Kiselev and Schwarzschild–de Sitter black holes in higher derivative theories of gravitation

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We consider the influence of the higher-order correction to the gravitational action inspired by the Goroff-Sagnotti term upon the Kiselev black hole with $\tilde{\omega} = -2/3$ and contrast the thus obtained results with the analogous results obtained for the Schwarzschild–de Sitter black hole. Expressing the perturbed solution in terms of the exact radius of the event horizon and the radius of the cosmological horizon of the unperturbed black hole we calculate corrections to the line element, to the cosmological horizon, and to the surface gravity. It is shown that although in both cases the lukewarm configuration does not exist classically (the equality of the surface gravities is possible only for the merged horizons), the sixth-order term removes the degeneracy of the classical solution and simultaneously shifts the degenerate horizon to a new place in the space of parameters. The lukewarm configuration is characterized by the value of the parameters that classically characterize the extremal solution. It is shown that the Karlhede scalar still may serve as the detector of the event and the cosmological horizon. Finally, we study the complex frequencies of the low-lying fundamental modes of the quasinormal oscillations and argue that they are the best candidates (at least theoretically) to distinguish between different black hole configurations.

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I. INTRODUCTION

It is widely accepted that the Einstein field equations, being classical, are valid only in the low-energy regime, where spacetime curvature is small and they should be replaced by a more fundamental theory, such as (not yet formulated) quantum gravity. Among the various proposals a prominent role is played by a class of theories in which the classical Einstein-Hilbert action functional is modified by the presence of the higher derivative terms. This is an old idea that can be traced back to the early days of general relativity. The reasons for such modifications are numerous but mostly related to our attempts to search for the imprints of quantum gravity (or even more general theory) on the classical solutions of the Einstein field equations. In the vast majority of approaches, the terms added to the classical gravitational action are constructed from the Riemann tensor, its covariant derivatives, and contractions (see for example [1,2] and the references cited therein). Good examples of this are the low-energy limit of the string theory [3,4], the Lovelock Lagrangians [5,6], and the quadratic (or higher order) gravity [7–9]. Among various alternatives to the general relativity a class of theories usually designated as $f(\mathcal{R})$ or their generalizations have received increasing attention [10–12]. Moreover, the (purely geometric) action functional of the quantized fields in the large mass limit can also be, after reformulation of the problem, included in this group [13–16]. Although the general relativity is the best and most accurate theory of gravitation we have and passes all the tests with flying colors [17–19], it is quite possible that we are on the threshold of a new, exciting era of discoveries [20].

In this paper, we will adopt a rather pragmatic point of view, which treats seriously the possibility of generalizing the classical gravitational action by adding higher-derivative terms. A somewhat idealized "observational" problem that we have in mind is the following: We have black holes characterized by the same radius of the event horizon. We also assume that in both cases there exists a second (cosmological) horizon. One of the configurations is accurately described by a classical solution of the Einstein field equations, whereas the second one is influenced by the higher derivative terms. Our task is to decide which black hole is which. A simple way to answer this question is to compare the outcome of a few measurements, such as the motion of test particles or the analysis of the complex frequencies of the quasinormal modes.

Typically, the new equations of motion are far more complicated than the classical ones and, except for the

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simplest configurations, they cannot be solved exactly. Roughly speaking, one has to either utilize numerical methods (the hybrid methods in which one starts with the analytic expressions and evaluates some of the terms numerically will be classified as numerical) or make use of approximations. In this paper, we will follow the latter method and propose an approach to analyze modifications of the "classical" Kiselev [21] and Schwarzschild–de Sitter (Kottler) [22] black holes caused by the higher-derivative gravity. Guided by the expected smallness of the coupling parameters we will treat the higher-derivative terms as perturbations of the "classical" action functional. Although the analysis may be carried out for any type of the Kiselev black hole, we focused on the $\tilde{\omega} = -2/3$ case. In this case the solution has two horizons, the event horizon and the cosmological horizon. There are, however, profound differences between the Schwarzschild–de Sitter and the Kiselev black hole. Indeed, the former is a solution to the vacuum Einstein field equations with the positive cosmological constant ($\Lambda > 0$), whereas the latter is a solution of the Einstein equations with the nonvanishing stress-energy tensor and $\Lambda = 0$. However, what do they have in common are their nice features that allow us to express the solutions solely in terms of the radii of the horizons.

Because of the great number of various formulations of extended gravity [11], each of which has its own aims, methods, and folklore, we have to restrict ourselves to some particular theory and our choice is dictated by the three conditions: (i) The modification should be physically motivated, (ii) its consequences should be relatively simple to calculate, and (iii) it should be a good representative of the whole class of theories. Inspired by the important and already classical result of Goroff and Sagnotti [23,24] that the divergent part of the on-shell two-loop effective gravitational action is given by

$$S_{\mathrm{div}} = \frac{209}{2880(4\pi)^2(D-4)} S_6, \tag{1}$$

where

$$S_6 = \int d^4x \sqrt{-g} \mathcal{R}_{ab}{}^{cd} \mathcal{R}_{cd}{}^{ef} \mathcal{R}_{ef}{}^{ab}. \tag{2}$$

$\mathcal{R}_{ab}{}^{cd}$ is the Riemann tensor and D is the dimension, and we will add to the gravitational action the term proportional to S_6. It can be thought of as the simplest modification of the pure gravity functional that absorbs the divergent term. Of course, there are other six-derivative (or higher) terms that can be taken into account. We decided, however, not to include them here. The reason for it is twofold: first, inclusion of the additional curvature terms, each of which has its own coupling parameter, would unnecessarily complicate the final results. Second, as the additional terms in the gravitational action can be treated exactly in the same manner as the "Goroff-Sagnotti term," one can easily generalize the results presented here. As long as the additional terms are purely geometric, we do not expect any technical complications. Thus our three requirements are fulfilled: the Goroff-Sagnotti term is certainly physically well motivated, its consequences are relatively easy to calculate and interpret (we have only one coupling constant) and the techniques needed in the calculation are universal.

The paper is organized as follows. In Sec. II we give an overview of that features of the Kiselev and Schwarzschild–de Sitter black holes that will be needed in the subsequent sections. The equations of motion and the low energy effective Lagrangians are discussed in Secs. III and IV. In Sec. V the first-order corrections to the Kielev's black hole are computed and discussed, and the analogous discussion for the Schwarzschild–de Sitter black hole is presented in Sec. VI. In Sec. VII we study the fundamental modes of the low-lying quasinormal oscillations. Finally, Sec. VIII contains additional remarks.

Throughout the paper we use natural units $c = G = 1$. The signature of the metric is taken to be "mainly positive," i.e., $+2$, and the conventions for the curvature tensor are $\mathcal{R}_a{}^{bcd} = \partial_c \Gamma^a{}_{bd} \ldots$ and $\mathcal{R}^a{}_{bac} = \mathcal{R}_{bd}$.

II. KISELEV AND SCHWARZSCHILD–de Sitter BLACK HOLES

The geometry of the Kiselev black hole [21] (see also [25,26]) is described by a simple line element

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2, \tag{3}$$

where

$$f(r) = h^{-1}(r) = 1 - \frac{r_g}{r} - \frac{r_n}{r^{3\tilde{\omega}+1}}, \tag{4}$$

$r_g = 2M$ (M is the black hole mass), r_n is the dimensional normalization constant, and $\tilde{\omega}$ is the state parameter. It is the exact solution of the Einstein field equations with the source term given by

$$T^t_t = T^r_r = \rho, \tag{5}$$

$$T^\theta_\theta = T^\phi_\phi = -\frac{1}{2}(3\tilde{\omega} + 1)\rho, \tag{6}$$

where

$$\rho = \frac{3\tilde{\omega} r_n}{8\pi} \frac{1}{r^{3(1+\tilde{\omega})}}. \tag{7}$$

The form of the stress-energy tensor can be deduced from Kiselev's paper, however, because of a different normalization it slightly differs from the tensor adopted here. Integration of the Einstein field equations and making use of the condition $f(r)h(r) = 1$ (equivalent to appropriate rescaling of t) gives the line element (3), where r_g is an integration constant. Even a brief look at the Kiselev solution reveals its generality: Indeed, depending on a choice of the parameters one has the Schwarzschild solution, the Reissner-Nordström solution, the Schwarzschild-(anti)–de Sitter solution and the solution that resembles the de Sitter solution. We have also an infinite family of new solutions that smoothly vary between this well-known "classical" solutions.

In this paper we shall concentrate on the special case $\tilde{\omega} = -2/3$. It has at most two horizons: the black hole horizon and the cosmological horizon. Moreover, for a certain set of the parameters both horizons may merge leading to the extreme configuration. First, observe that with such a choice of the parameter $\tilde{\omega}$ the line element (3) reduces to

$$ds^2 = -\left(1 - \frac{r_g}{r} - r_n r\right)dt^2 + \left(1 - \frac{r_g}{r} - r_n r\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{8}$$

If $r_n > 0$ and $r_n < 1/4r_g$ (the case considered in this paper), then the event and the cosmological horizons are located at

$$r_+ = \frac{1 - \sqrt{1 - 4r_g r_n}}{2r_n} \tag{9}$$

and

$$r_c = \frac{1 + \sqrt{1 - 4r_g r_n}}{2r_n}, \tag{10}$$

respectively, whereas the condition $r_n = 1/4r_g$ corresponds to the extremal configuration. The degenerate event horizon, r_d, is now located at

$$r_d = r_+ = r_c = 2r_g. \tag{11}$$

On the other hand, for small r_n ($r_n r_g \ll 1$) one has

$$r_+ \simeq r_g(1 + r_n r_g), \qquad r_c \simeq \frac{1}{r_n} - r_+. \tag{12}$$

The relations (9) and (10) can be inverted to (r_+, r_c) parametrization,

$$M = \frac{r_+ r_c}{2(r_+ + r_c)} \quad \text{and} \quad r_n = \frac{1}{r_+ + r_c}, \tag{13}$$

in which the metric reads

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_+ r_c}{(r_+ + r_c)r} - \frac{r}{r_+ + r_c}\right)dt^2 \\ &+ \left(1 - \frac{r_+ r_c}{(r_+ + r_c)r} - \frac{r}{r_+ + r_c}\right)^{-1} dr^2 + r^2 d\Omega^2. \end{aligned} \tag{14}$$

Equally well, we can solve the system $f(r_+) = 0$ and $f(r_c) = 0$ with respect to r_g and r_n and substitute the thus constructed solutions into the line element (8).

The above line element can be compared with the Schwarzschild–de Sitter solution expressed in the same parametrization

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{15}$$

where

$$f(r) = 1 - \frac{r_+}{r} - \frac{r^2}{r_+^2 + r_+ r_c + r_c^2} + \frac{r_+^3}{r(r_+^2 + r_+ r_c + r_c^2)}. \tag{16}$$

It is the solution of the vacuum Einstein field equations

$$R_{ab} = \Lambda g_{ab} \tag{17}$$

with a positive cosmological constant

$$\Lambda = \frac{3}{r_+^2 + r_+ r_c + r_c^2}. \tag{18}$$

The third (unphysical) root of the equation $f(r) = 0$ is given by $r_- = -(r_+ + r_c)$. For our purposes we prefer this representation of the Schwarzschild–de Sitter solution over the standard one, in which the function $f(r)$ is expressed in term of the mass

$$M = \frac{r_+ r_c (r_+ + r_c)}{2(r_+^2 + r_+ r_c + r_c^2)} \tag{19}$$

and the (positive) cosmological constant Λ. If $0 < r_+ \leq r_c$ and

$$\frac{27}{4} \frac{r_+^2 r_c^2 (r_+ + r_c)^2}{(r_+^2 + r_+ r_c + r_c^2)^3} \leq 1, \tag{20}$$

then we interpret r_+ and r_c as the radii of the event horizon and the cosmological horizon, respectively. In the standard representation the roots (two of which describe radii of the horizons) are expressed in terms of the trigonometric functions.

Equations (12) and (18) give the functions of the physical horizon areas A_i that are independent of mass [27,28]. Simple manipulations give

$$A_{+} + A_{c} + 2\sqrt{A_{+}A_{c}} = \frac{4\pi}{r_n^2} \tag{21}$$

and

$$A_{+} + A_{c} + \sqrt{A_{+}A_{c}} = \frac{12\pi}{\Lambda} \tag{22}$$

for the Kiselev and the Schwarzschild–de Sitter black holes, respectively.

It can be easily demonstrated that the geometry of the closest vicinity of the degenerate horizon is of the Nariai type. Indeed, consider a black hole configuration with the horizons located very closely each other. For $r_{+} \leq r \leq r_{c}$ the function $f(r)$ may be approximated by a parabola $\beta(r - r_{+})(r - r_{c})$. Assuming (approximately) $x_{+} = x_{d} - \varepsilon$ and $x_{c} = x_{d} + \varepsilon$, one obtains $\beta = -1/2r_d^2$. Now, introducing new coordinates $t = T/(\varepsilon B)$, $r = r_d + \varepsilon \cos y$ and subsequently taking the limit $\varepsilon \to 0$, one obtains

$$ds^2 = 2r_d^2(-dT^2 \sin^2 y + dy^2) + r_d^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{23}$$

provided $B = -\beta$. Topologically, it is $dS_2 \times S^2$, i.e., a product of the two-dimensional de Sitter space and the two-dimensional round two-sphere. The total curvature, $\mathcal{R}$, can be decomposed into the curvature of the two-dimensional de Sitter space, $\mathcal{R}_{dS} = 1/r_d^2$, and the curvature of the two-sphere $\mathcal{R}_{S^2} = 2/r_d^2$. The same result can be obtained calculating $\mathcal{R}$ of the line element (8) and restricting to the extreme configuration at the degenerate event horizon. Indeed, simple calculations give $\mathcal{R} = \mathcal{R}_{dS} + \mathcal{R}_{S^2} = 3/r_d^2$. On the other hand, for the Schwarzschild–de Sitter black hole the geometry of the closest vicinity of the degenerate horizon is described by the Nariai line element and $\mathcal{R}_{dS} = \mathcal{R}_{S^2} = 2/r_d^2$.

Finally, we observe that $\tilde{\omega} = -2/3$ case does not allow for the lukewarm configurations, i.e., the solutions for which the surface gravity of the event horizon and the cosmological horizon are equal. It can be easily demonstrated that the only solution with this property is degenerate, i.e., it has vanishing surface gravity of the horizon. Indeed, the condition

$$f'(r_{+}) = -f'(r_{c}) \tag{24}$$

gives $r_c = r_{+}$ and the line element takes the form (3) with

$$f(r) = h^{-1}(r) = 1 - \frac{r_{+}}{2r} - \frac{r}{2r_{+}}. \tag{25}$$

Similarly, for the Schwarzschild–de Sitter black hole, one has $r_c = r_{+}$ and

$$f(r) = 1 - \frac{2r_{+}}{3r} - \frac{r^2}{3r_{+}^2}. \tag{26}$$

III. THE FIELD EQUATIONS

Thus far, our discussion has been purely classical, even if the stress-energy tensor given by Eqs. (5)–(7) is somewhat nonstandard. Now, let us assume that the total gravitational action, S_{total}, may be decomposed into the Einstein action functional, S_g, and the higher derivative corrections, S_H

$$S_{\text{total}} = \frac{1}{16\pi}\int d^4x\sqrt{g}\mathcal{R} + S_H. \tag{27}$$

We shall not specify the exact form of the latter here but postpone this to the next section. Instead, we merely assume that (i) it is constructed solely from the Riemann tensor (and as such it depends functionally on the metric tensor), (ii) its contribution to the total gravitational action can be regarded as small, and (iii) the resulting equations cannot be solved exactly, except maybe some simple cases. Quite a number of various interesting and physically well motivated Lagrangians fit into this class. For example, the Lovelock gravity, the Lagrangians of quadratic gravity, the Lagrangians constructed to absorb the divergent Goroff-Sagnotti term, or the Lagrangians constructed from all possible time-reversal-invariant operators of dimension six, to name a few. Moreover, the semiclassical theory with the effective action of the quantized massive fields in a large mass limit may also be reformulated in this way.

Having specified the action, the field equations can be obtained by functional differentiation of the total gravitational action (possibly supplemented with the matter term) with respect to the metric tensor. The resulting equations can be written as

$$G^{ab} + \Lambda g^{ab} = 8\pi(\mathfrak{T}^{ab} + T^{ab}), \tag{28}$$

where

$$\mathfrak{T}^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}S_H \tag{29}$$

and T^{ab} is a stress-energy tensor of the matter.

Arguably the most interesting and important application of the higher derivative theories is the search for new effects and modifications of the classical solutions. Unfortunately, in most cases the complexity of the problem practically excludes construction of the exact solutions and to obtain valuable information one has to either devise some approximation scheme or treat the problem numerically. Here we shall choose the first option. To illustrate the procedure, we consider the simplest case of the spacetime generated by the spherically symmetric matter distribution with and without the cosmological constant. As we are interested in the effect of the higher order terms upon the classical solutions, which will be referred to as a back-reaction problem, we use a more general static and spherically symmetric line element [29]

$$ds^2 = -e^{2\psi(r)}\left(1-\frac{2m(r)}{r}\right)dt^2 + \left(1-\frac{2m(r)}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{30}$$

where $\psi(r)$ and $m(r)$ are functions that are to be determined. For example, the Kiselev black hole is given by $\psi(r) = 0$ and

$$m(r) = \frac{r_g}{2} + \frac{r_n}{2r^{3\tilde{\omega}}}. \tag{31}$$

Similarly, the Schwarzschild–de Sitter black hole is described by $\psi(r) = 0$ and

$$m(r) = M + \frac{\Lambda r^3}{6}. \tag{32}$$

The backreaction problem is now described by the differential equations:

$$-\frac{2}{r^2}\frac{d}{dr}m(r) + \Lambda = 8\pi(\mathfrak{T}_t^t + T_t^t) \tag{33}$$

and

$$-\frac{2}{r^2}\frac{d}{dr}m(r) + \frac{2}{r}\left(1-\frac{2m(r)}{r}\right)\frac{d}{dr}\psi(r) + \Lambda = 8\pi(\mathfrak{T}_r^r + T_r^r), \tag{34}$$

where the cosmological constant should be set to zero for the Kiselev solution. In practice, it is easier to study Eq. (33) and the equation obtained by subtracting (33) from (34).

According to our assumptions, the higher-derivative term in the right-hand side of Eq. (28) is much smaller than the classical term. It means that we can introduce a small (dimensionless) parameter ε

$$\mathfrak{T}_a^b \to \varepsilon \mathfrak{T}_a^b, \tag{35}$$

and expand the functions $m(r)$ and $\psi(r)$ as

$$m(r) = m_0(r) + \varepsilon m_1(r) + \mathcal{O}(\varepsilon^2) \tag{36}$$

and

$$\psi(r) = \varepsilon \psi_1(r) + \mathcal{O}(\varepsilon^2). \tag{37}$$

A role played by ε is to keep track of the order of terms in the complicated expansions and it is especially useful in computer algebra calculations. It should be set to 1 at the final stage of calculations.

Now, expanding the equations with respect to the parameter ε and retaining only the linear terms, one obtains the system of three differential equations for unknown $m_0(r)$, $m_1(r)$, and $\psi_1(r)$. Let us return to the Eq. (33). Our task is to integrate (formally) this equation with the initial condition $m(r_+) = r_+/2$. We rewrite this condition in the form suitable for further analysis: $m_0(r_+) = r_+/2$ and $m_1(r_+) = 0$. The first condition means that we work with the exact [to $\mathcal{O}(\varepsilon^2)$] radius of the event horizon from the very beginning, whereas the second one means that there is no room for the additional corrections of the location event horizon in the higher-order calculations, as they were already taken into account in its definition. Simple calculations with the Kiselev stress-energy tensor and unspecified $\tilde{\omega}$ give

$$m_0(r) = \frac{r_+}{2} - \frac{1}{2}\frac{r_n}{r_+^{3\tilde{\omega}}} + \frac{1}{2}\frac{r_n}{r^{3\tilde{\omega}}} \tag{38}$$

and

$$m_1(r) = -4\pi \int_{r_+}^{r} dr'(r')^2 \mathfrak{T}_t^t(r'). \tag{39}$$

As $\mathfrak{T}_t^b$ is constructed solely from the curvature tensor and its derivatives and contractions, the last term of the above equation is calculated for the zeroth-order solution.

For the vacuum field equations with the cosmological term one has

$$m_0(r) = \frac{r_+}{2} + \frac{1}{6}\Lambda r^3 - \frac{1}{6}\Lambda r_+^3. \tag{40}$$

Denoting the sum of the first and last term in the right-hand side of the above equation by M one obtains the standard form of the Schwarzschild–de Sitter solution, with the condition

$$1 - \frac{2M}{r_+} - \frac{\Lambda r_+^2}{3} = 0. \tag{41}$$

The correction $m_1(r)$ is still (formally) given by the analog of Eq. (39).

The second independent equation has the following formal solution

$$\psi_1(r) = 4\pi \int dr'(r')^2 \frac{\mathfrak{T}_r^r - \mathfrak{T}_t^t}{r - 2m(r)} + C_1, \tag{42}$$

where C_1 is an integration constant that can be determined with the aid of the condition $g_{tt}(r_\infty) g_{rr}(r_\infty) = -1$. In general, r_∞ is either infinity or the radius of the cosmological horizon or the radius of a cavity in which the spherically symmetric black hole is located [30,31]. Once again, to construct a solution it suffices to know $m_0(r)$. Finally, observe that to secure the finiteness of the function $\psi_1(r)$ one requires regularity of $\mathfrak{T}_a^b$ in a physical sense. Let us elaborate on this. The components of some tensor $\mathfrak{B}^{ab}$ in a freely falling frame are regular in a physical sense if $\mathfrak{B}_a^b$ and $(\mathfrak{B}_r^r - \mathfrak{B}_t^t)/(r - 2m_0(r))$ are finite at the event horizon. All the types of $\mathfrak{T}_a^b$ tensors considered in this paper do satisfy these requirements.

IV. THE LOW-ENERGY EFFECTIVE LAGRANGIAN

The next step requires specification of the action S_H. First, let us focus on the pure gravity. The one-loop corrections to the pure classical gravity are quadratic, i.e., the Lagrangian is built from the four-derivative terms $\mathcal{R}_{ab}{}^{cd}R_{cd}{}^{ab}$, $\mathcal{R}_{ab}\mathcal{R}^{ab}$, and $\mathcal{R}^2$. It should be noted that the last two terms vanish on shell. The divergent term calculated by 't Hooft and Veltman have the form [32]

$$\frac{1}{(4\pi)^2(D-4)}\left(\frac{1}{120}\mathcal{R}_{ab}\mathcal{R}^{ab} + \frac{7}{20}\mathcal{R}^2\right), \tag{43}$$

where D is the dimension, and hence the one-loop divergences of pure gravity vanish on shell. The Kretschmann scalar does not introduce any additional term in the equations of motion since the Gauss-Bonnet invariant

$$\mathcal{R}_{abcd}\mathcal{R}^{abcd} - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}^2 \tag{44}$$

is a total derivative.

On general grounds one expects that at the two-loop level the off-shell divergences should be constructed from 10 curvature invariants, which may be divided into the three groups. To the first group belong $\mathcal{R}^3, \mathcal{R}_{;a}\mathcal{R}^{;a}$, $\mathcal{R}_{ab;c}\mathcal{R}^{ab;c}, \mathcal{R}\mathcal{R}_{ab}\mathcal{R}^{ab}$, to the second $\mathcal{R}_{ab}\mathcal{R}_{cd}\mathcal{R}^{acbd}$, $\mathcal{R}^b_a\mathcal{R}^c_b\mathcal{R}^a_c$, $\mathcal{R}\mathcal{R}_{abcd}\mathcal{R}^{abcd}$, $\mathcal{R}_{abcd}\mathcal{R}^{abce}\mathcal{R}^d_e$, and finally the third group comprises of the two terms $\mathcal{R}_{ab}{}^{cd}\mathcal{R}_{cd}{}^{ef}\mathcal{R}_{ef}{}^{ab}$ and $\mathcal{R}_{abcd}\mathcal{R}^a{}_e{}^c{}_f\mathcal{R}^{bedf}$. The invariants of the first two groups vanish on shell. In their seminal papers, Goroff and Sagnotti [23,24] showed that the divergent part of the on-shell effective action is given by

$$\frac{209}{2880(4\pi)^2(D-4)}\int d^4x\sqrt{-g}\mathcal{R}_{ab}{}^{cd}\mathcal{R}_{cd}{}^{ef}\mathcal{R}_{ef}{}^{ab}, \tag{45}$$

and thus the Einstein theory of gravitation is not renormalizable. This is a very interesting and important result, though a bit pessimistic, which suggests a simple modification of the Einstein gravity. For example, one can add to the gravitational action functional the term

$$S_H = \beta\int d^4x\sqrt{-g}\mathcal{R}_{ab}{}^{cd}\mathcal{R}_{cd}{}^{ef}\mathcal{R}_{ef}{}^{ab}. \tag{46}$$

It may be regarded as a simplest generalization of the pure Einstein gravity that absorbs the two-loop divergent term. The functional derivative of S_H with respect to the metric tensor introduces into the field equations the term proportional to

$$\begin{aligned}\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}S_H = \mathfrak{T}^{ab}_{(6)} = 2\beta\Big(&12\mathcal{R}_c{}^b{}_{;d}\mathcal{R}^{da;c} - 12\mathcal{R}_c{}^b{}_{;d}\mathcal{R}^{ca;d}\\ &- 6\mathcal{R}_{cde}{}^b{}_{;i}\mathcal{R}^{cdia;e} + 12\mathcal{R}_c{}^b{}_{;de}\mathcal{R}^{cdea}\\ &+ 12\mathcal{R}_c{}^a{}_{;de}\mathcal{R}^{cdeb} - 12\mathcal{R}_{cdei}\mathcal{R}_j{}^{ceb}\mathcal{R}^{djia}\\ &- 6\mathcal{R}_{cd}\mathcal{R}_{ei}{}^{cb}\mathcal{R}^{daei} + \frac{1}{2}g^{ab}\mathcal{R}_{cdei}\mathcal{R}_{jk}{}^{cd}\mathcal{R}^{eijk}\Big).\end{aligned} \tag{47}$$

If the matter fields are present, then the gravitational part of the action has to be modified and the simplest nontrivial modification would include the quadratic terms. Their role is to absorb all possible one-loop divergences. With such a choice of the action functional the tensor $\mathfrak{T}^{ab}_{(4)}$ has the form

$$\begin{aligned}\mathfrak{T}^{ab}_{(4)} = \beta_1&\left(2\mathcal{R}^{;ab} + \frac{1}{2}\mathcal{R}^2g^{ab} - 2\Box\mathcal{R}g^{ab} - 2\mathcal{R}\mathcal{R}^{ab}\right)\\ &+ \beta_2\left(\mathcal{R}^{;ab} - \Box\mathcal{R}^{ab} - \frac{1}{2}\Box\mathcal{R}g^{ab} + \frac{1}{2}\mathcal{R}_{cd}\mathcal{R}^{cd}g^{ab}\right.\\ &\left. - 2\mathcal{R}_{cd}\mathcal{R}^{cadb}\right),\end{aligned} \tag{48}$$

where β_1 and β_2 are the parameters that should be determined from observations. At the two loops one may include the terms constructed from the six-derivative curvature invariants, as has been done for example, in Refs. [33–35]. Following Ref. [36], we will restrict ourselves to the simplest case of the low-energy effective action functional inspired by the Goroff-Sagnotti term, i.e., we neglect the four-derivative terms and retain only the six-derivative contribution given by (46). It should be noted that our six-derivative Lagrangian is sufficient to demonstrate the main features of the problem. Indeed, if there are some technical, mathematical, or even conceptual problems in the calculations or interpretation of the results within the framework of our simplified theory, the same is expected for its more complicated version. Similarly, if the calculation goes smoothly, then the same is expected for a more complex Lagrangian and the only difference (except the results) is the scale of the calculations.

V. FRIST-ORDER CORRECTIONS TO THE KISELEV BLACK HOLE WITH $\tilde{\omega} = -2/3$

In this section we shall restrict ourselves to the class of the Kiselev black holes with the event and the cosmological horizon. The extremal configuration as well as the configuration with one nondegenerate horizon can be obtained as a result of some limit procedure. Our general strategy is as follows. First, for a general spherically symmetric line element (30) we construct the field equations with $\mathfrak{T}^{ab}_{(6)}$ and the Kiselev stress-energy tensor. In the next step, we insert (36) and (37) into the (33) and (34), and linearize the thus obtained result with respect to the ε parameter. As has been discussed earlier, this procedure gives a system of the three differential equations for m_0, m_1, and ψ_1 that have to be solved with the appropriate conditions. The remaining equations are automatically satisfied once those three functions are known.

The zeroth-order equation has already been solved in Sec. III. Putting $\tilde{\omega} = -2/3$ in Eq. (38) gives

$$m_0(r) = \frac{r_+}{2} - \frac{r_n r_+^2}{2} + \frac{r_n r^2}{2}, \tag{49}$$

or, equivalently, in (r_+, r_c) parametrization

$$m_0(r) = \frac{r^2}{2(r_+ + r_c)} + \frac{r_+ r_c}{2(r_+ + r_c)}, \tag{50}$$

where, as before, r_c is the radius of the cosmological horizon of the unperturbed Kiselev black hole. With $r_c = \alpha r_+$ and $r = x r_+$ the above equation can be rewritten in a more transparent form

$$m_0 = \frac{\alpha r_+}{2(\alpha+1)} + \frac{r_+ x^2}{2(\alpha+1)}. \tag{51}$$

The line element (30) with (50) gives (14).

The (r_+, r_c) parametrization has some advantages: (i) it has a clear physical interpretation, (ii) for a given $\alpha = r_c/r_+$ the components of tensors constructed from the curvature depend solely on a rescaled radial coordinate $x = r/r_+$ $(1 \le x \le \alpha)$, and (iii) with our choice of the initial conditions the "exact" radius of the event horizon, r_+, enters both the zeroth and the first order equations. It should be noted however, that r_c is the radius of the cosmological horizon of the unperturbed Kiselev black hole and the corrected extreme configuration is no longer described by the condition $r_c/r_+ = 1$. In what follows, we denote the radius of the cosmological horizon of the perturbed black hole by $\tilde{r}_c$.

The structure of Eq. (39) shows that it is relatively easy to calculate the first-order correction $m_1(r)$ as the problem reduces to simple quadratures. The second equation, (42), is slightly more complicated. First, observe that it requires that the difference between the radial and time components of $\mathfrak{T}_a^b$ divided by the zeroth-order approximation of g_{tt} be finite as $r \to r_+$. Such a behavior is required to secure the regularity of the stress-energy tensor. On general grounds one expects that the stress-energy tensor $\mathfrak{T}_a^b$ should be regular for regular geometries, and, consequently, this property should be satisfied in a natural way. It is relatively easy to show that this is indeed the case. It can be demonstrated that the tensors $\mathfrak{T}_a^b$ are covariantly conserved and the difference between the radial and time components factorizes as

$$\mathfrak{T}_r^r - \mathfrak{T}_t^t = \left(1 - \frac{2m_0(r)}{r}\right) F(r), \tag{52}$$

where $F(r)$ is a regular function of r. The second subtlety is related to the integration constant C_1, which, according to our previous discussion should be determined at the cosmological horizon $\tilde{r}_c$ rather than r_c.

Let us return to the semiclassical line element. Inserting the components of the stress-energy tensor into the first order equations (39) and (42) one obtains, after some algebra and massive simplifications, an amazingly simple result:

$$\begin{aligned} m_1(r) = \frac{4\pi\beta}{r_+^3(1+\alpha)^3}\bigg(&10\alpha^3 - 21\alpha^2 - 6\alpha + \frac{98\alpha^3}{x^6} - \frac{108\alpha^3}{x^5} \\ &- \frac{108\alpha^2}{x^5} + \frac{105\alpha^2}{x^4} + \frac{24\alpha^2}{x^3} + \frac{24\alpha}{x^3} - \frac{6\alpha}{x^2} - \frac{12\alpha}{x} - \frac{12}{x} \\ &- 12\ln x + 12\bigg) \end{aligned} \tag{53}$$

and

$$\psi_1(r) = \frac{48\pi\beta}{r_+^4(1+\alpha)^2}\left(\frac{9}{\alpha^4} - \frac{1}{\alpha^3} + \frac{3}{\alpha^2} - \frac{9\alpha^2}{x^6} + \frac{\alpha}{x^4} - \frac{3}{x^2}\right). \tag{54}$$

The line element (30) together with (51) and (53)–(54) provide a complete solution to our problem. In (r_+, r_c) the representation the radius of the event horizon r_+ is exact, whereas r_c acquires a small correction r_1. Indeed, simple calculation gives

$$\tilde{r}_c = r_c + \varepsilon r_1, \tag{55}$$

where

$$\begin{aligned} r_1 = &-\frac{8\pi\beta}{r_+^3\alpha^3(1+\alpha)}(10\alpha^4 - 21\alpha^3 + 4\alpha^2 - 21\alpha + 10) \\ &+ \frac{96\pi\beta}{r_+^3(1+\alpha)^2(\alpha-1)}\ln\alpha. \end{aligned} \tag{56}$$

Note, that it has a well-defined limit

$$\lim_{\alpha\to 1} r_1 = \frac{96\pi\beta}{r_+^3}. \tag{57}$$

Inspection of Eq. (56) shows that for the Kiselev black hole the correction r_1 is positive for $1 < \alpha \le 2.43684$ and negative for remaining admissible values of α.

Our next task is the construction of some important characteristics of the corrected Kiselev solution. Since the calculations are complicated and the results are long and hard to interpret, we shall not display them here.

Instead, we briefly discuss the main features of the considered problems. In the next subsections, we shall briefly analyze the Karlhede scalar, the proper acceleration of test particles, and the surface gravity (temperature).

A. Karlhede scalar

First, let us consider the Karlhede scalar

$$I_K = \mathcal{R}_{abcd;e}\mathcal{R}^{abcd;e}, \tag{58}$$

which, for a certain class of metrics can act as the detector of the event horizon [37–41]. For the unperturbed black hole one has

$$I_K^{(0)} = \frac{16(\alpha - x)(x-1)}{r_+^6 x^9 (1+\alpha)^3}\left(x^4 + \frac{45}{4}\alpha^2\right), \tag{59}$$

which vanishes at the event ($x = 1$) and the cosmological horizon ($x = \alpha$). Moreover, it changes sign both on r_+ and r_c. Now, making use of the components of the metric tensor of the corrected black hole it can be shown that the total $I_K = I_K^{(0)} + I_K^{(1)}$ is zero at r_+ and its radial derivative is positive for $\alpha > 1$. The case of the corrected cosmological horizon is slightly more involved, as the calculations have to be carried out for $\tilde{r}_c$ rather than r_c. After some algebra, it can be shown that the total Karlhede scalar vanishes at $\tilde{r}_c$ and changes sign there. It should be noted however that the correction $I_K^{(1)}$ is positive for $\alpha \in (1.316, 2.267)$.

B. Acceleration of the massive test particle

It is of some interest to analyze the acceleration of the massive test particle that is initially at rest in the corrected spacetime. The four-velocity of the test particle parametrized by the proper time τ and its acceleration is given by

$$U^a = \left[\frac{dt}{d\tau}, 0, 0, 0\right] \tag{60}$$

and

$$a^a = \frac{dU^a}{d\tau} + \Gamma^a{}_{bc}U^a U^b, \tag{61}$$

respectively. In the particle's proper rest frame the acceleration is given by

$$a^{\hat{r}} = \frac{1}{2}(g^{rr})^{1/2}\frac{d}{dr}\ln(-g_{tt}). \tag{62}$$

For the unperturbed black hole one has

$$a^{\hat{r}} = \frac{\alpha - x^2}{2x^{3/2} r_+}\frac{1}{\sqrt{(\alpha - x)(x-1)(1+\alpha)}}, \tag{63}$$

and the gravitational attraction is balanced by the cosmological repulsion at

$$x = \alpha^{1/2}. \tag{64}$$

As the corrections to the acceleration are rather complicated and not very illuminating we shall not display them here. Instead, we shall briefly summarize their main features. Assume $\beta > 0$ (the case $\beta < 0$ can be studied in a similar way). Our calculations suggest that for $\alpha < 2.437$ the higher derivative term makes the Kiselev black hole more attractive for $x < x_{\rm crit}$ and less repulsive for $x > x_{\rm crit}$, where $x_{\rm crit}$ depends on α and defines points at which the acceleration of the massive test particle vanishes. The leading behavior of $x_{\rm crit}$ is, of course, given by (64). On the other hand, for $\alpha > 2.437$, it becomes gradually more repulsive near the cosmological horizon.

C. Surface gravities

Our next task is to construct the surface gravities of the horizons. It is a nontrivial problem. Frequently used in this regard is the well-known formula

$$\kappa^2 = g_{ab}\frac{(K^c\nabla_c K^a)(K^d\nabla_d K^b)}{-K^2}, \tag{65}$$

where $K = k\partial/\partial t$ is the Killing vector and k is a normalization factor. We have two natural choices of k. The first one requires the Killing vector to be null on both the black hole and the cosmological horizons, and, additionally, that its norm approaches $\sqrt{r/(r_+ + r_c)}$ as r goes to infinity, whereas the second one requires that the condition $K^2 = -1$ should be satisfied at points at which the black hole attraction and the cosmological repulsion cancel out [42]. In what follows, we have decided to work with the first definition of the surface gravity as the corresponding results for the surface gravities of the event and cosmological horizon constructed with the aid of the second definition can easily be obtained by multiplication with a constant numerical factor.

One can also find the surface gravity by defining two null vector fields β^a and l^a in the spacetime described by the Bardeen-like type of the line element. Indeed, let us transform the line element (30) to the ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ), where

$$dv = dt + dr_*, \tag{66}$$

and

$$dr_* = \frac{dr}{e^{\psi}(1 - 2mr^{-1})}. \tag{67}$$

Simple manipulations give

$$ds^2 = -e^{2\psi}\left(1-\frac{2m}{r}\right)dv^2 + 2e^{\psi}dvdr + r^2 d\theta^2 + r^2 d\theta^2 \sin^2\theta d\phi^2, \tag{68}$$

where $\psi = \psi(r)$ and $m = m(r)$ because the perturbed spacetime should be spherically symmetric and static. Now, define two future-directed vector fields: the outgoing null field l^a defined as

$$[l^v, l^r, l^\theta, l^\phi] = [1, e^{\psi}(1-2mr^{-1})/2, 0, 0] \tag{69}$$

and the ingoing field β^a

$$[\beta^v, \beta^r, \beta^\theta, \beta^\phi] = [0, -e^{-\psi}, 0, 0], \tag{70}$$

normalized according to the condition $l^a\beta_a = -1$. The quantity

$$\kappa = -\beta^a l^b \nabla_b l_a, \tag{71}$$

when evaluated at the horizon, defines the surface gravity. The expansion of the outgoing rays is given by

$$\theta = \nabla_a l^a - \kappa. \tag{72}$$

For the line element (68) and the vector fields l^a and β^a one has

$$\kappa = e^{\psi}\left(\frac{d}{dr}\psi\right)\left(1-\frac{2m}{r}\right) - \frac{e^{\psi}}{r}\left(\frac{d}{dr}m\right) + \frac{e^{\psi}}{r^2}m \tag{73}$$

and

$$\theta = e^{\psi}\left(1-\frac{2m}{r}\right). \tag{74}$$

As is well known the apparent horizon is determined by the condition $\theta = 0$ and since in the static case both the event and the apparent horizon coincide it is also a condition for calculation of the radius of the event horizon.

The identical result for the surface gravity can be constructed with the aid of the Wick rotation. The complexified (Euclidean) form of the line element (30) obtained from the rotation $(t \to -it)$ has no conical singularity as r approaches horizon, r_H, provided the time coordinate is periodic, with a period β given by

$$\beta = 4\pi \lim_{r\to r_H} (g_{tt}g_{rr})^{1/2}\left(\frac{d}{dr}g_{tt}\right)^{-1}, \tag{75}$$

and, consequently, the surface gravity is related to β by

$$\frac{\kappa}{2\pi} = \beta^{-1}. \tag{76}$$

Now, making use of the general formulas for the surface gravities it can be easily shown that to $\mathcal{O}(\varepsilon)$ one has

$$\kappa_H = \frac{\alpha-1}{2r_+(\alpha+1)} - \frac{24\pi\varepsilon\beta}{r_+^5\alpha^4(1+\alpha)^3}(\alpha^7 - 2\alpha^6 - 4\alpha^5 - 3\alpha^4 - 3\alpha^3 + 4\alpha^2 - 10\alpha + 9) \tag{77}$$

and

$$\kappa_C = \frac{\alpha-1}{2r_+\alpha(\alpha+1)} + \frac{48\pi\varepsilon\beta}{r_+^5\alpha^2(\alpha-1)(1+\alpha)^2}\ln\alpha - \frac{4\pi\varepsilon\beta}{r_+^5\alpha^5(1+\alpha)^3}(10\alpha^6 - \alpha^5 - 28\alpha^4 - 34\alpha^3 + 20\alpha^2 - 49\alpha + 58). \tag{78}$$

Numerically, the correction to the temperature of the event horizon is positive for $1 < \alpha \le 3.464$ and negative for greater values of α, whereas the correction to temperature of the cosmological horizon is negative for $1 < \alpha \le 2.326$ and positive everywhere else.

In what follows, we assume that the condition $r_+ \le \tilde{r}_c$ holds. The lukewarm configuration corresponds to a nonextreme black hole that has the same surface gravity (temperature) of the event horizon as the cosmological horizon. It can easily be shown that for $\beta > 0$ and $\alpha = 1$ one has

$$\kappa_C = \kappa_H = \frac{24\pi\beta}{r_+^5} \tag{79}$$

and

$$\tilde{r}_c = r_+ + \varepsilon\frac{96\pi\beta}{r_+^3}. \tag{80}$$

The lukewarm configuration is characterized by the values of the parameters that classically characterize the extremal solution. On the other hand, the extremal configuration is now characterized by the conditions $\beta < 0$,

$$\tilde{r}_c = r_+, \tag{81}$$

$$\kappa_C = \kappa_H = 0, \tag{82}$$

and

$$\alpha = 1 + \varepsilon\frac{96\pi|\beta|}{r_+^4}. \tag{83}$$

Finally, we investigate the Schwarzschild limit $r_c \to \infty$ $(\alpha \to \infty)$. As the spacetime becomes asymptotically flat, it is possible to define the mass $M_\infty = m(\infty)$ seen by a distant observer and relate it to the radius of the event horizon. Making use of our definition, one has

$$r_+ = 2M_\infty\left(1 - \frac{5\pi\varepsilon\beta}{M_\infty^4}\right), \tag{84}$$

which is a well-known result [34,36,43]. From Eqs. (77) and (78) one sees that the surface gravity of the cosmological horizon decreases to zero as its radius goes to infinity and the temperature of the event horizon is given by

$$T_H = \frac{1}{4\pi r_+} - \frac{12\varepsilon\beta}{r_+^5} \tag{85}$$

and when expressed in terms of M_∞

$$T_H = \frac{1}{8\pi M_\infty}\left(1 + \frac{\varepsilon\beta}{4M_\infty}\right), \tag{86}$$

where we have used the standard relation $T_H = \kappa_H/2\pi$.

VI. FIRST-ORDER CORRECTIONS TO THE SCHWARZSCHILD–de Sitter BLACK HOLE

In this section, we shall briefly analyse the influence of the higher-derivative term (46) upon the classical Schwarzschild–de Sitter solution. We closely follow the methods described in the previous section, and the organization of the material will be the same. First, let us consider the functions $m_1(r)$ and $\psi_1(r)$. It can be shown that in (r_+, α) parametrization one has

$$m_1(r) = \frac{8\pi\beta}{r_+^3(\alpha^2+\alpha+1)^3}\tilde{m}_1(r), \tag{87}$$

where

$$\begin{aligned}\tilde{m}_1(r) = {} & 5\alpha^6 + 15\alpha^5 + 3\alpha^4 - 19\alpha^3 + 4x^3 - 12\alpha^2 - \frac{54\alpha^6}{x^5} \\ & + \frac{66\alpha^4}{x^3} + \frac{49\alpha^6}{x^6} - \frac{162\alpha^5}{x^5} + \frac{132\alpha^3}{x^3} - 4 + \frac{147\alpha^5}{x^6} \\ & - \frac{216\alpha^4}{x^5} + \frac{66\alpha^2}{x^3} + \frac{147\alpha^4}{x^6} - \frac{162\alpha^3}{x^5} + \frac{49\alpha^3}{x^6} - \frac{54\alpha^2}{x^5}\end{aligned} \tag{88}$$

and

$$\psi_1(r) = -\frac{432\pi\beta}{\alpha^4 r_+^4(\alpha^2+\alpha+1)^2}\left(\frac{\alpha^8}{x^6} - \alpha^2 + \frac{2\alpha^7}{x^6} - 2\alpha + \frac{\alpha^6}{x^6} - 1\right). \tag{89}$$

Similarly, repeating the calculations for the radius of the corrected cosmological horizon $\tilde{r}_c$, one obtains

$$\tilde{r}_c = r_c + \varepsilon r_1, \tag{90}$$

where

$$\begin{aligned} r_1 = {} & -\frac{16\pi\beta}{r_+^3(\alpha^2+\alpha+1)(2\alpha+1)} \\ & \times\left(5\alpha^3 + 15\alpha^2 + 3\alpha - 10 + \frac{3}{\alpha} + \frac{15}{\alpha^2} + \frac{5}{\alpha^3}\right).\end{aligned} \tag{91}$$

Inspection of the above formula shows that the correction r_1 is always negative for positive β.

A. Karlhede scalar

Now, let use return to the Karlhede scalar discussed in the previous section. For the classical Schwarzschild–de Sitter black hole one has

$$I_K^{(0)} = \frac{180(x-1)(\alpha-x)(\alpha+x+1)\alpha^2(\alpha+1)^2}{r_+^6 x^9(\alpha^2+\alpha+1)^3}. \tag{92}$$

Inspection of $I_K^{(0)}$ shows that it vanishes at the event and at the cosmological horizon. It also changes a sign at r_+ and r_c, and, consequently, the Karlshede scalar may serve as a detector of the horizons. When the higher-derivative term is present, the Karlhede scalar still vanishes at r_+ and $\tilde{r}_c$. Our calculations (to complicated to be presented here) indicate that at the event horizon the radial derivative of the classical part of the Karlhede scalar is positive, whereas the derivative of $I_K^{(1)}$ is negative for $\alpha \in (1, 1.242)$ and positive everywhere else. On the other hand, at the cosmological horizon, the classical part of the derivative of the Karlhede scalar is negative and its correction is positive for $\alpha \in (1, 1.173)$.

B. Acceleration of the massive test particle

The acceleration of the massive test particle defined in Sec. V B now reads

$$a^{\hat{r}} = \frac{\alpha(\alpha+1) - 2x^3}{2r_+ x^{3/2}\sqrt{\alpha(\alpha+1)+1}\sqrt{(\alpha-x)(x-1)(\alpha+x-1)}} \tag{93}$$

and the gravitational attraction is balanced by the cosmological repulsion at

$$x = \sqrt[3]{\frac{\alpha(\alpha+1)}{2}}. \tag{94}$$

The higher derivative correction to the acceleration is always negative for $\alpha \in (1, 1.3176)$. For $\alpha \geq 1.1.317$ it is positive for $x < x_{\text{crit}}$ and negative for greater values of r. Here, as before, r_{crit} is defined as the point at which the acceleration of the test particle is zero and its leading behavior is given by (94).

C. Surface gravities

The surface gravity of the event horizon is given by

$$\kappa_H = \frac{(\alpha+2)(\alpha-1)}{2r_+(\alpha^2+\alpha+1)} - \frac{24\pi\varepsilon\beta}{r_+^5(\alpha^2+\alpha+1)^3\alpha^4} \times (\alpha^{10}+3\alpha^9+9\alpha^8+13\alpha^7+6\alpha^6-5\alpha^4-27\alpha^3 - 9\alpha^2+27\alpha+18), \tag{95}$$

whereas κ_C (defined to be positive)

$$\kappa_C = \frac{(2\alpha+1)(\alpha-1)}{2r_+\alpha(\alpha^2+\alpha+1)} - \frac{8\pi\varepsilon\beta}{r_+^5\alpha^5(2\alpha+1)(\alpha^2+\alpha+1)^3} \times (20\alpha^{10}+85\alpha^9+117\alpha^8+51\alpha^7+3\alpha^6-108\alpha^5 - 216\alpha^4-9\alpha^3+207\alpha^2+145\alpha+29). \tag{96}$$

Numerical calculations show that the corrections to the surface gravities of the event horizon and the cosmological horizon are always negative for positive β.

Now, let us consider a few interesting limits. From (95) and (96) one concludes that the extreme configuration is possible for $\beta > 0$. Indeed, a simple calculation shows that, for

$$\alpha = 1 + \varepsilon\frac{64\pi\beta}{r_+^4}, \tag{97}$$

one has $\tilde{r}_c = r_+$ and $\kappa_H = \kappa_c = 0$. On the other hand, for $\beta < 0$, one has the lukewarm configuration with

$$\kappa_H = \kappa_C = \frac{32\pi|\beta|}{r_+^5}. \tag{98}$$

The cosmological horizon is now defined by

$$\tilde{r}_c = r_+ + \varepsilon\frac{64\pi|\beta|}{r_+^3}. \tag{99}$$

It should be noted that the temperatures T_H and T_C have proper Schwarzschild asymptotics. Indeed, as $\alpha \to \infty$ the temperature T_C goes to zero and T_H approaches its Schwarzschild value and the formulas presented in the last paragraph of the Sec. V C remain intact.

VII. QUASINORMAL MODES

In this section we shall briefly analyze the quasinormal modes of the axial gravitational perturbations of the Kiselev and the Schwarzschild–de Sitter black holes. Let us imagine, once again, the idealized situation in which the Kiselev and the Schwarzschild–de Sitter black holes have the same radii of the cosmological and the event horizon. Our first task is to tell, solely on the basis of the frequencies of the quasinormal modes (labeled by the multipole number ℓ and the overtone number n), which black hole is which. We will concentrate on the low-lying fundamental modes. It is a natural choice motivated by the fact that the fundamental modes are astrophysically most important. For the black holes considered in this paper the axial perturbations satisfy the ordinary differential equation of the Schrödinger type

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right)\psi(r) = 0, \tag{100}$$

where

$$dr_* = \frac{dr}{f(r)} \tag{101}$$

is the tortoise (Regge-Wheeler) coordinate and $V(r)$ is the potential. In the (r_+, r_c) representation the potentials are

$$V_K(r) = \left(1 - \frac{r_+r_c}{r(r_++r_c)} - \frac{r}{r_++r_c}\right) \times \left(\frac{\ell(\ell+1)}{r^2} - \frac{1}{r(r_++r_c)} - \frac{3r_+r_c}{r^3(r_++r_c)}\right) \tag{102}$$

and

$$V_{\mathrm{SdS}}(r) = \left(1 - \frac{r_+r_c(r_++r_c)}{r(r_+^2+r_cr_++r_c^2)} - \frac{r^2}{r_+^2+r_+r_cr_+^2}\right) \times \left(\frac{\ell(\ell+1)}{r^2} - \frac{3r_+r_c(r_++r_c)}{r^3(r_+^2+r_+r_c+r_c^2)}\right), \tag{103}$$

for the Kiselev and the Schwarzschild–de Sitter black holes, respectively. Since the expected differences in the complex frequencies of the quasinormal modes may be quite small, one needs the accurate method that allows to work with as many decimal places of ω as needed. Here we will employ the higher-order Wentzel-Kramers-Brillouin (WKB)-Padé method [44,45] (see also [46]), which is a major modification and improvement of the original WKB method [47–49]. It has been demonstrated that, depending on the number of the terms taken into account in the WKB expansion, one can achieve, within the domain of applicability, highly accurate results [44,45,50]. The essence of the WKB-Padé method is to introduce an expansion parameter $\tilde{\varepsilon}$ into the expression relating the quasinormal frequencies and the derivatives of $Q(r) = \omega^2 - V(r)$ calculated at the maximum of the potential

$$\frac{iQ_0}{\sqrt{2Q_0''}\tilde{\varepsilon}} - \sum_{k=2}^{N}\tilde{\varepsilon}^{k-1}\Lambda_k = n + \frac{1}{2} \tag{104}$$

and instead of summing up the terms of the expression for ω^2

$$\omega^2 = V(x_0) - i\left(n+\frac{1}{2}\right)\sqrt{2Q_0''}\tilde{\varepsilon} - i\sqrt{2Q_0''}\sum_{i=2}^{N}\tilde{\varepsilon}^j\Lambda_j$$
$$\equiv V(x_0) + \sum_{i=1}^{N}\tilde{\varepsilon}^i\tilde{\Lambda}_i \quad (105)$$

to construct its Padé approximants. Here and in what follows the subscript 0 denotes quantities evaluated at the maximum of the potential. The functions Λ_k are constructed from the derivatives of Q and their complexity grows rapidly with k. Their general analytic form is known for $k \leq 16$. On the other hand, for a given potential with prescribed ℓ and n they can be calculated numerically. It should be emphasized that in order to obtain highly accurate results one has to take sufficiently great N. Our results for the diagonal Padé approximants $P^{N/2}_{N/2}$ of the fundamental $(\ell = 2, \ldots, 7, n = 0)$ complex frequencies of the quasinormal modes of the Kiselev and the Schwarzschild–de Sitter black holes are tabulated in Table I. The radial coordinate of the event horizon is always $r_+ = 1$ and the cosmological horizon is located at $\alpha = 50, 10^2, 10^3$, and 10^4. Because of the number of terms retained in the (formal) series (105) we believe that our results are accurate to 10 decimal places. Inspection of the Table I shows that although the line elements (and hence the potentials) of the Kiselev and the Schwarzschild–de Sitter black holes are quite different, the calculated frequencies of the quasinormal modes follow a similar pattern and for a given ℓ are numerically very close. They are, as expected, noticeably different for larger values of the cosmological constant (smaller radii of the cosmological horizon) and practically equal for $r_c \gg r_+$, approaching their Schwarzschild values as $\alpha \to \infty$. For the real part of the modes one has $\mathfrak{Re}(\omega_K) < \mathfrak{Re}(\omega_{\mathrm{SdS}})$, whereas the imaginary part of ω satisfies $\mathfrak{Im}(\omega_K) > \mathfrak{Im}(\omega_{\mathrm{SdS}})$, that means that the fundamental quasinormal modes of the Schwarzschild–de Sitter black hole decay slightly faster. The gravitational quasinormal modes of the Kiselev black hole for a few exemplary values of r_n and $\tilde{\omega}$ has been calculated in Ref. [51] within the framework of the sixth-order WKB method. The $\tilde{\omega} = -2/3$ case has not been studied. Our results are in concord with the general tendency clearly visible in their data. The same method has been adapted in Ref. [52] to calculate the quasinormal modes of the Schwarzschild–de Sitter black holes.

Now, let us consider the influence of the Goroff-Sagnotti term upon the complex frequencies of the quasinormal modes. In our previous paper it has been shown that the WKB-Padé method can be used to construct the quasinormal modes of the corrected Schwarzschild black hole, and because we know the fundamental modes to the

TABLE I. The complex frequencies of the gravitational axial quasinormal modes of the Kiselev and Schwarzschild–de Sitter black holes calculated within the framework of the WKB-Padé method. N is the number of $\tilde{\Lambda}_i$ terms retained in Eq. (105) and $\alpha = r_c/r_+$. As $\alpha \to \infty$ the frequencies tend to their Schwarzschild values.

ℓ	n	N	α	$\omega_K r_+$	$\omega_{\mathrm{SdS}} r_+$
2	0	200	50	$0.7289275752 - 0.1716418097i$	$0.7466492671 - 0.1777973964i$
			10^2	$0.7380909713 - 0.1747565012i$	$0.7471681067 - 0.1778925163i$
			10^3	$0.7464140302 - 0.1776053798i$	$0.7473416004 - 0.1779243074i$
			10^4	$0.7472503935 - 0.1778926817i$	$0.7473433511 - 0.1779246281i$
3	0	200	50	$1.1695549479 - 0.1785000505i$	$1.1977800888 - 0.1852509921i$
			10^2	$1.1841580941 - 0.1819199403i$	$1.1986071842 - 0.1853669366i$
			10^3	$1.1974079773 - 0.1850544491i$	$1.1988837577 - 0.1854057008i$
			10^4	$1.1987386588 - 0.1853709006i$	$1.1988865487 - 0.1854060919i$
4	0	200	50	$1.5786131176 - 0.1811632574i$	$1.6168592499 - 0.1881626307i$
			10^2	$1.5983999580 - 0.1847095520i$	$1.6179786267 - 0.1882861878i$
			10^3	$1.6163532807 - 0.1879627845i$	$1.6183529396 - 0.1883275009i$
			10^4	$1.6181563289 - 0.1882913749i$	$1.6183567169 - 0.1883279178i$
5	0	100	50	$1.9747382807 - 0.1824497923i$	$2.0227137109 - 0.1895709118i$
			10^2	$1.9995571002 - 0.1860579308i$	$2.0241166917 - 0.1896980775i$
			10^3	$2.0220774260 - 0.1893692876i$	$2.0245858421 - 0.1897405987i$
			10^4	$2.0243392050 - 0.1897038230i$	$2.0245905764 - 0.1897410278i$
6	0	100	50	$2.3642310617 - 0.1831693608i$	$2.4217697497 - 0.1903588957i$
			10^2	$2.3939958874 - 0.1868122307i$	$2.4234515283 - 0.1904880607i$
			10^3	$2.4210053922 - 0.1901562356i$	$2.4240139089 - 0.1905312514i$
			10^4	$2.4237180965 - 0.1904941106i$	$2.4240195839 - 0.1905316873i$
7	0	100	50	$2.7498507597 - 0.1836130202i$	$2.8168512541 - 0.1908448218i$
			10^2	$2.7845091578 - 0.1872773402i$	$2.8188089282 - 0.1909752129i$
			10^3	$2.8159602526 - 0.1906415100i$	$2.8194635683 - 0.1910188141i$
			10^4	$2.8191191022 - 0.1909814478i$	$2.8194701744 - 0.1910192541i$

TABLE II. The complex frequencies of the gravitational axial quasinormal modes of the corrected Kiselev and Schwarzschild–de Sitter black holes calculated within the framework of the WKB-Padé method. Only the first-order corrections are presented. N is the number of $\tilde{\Lambda}_i$ terms retained in Eq. (105), $\alpha = r_c/r_+$, and $\beta = 10^{-4}$.

ℓ	n	N	α	$\omega_K r_+$	$\omega_{\mathrm{SdS}} r_+$
2	0	200	50	$0.7143917774 - 0.1672726483i$	$0.7342646168 - 0.1798836544i$
3	0	200	50	$1.1446680223 - 0.1733416386i$	$1.1805148323 - 0.1887514359i$
4	0	200	50	$1.5442398767 - 0.1757022306i$	$1.5951707178 - 0.1921851300i$
5	0	100	50	$1.9312685366 - 0.1768469143i$	$1.9965723465 - 0.1938174699i$
6	0	100	50	$2.3118739188 - 0.1774900040i$	$2.3911284069 - 0.1947211152i$
7	0	100	50	$2.6887306934 - 0.1778880864i$	$2.7816783108 - 0.1952755306i$

accuracy of at least 32 digit places, this sets our current (lower) limit, say β_{crit}, for the admissible values of β. Smaller values of β do not influence our results. One expects a similar behavior for the Kiselev and Schwarzschild–de Sitter black holes. It should be emphasized, however, that in this case the calculations are much more complicated and pose high demands on the computer resources.

The gravitational quasinormal modes satisfy Eq. (100) with (see Ref. [53] and the references therein)

$$V(r) = e^{2\psi}\left(1 - \frac{2m}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3} + \frac{2}{r^2}\frac{dm}{dr} - \frac{1}{r}\left(1 - \frac{2m}{r}\right)\frac{d\psi}{dr}\right] \tag{106}$$

with

$$dr_* = e^{\psi(r)}\frac{dr}{1 - 2m(r)/r}. \tag{107}$$

Our preliminary results are presented in Table II suggest the following: (i) For a given β, the modulus of the real and the imaginary part of the first-order corrections to the ω_K and ω_{SdS} are bigger than the analogous Schwarzschild corrections (i.e., the frequency of the modes are quite sensitive to small deformations of the potential), (ii) they approach the Schwarzschild values as $\alpha \to \infty$, and (iii) the quasinormal modes of the corrected Kiselev black holes are slightly less damped whereas the analogous corrections to the modes of the Schwarzschild–de Sitter black holes make them more damped. The calculations of the frequencies of the low-lying fundamental modes $(2 \le \ell \le 7)$ have been carried out for $\alpha = 50$ and $\beta = 10^{-4}$. We chose such a large value of the parameter β because we wanted the corrections caused by the higher-order terms be easily visible in the numerical results. It should be emphasized however that a great care should be taken both in the calculations and the interpretation of the results. Indeed, our calculations include only the first-order corrections in $\tilde{\varepsilon}$. The second-order corrections to the quasinormal modes of the Schwarzschild black hole caused by the Goroff-Sagnotti term has been analyzed in Ref. [43]. It has been demonstrated that the second-order corrections to ω are a few orders of magnitude smaller than the first order corrections and we suspect that this pattern holds for the Kiselev and the Schwarzschild–de Sitter black holes. Finally, observe that the high sensitivity of the ω to the deformations of the potential caused by the higher order terms make the quasinormal modes an ideal (theoretical) tool for distinguishing various types of black holes.

VIII. FINAL REMARKS

Motivated by the fact that the low-energy generalizations of the Einstein-Hilbert action involve the higher curvature terms we have considered the influence of the theory inspired by the Goroff-Sagnotti results upon the Kiselev black hole with $\tilde{\omega} = -2/3$ and the Schwrzschild–de Sitter black hole. Expressing the resulting solution in terms of the exact radius of the event horizon and the radius of the cosmological horizon of the unperturbed black hole, we have calculated corrections to the cosmological horizon, to the surface gravities and to the acceleration of the massive test particles. We have also checked if the Karlhede scalar may serve as the detector of the horizons. We prefer (r_+, r_c) representation over the standard one, in which the location of the horizons of the Schwrzschild–de Sitter solution is expressed in terms of the trigonometric functions. It should be noted that with a little additional work all our final results can be expressed in terms of the exact radii of the event and the cosmological horizon. For the Kiselev and the Schwarzschild–de Sitter black holes the general conclusions are essentially the same or complementary, despite the fact that their asymptotic behavior as $r \to \infty$ is different. Indeed, one has $f(r) \sim -r/(r_+ + r_c)$ for the Kiselev black hole and $f(r) \sim -r^2/(r_+^2 + r_+ r_c + r_c^2)$ for the Schwarzschild–de Sitter black hole.

Classically, the parameter space of the solutions is too limited to allow for the lukewarm configurations. However, with the higher-order term present, such configurations are possible. It should be noted however, that the coordinate distance between the horizons is small. The extremal configuration is still possible but for a different values of the parameters.

The quasinormal modes are arguably the most promising candidates for the observational studies of black holes.

In the previous section, we have demonstrated their sensitivity to small changes in the shape of the potential function $V(r)$. Specifically, for the Kiselev and the Schwarzschild–de Sitter black holes characterized by the same values of radii of the event and the cosmological horizon, the differences between the frequencies of the quasinormal modes, $\Delta\omega_{\ell n}$, for a given ℓ and n are clearly visible (see Table I). Similarly, one can study the corrections to ω caused by the higher derivative terms. In a similar manner one can easily distinguish between the various types of the black holes, say, the classical ones, and the black holes made of or surrounded by some exotic form of matter. Although the low-lying fundamental modes are astrophysically most important, it is quite possible that a detailed knowledge of the overtones is necessary to reconstruct other important black hole characteristics.

The correction considered in this paper does not exhaust all possibilities. One can add, for example, other terms of the background dimensionality six constructed from the Riemann tensor, its covariant derivatives and contractions. These terms will certainly modify the results, but the general strategy of the calculations would remain intact. The Kiselev family of solutions is rich and it would be of some interest to repeat or extend our calculations to other members of the family also. We left out some important problems here, such as the entropy as they go beyond the scope of this paper. Presence of the second (cosmological) horizon and the absence of the unique approach to the entropy of such configurations make this problem exciting but conceptually complicated. This group of problems certainly deserves a separate study. We plan to return to these and other topics in subsequent publications.

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