Tolman-Ehrenfest's criterion of thermal equilibrium extended to conformally static spacetimes

Valerio Faraoni[*] and Robert Vanderwee[†]

Department of Physics and Astronomy, Bishop's University, 2600 College Street, Sherbrooke, Québec, Canada J1M 1Z7

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With insight from examples and physical arguments, the Tolman-Ehrenfest criterion of thermal equilibrium for test fluids in static spacetimes is extended to local thermal equilibrium in conformally static geometries. The temperature of the conformally rescaled fluid scales with the inverse of the conformal factor, reproducing the evolution of the cosmic microwave background in Friedmann universes, the Hawking temperature of the Sultana-Dyer cosmological black hole, and a heuristic argument by Dicke.

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I. INTRODUCTION

Thermal physics in relativity and in curved spacetime is more intriguing, and notoriously more difficult, than in nonrelativistic situations, and several results necessarily have limited validity. An example is the Tolman-Ehrenfest criterion for the thermal equilibrium of a fluid in a static spacetime [1–3]. In a coordinate system adapted to the time symmetry, in which the line element reads[1]

$$ds^2 = g_{00}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j \qquad (i,j,k = 1,2,3), \quad (1.1)$$

the temperature $\mathcal{T}$ of a *test* fluid at rest with respect to the static observers (i.e., those with four-velocity parallel to the timelike Killing vector k^a) obeys [1–3]

$$\mathcal{T}\sqrt{-g_{00}} = \mathcal{T}_0, \quad (1.2)$$

where $\mathcal{T}_0$ is a constant. Equation (1.2) is referred to as the Tolman-Ehrenfest criterion of thermal equilibrium. It expresses the fact that, since heat is mass energy, it will sink in a gravitational field and regions of stronger gravity will be hotter. As a result, a fluid at rest in a static gravitational field and in thermal equilibrium has a nonvanishing temperature gradient, a counterintuitive result. Klein formulated the analogous condition for the equilibrium of particles with respect to diffusion in a static spacetime by replacing temperature $\mathcal{T}$ with chemical potential μ [5]. *Caveats* on the standard presentations of the Tolman-Ehrenfest criterion have been discussed exhaustively in the recent works [6–8], in particular the generalization of this law to stationary (but nonstatic) geometries. The criterion has inspired also a connection between gravitational fields and thermal transport in materials: thermal transport, understood as the linear response of a material to a temperature gradient, was mimicked by Luttinger as a counterbalancing weak gravitational field restoring thermal equilibrium in the presence of this gradient [9]. The Tolman-Ehrenfest criterion (1.2) is applied to neutron stars [10–12]; equilibrium with respect to simultaneous heat conduction and particle diffusion has been discussed in [13,14], together with the corresponding criterion in Weyl-integrable geometries [15].

The Tolman-Ehrenfest criterion can be derived from Eckart's generalization of the Fourier law for heat conduction, a constitutive relation assumed in Eckart's first-order thermodynamics of dissipative fluids [16]. An imperfect fluid with four-velocity u^a is described by the energy-momentum tensor

$$T_{ab} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a u_b + q_b u_a, \quad (1.3)$$

where ρ is the energy density, P is the isotropic pressure, π_{ab} is the anisotropic stress tensor, q^a is the heat flux density, and $h_{ab} \equiv g_{ab} + u_a u_b$ is the Riemannian metric on the three-space orthogonal to u^a. π^{ab} and q^a are purely spatial with respect to u^a and π^{ab} is trace-free:

$$\pi_{ab}u^a = \pi_{ab}u^b = q^a u_a = 0, \qquad \pi^a{}_a = 0. \quad (1.4)$$

Eckart's theory assumes the three constitutive relations for this fluid [16],

$$q_a = -\mathcal{K} h_{ab}(\nabla^b \mathcal{T} + \mathcal{T}\dot{u}^b), \quad (1.5)$$

$$P = P_{\text{nonviscous}} + P_{\text{viscous}} = P_{\text{nonviscous}} - \zeta\Theta, \quad (1.6)$$

$$\pi_{ab} = -2\eta\sigma_{ab}, \quad (1.7)$$

[*] vfaraoni@ubishops.ca
[†] rvanderwee20@ubishops.ca

[1] We follow the notation of Ref. [4].

where $\mathcal{T}$ is the temperature, $\mathcal{K}$ is the thermal conductivity, $\Theta = \nabla_c u^c$ is the expansion scalar, the shear tensor σ_{ab} is the symmetric, trace-free part of $h_a{}^c h_b{}^d \nabla_d u_c$ [17], while ζ and η are the bulk and shear viscosity coefficients, respectively. $\dot{u}^c \equiv a^c \equiv u^b \nabla_b u^c$ is the fluid's four-acceleration, which contributes an inertial term to the heat flux (1.5) [16].

The derivation of the Tolman-Ehrenfest criterion from Eq. (1.5), which generalizes the usual nonrelativistic Fourier law of heat conduction, appears in [[18], Exercise 22.7, p. 567] and, more recently, in Ref. [7]. For the reader's convenience, we reproduce this derivation in the Appendix.

We define *thermal equilibrium* in a static spacetime (and, later, local thermal equilibrium in time-dependent ones) as the absence of heat fluxes, $q^a = 0$. It is clear that, if a fluid is in thermal equilibrium in a certain frame, any observer moving relatively to it will detect a heat flux (which lies at the origin of some of the subtleties in generalizing Eq. (1.2) to stationary geometries [6–8]). To make this observation quantitative, consider a perfect fluid seen from a noncomoving frame, in which it appears "tilted." Denote (momentarily) with a star quantities associated with the comoving frame; for example, u^{*a} is the fluid four-velocity. The stress-energy tensor T_{ab} of the perfect fluid (an observer-independent object) can be decomposed according to this frame as

$$T_{ab} = \rho^* u_a^* u_b^* + P^* h_{ab}^*, \tag{1.8}$$

where $h_{ab}^* \equiv g_{ab} + u_a^* u_b^*$ is the Riemannian three-metric on the three-space seen by the observers u_a^* comoving with the fluid.

The frame of an observer moving with respect to this fluid (in which the fluid appears to be moving) is characterized by a different four-velocity u^a related to u^{*a} by [17]

$$u^{*a} = \gamma(u^a + v^a), \tag{1.9}$$

where v^a is a purely spatial vector according to u_a^*, $v_a u^{*a} = 0$, with $0 \leq v^2 \equiv v_a v^a < 1$, and

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \tag{1.10}$$

is the corresponding Lorentz factor. The fluid stress-energy tensor can be decomposed according to the observers[2] u^a as

$$T_{ab} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a u_b + q_b u_a, \tag{1.11}$$

where $h_{ab} \equiv u_a u_b + g_{ab}$ and [17]

$$\rho = \rho^* + \gamma^2 v^2 (\rho^* + P^*) = \gamma^2 (\rho^* + v^2 P^*), \tag{1.12}$$

$$P = P^* + \frac{\gamma^2 v^2}{3} (\rho^* + P^*), \tag{1.13}$$

$$q^a = (1 + \gamma^2 v^2)(\rho^* + P^*) v^a = \gamma^2 (\rho^* + P^*) v^a, \tag{1.14}$$

$$\pi_{ab} = \gamma^2 (\rho^* + P^*) \left(v^a v^b - \frac{v^2}{3} h^{ab} \right). \tag{1.15}$$

In the frame u^a, the fluid cannot be in equilibrium since $q^a \neq 0$: indeed, $q^a = 0$ implies $v^c = 0$ and $u^a = u^{*a}$. A perfect fluid is in thermal equilibrium in its comoving frame (i.e., $q^{*a} = 0$), but any other frame moving with respect to it ($v^2 > 0$) will experience a (purely convective) heat flux with density $q^a \neq 0$ given by Eq. (1.14), and there cannot be thermal equilibrium.

Before proceeding, let us be clear on the motivations of this work: the most interesting applications of the new generalized Tolman-Ehrenfest criterion that we present are about conformally invariant systems (the cosmic microwave background in cosmology, a blackbody gas of Hawking radiation, or a massless conformally coupled scalar field). It is possible that useful applications of the new criterion will be limited to conformally invariant systems, although this is not, by all means, established. However, even if this potential limitation turns out to be real, the generalized Tolman-Ehrenfest criterion of local thermal equilibrium presented here is very interesting because (1) it still allows one to discuss interesting (and varied) physics, and (2) it deepens our understanding of thermal physics in relativity. The first point will be elaborated in the following sections. As for the second point, one should keep in mind that the original Tolman-Ehrenfest criterion, which has not been applied widely to theoretical physics and astrophysics, is still a valuable contribution to the understanding of thermal physics in relativity. The latter is definitely incomplete, on par with the understanding of general nonequilibrium thermodynamics. In this sense, generalizing the Tolman-Ehrenfest criterion as done here seems valuable for the understanding of local thermal equilibrium in nonstatic spacetimes.

The rest of this article proceeds as follows. Section II discusses two examples showing how to generalize the Tolman-Ehrenfest criterion to conformally static spacetimes. Section III derives the generalized formula $\tilde{\mathcal{T}} = \mathcal{T}/\Omega$ for conformally static geometries $\tilde{g}_{ab} = \Omega^2 g_{ab}$ in two independent ways, while Sec. IV discusses an application to geometries conformal to the Schwarzschild black hole and Sec. V contains a discussion and the conclusions.

[2]There is only one stress-energy tensor T_{ab} but it can be decomposed in infinitely many ways according to the possible timelike observers u^a.

II. EXAMPLES

In this section we examine examples leading to a way of generalizing the Tolman-Ehrenfest criterion to conformally static spacetimes characterized by the metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where the conformal factor $\Omega(x^\alpha)$ is a regular and nowhere-vanishing function of the spacetime coordinates.

A. Example 1: Static conformal factor

The first example is almost trivial but points to the way to proceed in more interesting situations. Assume that the metric $\tilde{g}_{ab}$ is also static; then there is a timelike Killing[3] vector $\tilde{k}^a$ and, in coordinates adapted to this time symmetry,

$$\partial_t \tilde{g}_{\mu\nu} = 0, \qquad \tilde{g}_{0i} = 0 \quad (i = 1, 2, 3). \tag{2.1}$$

The conformal factor is static, $\Omega = \Omega(x^i)$; hence $\partial_t \Omega = 0$ and

$$\partial_t \tilde{g}_{00} = \partial_t [\Omega^2(x^i) g_{00}(x^i)] = 0, \qquad \tilde{g}_{0i} = \Omega^2 g_{0i} = 0; \tag{2.2}$$

applying the Tolman-Ehrenfest criterion directly to the static geometry $\tilde{g}_{ab}$, one obtains

$$\tilde{\mathcal{T}} \sqrt{-\tilde{g}_{00}} = \Omega \tilde{\mathcal{T}} \sqrt{-g_{00}} = \text{const.} \tag{2.3}$$

Since in the geometry g_{ab} we have $\mathcal{T}\sqrt{-g_{00}} = \text{const}$, it follows that $\Omega\tilde{\mathcal{T}}/\mathcal{T} = \text{const}$. One can redefine the time coordinate to absorb the constant (or simply note that $\Omega = 1$ must reproduce the identity), obtaining

$$\tilde{\mathcal{T}} = \frac{\mathcal{T}}{\Omega}. \tag{2.4}$$

As we will see in the following, Eq. (2.4) relates the temperature between conformally related spacetimes also in more physically significant situations.

B. Example 2: Cosmic microwave background in FLRW universes

All Friedmann-Lemaître-Robertson-Walker (FLRW) universes are conformally flat [4], and hence conformally static. Consider, for simplicity, a spatially flat FLRW universe with line element

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \\ &= a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2) \end{aligned} \tag{2.5}$$

in comoving coordinates (t, x, y, z), or using the conformal time η defined by $dt = a(\eta)d\eta$. Consider a radiation fluid (the cosmic microwave background) in local thermal equilibrium[4] in an expanding, spatially flat, FLRW universe. After decoupling from baryons, the cosmic microwave background evolves as a radiation fluid independent of the other fluids present in the universe. It is well-known that, to maintain the blackbody distribution and thermal equilibrium, the temperature of the cosmic microwave background must scale according to $\mathcal{T} \sim 1/a$ [4], that is, in accordance with Eq. (2.4):

$$\mathcal{T}(t) = \frac{\mathcal{T}_0}{a(t)}, \tag{2.6}$$

where $\mathcal{T}_0 = \mathcal{T}(a(t_0) = 1) = \mathcal{T}(t_0)$ is constant and the instant t_0 is defined by $a(t_0) = 1$. In fact, the Planck distribution for the spectral energy density of a blackbody is

$$u(\nu, \mathcal{T}) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{K_B \mathcal{T}}} - 1}, \tag{2.7}$$

where ν is the photon frequency, $\mathcal{T}$ the absolute temperature, and h, c, and K_B are the Planck constant, speed of light, and Boltzmann constant, respectively. Since in a FLRW universe frequencies redshift with the cosmic expansion according to $\nu \sim 1/a$ (equivalently, the proper wavelength scales as λa, where λ is the comoving wavelength), it must be $K_B \mathcal{T} \sim 1/a$, or else the Planck distribution would be distorted by the cosmic expansion

$$\mathcal{T} \sim \frac{1}{a} \sim \frac{1}{\Omega}, \tag{2.8}$$

where $\Omega = a(\eta)$ is the conformal factor of the FLRW line element (2.5).

This result can be obtained in another way that highlights formulas useful in the following. Assuming the number of photons to be conserved (which is true after decoupling), the first law of thermodynamics for the radiation fluid reads

$$\mathcal{T} dS = dU + P dV, \tag{2.9}$$

where U is the internal energy, P is the radiation pressure, and V is the volume. The entropy density is

$$s \equiv \frac{dS}{dV} = \frac{\rho + P}{T}, \tag{2.10}$$

where $\rho = dU/dV$ is the energy density, while the entropy is (e.g., [20])

[3]If the conformal factor Ω is not static, there is only a conformal Killing vector $\tilde{k}^a$ in the conformally rescaled spacetime [4].

[4]Of course, a conformal transformation is just a mathematical operation and does not guarantee local thermal equilibrium, which must be assumed and depends on the microphysics (reaction rates must be faster than the Hubble rate to maintain equilibrium [19]).

$$S = \frac{32\pi^5 K_B}{45}\left(\frac{K_B \mathcal{T}}{hc}\right)^3 V, \tag{2.11}$$

implying that

$$\rho + P = \frac{32\pi^5 K_B^4}{45(hc)^3}\mathcal{T}^4. \tag{2.12}$$

For conformal transformations of perfect fluids in FLRW cosmology, the pressure and energy density transform as [21]

$$\tilde{\rho} = \Omega^{-4}\rho, \qquad \tilde{P} = \Omega^{-4}P, \tag{2.13}$$

and then

$$\tilde{\rho} + \tilde{P} = \Omega^{-4}(\rho + P) = \Omega^4 \frac{32\pi^5 K_B^4}{45(hc)^3}\mathcal{T}^4 = \frac{32\pi^5 K_B^4}{45(hc)^3}\tilde{\mathcal{T}}^4, \tag{2.14}$$

leading again to

$$\tilde{\mathcal{T}} = \frac{\mathcal{T}}{\Omega}, \tag{2.15}$$

where

$$\mathcal{T} = \left[\frac{45(hc)^3}{32\pi^5 K_B^4}\right]^{1/4} \tag{2.16}$$

and where ρ and $P = \rho/3$ are constant for blackbody radiation at rest in Minkowski spacetime.

In other words, one notes that for blackbody radiation

$$\rho = \frac{U}{V} = A\mathcal{T}^4, \qquad A = \frac{8\pi^5 K_B^4}{15h^3c^3}, \tag{2.17}$$

$$s = \frac{4\rho}{3\mathcal{T}} \sim \mathcal{T}^3. \tag{2.18}$$

Then, comparing the expressions of the rescaled energy density

$$\tilde{\rho} = A\tilde{\mathcal{T}}^4, \tag{2.19}$$

$$\tilde{\rho} = \Omega^{-4}\rho, \tag{2.20}$$

one obtains Eq. (2.4) with $T = (\rho/A)^{1/4}$. These calculations are appropriate to the physics at hand: in Minkowski spacetime a radiation fluid has $T = \text{const}$, while in FLRW spacetime

$$\mathcal{T}\sqrt{-g_{00}} = \frac{\mathcal{T}}{\Omega}\Omega\sqrt{-g_{00}} = \frac{\mathcal{T}\sqrt{-\tilde{g}_{00}}}{\Omega} = \tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}} \tag{2.21}$$

implies that $\tilde{\mathcal{T}} = \mathcal{T}/\Omega$. The reasoning works in coordinates in which $\tilde{g}_{ab}$ is explicitly conformally static, that is, comoving frame and conformal time are needed.

III. TEST FLUIDS IN CONFORMALLY STATIC SPACETIMES

We now generalize the Tolman-Ehrenfest criterion for thermal equilibrium to the local thermal equilibrium of fluids in conformally static spacetimes. That this is possible is suggested by the previous example of the cosmic microwave background in FLRW universes. Conformally static spacetimes are nontrivial because they can be dynamical (as the FLRW geometry), which is a significant deviation from the situation of a static fluid at rest in a static spacetime, to which the Tolman-Ehrenfest criterion has been confined since its inception [1–3] (only recently a proper description of stationary spacetimes has been given [6–8]). We provide two different derivations of the generalized Tolman-Ehrenfest criterion.

A. Derivation using perfect fluids

Consider a conformally static metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where g_{ab} is static, and use coordinates (t, x^i) adapted to the time symmetry, in which $\partial g_{\mu\nu}/\partial t = 0$ and $g_{0i} = 0$. The normalization of the four-velocity in the conformally rescaled world $-1 = \tilde{u}^c\tilde{u}_c = \Omega^2 g_{ab}\tilde{u}^a\tilde{u}^b$, in conjunction with $g_{ab}u^a u^b = -1$, gives

$$\tilde{u}^c = \frac{u^c}{\Omega}, \qquad \tilde{u}_c = \Omega u_c. \tag{3.1}$$

In the comoving frame of the fluid, assumed to coincide with the frame of the static observers, the components of the fluid's four-velocity are $u^\mu = (u^0, 0, 0, 0)$ and the conformal image of this frame is the comoving frame of the conformally transformed fluid because

$$\tilde{u}^\mu = \left(\frac{u^0}{\Omega}, 0, 0, 0\right). \tag{3.2}$$

Denoting with $\tilde{g}^{(3)}$ the determinant of the spatial three-metric induced by $\tilde{g}_{ab}$, the three-dimensional volume of a region of the rescaled three-space is

$$\tilde{V} = \int d^3\vec{x}\sqrt{\tilde{g}^{(3)}} = \int d^3\vec{x}\sqrt{\Omega^6 g^{(3)}} = \int d^3\vec{x}\,\Omega^3\sqrt{g^{(3)}}; \tag{3.3}$$

thus, if $\Omega = \Omega(t)$, then $\tilde{V} = \Omega^3 V$, but this is not true if $\Omega(x^\mu)$ depends on the spatial coordinates. However, it is always true that for infinitesimal volumes $d\tilde{V} = \sqrt{\tilde{g}^{(3)}}d^3x = \Omega^3\sqrt{g^{(3)}}d^3x = \Omega^3 dV$. The relations $\tilde{\rho} = \Omega^{-4}\rho$ and $\tilde{P} = \Omega^{-4}P$ valid for perfect fluids in FLRW spacetimes [21] can be generalized to test fluids in any conformally static spacetime. In fact, equivalent Lagrangian densities for a perfect fluid are $-\rho$ and P [22–25]. We can relate these equivalent actions for a perfect fluid to those of the conformally transformed fluid as follows:

$$\begin{aligned} J &\equiv \int d^4x\sqrt{-g}\mathcal{L}^{(\mathrm{m})}_{(1)} = -\int d^4x\sqrt{-g}\rho \\ &= -\int d^4x\sqrt{-\tilde{g}}\tilde{\rho} = \int d^4x\sqrt{-\tilde{g}}\tilde{\mathcal{L}}^{(\mathrm{m})}_{(1)}, \end{aligned} \tag{3.4}$$

where $\tilde{\rho} = \Omega^{-4}\rho$ and $\tilde{g} = \Omega^8 g$. Similarly, for the equivalent perfect fluid Lagrangian,

$$\begin{aligned} J &\equiv \int d^4x\sqrt{-g}\mathcal{L}^{(\mathrm{m})}_{(2)} = \int d^4x\sqrt{-g}P \\ &= \int d^4x\sqrt{-\tilde{g}}\tilde{P} = \int d^4x\sqrt{-\tilde{g}}\tilde{\mathcal{L}}^{(\mathrm{m})}_{(2)}, \end{aligned} \tag{3.5}$$

where $\tilde{P} = \Omega^{-4}P$. The perfect fluid remains a perfect fluid if we add the information that $\tilde{u}_a = \Omega u_a$. In fact,

$$\begin{aligned} \tilde{T}_{ab} &= \tilde{\rho}\tilde{u}_a\tilde{u}_b + \tilde{P}\tilde{h}_{ab} = \Omega^{-4}\rho\Omega u_a\Omega u_b + \Omega^{-4}P\Omega^2 h_{ab} \\ &= \Omega^{-2}(\rho u_a u_b + P h_{ab}) = \Omega^{-2}T_{ab}: \end{aligned} \tag{3.6}$$

the conformal transformation does not generate dissipative terms in the stress-energy tensor of a test perfect fluid. However, if T_{ab} sources the Einstein equations, then $\tilde{g}_{ab}$ is not a solution of the Einstein equations with the same source because these change to

$$\tilde{G}_{ab} = 8\pi(\tilde{T}_{ab} + T^{(\Omega)}_{ab}), \tag{3.7}$$

where

$$\begin{aligned} 8\pi T^{(\Omega)}_{ab} = &-\frac{2}{\Omega}(\nabla_a\nabla_b\Omega - g_{ab}\Box\Omega) \\ &+\frac{1}{\Omega^2}(4\nabla_a\Omega\nabla_b\Omega - g_{ab}\nabla_c\Omega\nabla^c\Omega) \end{aligned} \tag{3.8}$$

is generated by Ω and its first and second covariant derivatives. This fact is immaterial for our discussion, in which T_{ab} describes a test fluid and the Tolman-Ehrenfest criterion is purely kinematic [7]; hence we do not worry about the field equations.

Let us proceed with our reasoning. For a conformally static spacetime, the proper three-volume element is $d\tilde{V} = \Omega^3 dV \equiv \Omega^3 d\tilde{V}_{\mathrm{comoving}}$. For a perfect fluid the entropy is constant along the fluid lines, which means that there is no entropy generation (because there is no dissipation) in the comoving frame, or the entropy remains constant in time in the comoving frame and, in this frame, also the entropy density

$$\tilde{s}_{\mathrm{comoving}} \equiv \frac{d\tilde{S}}{d\tilde{V}_{\mathrm{comoving}}} = \mathrm{const.} \tag{3.9}$$

Then

$$\begin{aligned} \tilde{s}_{\mathrm{comoving}} &= \frac{d\tilde{S}}{d\tilde{V}_{\mathrm{comoving}}} = \frac{d\tilde{S}}{\Omega^{-3}d\tilde{V}} \\ &= \Omega^3\tilde{s} = \Omega^3\left(\frac{\tilde{\rho}+\tilde{P}}{\tilde{\mathcal{T}}}\right) = \mathrm{const.} \end{aligned} \tag{3.10}$$

Using the fact just proven that $\tilde{\rho} = \Omega^{-4}\rho$, $\tilde{P} = \Omega^{-4}P$, we can write

$$\Omega^{-1}\left(\frac{\rho+P}{\tilde{\mathcal{T}}}\right) = \mathrm{const}, \tag{3.11}$$

which implies that

$$\tilde{\mathcal{T}} = \mathrm{const}\left(\frac{\rho+P}{\Omega}\right) = \mathrm{const}\frac{\mathcal{T}}{\Omega}\left(\frac{\rho+P}{\mathcal{T}}\right). \tag{3.12}$$

Using now the fact that for the Minkowski space perfect fluid $s = (\rho+P)/\mathcal{T}$ is constant, we have $\tilde{\mathcal{T}} = \mathrm{const}\,\mathcal{T}/\Omega$. The multiplicative constant is determined by the fact that $\Omega = 1$ (or more generally, $\Omega = \mathrm{const}$) gives the identity, yielding

$$\tilde{\mathcal{T}} = \frac{\mathcal{T}}{\Omega}. \tag{3.13}$$

As is well-known (e.g., [21]), in general the stress-energy tensor of the conformally transformed fluid is not covariantly conserved but satisfies

$$\tilde{\nabla}_b\tilde{T}^{ab} = -\frac{\tilde{T}\nabla^a\Omega}{\Omega^3} \tag{3.14}$$

and is conserved only for a conformally invariant fluid with $T = \tilde{T} = 0$ (this is the case for the radiation fluid in FLRW universes just considered).

B. Derivation from Eckart's law of heat conduction

In the conformally static geometry $\tilde{g}_{ab} = \Omega^2 g_{ab}$, Eckart's law for heat conduction in a dissipative fluid reads [16]

$$\tilde{q}_a = -\tilde{K}\tilde{h}_{ab}(\tilde{\nabla}^b\tilde{\mathcal{T}} + \tilde{\mathcal{T}}\tilde{a}^b). \tag{3.15}$$

If the generalized Tolman-Ehrenfest criterion $\tilde{\mathcal{T}} = \mathcal{T}/\Omega$ holds in the conformally rescaled frame, one should be able to derive it directly from Eckart's law (1.5) written in this frame, which we do here. Essentially, we use again the *definition* of local thermal equilibrium $\tilde{q}^a = 0$, and the temperature $\mathcal{T}$ is not required to be time independent.

Indeed, even if $u^c\nabla_c\mathcal{T}=0$ in the static spacetime, in the rescaled world

$$\tilde{u}^c\tilde{\nabla}_c\tilde{\mathcal{T}}=\frac{u^b}{\Omega}\nabla_b\left(\frac{\mathcal{T}}{\Omega}\right)=\frac{u^b\nabla_b\mathcal{T}}{\Omega^2}-\frac{\mathcal{T}u^b\nabla_b\Omega}{\Omega^3}=-\frac{\mathcal{T}\dot{\Omega}}{\Omega^3}\neq 0 \tag{3.16}$$

unless the conformal factor Ω is time independent, which would bring us back to the rather trivial example 1 of Sec. II A.

Recall that, to derive the Tolman-Ehrenfest law in a static spacetime, one uses the Buchdahl identity [26] $a^c=\nabla^c\ln\sqrt{-g_{00}}$ (the Appendix). While, in general, it is not true that $\tilde{a}^c=\tilde{\nabla}^c\ln\sqrt{-\tilde{g}_{00}}$, the relation

$$\tilde{h}_{ab}\tilde{a}^b=\tilde{h}_{ab}\tilde{\nabla}^b\ln\sqrt{-\tilde{g}_{00}} \tag{3.17}$$

is valid and is all that is needed. This condition differs from the previous one if the four-acceleration has a component parallel to the four-velocity. Instances in which a four-force is parallel to the four-velocity of a particle and causes an *effective* acceleration with the same direction comprise particles with variable mass [27,28] (including rockets and solar sails [29–31]), interacting dark energy [32–37], timelike geodesic curves mapped to the Einstein frame of scalar-tensor or dilaton gravity [21,38–41], the worldlines of fluid elements in FLRW cosmology as seen by comoving observers when the cosmic fluid is not a dust [42], and nonaffinely parametrized geodesics [17]. The reason why a four-acceleration is not parallel to the corresponding four-velocity is simply because, in these situations, the proper time fails to be an affine parameter along the particle trajectory and does not contradict standard tenets of special relativity [42].

To prove Eq. (3.17), first compute

$$\begin{aligned}\tilde{a}_a&\equiv\tilde{u}^b\tilde{\nabla}_b\tilde{u}_a=\frac{u^b}{\Omega}\tilde{\nabla}_b(\Omega u_a)=\left(\frac{u^b\nabla_b\Omega}{\Omega}\right)u_a+u^b\tilde{\nabla}_bu_a=\frac{\dot{\Omega}}{\Omega}u_a+u^b(\partial_bu_a-\tilde{\Gamma}^c_{ab}u_c)\\&=\frac{\dot{\Omega}}{\Omega}u_a+u^b\left[\partial_bu_a-\Gamma^c_{ab}u_c-\frac{1}{\Omega}(\delta^c{}_b\partial_a\Omega+\delta^c{}_a\partial_b\Omega-g_{ab}\partial^c\Omega)\right]u_c\\&=\frac{\dot{\Omega}}{\Omega}u_a+u^b\nabla_bu_a-\frac{1}{\Omega}u^cu_c\partial_a\Omega=\frac{\dot{\Omega}}{\Omega}u_a+a_a+\frac{\nabla_a\Omega}{\Omega}\\&=a_a+\frac{1}{\Omega}(u_au_b\nabla^b\Omega+g_{ab}\nabla^b\Omega)\equiv a_a+h_{ab}\frac{\nabla^b\Omega}{\Omega},\end{aligned} \tag{3.18}$$

where we used [4,21]

$$\tilde{\Gamma}^c_{ab}=\Gamma^c_{ab}+\frac{1}{\Omega}(\delta^c{}_b\partial_a\Omega+\delta^c{}_a\partial_b\Omega-g_{ab}\partial^c\Omega). \tag{3.19}$$

The four-acceleration $\tilde{a}^c$ is still orthogonal to the four-velocity $\tilde{u}^c$:

$$\begin{aligned}\tilde{g}_{ab}\tilde{a}^a\tilde{u}^b&=\Omega^{-1}\tilde{a}_au^a=\Omega^{-1}\left(a_a+h_{ab}\frac{\nabla^b\Omega}{\Omega}\right)u^a\\&=\Omega^{-1}a_bu^b=0.\end{aligned} \tag{3.20}$$

We now compute

$$\begin{aligned}\tilde{h}_{ab}\tilde{a}^b&=\Omega^2h_{ab}\left(a^b+h^{bc}\frac{\nabla_c\Omega}{\Omega}\right)\\&=\Omega^2h_{ab}(\nabla^b\ln\sqrt{-g_{00}}+\nabla^b\ln\Omega)\\&=\Omega^2h_{ab}\nabla^b\ln(\Omega\sqrt{-g_{00}})\\&=\Omega^2h_{ab}\nabla^b\ln\sqrt{-\tilde{g}_{00}}\\&=\tilde{h}_{ab}\tilde{\nabla}^b\ln\sqrt{-\tilde{g}_{00}},\end{aligned} \tag{3.21}$$

which completes the proof[5] of Eq. (3.17). One then has

$$\begin{aligned}\tilde{q}_a&=-\tilde{\mathcal{K}}\tilde{h}_{ab}(\tilde{\nabla}^b\tilde{\mathcal{T}}+\tilde{\mathcal{T}}\tilde{a}^b)\\&=-\tilde{\mathcal{K}}\tilde{h}_{ab}\tilde{\mathcal{T}}\left(\tilde{\nabla}^b\ln\tilde{\mathcal{T}}+\tilde{\nabla}^b\ln\sqrt{-\tilde{g}_{00}}\right)\\&=-\tilde{\mathcal{K}}\tilde{h}_{ab}\tilde{\mathcal{T}}\tilde{\nabla}^b\ln\left(\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}}\right),\end{aligned} \tag{3.22}$$

and thermal equilibrium $\tilde{q}_a=0$ implies that $\tilde{\nabla}^b\ln(\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}})$ is parallel to $\tilde{u}^b$. Then $\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}}$ must depend only on time,

$$\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}}=f(t), \tag{3.23}$$

where $f(t)$ is an integration function, or

[5]Contrary to the proof of the analogous relation for static spacetimes (the Appendix), the Killing equation has not been used. Indeed, the conformally rescaled world, in general, has no timelike Killing vector, but only a conformal Killing vector [4].

$$\tilde{\mathcal{T}} = \frac{f(t)}{\Omega\sqrt{-g_{00}}} = \frac{f(t)\mathcal{T}}{\Omega\left(\mathcal{T}\sqrt{-g_{00}}\right)} = \mathrm{const}\frac{f(t)\mathcal{T}}{\Omega}. \tag{3.24}$$

The product const $\times f(t)$ is fixed by the fact that, if $\Omega \equiv 1$, the conformal transformation must reduce to the identity with $\tilde{\mathcal{T}} = \mathcal{T}$ and const $\times f(t) = 1$. We are left with Eq. (3.13) again.

IV. CONFORMALLY SCHWARZSCHILD GEOMETRIES

It is interesting to compare the generalized Tolman-Ehrenfest criterion (2.4) with spacetimes designed intentionally to be conformal to the (exterior) Schwarzschild black hole geometry

$$ds^2_{(0)} = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega^2_{(2)}, \tag{4.1}$$

where $d\Omega^2_{(2)} \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit two-sphere and the parameter m is the (constant) black hole mass.

The first such spacetime described here is the Sultana-Dyer solution of the Einstein equations, which is a Petrov type D, time-dependent, and spherically symmetric spacetime sourced by two noninteracting fluids, a null dust and an ordinary (timelike) dust [43]. It is interpreted as describing a black hole embedded in a spatially flat FLRW universe.

Since we need a test fluid at rest in the static seed spacetime to apply the criterion (2.4), we consider the region around the Schwarzschild event horizon, in which Hawking radiation creates a static blackbody radiation fluid at the Hawking temperature $\mathcal{T} = \frac{1}{8\pi m}$ (in geometrized units). The Tolman-Ehrenfest criterion clearly fails at horizons since, for the Schwarzschild black hole it would give $\mathcal{T} = \frac{\mathcal{T}_0}{1-2m/r}$, which diverges as $r \to 2m^+$. However, the cause is not that the criterion is inherently bad but it is restricted to static coordinates, and the latter fail at the Schwarschild event horizon. Hawking radiation is a quantum phenomenon and the proper calculation of the Hawking temperature requires quantum field theory in curved space, including a careful consideration of the vacuum state. Once this is done and the temperature appearing in the Tolman-Ehrenfest criterion is cured producing the Hawking result $\mathcal{T} = \frac{1}{8\pi m}$, one can consider the Schwarzschild geometry as a seed for constructing the Sultana-Dyer spacetime by a conformal transformation. The Sultana-Dyer line element is [43]

$$\begin{aligned} ds^2 &= a^2(\eta, r)\left[-\left(1 - \frac{2m}{r}\right)d\eta^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega^2_{(2)}\right] \\ &= a^2(\eta, r) ds^2_{(0)}, \end{aligned} \tag{4.2}$$

where

$$a(\eta, r) = \left(\eta + 2m \ln\left|\frac{r}{2m} - 1\right|\right)^2. \tag{4.3}$$

If $m = 0$, the line element reduces to the FLRW one written in conformal time. The coordinate change

$$\tau(\eta, r) = \eta + 2m \ln\left|\frac{r}{2m} - 1\right| \tag{4.4}$$

turns the line element into the original Sultana-Dyer form [43]

$$ds^2 = a^2(\tau)\left[-d\tau^2 + dr^2 + r^2 d\Omega^2_{(2)} - \frac{2m}{r}(d\tau + dr)^2\right], \tag{4.5}$$

with $a(\tau) = \tau^2$ [43]. The Tolman criterion applied to the Sultana-Dyer geometry yields the temperature

$$\mathcal{T} = \frac{\mathcal{T}_0}{a\sqrt{1 - 2m/r}}, \tag{4.6}$$

which, as usual, diverges at the event horizon where the static coordinates fail, and needs to be regularized. This has been done by Saida, Harada, and Maeda [44], who studied the Hawking radiation of a massless, conformally coupled scalar field ϕ in this geometry and computed the renormalized stress-energy tensor $\langle T_{ab}[\phi]\rangle$ taking into account the conformal anomaly and particle creation. The calculation, analogous to Hawking's calculation in the fixed Schwarschild geometry with constant mass m (that is, neglecting backreaction), is feasible only in an adiabatic approximation in which the black hole mass evolves very slowly, which is necessary to guarantee thermal equilibrium. (This condition is analogous to the condition that reaction rates exceed the Hubble expansion rate in a FLRW universe to maintain local thermal equilibrium.) The generalized Tolman-Ehrenfest criterion (2.4) then predicts that the temperature of the Sultana-Dyer black hole is $\mathcal{T} = \mathcal{T}_0/\Omega = \mathcal{T}_0/a$, where $\mathcal{T}_0$ is the Hawking temperature. The calculation of [44] produces the result

$$\mathcal{T} = \frac{1}{8\pi m a} + \cdots, \tag{4.7}$$

where the corrections omitted are negligible in the adiabatic approximation of a slowly evolving black hole [44].

An independent calculation using the method of chiral anomaly confirms the temperature (4.7) of the Sultana-Dyer black hole [45,46], which is supported also by previous heuristic dimensional reasoning [47]. The generalized Tolman-Ehrenfest criterion makes a definite prediction about the temperature of cosmological black holes conformal to Schwarzschild, and the conformal transformation is a popular technique to generate exact solutions of general relativity [48] and of alternative theories of gravity [49].

V. CONCLUSIONS

The applicability of the Tolman-Ehrenfest criterion (1.2) for the thermal equilibrium of a fluid is quite restricted. It requires a static spacetime and a fluid at rest with respect to the static observers of the latter, who have four-velocity parallel to the timelike Killing vector. Extending the Tolman-Ehrenfest criterion to more general geometries is, therefore, not an insignificant task. Here we have presented its generalization to conformally static spacetimes with metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where g_{ab} is static and the test fluid of temperature $\mathcal{T}$ is at rest in the frame associated with the static observers of g_{ab}. Then, assuming local thermal equilibrium, the generalization of the Tolman-Ehrenfest criterion is $\tilde{\mathcal{T}} = \mathcal{T}/\Omega$. The most obvious application of this generalized criterion is to the cosmic microwave background in FLRW universes, which reproduces the well-known scaling of its blackbody temperature $\mathcal{T} \sim 1/a$.

The temperature scaling $\tilde{\mathcal{T}} = \mathcal{T}/\Omega$ found is compatible with Dicke's heuristic argument on the scaling of physical quantities under conformal transformations [38] (cf. Ref. [47]) and is confirmed by precise calculations [45,46] in the particular case of the Sultana-Dyer black hole, as discussed in the previous section. With this argument, physical quantities themselves do not carry physical meaning, which is instead attributed to the ratios of physical quantities to their units, the only outcome of measurements. Usually the units are taken to be constant in spacetime, but a conformal rescaling amounts to a rescaling of physical units that depends on the spacetime point: lengths and times scale as Ω, masses scale as $1/\Omega$, and derived quantities scale accordingly to their dimensions [38]. Then, since $K_B\mathcal{T}$ is an energy, or a mass, and K_B remains constant, $\mathcal{T}$ should scale as $1/\Omega$, which is what we found. Dicke's argument, however, is rather heuristic and is known to become imprecise in the conformal transformation from Jordan to Einstein frame in scalar-tensor gravity. One must be precise in the discussion of what kind of fluid is considered, according to which observers, the definition of local thermal equilibrium, and the vanishing of q^a and $\tilde{q}^a$. It is interesting, however, that our finding agrees with Dicke's heuristic reasoning.

Already the generalization of the Tolman-Ehrenfest criterion to stationary, but nonstatic, spacetimes requires much care [6–8]. The extension of our generalization to conformally stationary spacetimes is problematic because, under conformal transformations, the nonunique timelike Killing vector of the stationary spacetime g_{ab} does not map into another timelike Killing vector of $\tilde{g}_{ab}$, but only into a conformal Killing vector [4]. In any case, the problems encountered in stationary but nonstatic spacetimes [6–8] are not going to be cured in conformally stationary ones.

The most interesting applications of the generalized Tolman-Ehrenfest criterion (2.4) uncovered here (the cosmic microwave background in FLRW universes and the Hawking temperature of the Sultana-Dyer black hole) are about conformally invariant systems (a blackbody gas of Hawking radiation or a massless conformally coupled scalar field, which is conformally invariant [4]). We suspect that the most useful applications of this criterion will involve conformally invariant systems, but other applications are not excluded at this stage. Even with this potential restriction, however, it appears that interesting physics can be tackled with the new criterion (2.4). Indeed, the phenomena discussed here are already quite varied, ranging from cosmology to time-dependent black holes. In any case, even the original Tolman-Ehrenfest criterion of thermal equilibrium now reported in textbooks [18] has not found widespread applications to theoretical physics and astrophysics, but it has intellectual value in itself as a contribution to the understanding of thermal physics in relativity, which is still fairly incomplete (as is the understanding of nonequilibrium thermodynamics in general), and its generalization to nonstatic situations appears to be valuable.

Finally, in our derivation we used the fact that the fluid is a test fluid. Although the original Tolman-Ehrenfest temperature gradient is a kinematic effect, relating solutions of the Einstein equations (or of the field equations of alternative theories of gravity) through conformal transformations spoils the reasoning of Sec. III (an exception is the radiation fluid which, due to its conformal invariance and the fact that photons are massless, is conserved after the conformal transformation). Further generalization of the Tolman-Ehrenfest criterion beyond conformally static spacetimes and test fluids seems difficult to achieve.

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APPENDIX: DERIVATION OF THE TOLMAN-EHRENFEST CRITERION FROM ECKART'S LAW (1.5)

Here we derive the Tolman-Ehrenfest criterion from Eckart's generalization of the Fourier law for heat conduction in imperfect fluids [16], using modern notation. We follow Ref. [7] step-by-step.

Consider a static test fluid at rest in a static spacetime and let $g_{\mu\nu}$ be the metric components in coordinates adapted to the time symmetry. The timelike Killing vector k^a satisfies the Killing equation

$$\nabla_{(a}k_{b)} = \frac{1}{2}(\nabla_a k_b + \nabla_b k_a) = 0 \qquad \text{(A1)}$$

and has components $k^\mu = (1, 0, 0, 0)$ in these coordinates, while $g_{00} = k_c k^c$.

The first step consists of a relation, due to Buchdahl [26], between the four-acceleration of a test particle at rest with respect to the static observers and g_{00},

$$a_c \equiv \dot{u}_c \equiv u^b \nabla_b u_c = \nabla_c \ln(\sqrt{-g_{00}}). \tag{A2}$$

To prove this relation, note that a fluid element has normalized four-velocity

$$u^a = \frac{k^a}{\sqrt{-k^b k_b}} \tag{A3}$$

and four-acceleration

$$\begin{aligned}
\frac{du_b}{d\tau} &\equiv u^c \nabla_c u_b = u^c \nabla_c \left(\frac{k_b}{\sqrt{-k^d k_d}} \right) \\
&= u^c \left[\frac{\nabla_c k_b}{\sqrt{-k^d k_d}} + k_b \left(\frac{-1}{2} \frac{1}{(-k^d k_d)^{3/2}} \right) \nabla_c (-k^d k_d) \right] \\
&= u^c \left[\frac{\nabla_c k_b}{\sqrt{-k^d k_d}} - \frac{k_b \nabla_c (-k^d k_d)}{2(-k^d k_d)^{3/2}} \right]
\end{aligned} \tag{A4}$$

(where τ is the proper time along the fluid lines). The second term in the last line vanishes since $k^a \nabla_a (k_b k^b) = 0$ because

$$\begin{aligned}
k^a \nabla_a (k_b k^b) &= k^a \nabla_a (g_{bc} k^b k^c) \\
&= k^a g_{bc} (k^c \nabla_a k^b + k^b \nabla_a k^c) \\
&= k^a k^c (\nabla_a k_c) + k^a k^c (\nabla_a k_c) \\
&= 2 k^a k^c \nabla_a k_c \\
&= 2 k^a k^c \nabla_{(a} k_{c)} = 0,
\end{aligned}$$

where, in the second to last step, we used the fact that since $k^a k^b$ is symmetric only the symmetric part of $\nabla_a k_b$ contributes, while the last step follows from the Killing equation. Then the fluid's four-acceleration is

$$\begin{aligned}
a_b &= \frac{u^c \nabla_c k_b}{\sqrt{-k^d k_d}} = \frac{k^c}{\sqrt{-k^d k_d}} \frac{\nabla_c k_b}{\sqrt{-k^d k_d}} \\
&= \frac{k^c \nabla_c k_b}{-k^d k_d} = \frac{-k^c \nabla_b k_c}{-k^d k_d},
\end{aligned} \tag{A5}$$

where we used again the Killing equation (A1). Since $\nabla_b (k^c k_c) = 2 k^c \nabla_b k_c$,

$$k^c \nabla_b k_c = \frac{1}{2} \nabla_b (k^c k_c) \tag{A6}$$

and the above identity yields

$$a_b = \frac{\nabla_c (-k^d k_d)}{-2 k^d k_d} = \frac{\nabla_b \ln(-k^d k_d)}{2} = \nabla_b \ln\left(\sqrt{-k^d k_d} \right). \tag{A7}$$

In the adapted coordinates $k^a k_a = g_{00}$, and hence

$$a_b = \nabla_b \ln \sqrt{-g_{00}}. \tag{A8}$$

Eckart's generalization of the Fourier law for heat conduction in dissipative fluids is then used to complete the derivation. The Tolman-Ehrenfest criterion refers to *perfect* fluids, described by the simpler stress-energy tensor

$$T_{ab} = \rho u_a u_b + P h_{ab}, \tag{A9}$$

but in Eckart's first-order thermodynamics an imperfect fluid at rest coincides with that of a perfect fluid because the imperfect fluid dissipative quantities are assumed to be linear in the gradient of the four-velocity [cf. Eqs. (1.5)–(1.7) [16]].

For a fluid at rest in a static spacetime, Θ and σ_{ab} vanish and $q^a = 0$ in thermal equilibrium; hence the stress-energy tensor (1.3) takes the perfect fluid form (A9). The temperature of such a fluid is time-independent,

$$\frac{d\mathcal{T}}{d\tau} \equiv u^a \nabla_a \mathcal{T} = 0, \tag{A10}$$

and then

$$h_{ab} \nabla^b \mathcal{T} \equiv (u_a u_b + g_{ab}) \nabla^b \mathcal{T} = \nabla_a \mathcal{T}. \tag{A11}$$

By definition there is no heat flow in thermal equilibrium, $q_a = 0$, and Eckart's law (1.5) gives

$$\begin{aligned}
\nabla_a \mathcal{T} + \mathcal{T} \nabla_a (\ln \sqrt{-g_{00}}) &= 0, \\
\nabla_a \ln \mathcal{T} + \nabla_a \ln \sqrt{-g_{00}} &= \text{const},
\end{aligned}$$

and finally

$$\mathcal{T} \sqrt{-g_{00}} = \text{const}. \tag{A12}$$

[1] R. C. Tolman, On the extension of thermodynamics to general relativity, Proc. Natl. Acad. Sci. U.S.A. **14**, 268 (1928).
[2] R. C. Tolman, On the weight of heat and thermal equilibrium in general relativity, Phys. Rev. **35**, 904 (1930).
[3] R. Tolman and P. Ehrenfest, Temperature equilibrium in a static gravitational field, Phys. Rev. **36**, 1791 (1930).
[4] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1987),
[5] O. Klein, On the thermodynamical equilibrium of fluids in gravitational fields, Rev. Mod. Phys. **21**, 531 (1949).
[6] J. Santiago and M. Visser, Tolman temperature gradients in a gravitational field, Eur. J. Phys. **40**, 025604 (2019).
[7] J. Santiago, On the connections between thermodynamics and general relativity, PhD thesis, Victoria University of Wellington, New Zealand, 2019.
[8] J. Santiago and M. Visser, Gravity's universality: The physics underlying Tolman temperature gradients, Int. J. Mod. Phys. D **27**, 1846001 (2018).
[9] J. M. Luttinger, Theory of thermal transport coefficients, Phys. Rev. **135**, A1505 (1964).
[10] P. Laskos-Patkos, P. S. Koliogiannis, A. Kanakis-Pegios, and C. C. Moustakidis, Thermodynamics of hot neutron stars and universal relations, Universe **8**, 395 (2022).
[11] H. C. Kim and Y. Lee, Heat conduction in general relativity, Classical Quantum Gravity **39**, 245011 (2022).
[12] J. Li, T. Guo, J. Zhao, and L. He, Do we need dense matter equation of state in curved spacetime for neutron stars?, Phys. Rev. D **106**, 083021 (2022).
[13] J. A. S. Lima, A. Del Popolo, and A. R. Plastino, Thermodynamic equilibrium in general relativity, Phys. Rev. D **100**, 104042 (2019).
[14] H. C. Kim and Y. Lee, Local temperature in general relativity, Phys. Rev. D **105**, L081501 (2022).
[15] J. A. S. Lima and J. Santos, Tolman-Ehrenfest-Klein law in non-Riemannian geometries, Phys. Rev. D **104**, 124089 (2021).
[16] C. Eckart, The thermodynamics of irreversible processes. 3. Relativistic theory of the simple fluid, Phys. Rev. **58**, 919 (1940).
[17] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology* (Cambridge University Press, Cambridge, England, 2012).
[18] C. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[19] E. W. Kolb and M. S. Turner, *The Early Universe* (AddisonWesley, Redwood City, CA, 1990).
[20] A. H. Carter, *Classical and Statistical Thermodynamics* (Prentice Hall, Upper Saddle River, NJ, 2001).
[21] V. Faraoni, *Cosmology in Scalar-Tensor Gravity* (Kluwer Academic, Dordrecht, 2004).
[22] R. L. Seliger and G. B. Whitham, Variational principles in continuum mechanics, Proc. R. Soc. A **305**, 1 (1968).
[23] B. F. Schutz, Perfect fluids in general relativity: Velocity potentials and a variational principle, Phys. Rev. D **2**, 2762 (1970).
[24] J. D. Brown, Action functionals for relativistic perfect fluids, Classical Quantum Gravity **10**, 1579 (1993).
[25] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973), ISBN 978-0-521-20016-5, 978-0-521-09906-6, 978-0-511-82630-6, 978-0-521-09906-6.
[26] H. A. Buchdahl, Temperature equilibrium in a stationary gravitational field, Phys. Rev. **76**, 427 (1949).
[27] J. P. Mbelek, Motion of a test body in the presence of an external scalar field which respects the weak equivalence principle, Acta Cosmol. **24**, 127 (1998).
[28] J. P. Mbelek, Modelling the rotational curves of spiral galaxies with a scalar field, Astron. Astrophys. **424**, 761 (2004).
[29] R. L. Forward, Roundtrip interstellar travel using laser-pushed light-sails, J. Spacecraft Rockets **21**, 187 (1984).
[30] A. Füzfa, Interstellar travels aboard radiation-powered rockets, Phys. Rev. D **99**, 104081 (2019).
[31] A. Füzfa, W. Dhelonga-Biarufu, and O. Welcomme, Sailing towards the stars close to the speed of light, Phys. Rev. Res. **2**, 043186 (2020).
[32] W. Zimdahl, D. J. Schwarz, A. B. Balakin, and D. Pavon, Cosmic anti-friction and accelerated expansion, Phys. Rev. D **64**, 063501 (2001).
[33] D. J. Schwarz, W. Zimdahl, A. B. Balakin, and D. Pavon, Cosmic acceleration from effective forces?, arXiv:astro-ph/0110296.
[34] W. Zimdahl, J. Triginer, and D. Pavon, Collisional equilibrium, particle production and the inflationary universe, Phys. Rev. D **54**, 6101 (1996).
[35] W. Zimdahl, Cosmological particle production and generalized thermodynamic equilibrium, Phys. Rev. D **57**, 2245 (1998).
[36] W. Zimdahl and A. B. Balakin, Kinetic theory for non-geodesic particle motion: Self-interacting equilibrium states and effective viscous fluid pressures, Classical Quantum Gravity **15**, 3259 (1998).
[37] W. Zimdahl and A. B. Balakin, Inflation in a self-interacting gas universe, Phys. Rev. D **58**, 063503 (1998).
[38] R. H. Dicke, Mach's principle and invariance under transformation of units, Phys. Rev. **125**, 2163 (1962).
[39] T. Damour, G. W. Gibbons, and C. Gundlach, Dark Matter, Time Varying *G*, and a Dilaton Field, Phys. Rev. Lett. **64**, 123 (1990).
[40] J. A. Casas, J. Garcia-Bellido, and M. Quiros, Scalar-tensor theories of gravity with phi dependent masses, Classical Quantum Gravity **9**, 1371 (1992).
[41] J. Garcia-Bellido, Dark matter with variable masses, Int. J. Mod. Phys. D **02**, 85 (1993).
[42] V. Faraoni and G. Vachon, Quasi-geodesics in relativistic gravity, Eur. Phys. J. C **81**, 22 (2021).
[43] J. Sultana and C. C. Dyer, Cosmological black holes: A black hole in the Einstein-de Sitter universe, Gen. Relativ. Gravit. **37**, 1347 (2005).
[44] H. Saida, T. Harada, and H. Maeda, Black hole evaporation in an expanding universe, Classical Quantum Gravity **24**, 4711 (2007).
[45] B. R. Majhi, Thermodynamics of Sultana-Dyer black hole, J. Cosmol. Astropart. Phys. 05 (2014) 014.

[46] K. Bhattacharya and B. R. Majhi, Temperature and thermodynamic structure of Einstein's equations for a cosmological black hole, Phys. Rev. D **94**, 024033 (2016).
[47] V. Faraoni, The Hawking temperature of expanding cosmological black holes, Phys. Rev. D **76**, 104042 (2007).
[48] V. Faraoni, *Cosmological and Black Hole Apparent Horizons*, Lectures notes Physics (Springer, New York, 2015), Vol. 907 ISBN 978-3-319-19239-0, 978-3-319-19240-6.
[49] V. Faraoni, A. Giusti, and B. H. Fahim, Spherical inhomogeneous solutions of Einstein and scalar–tensor gravity: A map of the land, Phys. Rep. **925**, 1 (2021).