

Can two ultrarelativistic objects lose almost all their energy to gravitational radiation?

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In 2007, Pretorius and Khurana [Black hole mergers and unstable circular orbits, *Classical Quantum Gravity* **24**, S83 (2007).] did “speculate that at threshold [at a critical impact parameter], all of the kinetic energy of the system [two ultrarelativistic black holes] is converted to gravitational waves, which can be an arbitrarily large fraction of the total energy.” However, in 2012, Sperhake *et al.* [Universality, Maximum Radiation and Absorption in High-Energy Collisions of Black Holes with Spin, *Phys. Rev. Lett.* **111**, 041101 (2013).] performed numerical calculations that led them to the contrary conclusion: “An extrapolation of our results to the limit $\gamma \rightarrow \infty$ suggests that about half of the center-of-mass energy of the system can be emitted in gravitational radiation, while the rest must be converted into rest-mass and spin energy.” Here I present arguments against this latter conclusion and in support of the earlier speculation that for sufficiently large γ , all but an arbitrarily small fraction of the total energy can be radiated away as gravitational waves.

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I. INTRODUCTION

LIGO and Virgo have had enormous success in detecting the gravitational waves from astrophysical inspiraling black holes that coalesce to form a larger black hole [1–28]. In these cases the black holes have been inspiraling long before their gravitational wave emission is strong enough to be detected by LIGO and Virgo, so that well before the final coalescence, the two black holes are moving with non-relativistic velocities $v \ll c$ relative to each other and are gravitationally bound to each other.

Astrophysically, it seems that it would be very rare (and not yet observed) for two black holes or other macroscopic objects to approach at relativistic velocities $v \sim c$ before coalescing. However, it is an interesting question what would happen if this occurred. Indeed, there has been an enormous amount of work [29–116] calculating what happens for compact objects approaching each other at arbitrary relative velocities in 4-dimensional spacetime (not including considerable additional work that has also been done in higher dimensions that I shall not discuss here). Much of this work has shown essentially that if the Schwarzschild radius corresponding to the total center-of-momentum (COM) energy E , that is, $R_S = 2GE/c^4 = 2E$ (where here and henceforth I shall use units with $G = c = 1$), is much larger than the sum R of the maximum intrinsic linear sizes of the two objects, then the composition of each object does not matter significantly

for how much gravitational wave energy ΔE is emitted, which, for fixed total COM energy E instead depends almost entirely on the impact parameter $b \equiv J/p \equiv \beta E$, where J is the total angular momentum in the COM and p is the magnitude of the initial spatial momentum of each object in the COM (equal magnitudes but opposite directions, since the total spatial momentum is zero in the COM). Here

$$\beta \equiv \frac{b}{E} \equiv \frac{J}{pE} \approx \frac{2J}{E^2} = \frac{8J}{R_S^2} \quad (1)$$

is a dimensionless measure of the impact parameter (not to be confused with v/c). The approximation of the last two terms occurs in the limit that the COM energy E is much greater than the sum $M = m_1 + m_2$ of the rest masses m_1 and m_2 of the two objects, so that then $p \approx E/2$, giving $b \approx 2J/E$. (Note that I am *not* using M for the total ADM mass of the spacetime as many others do, but rather E .)

For fixed $E \gg M$ and $R_S = 2E \gg R$, there is at least one critical impact parameter $b_c = \beta_c E = \beta_c R_S/2$ (with β_c some dimensionless number of the order of unity, which Shibata, Okawa, and Yamamoto [68] have estimated to be about $2.5/v$), that should be independent of E , say with value $b_c \approx 2.5E$, in the limit $E/M \rightarrow \infty$ and $R_S/R \rightarrow \infty$, such that for impact parameter b infinitesimally on one side of b_c , the two objects will, at least eventually, form a single black hole, but for b infinitesimally on the other side of b_c , the two objects will scatter separately to infinity (though leaving open the possibility that either of the objects might

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separately become a black hole even if it were not one originally, or that both might become black holes that never merge).

It is perhaps simplest to assume that for fixed $E \gg M$ and $R_S = 2E \gg R$, there is a unique critical impact parameter b_c such that for all smaller impact parameters, $b < b_c$, the two objects eventually merge and form a single black hole, but for all larger impact parameters, $b > b_c$, the two objects scatter separately to infinity. However, near the end of this paper I shall discuss the possibility that this might not be true, and that there might instead be a larger odd number of critical impact parameters b_c . Counting with positive integers n , with $n = 1$ for the smallest critical impact parameter $b_c = b_{c,1}$ (so that at all lower values of b , that is for all $0 \leq b < b_{c,1}$, a single black hole forms), each odd critical impact parameter, $b_{c,2n-1}$, has, for b infinitesimally smaller than $b_{c,2n-1}$, the two objects merging to form a single black hole and has, for b infinitesimally larger than $b_{c,2n-1}$, the two objects scattering separately to infinity. Conversely, each even critical impact parameter (if any such exist), $b_{c,2n}$, has, for b infinitesimally smaller than $b_{c,2n}$, the two objects scattering separately to infinity and has, for b infinitesimally larger than $b_{c,2n}$, the two objects merging to form a single black hole.

In the following, when I discuss the impact parameter b_c , if in fact there are more than one, I shall mean by b_c the particular critical impact parameter at which the largest fraction f of the fixed total energy E is emitted into gravitational waves of energy $\Delta E = fE$. Similarly, $\beta_c \equiv b_c/E$ will denote the dimensionless quantity analogous to that particular critical impact parameter.

When b is much smaller than $b_c = \beta_c E$, it appears that the fraction f of the initial COM energy E that is radiated into gravitational waves of energy $\Delta E = fE$ is significantly less than unity (approximately 0.14 ± 0.03 for a head-on collision, $b = 0$, when $E \gg M$ [67]), and it seems that increasing b increases $f = \Delta E/E$. On the other hand, when b is much larger than $b_c = \beta_c E$, the two objects have a small scattering angle and also do not emit a large fraction of their energy into gravitational radiation, and ΔE , the amount radiated, goes down with increasing b as the scattering angle also decreases. Therefore, there should be some value in between for b , say $b_m = \beta_m E$, at which the maximum fraction $f = \Delta E/E$, say f_m , of the total COM energy E (which is to be kept fixed when b is varied) is emitted as gravitational wave energy ΔE . If there are in fact more than one critical impact parameters, it also seems possible that f does not decrease monotonically from f_m as b is moved in either direction away from b_m , so that there may be more than one local maximum for f (perhaps one for each odd critical impact parameter $b_{c,2n-1}$), but I am taking f_m to be the global maximum for f . It seems plausible that b_m would be close to the critical impact parameter b_c (so β_m is close to β_c), especially in the limit $E/M \rightarrow \infty$ and $R_S/R \rightarrow \infty$, but here I shall not assume that this needs to be true.

Now Conjecture 1, a slight generalization of the 2007 speculation of Pretorius and Khurana [61] to an arbitrary pair of initial objects of finite size, is that in the limit $\gamma \equiv E/M \rightarrow \infty$ and $R_S/R \rightarrow \infty$, $f_m \rightarrow 1$.

Slightly stronger is Conjecture 2, that at the critical impact parameter b_c that has the largest $f = \Delta E/E$ for finite E/M and finite R_S/R , when one takes the limit $\gamma \equiv E/M \rightarrow \infty$ and $R_S/R \rightarrow \infty$, $f = \Delta E/E$ at this γ -dependent b_c approaches 1.

However, in 2012 Sperhake, Berti, Cardoso, and Pretorius (SBCP) [81] performed numerical calculations that led them to the contrary conclusion: ‘‘An extrapolation of our results to the limit $\gamma \rightarrow \infty$ suggests that about half of the center-of-mass energy of the system can be emitted in gravitational radiation, while the rest must be converted into rest-mass and spin energy.’’

Here I wish to revisit the question of what fraction $f = \Delta E/E$ of the initial COM energy E can be emitted in gravitational wave energy ΔE in the limit that $\gamma \equiv E/M$ is taken to infinity. Despite the numerical evidence presented by [81], I shall argue that it seems to be more plausible that f can be made arbitrarily near 1 by taking γ to infinity while fine tuning the impact parameter b appropriately.

II. ULTRARELATIVISTIC COLLISIONS OF BLACK HOLES

My arguments should generalize to apply for any pair of objects with total linear size R and with M the sum of their rest masses (here *not* the total ADM mass of the spacetime, which is what I shall denote by the total energy E), when one separately boosts the two objects so that the center-of-momentum (COM) energy E is much greater than both R and M (in units with $G = c = 1$). However, for simplicity and concreteness let me assume that the two objects are initially Schwarzschild (nonrotating) black holes having positive rest masses m_1 and m_2 , with $M = m_1 + m_2$ and $R = 2m_1 + 2m_2 = 2M$ if one uses the Schwarzschild radius of each black hole as its linear size.

Then the critical impact parameter that gives the largest $f = \Delta E/E$ out of all critical impact parameters (if there are more than one) should be

$$b_c = b_c(m_1, m_2, E) = \beta_2(m_1/E, m_2/E)E, \quad (2)$$

with the dimensionless function $\beta_2(m_1/E, m_2/E)$, a symmetric function of its two dimensionless arguments, going to a constant, which I shall call β_0 , in the limit that $m_1/E \rightarrow 0$ and $m_2/E \rightarrow 0$ (which is implied by the single limit $M/E \rightarrow 0$):

$$\beta_0 \equiv \lim_{M/E \rightarrow 0} \frac{b_c}{E}. \quad (3)$$

For impact parameter b just infinitesimally larger than this particular critical impact parameter b_c , the two black

holes (each initially much smaller than the impact parameter, since their Schwarzschild radii, $2m_1$ and $2m_2$, are much smaller than $b_c \sim E$, leaving out the dimensionless factor $\beta_2(m_1/E, m_2/E)$ that is expected to be of the order of unity) do not merge into a larger black hole. On the other hand, for impact parameter b just infinitesimally smaller than the critical impact parameter b_c , the two black holes would merge into a larger black hole.

In this example, there are four independent quantities that with $G = c = 1$ have the dimension of mass or length or time, namely m_1 , m_2 , E , and b , and hence there are three independent dimensionless ratios, namely m_1/E , m_2/E , and b/E . At the critical impact parameter b_c given by Eq. (2), there are only two independent dimensionless ratios, namely m_1/E and m_2/E .

It is also interesting to restrict to an even simpler model in which each initially Schwarzschild black hole has the same mass, $m_1 = m_2 = M/2$, so that then there are only three independent quantities that have the dimension of mass or length or time, namely M , E , and b , and hence there are only two independent dimensionless ratios, namely $\gamma \equiv E/M$ and $\beta \equiv b/E$ (here *not* equal to $v/c = \sqrt{1 - 1/\gamma^2}$). At the particular critical impact parameter b_c that gives the maximum $f = \Delta E/E$ out of all critical impact parameters if there are more than one, which now has the form

$$b_c = b_c(M/2, M/2, E) = \beta_1(\gamma)E, \quad (4)$$

there is only one independent dimensionless ratio, namely $\gamma \equiv E/M$. The fraction $f = \Delta E/E$ of the initial energy E that is radiated into gravitational waves with energy ΔE at the critical impact parameter would then be a function only of γ , and the question under dispute is how $f(\gamma)$ depends on γ in the limit that γ is taken to infinity.

III. QUALITATIVE ARGUMENT FOR ULTRARELATIVISTIC SCATTERING NEAR THE CRITICAL IMPACT PARAMETER

Here I shall restrict to the simplest model, with two initially incoming Schwarzschild black holes, each of mass $M/2$, which in the COM have opposite spatial momenta each of magnitude $p = (M/2)\gamma v$ with equal gamma-factors, $\gamma = 1/\sqrt{1 - v^2}$, so that the total energy in the COM frame is $E = M\gamma$, and with total angular momentum $J = bp$ with impact parameter $b = J/p = 2J/(Ev)$ that is near the critical impact parameter $b_c = \beta_1(\gamma)E$ with the maximum $f = \Delta E/E$ (out of all critical impact parameters, if there are more than one, not out of all possible values of the impact parameter b). I shall describe the picture Pretorius and Khurana [61] painted and then argue that it seems generally qualitatively correct, despite the opposing arguments of Spherhake, Berti, Cardoso, and Pretorius (SBCP) [81].

In particular, the idea is that near the critical impact parameter $b_c = \beta_1(\gamma)E$, the two black holes, each of size much smaller than the critical impact parameter b_c when $\gamma = E/M \gg 1$ as I shall always assume, spiral inward while radiating away almost all their initial energy, until they coalesce at the center to form a black hole that on a logarithmic scale has a mass closer to the original total rest mass M than to the initial COM energy $E = M\gamma$. At each time t that is measured by a static observer in the COM frame at radial infinity, let $r(t)$ be some measure of the distance of each black hole from the center of momentum, such as the proper distance along a spacelike geodesic from the center of momentum that at this center is orthogonal to the worldline of the center of momentum.

With the black holes orbiting around the COM at very nearly the speed of light, one would expect, in analogy with the quadrupole formula, that the sum of the rest masses and kinetic energies of the black holes, which I shall call E_h (not including the gravitational potential energy, which would be expected to be much smaller than E_h for the ultrarelativistic motion, and not including the gravitational radiation energy $E_{GW} \approx E - E_h$, which would carry away significant amounts of energy to cause E_h to decrease) would decrease at a rate per time that would be roughly proportional to $(E_h/r)^2$, assuming that the shape of the orbit is nearly a self-similar equiangular spiral. Dividing this radiation power outflow from the two black holes by the negative radial velocity $v_r = dr/dt$, which would be expected to have a magnitude 1–2 orders of magnitude smaller than unity but still be of the general order of unity (i.e., not having any strong dependence on γ when $\gamma \gg 1$), and including an unknown numerical factor of α , one gets (here initially ignoring the effect of the absorption of gravitational waves by the black holes)

$$\frac{dE_h}{dr} \sim \frac{1}{\alpha} \frac{E_h^2}{r^2}, \quad (5)$$

which leads to a solution that asymptotically, as r decreases far below the original impact parameter b , has the form $E_h \sim ar$ for the total rest mass plus kinetic energy of the two black holes.

Spherhake, Berti, Cardoso, and Pretorius [81] argue that, even in the limit $\gamma \rightarrow \infty$, a significant fraction of the kinetic energy of the two black holes is not radiated away but falls into the black holes to increase their sizes significantly. Here I shall argue that the fraction of the total energy that falls into the black holes plausibly goes to zero in the limit that one takes γ to infinity, so long as at each γ one chooses the impact parameter b to be near the critical impact parameter b_c giving the largest $f = \Delta E/E$ out of all critical impact parameters (if there are more than one). Alternatively, one could choose b to be near b_m , the value of b that maximizes f without restricting to critical impact

parameters, though I would expect b_m to be near b_c for each value of γ , at least for $\gamma \gg 1$.

The argument is that during most of the inspiral, when most of the energy is radiated into gravitational waves, the black holes, initially each of mass $m = M/2 = E/(2\gamma) \ll b_c \sim E$ and hence each of Schwarzschild radii $2m = M = E/\gamma$, are so much smaller than the region of linear size $\sim r \sim E_h$ where the gravitational wave energy is localized near the black hole orbits before it flows outward, that only a tiny fraction of this energy will be absorbed by the black holes.

Let me make a crude model to estimate the order of magnitude of the fraction $F \equiv 1 - f$ of the initial total COM energy E that is absorbed by the black holes, as a function of $\gamma = E/M$ when it is large, and when b is sufficiently close to either b_c or to b_m for that γ . Each black hole of rest mass m has an effective cross section for absorbing gravitational radiation that is m^2 multiplied by some dimensionless number (depending on the frequency spectrum of the gravitational radiation in the frame of the black hole and which is 27π for gravitational radiation of wavelength very short compared with the Schwarzschild radius $2m$ of the black hole when it is nonrotating, which is the geometric optics limit in which null geodesics fall into a Schwarzschild black hole when they have impact parameter $b < \sqrt{27}m$ [117,118]). Therefore, the rate of gravitational wave energy absorbed by each black hole is some dimensionless number multiplying $m^2\rho$, where ρ is the energy density of the gravitational waves at the location of the black hole. One can crudely approximate ρ to be the energy in gravitational waves within the distance r of the COM origin that is the distance each black hole is from that origin at the time t of absorption, divided by the volume $\sim 4\pi r^3/3$ inside that region. Since the gravitational wave energy is produced at a rate $\approx -dE_h/dt \sim dE_h/dr$ and is flowing generally outward with an effective radial velocity near the speed of light ($c = 1$), the energy within the distance r of the COM origin is of the order of rdE_h/dr , giving an effective gravitational wave energy density $\rho \sim (rdE_h/dr)/r^3 = (1/r^2)dE_h/dr$, dropping factors of the order of unity such as $4\pi/3$ and various velocities in units of the speed of light.

For the total rate of the increase of E_h , the time-dependent sum of the rest masses plus kinetic energies of the two black holes, if we now add the positive contribution of the absorption of gravitational waves to the negative contribution from the emission of the gravitational waves, and divide by the negative $v_r = dr/dt$, we get, as an improvement over Eq. (5),

$$\frac{dE_h}{dr} \sim \frac{1}{\alpha} \frac{E_h^2}{r^2} - \delta \frac{m^2}{r^2} \frac{dE_h}{dr}, \quad (6)$$

where I have introduced a second unknown dimensionless factor δ to take into account the unknown factors of the order of unity that were dropped in the previous paragraph. Of course, Eq. (6) can readily be rearranged to give

$$\frac{dE_h}{dr} \sim \left(1 + \delta \frac{m^2}{r^2}\right)^{-1} \frac{1}{\alpha} \frac{E_h^2}{r^2}. \quad (7)$$

Next, we need to get an approximate differential equation for how m , the rest mass of each black hole, evolves as r decreases. If each black hole were absorbing radiation whose center-of-momentum energy were at rest in the COM frame (i.e., if there were no energy flux in that frame), the mass would increase at a rate that is $\gamma_h \equiv (1/2)E_h/m$ times the rate of its energy increase from the absorption of gravitational waves that was approximated above by $-(m^2/r^2)(dE_h/dt)$, so then

$$\frac{dm}{dt} \sim -\frac{1}{2} \frac{E_h}{m} \frac{m^2}{r^2} \frac{dE_h}{dt}. \quad (8)$$

Replacing the factor of $1/2$ by a third unknown dimensionless factor of ϵ to take into account the uncertainty in the mean effective velocity of the gravitational waves, rearranging Eq. (8), and combining it with Eq. (7) gives

$$\frac{r}{m} \frac{dm}{dr} \sim -\epsilon \frac{E_h}{r} \frac{dE_h}{dr} \sim -\left(1 + \delta \frac{m^2}{r^2}\right)^{-1} \frac{\epsilon}{\alpha} \frac{E_h^3}{r^3}. \quad (9)$$

One can then combine Eqs. (7) and (9) to get the rate at which the logarithm of m increases when the logarithm of E_h decreases:

$$\frac{d \ln m}{d \ln E_h} \equiv \frac{E_h}{m} \frac{dm}{dE_h} \sim -\epsilon \frac{E_h^2}{r^2}. \quad (10)$$

One can see from Eq. (7) that when $\delta m^2/r^2 \ll 1$, as r decreases with time as the black holes spiral closer together, say with initial conditions $E_h \sim E$ at $r \sim r_0$,

$$E_h \sim \frac{\alpha r}{1 + \alpha r/E - r/r_0}, \quad (11)$$

so that as r drops far below r_0 and E, E_h approaches αr . Therefore, during this stage of the evolution, when $E_h \sim \alpha r$, Eq. (10) gives

$$\frac{d \ln m}{d \ln E_h} \sim -\epsilon \alpha^2, \quad (12)$$

so

$$\frac{m}{m_0} \sim \left(\frac{E_h}{E}\right)^{-\alpha^2 \epsilon} \sim \left(\frac{\alpha r}{E}\right)^{-\alpha^2 \epsilon}, \quad (13)$$

where $m_0 = M/2 = E/(2\gamma)$ is the initial rest mass of each black hole (with the sum of the two initial black hole rest masses being $2m_0 = M = E/\gamma$), and where the initial value of E_h , the total rest mass plus kinetic energies of the two black holes, is the constant total COM energy $E = \gamma M = 2\gamma m_0$,

and where the initial value of r at the beginning of inspiral is $r_0 \sim b \sim E$, assuming $b \sim b_c = \beta E$ with $\beta \sim \beta_c \sim 1$.

The black holes will merge when $m \sim r$, so that then

$$1 \sim \frac{m}{r} = m_0 \frac{1}{r} \frac{m}{m_0} \sim \frac{E}{2\gamma E_h} \left(\frac{E_h}{E} \right)^{-\alpha^2 \epsilon} = \frac{\alpha}{2\gamma} \left(\frac{E_h}{E} \right)^{-1-\alpha^2 \epsilon}. \quad (14)$$

Therefore, one gets that the fraction F of the initial total COM energy R that is *not* radiated into gravitational waves is approximately

$$F \equiv 1 - f \sim \frac{E_h}{E} \sim \left(\frac{\alpha}{2\gamma} \right)^{\frac{1}{1+\alpha^2 \epsilon}} \sim \left(\frac{\alpha}{2\gamma} \right)^p, \quad (15)$$

with exponent

$$p \sim \frac{1}{1 + \alpha^2 \epsilon}. \quad (16)$$

Therefore, for large $\gamma \equiv E/M = E/(2m_0)$, $F = 1 - f$ goes to zero as a positive power of $1/\gamma$, though without numerical calculations the exponent, $p \sim 1/(1 + \alpha^2 \epsilon)$, which lies in the range between 0 and 1, apparently cannot be predicted precisely.

The numerical prefactor of the power of $1/\gamma$, here crudely estimated to be $(\alpha/2)^p$, would also be expected to have relatively small corrections from ignoring nonzero values of δ and $E/r_0 - \alpha$ and from also ignoring various departures from the power-law relations assumed above. However, in any case the evidence seems strong that the fraction of energy radiated away goes to zero as a positive power of $1/\gamma$ when γ is taken to infinity, with the exponent apparently being positive but less than one.

IV. POSSIBLE REASONS FOR THE CONTRARY CONCLUSION OF SPERHAKE, BERTI, CARDOSO, AND PRETORIUS

Now I wish to examine some of the possible reasons for the contrary conclusion of Spherhake, Berti, Cardoso, and Pretorius [81], “An extrapolation of our results to the limit $\gamma \rightarrow \infty$ suggests that about half of the center-of-mass energy of the system can be emitted in gravitational radiation, while the rest must be converted into rest-mass and spin energy.”

First, the numerical data SBCP used only went up to $\gamma = 2.49$, which is somewhat large compared with unity, but hardly large enough to form definite conclusions about what happens when γ is taken to infinity. For example, Spherhake, Cardoso, Pretorius, Berti, and González [67] found that a good fit to their head-on collision data of the fraction, which I am hereby labeling $f_0(\gamma) = \Delta E/E$, of the gravitational wave energy radiated, ΔE , out of the total energy, E , for impact parameter $b = 0$ as a function of

$\gamma \equiv E/M$ (with my M and E , not their M which is the total ADM mass that is my E , and not their E which is my ΔE) is given by the zero frequency limit (ZFL) formula [38], the right-hand side of Eq. (3) in [67], with the energy cutoff for the ZFL evaluated as $E_\infty = 0.14 \pm 0.03$, say $E_\infty = 0.14$ for the numerical values below, so that Eq. (3) in [67] can be written as

$$\begin{aligned} f_0(\gamma) &= 0.14 \left(1 + \frac{1}{2\gamma^2} - \frac{(4\gamma^2 - 1) \ln(\gamma + \sqrt{\gamma^2 - 1})}{2\gamma^3 \sqrt{\gamma^2 - 1}} \right) \\ &= 0.14 \left(\frac{3 - v^2}{2} - (1 - v^2)(3 + v^2) \frac{1}{4v} \ln \frac{1 + v}{1 - v} \right) \\ &= 0.14 \sum_{n=2}^{\infty} \frac{8(n-1)v^{2n}}{(2n-3)(2n-1)(2n+1)} \\ &= 0.14 \left(\frac{8}{15} v^4 + \frac{16}{105} v^6 + \frac{8}{105} v^8 + \frac{32}{693} v^{10} \right. \\ &\quad \left. + \frac{40}{1287} v^{12} + \frac{16}{715} v^{14} + \dots \right). \end{aligned} \quad (17)$$

For $\gamma = 2.49$, this formula gives a fraction $f_0(2.49) = 0.077$ that is only 55% as large as the $\gamma \rightarrow \infty$ limit of 0.14, which does not seem sufficient for obtaining firm conclusions about the $\gamma \rightarrow \infty$ limit.

Second, SBCP [81] fit to a formula that, as $1/\gamma \rightarrow 0$, approaches a constant with a deviation proportional to $1/\gamma$, rather than to $(1/\gamma)^p$ with the smaller exponent $p = 1/(1 + \alpha^2 \epsilon)$ that my analysis gives. Therefore, it is perhaps not surprising that they did not see the slower decrease of $1 - f = (E - \Delta E)/E$ to zero for $\gamma \rightarrow \infty$.

In particular, SBCP fit their numerical estimates for what I am calling $f(\gamma) = \Delta E/E$, the fraction of the total energy E radiated as gravitational waves of energy ΔE at the γ -dependent critical impact parameter $b = b_c = \beta_c E$, with the following formula for what I shall label as $f_{\text{SBCP}}(\gamma)$, in terms of the fraction $f_0(\gamma)$ at $b = 0$:

$$f_{\text{SBCP}}(\gamma) = \frac{f_0(\gamma)}{\mathcal{R}_{\text{SBCP}}(\gamma)}, \quad \mathcal{R}_{\text{SBCP}}(\gamma) = 0.34(1 - 1/\gamma). \quad (18)$$

Let me compare different fits to the data given for initially nonrotating black holes in Table I of [81] for $f(\gamma) = \Delta E/E$, which is the product of what is labeled in Table I as $K/M = (\gamma - 1)/\gamma$ and what is labeled as E_{rad}/K , with their E_{rad} being the same as my ΔE , the total gravitational wave energy radiated. These data for $f(\gamma)$ are (here retaining all 6 digits from multiplying the 3-digit values of K/M and E_{rad}/K in Table I, leaving rounding to fewer digits until later)

$$\begin{aligned} f(1.22) &= 0.160921, & f(1.88) &= 0.298116, \\ f(2.49) &= 0.343252. \end{aligned} \quad (19)$$

More relevant for my arguments that $f(\gamma) \rightarrow 1$ as $\gamma \rightarrow \infty$ are the values for

$$\begin{aligned} F(\gamma) &\equiv 1 - f(\gamma) = M_f/E \\ &= (\text{final rest mass } M_f)/(\text{total energy } E), \end{aligned} \quad (20)$$

where M_f is the final mass of the black hole that forms for impact parameter b infinitesimally below the critical impact parameter b_c . Table I of [81] then implies

$$\begin{aligned} F(1.22) &= 0.839079, & F(1.88) &= 0.701884, \\ F(2.49) &= 0.656748. \end{aligned} \quad (21)$$

The SBCP fitting function $f_{\text{SBCP}}(\gamma)$ of Eqs. (17) and (18) then gives what I call

$$\begin{aligned} F_{\text{SBCP}}(\gamma) &= 1 - \frac{0.14/0.34}{(1 - 1/\gamma)} \\ &\times \left(1 + \frac{1}{2\gamma^2} - \frac{(4\gamma^2 - 1) \ln(\gamma + \sqrt{\gamma^2 - 1})}{2\gamma^3 \sqrt{\gamma^2 - 1}} \right), \end{aligned} \quad (22)$$

with values

$$\begin{aligned} F_{\text{SBCP}}(1.22) &= 0.854 = 1.018F(1.22), \\ F_{\text{SBCP}}(1.88) &= 0.676 = 0.963F(1.88), \\ F_{\text{SBCP}}(2.49) &= 0.620 = 0.944F(2.49), \end{aligned} \quad (23)$$

which have an rms error relative to the numerical values of $F(\gamma)$ of about 4.0%.

However, since my Eq. (15) for the asymptotic behavior of $F(\gamma)$ has two unknown parameters, α and ϵ (ignoring weak dependencies on the other unknown dimensionless parameters such as δ and $E/r_0 - \alpha$), for a fair comparison I should replace the one-parameter SBCP fitting function $f_{\text{SBCP}}(\gamma) = f_0(\gamma)/\mathcal{R}_{\text{SBCP}}(\gamma)$, having its one fitted constant 0.34 in $\mathcal{R}_{\text{SBCP}}(\gamma) = 0.34(1 - 1/\gamma)$, with a two-parameter fitting function $f_2(\gamma) = f_0(\gamma)/\mathcal{R}_2(\gamma)$ with $\mathcal{R}_2(\gamma) = a - b/\gamma$, having two fitting parameters, a and b . Then a least-squares fit to the numerical data for $f(\gamma)$ given in Eq. (19) yields $a = 0.389$ and $b = 0.406$ after rounding to 3 digits, and hence

$$\begin{aligned} F_2(\gamma) &= 1 - \frac{f_0(\gamma)}{(0.389 - 0.406/\gamma)} \\ &= 1 - \frac{0.14}{(0.389 - 0.406/\gamma)} \\ &\times \left(1 + \frac{1}{2\gamma^2} - \frac{(4\gamma^2 - 1) \ln(\gamma + \sqrt{\gamma^2 - 1})}{2\gamma^3 \sqrt{\gamma^2 - 1}} \right). \end{aligned} \quad (24)$$

This fit gives

$$\begin{aligned} F_2(1.22) &= 0.8408 = 1.0021F(1.22), \\ F_2(1.88) &= 0.7018 = 0.9999F(1.88), \\ F_2(2.49) &= 0.6577 = 1.0014F(2.49), \end{aligned} \quad (25)$$

which have an rms error relative to the numerical values of $F(\gamma)$ of only about 0.14%, a remarkably good fit, but perhaps the extreme excellency of this fit is somewhat spurious because of the numerical uncertainties of the values of $F(\gamma)$ given in Eq. (21) derived from the SBCP Table I of [81].

In contrast to these fits that have $1/\gamma$ appearing linearly in $\mathcal{R}_{\text{SBCP}}(\gamma)$ and $\mathcal{R}_2(\gamma)$ (though there is a term asymptotically going as $(\ln \gamma)/\gamma^2$ in $f_0(\gamma)$ given by Eq. (17) from the right-hand side of the ZFL Eq. (3) of [67] that originally came from Eq. (2.20) of [38]), the derivation in Sec. III above of the large- γ asymptotic form given by Eq. (15) suggests a fit by $F_p(\gamma) = A\gamma^{-p}$ with two parameters, $p \sim 1/(1 + \alpha^2\epsilon)$ and $A \sim (\alpha/2)^p$. Then a least-squares fit of the three numerical values of $\ln F(\gamma)$ by $\ln A - p \ln \gamma$ gives, to three digits each, $A = 0.892$ and $p = 0.343 = (0.7)^3$, or

$$F_p(\gamma) = 0.892\gamma^{-0.343}. \quad (26)$$

This fit gives $\alpha \sim 1.44$, $\epsilon \sim 0.93$ (though I would not assign much precision to these estimated values, because of the uncertainty of the extrapolation to $\gamma \gg 1$).

Then $F_p(\gamma) = 0.892\gamma^{-0.343}$ gives

$$\begin{aligned} F_p(1.22) &= 0.833 = 0.993F(1.22), \\ F_p(1.88) &= 0.718 = 1.023F(1.88), \\ F_p(2.49) &= 0.652 = 0.993F(2.49), \end{aligned} \quad (27)$$

which has an rms relative error of about 1.46%, a bit over 10 times the rms relative error of $F_2(\gamma)$. However, since the numerical error of the data for $F(\gamma)$ in Eq. (21) is likely to be a few percent, it seems that both formulas fit the three data points of Eq. (21) quite well. Certainly Eq. (26) for $F_p(\gamma)$ looks simpler than Eq. (24) for $F_2(\gamma)$, and appears even simpler than Eq. (22) for $F_{\text{SBCP}}(\gamma)$ that does not fit quite so well, though since the complication arises mostly from the complexity of Eq. (17) for $f_0(\gamma)$, the fraction of energy radiated in a head-on collision, which is theoretically determined by the zero frequency limit of [38] with

only the one free parameter E_∞ from the frequency cutoff, there is actually only one free parameter that is fit to the data in Eq. (22), namely the fraction $0.14/0.34 = 7/17 \approx 0.412$.

The main point is that Eq. (15) has the theoretical justification given in Sec. III, as an approximate asymptotic formula for large $\gamma = E/M = (\text{total energy})/(\text{rest mass})$, whereas the factors of $1 - 1/\gamma$ and $0.389 - 0.406/\gamma$ in Eqs. (22) and (24) appear to be rather *ad hoc*. And even if Eq. (15) is not quite the correct asymptotic form for the fraction of energy not radiated when $\gamma \gg 1$, the fact that during most of the evolution the black holes are expected to be much smaller than their separations very strongly suggests that they cannot absorb a large fraction of the initial energy.

A third possible reason for the disputed conclusion of Sperhake, Berti, Cardoso, and Pretorius [81], that $F = 1 - f = (E - \Delta E)/E$ remains bounded below by a positive number when γ is taken to infinity, is that the 2007 results of Pretorius and Khurana [61] suggest that $f = \Delta E/E$, the fraction of the initial total energy E going into the energy ΔE , is a very sensitive function of the impact parameter b when it is near the critical impact parameter b_c . Therefore, it might be the case that for the crucial data SBCP calculated at $\gamma = 2.49$, which was done with impact parameter $b = 2.749E$ (in my notation; they use M for the total ADM mass that I call E), might not have been near enough to the impact parameter $b_m = \beta_m E$ that would maximize the fraction $f = \Delta E/E$ going into gravitational waves at that value of $\gamma = E/M$ (where my M is the sum of the initial two rest masses of the black holes).

An observation supporting the hypothesis that with the limited computational resources available, SBCP were not able to tune the impact parameter close enough to the value b_m that would give the maximum energy radiated to correctly evaluate what that maximum is for each γ , is the fact that Fig. 1 of [81] shows that the example they calculated for their maximum γ , $\gamma = 2.49$, does not have the black hole separation decreasing monotonically before they reach their minimum separation and then move apart. I would expect that at $b = b_m$, and also at $b = b_c$, the critical impact parameter that maximizes $f = \Delta E/E$ out of all critical impact parameters, the separation between the two black holes would decrease monotonically until the black holes merge. Therefore, Fig. 1 seems to be evidence that SBCP were not able to tune the impact parameter sufficiently precisely to obtain the maximum fraction f of energy radiated. Of course, this is not a criticism of their heroic effort to learn what they could at as large a value of γ that was feasible with their calculational resources, but it does suggest that even more resources will be needed to find numerical evidence convincingly suggesting whether or not the final black hole (if the impact parameter b is infinitesimally smaller than the critical impact parameter $b_c = \beta_c E$ for fixed total energy E that maximizes $f = \Delta E/E$ at that E) can have a mass M_f such that $M_f/E \rightarrow 0$ as $M/E \rightarrow 0$.

Actually, the “zoom-whirl” behavior of the two black holes SBCP calculated for $\gamma \equiv E/M = 2.49$ (giving $v \approx 0.9158$) and impact parameter $b = 2.755E \approx 1.009E(2.5/v)$ (incidentally fitting very closely to the estimate of Shibata, Okawa, and Yamamoto [68] that $b_c \approx 2.5E/v$) suggests to me that perhaps that value of b SBCP used is closer to a critical dimensionless impact parameter that is different from the one that maximizes $f = \Delta E/E$ and which I would expect would lead to something close to an equiangular spiral in which the separation of the two black holes decreases monotonically until they merge (if b is infinitesimally below the b_c that maximizes f). So far the data gives me only a weak hint of the possibility that there may be more than one critical impact parameter, and that the fraction $f = \Delta E/E$ of energy radiated into gravitational waves might not have a single local maximum as a function of $\gamma = E/M$. However, it would be very interesting to explore this intriguing possibility.

I might initially *guess* that the number N of local maxima of f might grow approximately linearly with γ , and that the number of critical impact parameters might be $2N - 1$ (say if there is one local maximum of f for each odd-numbered critical impact parameter). This would suggest that for fixed total energy $E \gg M$ and fixed initial total rest mass $M = E/\gamma$, although the global maximum for f would occur for $b = \beta_m E \sim E$, $f(b)$ might have some oscillatory structure with a period of the order of $\delta b \sim M = E/\gamma$. On the other hand, this would give a range for all $2N - 1$ critical impact parameters of $\Delta b \sim N\delta b \sim E$, whereas I would expect the ratio of the range of the oscillatory behavior to the value of b_m (with b_m giving the global maximum for f for the fixed values of E and M) to decrease as γ is increased, so perhaps the typical period of the oscillations, say δb , goes as $b_m \sim E$ multiplied by a power of $1/\gamma = M/E$ that is larger than one, so that δb becomes far smaller than M for large γ . Or, perhaps the number N of local maxima of f might increase slower than linearly with γ , so that the typical period of the oscillations, δb , could remain of the order of M but the range, $\Delta b \sim N\delta b$, could become much smaller than b_m for large γ . In any case, it would be very interesting to see what the behavior of $f(\gamma)$ is as a function of $\gamma = E/M$ when it becomes large, and to check whether it exhibits any oscillations, giving more than one local maximum.

Perhaps when the computational resources become available for making many more calculations at slightly different impact parameters for the same γ , and ideally also for larger γ , it might become more apparent whether or not the fraction F of the total energy that does not get radiated does approach zero as γ is increased indefinitely with $\beta \equiv b/E$ tuned to the γ -dependent value $\beta_m(\gamma)$ that maximizes the fraction $f = 1 - F$ of the total energy that is radiated for each $\gamma = E/M$.

Another argument for $F(\gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$ is that in the limit $1/\gamma = 0$ at fixed total energy E , the rest mass of each

black hole, $M/2 = E/(2\gamma)$, goes to zero, so initially one has two incoming Aichelburg-Sexl metrics [37], corresponding to the gravitational fields of two classical massless point particles. In this case there are, at least initially, no black holes to absorb any of the energy, so when SBCP argue [81] that “absorption sets an upper bound on the maximum energy that can be radiated,” it would seem that no absorption would occur in the $M = 0$ limit, since there are no black holes to absorb the energy (until the two massless particles form a single black hole).

One weakness of this last argument is that it might occur that in the Aichelburg-Sexl limit, as massless particle A collides with the gravitational shock wave of massless particle B, the gravitational shock wave field of A might focus the gravitational shock wave field of B to form a black hole separate from a black hole that might form from the field of B focusing the shock wave of A. Thus *perhaps* two black holes *might* form that conceivably each could have its rest mass be a fraction of the total energy that is bounded below by a positive number. In this case that positive number would give a lower bound on the ratio M_f/E for the final black hole that forms with impact parameter b slightly below the critical impact parameter b_c .

This conceptual possibility raises the question of whether these possible two distinct holes each engulf the corresponding massless particle, which might limit the number of black holes that form to that pair, or whether the two holes that might form from the focusing of the gravitational shock wave of one massless particle by the shock wave of the other might form behind the massless particles rather than engulfing them. This latter possibility raises the question of whether the gravitational fields of the two persisting massless particles might focus to produce even more pairs of black holes. Although I am rather skeptical that the gravitational fields of the two massless particles can form even one pair of black holes that persist before merging to form one final black hole (if $b < b_c$) or eventually flying apart (if $b > b_c$), if two black holes can form without engulfing the massless particles, it would seem conceivable that an arbitrarily large number of pairs of black holes could form from suitable tuning of $\beta = b/E$, and even that an arbitrarily large number could survive the encounter rather than merging if $b < b_c$.

However, the more plausible possibility for $1/\gamma = M/E = 0$ seems to be that the two massless particles that are the source of the original pair of Aichelburg-Sexl metrics cannot form separate black holes but only a single black hole if they merge. In this case, there are never any separate black holes to absorb any of the radiation, and it seems most plausible that by fine tuning the ratio $\beta = b/E$ of the impact parameter b to the total energy E to the critical value β_0 , all of the energy would be radiated away before the two particles merge and disappear at a naked singularity

corresponding to a final single black hole in the limit that its mass goes to zero.

An anonymous referee suggested that “a third logical possibility exists whereby, at the critical impact parameter, a vanishingly small mass black-hole forms from the collapse of a vanishingly small amount of radiation while the remaining (i.e., 100%) of the energy reaches infinity but is shared between radiation and the two colliding particles. Only at a smaller impact parameter the colliding particles will be swallowed by the collapse but, by then, a finite fraction of the energy will remain in the collapsing region(s). In this case a 100% efficiency will never be reached.”

Although this also seems logically possible, I would find it more surprising than the possibility that all but a vanishingly small fraction of the initial energy can be radiated as gravitational waves in the limit that $\gamma = E/M$ is taken to infinity with the impact parameter suitably tuned at each finite γ as one takes this limit.

V. CONCLUSIONS

In this paper I have given arguments that the 2007 conjecture of Pretorius and Khurana [61] is essentially correct, that all but an arbitrarily small fraction of the total energy E of ultrarelativistic objects of original total rest mass M can be converted to gravitational waves if M/E can be taken small enough. (I do not argue that *all* of the kinetic energy can be converted, since the final total rest mass M_f could be larger than M , but only that as M/E is taken to zero, M_f/E also approaches zero with the proper fine tuning of the impact parameter.) The argument for two original black holes with $M/E \ll 1$ is that during most of the gravitational wave emission, the black holes are so much smaller than the region where most of the gravitational wave energy is located that they can absorb only a tiny fraction of it, with this fraction going to zero as M/E is taken to zero.

Contrary arguments in 2012 by Sperhake, Berti, Cardoso, and Pretorius do not seem sufficient to refute this argument, since (1) they could only calculate the emission near the critical impact parameter for $\gamma \leq 2.49$, (2) they used a rather *ad hoc* fitting function that does not agree with the theoretical expectations put forward in this paper, and (3) they do not seem to have been able to tune the impact parameter close enough to the critical value to get a convincingly good estimate of the maximum fraction of energy that could be emitted for their finite values of E/M .

I had posted on the arXiv a previous version of my conjecture that by suitably tuning the impact parameter, in the limit that the total energy of two colliding small objects is taken to infinity, all but an infinitesimally small fraction can be radiated as gravitational waves [119], before

I learned that Pretorius and Khurana [61] had earlier made essentially the same conjecture and that Sperhake, Berti, Cardoso, and Pretorius [81] had presented numerical evidence against that conjecture. That earlier paper of mine does give a lot of details not included in the present paper about the conjectured limiting case of an asymptotically self-similar solution with a homothetic Killing vector, so after the present paper is published, I plan to revise that earlier paper in light of this present one and resubmit it also for publication.

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- [1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, *Phys. Rev. Lett.* **116**, 061102 (2016).
- [2] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW150914: The Advanced LIGO Detectors in the Era of First Discoveries, *Phys. Rev. Lett.* **116**, 131103 (2016).
- [3] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW150914: First results from the search for binary black hole coalescence with Advanced LIGO, *Phys. Rev. D* **93**, 122003 (2016).
- [4] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Properties of the Binary Black Hole Merger GW150914, *Phys. Rev. Lett.* **116**, 241102 (2016).
- [5] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Tests of General Relativity with GW150914, *Phys. Rev. Lett.* **116**, 221101 (2016); **121**, 129902(E) (2018).
- [6] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), The rate of binary black hole mergers inferred from Advanced LIGO observations surrounding GW150914, *Astrophys. J. Lett.* **833**, L1 (2016).
- [7] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Astrophysical implications of the binary black-hole merger GW150914, *Astrophys. J. Lett.* **818**, L22 (2016).
- [8] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, *Phys. Rev. Lett.* **116**, 241103 (2016).
- [9] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Binary Black Hole Mergers in the first Advanced LIGO Observing Run, *Phys. Rev. X* **6**, 041015 (2016); **8**, 039903(E) (2018).
- [10] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, *Phys. Rev. Lett.* **118**, 221101 (2017); **121**, 129901(E) (2018).
- [11] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence, *Phys. Rev. Lett.* **119**, 141101 (2017).
- [12] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170608: Observation of a 19-solar-mass binary black hole coalescence, *Astrophys. J. Lett.* **851**, L35 (2017).
- [13] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Tests of General Relativity with GW170817, *Phys. Rev. Lett.* **123**, 011102 (2019).
- [14] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs, *Phys. Rev. X* **9**, 031040 (2019).
- [15] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Binary black hole population properties inferred from the first and second observing runs of Advanced LIGO and Advanced Virgo, *Astrophys. J. Lett.* **882**, L24 (2019).
- [16] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Open data from the first and second observing runs of Advanced LIGO and Advanced Virgo, *SoftwareX* **13**, 100658 (2021).
- [17] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Tests of general relativity with the binary black hole signals from the LIGO-Virgo catalog GWTC-1, *Phys. Rev. D* **100**, 104036 (2019).
- [18] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW190425: Observation of a compact binary coalescence with total mass $\sim 3.4M_{\odot}$, *Astrophys. J. Lett.* **892**, L3 (2020).
- [19] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW190412: Observation of a binary-black-hole coalescence with asymmetric masses, *Phys. Rev. D* **102**, 043015 (2020).
- [20] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW190814: Gravitational waves from the coalescence of a 23 solar mass black hole with a 2.6 solar mass compact object, *Astrophys. J. Lett.* **896**, L44 (2020).
- [21] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW190521: A Binary Black Hole Merger with a Total Mass of $150M_{\odot}$, *Phys. Rev. Lett.* **125**, 101102 (2020).
- [22] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Properties and astrophysical implications of the $150 M_{\odot}$ binary black hole merger GW190521, *Astrophys. J. Lett.* **900**, L13 (2020).

- [23] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run, *Phys. Rev. X* **11**, 021053 (2021).
- [24] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Tests of general relativity with binary black holes from the second LIGO-Virgo gravitational-wave transient catalog, *Phys. Rev. D* **103**, 122002 (2021).
- [25] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-2.1: Deep extended catalog of compact binary coalescences observed by LIGO and Virgo during the first half of the third observing run, [arXiv:2108.01045](https://arxiv.org/abs/2108.01045).
- [26] R. Abbott *et al.* (LIGO Scientific, Virgo and KAGRA Collaborations), GWTC-3: Compact binary coalescences observed by LIGO and Virgo during the second part of the third observing run, [arXiv:2111.03606](https://arxiv.org/abs/2111.03606).
- [27] R. Abbott *et al.* (LIGO Scientific, Virgo and KAGRA Collaborations), Tests of general relativity with GWTC-3, [arXiv:2112.06861](https://arxiv.org/abs/2112.06861).
- [28] A. Ghosh (LIGO Scientific–Virgo–Kagra Collaborations), Summary of tests of general relativity with GWTC-3, [arXiv:2204.00662](https://arxiv.org/abs/2204.00662).
- [29] B. Bertotti, On gravitational motion, *Nuovo Cimento* **4**, 898 (1956).
- [30] B. Bertotti and J. Plebanski, Theory of gravitational perturbations in the fast motion approximation, *Ann. Phys. (N.Y.)* **11**, 169 (1960).
- [31] S. Weinberg, Infrared photons and gravitons, *Phys. Rev.* **140**, B516 (1965).
- [32] R. Penrose, General relativistic energy flux and elementary optics, in *Perspectives in Geometry and Relativity*, edited by B. Hoffman (Indiana University Press, Bloomington, 1966).
- [33] K. A. Khan and R. Penrose, Scattering of two impulsive gravitational plane waves, *Nature (London)* **229**, 185 (1971).
- [34] S. W. Hawking, Gravitational Radiation from Colliding Black Holes, *Phys. Rev. Lett.* **26**, 1344 (1971).
- [35] K. S. Thorne, Nonspherical gravitational collapse: A short review, in *Magic Without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco), p. 231.
- [36] P. D. D’Eath, High speed black hole encounters and gravitational radiation, *Phys. Rev. D* **18**, 990 (1978).
- [37] P. C. Aichelburg and R. U. Sexl, On the gravitational field of a massless particle, *Gen. Relativ. Gravit.* **2**, 303 (1971).
- [38] L. Smarr, Gravitational radiation from distant encounters and from headon collisions of black holes: The zero frequency limit, *Phys. Rev. D* **15**, 2069 (1977).
- [39] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 3. Derivation of bremsstrahlung formulas, *Astrophys. J.* **217**, 252 (1977).
- [40] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 4. bremsstrahlung, *Astrophys. J.* **224**, 62 (1978).
- [41] T. Damour and N. Deruelle, Radiation reaction and angular momentum loss in small angle gravitational scattering, *Phys. Lett.* **87A**, 81 (1981).
- [42] L. Bel, T. Damour, N. Deruelle, J. Ibanez, and J. Martin, Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity, *Gen. Relativ. Gravit.* **13**, 963 (1981).
- [43] G. ’t Hooft, Graviton dominance in ultrahigh-energy scattering, *Phys. Lett. B* **198**, 61 (1987).
- [44] D. Amati, M. Ciafaloni, and G. Veneziano, Superstring collisions at Planckian energies, *Phys. Lett. B* **197**, 81 (1987).
- [45] D. Amati, M. Ciafaloni, and G. Veneziano, Classical and quantum gravity effects from Planckian energy superstring collisions, *Int. J. Mod. Phys. A* **03**, 1615 (1988).
- [46] I. J. Muzinich and M. Soldate, High-energy unitarity of gravitation and strings, *Phys. Rev. D* **37**, 359 (1988).
- [47] B. Sundborg, High-energy asymptotics: The one loop string amplitude and resummation, *Nucl. Phys.* **B306**, 545 (1988).
- [48] D. Amati, M. Ciafaloni, and G. Veneziano, Higher order gravitational deflection and soft Bremsstrahlung in Planckian energy superstring collisions, *Nucl. Phys.* **B347**, 550 (1990).
- [49] P. D. D’Eath and P. N. Payne, Gravitational radiation in high speed black hole collisions. 1. Perturbation treatment of the axisymmetric speed of light collision, *Phys. Rev. D* **46**, 658 (1992).
- [50] P. D. D’Eath and P. N. Payne, Gravitational radiation in high speed black hole collisions. 2. Reduction to two independent variables and calculation of the second order news function, *Phys. Rev. D* **46**, 675 (1992).
- [51] P. D. D’Eath and P. N. Payne, Gravitational radiation in high speed black hole collisions. 3. Results and conclusions, *Phys. Rev. D* **46**, 694 (1992).
- [52] D. Amati, M. Ciafaloni, and G. Veneziano, Planckian scattering beyond the semiclassical approximation, *Phys. Lett. B* **289**, 87 (1992).
- [53] M. W. Choptuik, Universality and Scaling in Gravitational Collapse of a Massless Scalar Field, *Phys. Rev. Lett.* **70**, 9 (1993).
- [54] T. Banks and W. Fischler, A model for high-energy scattering in quantum gravity, [arXiv:hep-th/9906038](https://arxiv.org/abs/hep-th/9906038).
- [55] J. J. Levin, Gravity Waves, Chaos, and Spinning Compact Binaries, *Phys. Rev. Lett.* **84**, 3515 (2000).
- [56] D. M. Eardley and S. B. Giddings, Classical black hole production in high-energy collisions, *Phys. Rev. D* **66**, 044011 (2002).
- [57] E. Kohlprath and G. Veneziano, Black holes from high-energy beam-beam collisions, *J. High Energy Phys.* **06** (2002) 057.
- [58] H. Yoshino and Y. Nambu, Black hole formation in the grazing collision of high-energy particles, *Phys. Rev. D* **67**, 024009 (2003).
- [59] S. B. Giddings and V. S. Rychkov, Black holes from colliding wavepackets, *Phys. Rev. D* **70**, 104026 (2004).
- [60] H. Yoshino and V. S. Rychkov, Improved analysis of black hole formation in high-energy particle collisions, *Phys. Rev. D* **71**, 104028 (2005); **77**, 089905(E) (2008).
- [61] F. Pretorius and D. Khurana, Black hole mergers and unstable circular orbits, *Classical Quantum Gravity* **24**, S83 (2007).
- [62] S. B. Giddings, D. J. Gross, and A. Maharana, Gravitational effects in ultrahigh-energy string scattering, *Phys. Rev. D* **77**, 046001 (2008).
- [63] C. Gundlach and J. M. Martin-Garcia, Critical phenomena in gravitational collapse, *Living Rev. Relativity* **10**, 5 (2007).

- [64] S. B. Giddings and M. Srednicki, High-energy gravitational scattering and black hole resonances, *Phys. Rev. D* **77**, 085025 (2008).
- [65] D. Amati, M. Ciafaloni, and G. Veneziano, Towards an S-matrix description of gravitational collapse, *J. High Energy Phys.* **02** (2008) 049.
- [66] G. Veneziano and J. Wosiek, Exploring an S-matrix for gravitational collapse. II. A momentum space analysis, *J. High Energy Phys.* **09** (2008) 024.
- [67] U. Sperhake, V. Cardoso, F. Pretorius, E. Berti, and J. A. Gonzalez, The High-Energy Collision of Two Black Holes, *Phys. Rev. Lett.* **101**, 161101 (2008).
- [68] M. Shibata, H. Okawa, and T. Yamamoto, High-velocity collision of two black holes, *Phys. Rev. D* **78**, 101501(R) (2008).
- [69] J. Healy, J. Levin, and D. Shoemaker, Zoom-Whirl Orbits in Black Hole Binaries, *Phys. Rev. Lett.* **103**, 131101 (2009).
- [70] U. Sperhake, V. Cardoso, F. Pretorius, E. Berti, T. Hinderer, and N. Yunes, Cross Section, Final Spin and Zoom-Whirl Behavior in High-Energy Black Hole Collisions, *Phys. Rev. Lett.* **103**, 131102 (2009).
- [71] M. W. Choptuik and F. Pretorius, Ultra Relativistic Particle Collisions, *Phys. Rev. Lett.* **104**, 111101 (2010).
- [72] U. Sperhake, V. Cardoso, F. Pretorius, E. Berti, T. Hinderer, and N. Yunes, Ultra-relativistic grazing collisions of black holes, [arXiv:1003.0882](https://arxiv.org/abs/1003.0882).
- [73] E. Berti, V. Cardoso, T. Hinderer, M. Lemos, F. Pretorius, U. Sperhake, and N. Yunes, Semianalytical estimates of scattering thresholds and gravitational radiation in ultra-relativistic black hole encounters, *Phys. Rev. D* **81**, 104048 (2010).
- [74] S. B. Giddings, M. Schmidt-Sommerfeld, and J. R. Andersen, High energy scattering in gravity and supergravity, *Phys. Rev. D* **82**, 104022 (2010).
- [75] V. Cardoso, L. Gualtieri, C. Herdeiro, U. Sperhake, P. M. Chesler, L. Lehner, S. C. Park, H. S. Reall, C. F. Sopuerta, D. Alic *et al.*, NR/HEP: Roadmap for the future, *Classical Quantum Gravity* **29**, 244001 (2012).
- [76] C. Gundlach, S. Akcay, L. Barack, and A. Nagar, Critical phenomena at the threshold of immediate merger in binary black hole systems: The extreme mass ratio case, *Phys. Rev. D* **86**, 084022 (2012).
- [77] S. Akcay, L. Barack, T. Damour, and N. Sago, Gravitational self-force and the effective-one-body formalism between the innermost stable circular orbit and the light ring, *Phys. Rev. D* **86**, 104041 (2012).
- [78] L. Rezzolla and K. Takami, Black-hole production from ultrarelativistic collisions, *Classical Quantum Gravity* **30**, 012001 (2013).
- [79] W. E. East and F. Pretorius, Ultrarelativistic Black Hole Formation, *Phys. Rev. Lett.* **110**, 101101 (2013).
- [80] D. Bini and T. Damour, Gravitational radiation reaction along general orbits in the effective one-body formalism, *Phys. Rev. D* **86**, 124012 (2012).
- [81] U. Sperhake, E. Berti, V. Cardoso, and F. Pretorius, Universality, Maximum Radiation and Absorption in High-Energy Collisions of Black Holes with Spin, *Phys. Rev. Lett.* **111**, 041101 (2013).
- [82] C. R. Galley and R. A. Porto, Gravitational self-force in the ultra-relativistic limit: The ‘large- N ’ expansion, *J. High Energy Phys.* **11** (2013) 096.
- [83] T. Damour, F. Guercilena, I. Hinder, S. Hopper, A. Nagar, and L. Rezzolla, Strong-field scattering of two black holes: Numerics versus analytics, *Phys. Rev. D* **89**, 081503(R) (2014).
- [84] A. Gruzinov and G. Veneziano, Gravitational radiation from massless particle collisions, *Classical Quantum Gravity* **33**, 125012 (2016).
- [85] M. Ciafaloni, D. Colferai, and G. Veneziano, Emerging Hawking-Like Radiation from Gravitational Bremsstrahlung Beyond the Planck Scale, *Phys. Rev. Lett.* **115**, 171301 (2015).
- [86] J. Healy, I. Ruchlin, C. O. Lousto, and Y. Zlochower, High energy collisions of black holes numerically revisited, *Phys. Rev. D* **94**, 104020 (2016).
- [87] U. Sperhake, E. Berti, V. Cardoso, and F. Pretorius, Gravity-dominated unequal-mass black hole collisions, *Phys. Rev. D* **93**, 044012 (2016).
- [88] M. Ciafaloni, D. Colferai, F. Coradeschi, and G. Veneziano, Unified limiting form of graviton radiation at extreme energies, *Phys. Rev. D* **93**, 044052 (2016).
- [89] E. Berti, V. Cardoso, L. C. B. Crispino, L. Gualtieri, C. Herdeiro, and U. Sperhake, Numerical relativity and high energy physics: Recent developments, *Int. J. Mod. Phys. D* **25**, 1641022 (2016).
- [90] T. Damour, Gravitational scattering, post-Minkowskian approximation and effective one-body theory, *Phys. Rev. D* **94**, 104015 (2016).
- [91] A. Addazi, M. Bianchi, and G. Veneziano, Glimpses of black hole formation/evaporation in highly inelastic, ultra-Planckian string collisions, *J. High Energy Phys.* **02** (2017) 111.
- [92] D. Bini and T. Damour, Gravitational scattering of two black holes at the fourth post-Newtonian approximation, *Phys. Rev. D* **96**, 064021 (2017).
- [93] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, *Phys. Rev. D* **97**, 044038 (2018).
- [94] F. Pretorius and W. E. East, Black hole formation from the collision of plane-fronted gravitational waves, *Phys. Rev. D* **98**, 084053 (2018).
- [95] M. Ciafaloni, D. Colferai, and G. Veneziano, Infrared features of gravitational scattering and radiation in the eikonal approach, *Phys. Rev. D* **99**, 066008 (2019).
- [96] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, *Phys. Rev. Lett.* **122**, 201603 (2019).
- [97] A. Addazi, M. Bianchi, and G. Veneziano, Soft gravitational radiation from ultra-relativistic collisions at sub- and sub-sub-leading order, *J. High Energy Phys.* **05** (2019) 050.
- [98] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, *J. High Energy Phys.* **10** (2019) 206.

- [99] D. Bini, T. Damour, and A. Geralico, Novel Approach to Binary Dynamics: Application to the Fifth Post-Newtonian Level, *Phys. Rev. Lett.* **123**, 231104 (2019).
- [100] L. Barack, M. Colleoni, T. Damour, S. Isoyama, and N. Sago, Self-force effects on the marginally bound zoom-whirl orbit in Schwarzschild spacetime, *Phys. Rev. D* **100**, 124015 (2019).
- [101] T. Damour, Classical and quantum scattering in post-Minkowskian gravity, *Phys. Rev. D* **102**, 024060 (2020).
- [102] Z. Bern, H. Ita, J. Parra-Martinez, and M. S. Ruf, Universality in the Classical Limit of Massless Gravitational Scattering, *Phys. Rev. Lett.* **125**, 031601 (2020).
- [103] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian nonlocal-in-time dynamics of binary systems, *Phys. Rev. D* **102**, 084047 (2020).
- [104] D. Bini, T. Damour, A. Geralico, S. Laporta, and P. Mastrolia, Gravitational dynamics at $O(G^6)$: Perturbative gravitational scattering meets experimental mathematics, [arXiv:2008.09389](https://arxiv.org/abs/2008.09389).
- [105] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Universality of ultra-relativistic gravitational scattering, *Phys. Lett. B* **811**, 135924 (2020).
- [106] T. Damour, Radiative contribution to classical gravitational scattering at the third order in G , *Phys. Rev. D* **102**, 124008 (2020).
- [107] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Radiation reaction from soft theorems, *Phys. Lett. B* **818**, 136379 (2021).
- [108] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C. H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and Conservative Binary Dynamics at $O(G^4)$, *Phys. Rev. Lett.* **126**, 171601 (2021).
- [109] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, Gravitational Bremsstrahlung from Reverse Unitarity, *Phys. Rev. Lett.* **126**, 201602 (2021).
- [110] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(G^3)$, *J. High Energy Phys.* **07** (2021) 169.
- [111] E. Barausse, E. Berti, V. Cardoso, S. A. Hughes, and G. Khanna, Divergences in gravitational-wave emission and absorption from extreme mass ratio binaries, *Phys. Rev. D* **104**, 064031 (2021).
- [112] D. Bini, T. Damour, and A. Geralico, Radiative contributions to gravitational scattering, *Phys. Rev. D* **104**, 084031 (2021).
- [113] G. Veneziano and G. A. Vilkovisky, Angular momentum loss in gravitational scattering, radiation reaction, and the Bondi gauge ambiguity, *Phys. Lett. B* **834**, 137419 (2022).
- [114] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal operator at arbitrary velocities I: The soft-radiation limit, *J. High Energy Phys.* **07** (2022) 039.
- [115] Gabriele Veneziano, Lessons from the Ultra-Relativistic Frontier, *KITP Conference: Storming the Gravitational Wave Frontier* (2022), <https://online.kitp.ucsb.edu/online/gwaves-c22/veneziano/>.
- [116] Frans Pretorius, The classical ultra-relativistic scattering problem, KITP Program: High-Precision Gravitational Waves (2022), <https://online.kitp.ucsb.edu/online/gwaves22/pretorius/>.
- [117] Y. Hagiwara, Theory of the relativistic trajectories in a gravitational field of Schwarzschild, *Jpn. J. Astron. Geophys.* **8**, 67 (1931).
- [118] C. G. Darwin, The gravity field of a particle, *Proc. R. Soc. A (London)* **249**, 180 (1958).
- [119] D. N. Page, Critical gravitational inspiral of two massless particles, [arXiv:2209.15014](https://arxiv.org/abs/2209.15014).