown in Sec. III that the phase shifts δ_{mn} depend only on the dimensionless combination $\tilde{k} = k\lambda$. Equations (55) and (59) then tell us that the dimensionless combinations $\sqrt{2\pi k}f_{mn}$ also depend only on $\tilde{k}$. Figure 3 shows the dependences of $\sqrt{2\pi k}\mathrm{Re}[f_{mn}]$ on $\tilde{k}$ for the first few positive eigenvalues m of the angular momentum and the soliton winding number $n = 1$. In Fig. 3, the solid curves correspond to the exact solution (40), and the dashed curves correspond to the Born approximation (64). Similar curves for negative eigenvalues m are shown in Fig. 4.

From Figs. 3 and 4, it follows that the accuracy of the Born approximation improves with an increase in $\tilde{k}$. At the same time, a comparison of Eqs. (73) and (81) shows that even for large $\tilde{k}$, the Born phases differ from those obtained numerically (or within the quasiclassical approximation) by a shift of $\pi/(8\tilde{k})$. This difference is due to the violation of unitarity in the Born approximation, and becomes insignificant with an increase in $|m|$. Note that the Born partial amplitudes change sign under the replacement $m \to -m$, which is a consequence of Eq. (70). This property, however, is true only in the Born approximation, and is lost when we pass to the exact partial amplitudes. Instead, the exact partial amplitudes satisfy the condition $f_{mn}(k) = f_{-m-n}(k)$, which is a consequence of the general symmetry relation (60).

VI. CONCLUSION

In this paper, we have investigated fermion scattering on topological solitons of the $(2+1)$-dimensional $\mathbb{CP}^{N-1}$ model in the framework of the background field approximation. In particular, we found exact solutions to the Dirac equation describing fermionic states in the background fields of $\mathbb{CP}^{N-1}$ solitons with winding numbers $n = \pm 1$. It turns out that these exact solutions can be expressed in terms of the confluent Heun functions. The symmetry properties of the fermionic wave functions under discrete transformations of the Dirac equation were found, which allowed us to establish the discrete symmetry property of the phase shifts. We studied the presence of fermionic bound states in the background fields of the $\mathbb{CP}^{N-1}$ solitons with winding numbers $n = \pm 1$, and came to the conclusion that there are no such states.

Within the framework of the background field approximation, the process of fermion-soliton scattering is elastic, and can therefore be fully described in terms of phase shifts. However, the absence of analytical asymptotics for the confluent Heun functions makes it impossible to obtain analytical expressions for the phase shifts. In view of this, we studied the fermion-soliton scattering in the Born approximation, which gave us the opportunity to obtain an approximate analytical expressions for the phase shifts, scattering amplitudes, and differential cross sections, and to study their asymptotic forms. We found that the total cross section of the fermion-soliton scattering diverges due to the long-range character of the soliton field. However, the transport cross section of the fermion-soliton scattering turns out to be finite, and can be expressed in terms of the Meijer G-functions.

We have also performed a numerical study of fermion scattering in the background fields of the $\mathbb{CP}^{N-1}$ solitons with winding numbers $n = \pm 1$. In particular, it was found that the phase shifts δ_{mn} depend only on the dimensionless combination $\tilde{k} = k\lambda$, and the curves $\delta_{m\pm 1}(\tilde{k})$ were obtained for $|m| \leq 13/2$. The main feature of the curves $\delta_{mn}(\tilde{k})$ is that they tend to a nonzero value $\delta_{mn}(0) = \mathrm{sign}(mn)\pi/2$ as $\tilde{k} \to 0$. At the same time, the phase shifts $\delta_{mn}(\tilde{k})$ tend to zero $\propto \tilde{k}^{-1}$ as $\tilde{k} \to \infty$. The nonzero value of the difference $\delta_{mn}(0) - \delta_{mn}(\infty)$ in spite of the absence of bound fermionic states is related to the long-range gauge field (9) of the $\mathbb{CP}^{N-1}$ soliton.

We have found that as $|m|$ increases, the curves $\delta_{mn}(\tilde{k})$ shift to the region of larger $\tilde{k}$. Using the quasiclassical approximation, we have shown that the phase shifts $\delta_{mn}(\tilde{k})$ tend to $\mathrm{sign}(mn)\pi/2$ as $|m|$ tends to ∞, which is consistent with our numerical results. It follows that partial waves with arbitrarily large $|m|$ make a significant contribution to fermion scattering at any value of $\tilde{k}$ (including small values). This feature of fermion-soliton scattering is also related to the long-range character of the gauge field in Eq. (9).

The nonlinear $O(N+1)$ sigma models and the $\mathbb{CP}^{N-1}$ models have one specific model in common, namely the $O(3)$ sigma model, which is equivalent to the $\mathbb{CP}^1$ model. The equivalence is realized via the identification $\phi^a = n_k^* \sigma_{kl}^a n_l$, where ϕ^a are the components of the scalar isotriplet in the $O(3)$ sigma model, and n_k are the components of the scalar isodoublet in the $\mathbb{CP}^1$ model. In Ref. [42], the scattering of fermions on topological solitons of the nonlinear $O(3)$ sigma model with winding numbers $n = \pm 1$ is studied within the framework of the background field approximation. In this model, the fermionic isodoublet ψ interacts with the scalar isotriplet $\boldsymbol{\phi}$ via the Yukawa term $h\boldsymbol{\phi} \cdot \bar{\psi}\boldsymbol{\sigma}\psi$, where h is the Yukawa coupling constant.

After the identification $\phi^a = n_k^* \sigma_{kl}^a n_l$, the Yukawa term is transformed to the linear combination $2h(\bar{\psi}_a n_a)(n_b^* \psi_b) - h\bar{\psi}_a \psi_a$ of the non-Yukawa scalar-fermion interaction term and the mass term. We see that this scalar-fermion interaction differs significantly from the minimal scalar-fermion interaction in the Lagrangian (1). In particular, the Lagrangian (1) is diagonal with respect to the index of the internal $SU(N)$ symmetry, which leads to the conservation of the fermion isospin when a fermion is scattered in the background field of the $\mathbb{CP}^1$ soliton. In this case, the fermion scattering is elastic, and can therefore be described in terms of the phase shifts (59). The trivial isospin structure of the Lagrangian (1) makes it possible to reduce the Dirac equation (18) to the second-order linear differential equation (29), whose solution can be expressed in terms of the confluent Heun functions.

In contrast, the Lagrangian of the model considered in Ref. [42] is not diagonal with respect to the indices of the internal $SU(2)$ symmetry, and therefore the fermion isospin is not conserved. In this case, the fermion-soliton scattering can occur both without a change of the fermion isospin (elastic channel) and with a change of the fermion isospin (inelastic channel), and therefore a description of the fermion-soliton scattering in terms of the phase shifts becomes impossible. The nontrivial isospin structure of the Lagrangian makes it impossible to reduce the corresponding Dirac equation to a second-order differential equation and find the fermionic wave functions in an analytical form.

Significant differences in the scalar-fermion interaction between the nonlinear $O(3)$ sigma model and the $\mathbb{CP}^1$ model lead to differences in the properties of the corresponding fermion-soliton systems. In particular, if fermions are scattered on the $\mathbb{CP}^1$ soliton, the elastic partial S-matrix elements tend to the universal limit of minus one as the fermion momentum tends to zero. At the same time, if fermions are scattered on the $O(3)$ soliton, the elastic partial S-matrix elements do not tend to any universal limit when the fermion momentum vanishes. Furthermore, the fermion-soliton system of the $\mathbb{CP}^1$ model considered in this paper has no bound states, whereas the fermion-soliton system of the nonlinear $O(3)$ sigma model considered in Ref. [42] has bound states if the Yukawa coupling constant is sufficiently large.

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[1] N. Manton and P. Sutclffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2004).
[2] E. J. Weinberg, *Classical Solutions in Quantum Field Theory: Solitons and Instantons in High Energy Physics* (Cambridge University Press, Cambridge, England, 2012).
[3] W. J. Zakrzewski, *Low Dimensional Sigma Models* (Taylor & Francis, London, 1989).
[4] A. A. Abrikosov, Zh. Exp. Teor. Fiz. **32**, 1442 (1957) [Sov. Phys. JETP **5**, 1174 (1957)].
[5] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[6] A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Exp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].
[7] E. Cremmer and J. Scherk, Phys. Lett. **74B**, 341 (1978).
[8] H. Eichenherr, Nucl. Phys. **B146**, 215 (1978).
[9] V. L. Golo and A. M. Perelomov, Lett. Math. Phys. **2**, 477 (1978).
[10] V. L. Golo and A. M. Perelomov, Phys. Lett. **79B**, 112 (1978).
[11] A. D'Adda, M. Lüscher, and P. Di Vecchia, Nucl. Phys. **B146**, 63 (1978).
[12] E. Witten, Nucl. Phys. **B149**, 285 (1979).
[13] A. M. Polyakov, Phys. Lett. **59B**, 79 (1975).
[14] A. Hanany and D. Tong, J. High Energy Phys. 07 (2003) 037.
[15] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Nucl. Phys. **B673**, 187 (2003).
[16] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004).
[17] A. Hanany and D. Tong, J. High Energy Phys. 04 (2004) 066.
[18] M. Shifman and A. Yung, Rev. Mod. Phys. **79**, 1139 (2007).
[19] D. Tong, Ann. Phys. (N.Y.) **324**, 30 (2009).
[20] A. M. Tsvelik, *Quantum Field Theory in Condensed Matter* (Cambridge University Press, Cambridge, England, 1995).
[21] E. Mottola and A. Wipf, Phys. Rev. D **39**, 588 (1989).
[22] E. Abdalla, M. C. B. Abdalla, and M. Gomes, Phys. Rev. D **25**, 452 (1982).
[23] T. Birkandan and M. Hortaçsu, Rep. Math. Phys. **79**, 81 (2017).
[24] A. Yu. Loginov, Eur. Phys. J. C **82**, 662 (2022).
[25] A. Yu. Loginov, Nucl. Phys. **B984**, 115964 (2022).
[26] A. M. Din and W. J. Zakrzewski, Nucl. Phys. **B174**, 397 (1980).
[27] A. M. Din and W. J. Zakrzewski, Phys. Lett. **95B**, 426 (1980).
[28] A. M. Din and W. J. Zakrzewski, Nucl. Phys. **B182**, 151 (1981).
[29] R. Dabrowski and G. V. Dunne, Phys. Rev. D **88**, 025020 (2013).
[30] G. Basar, G. V. Dunne, and M. Unsal, J. High Energy Phys. 10 (2013) 041.
[31] A. Cherman, D. Dorigoni, G. V. Dunne, and M. Unsal, Phys. Rev. Lett. **112**, 021601 (2014).
[32] S. Bolognesi and W. Zakrzewski, Phys. Rev. D **89**, 065013 (2014).
[33] S. Y. Slavyanov and W. Lay, *Special Functions: A Unified Theory Based on Singularities* (Oxford University Press, Oxford, 2000).
[34] *Heun's Differential Equations*, edited by A. Ronveaux (Oxford University Press, Oxford, 1995).
[35] *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Cambridge University Press, Cambridge, England, 2010).
[36] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory. Vol. 3* (3rd ed.) (Pergamon Press, Oxford, 1977).
[37] J. R. Taylor, *Scattering Theory: Quantum Theory on Nonrelativistic Collisions* (John Wiley & Sons, New York, 1972).
[38] A. Prudnikov, Y. A. Brychkov, and O. Marichev, *Integrals and Series. Vol. 3* (Gordon and Breach Science Publishers, New York, 1990).
[39] Wolfram Research, Inc., *Mathematica*, Version 12.2, Champaign, IL, 2020.
[40] Maple User Manual, Maplesoft, Waterloo, Canada, 2019.
[41] N. Levinson, Dan. Vidensk. Selsk. K. Mat.-Fys. Medd. **25**, 9 (1949).
[42] A. Yu. Loginov, Phys. Rev. D **104**, 045011 (2021).