Negative mass black holes in de Sitter space

Brayden R. Hull[1,3,*] and Robert B. Mann[2,3,†]
[1]*Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2G3G1, Canada*
[2]*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2G3G1, Canada*
[3]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*

(Received 1 December 2022; accepted 3 January 2023; published 13 March 2023)

We show that asymptotically de Sitter black holes of negative mass can exist in Lovelock gravity. Such black holes have horizon geometries with nonconstant curvature and are known as exotic black holes. We explicitly examine the case of Gauss-Bonnet gravity. We briefly discuss the positive mass case where we show how the transverse space geometry affects whether a black hole will exist or not. For negative mass solutions we show how three different black hole spacetimes are possible depending on the transverse space geometry. We also provide closed form bounds for the geometric parameters to ensure that a black hole spacetime is observed. We close with a discussion of the massless case, where there are many different spacetimes that are permitted.

DOI: 10.1103/PhysRevD.107.064027

I. INTRODUCTION

In general relativity, the simplest black hole solution is that provided by the Schwarzschild metric [1], where the only parameter of the solution is the mass m. Less than half a century later a more general solution was obtained [2], which describes a black hole with three parameters: mass, angular momentum, and charge. The introduction of a cosmological constant admits further solutions, as do other matter sources.

In four dimensions the assumption that the metric has radial symmetry restricts black hole solutions to have horizons of constant curvature in Einstein gravity. These can be spherical, planar, or hyperbolic if a negative cosmological constant $\Lambda < 0$ is introduced [3–5]. If the geometry is hyperbolic then these black holes can have negative mass [6] and can form from gravitational collapse [7]. However if $\Lambda \geq 0$ then the situation is notably different. While it has recently been shown that de Sitter–Schwarzschild spacetimes with negative mass in [8,9] can exist in Einstein gravity, these solutions consist of spacetime bubbles. No negative mass black hole vacuum solutions have been obtained.

We show in this paper that asymptotically de Sitter black hole vacuum solutions of negative mass exist in Lovelock gravity. This somewhat unexpected finding arises from a feature recently pointed out for black holes in higher-curvature gravity: Their event horizons need not be of constant curvature [10–16]. The transverse space of the (potential) black hole solution need only be an Einstein space admitting horizon geometries of increasing complexity as the spacetime dimension increases. For this reason these have been referred to as exotic black holes (EBHs).

We demonstrate that static asymptotically de Sitter Lovelock EBHs can have negative mass. We explicitly illustrate this for Gauss-Bonnet gravity, but it is clear that this will occur in higher-order Lovelock gravity and we expect in more general higher-curvature theories as well. Lovelock gravity is a particular subclass of higher-curvature gravity theories, which have garnered much attention since quantum gravity induces such higher-order corrections to the standard Einstein-Hilbert gravitational action [17]. Furthermore, renormalization properties are improved for such theories [18]. The Lovelock class [19] is of particular interest: it is regarded as the most natural higher-curvature generalization of Einstein gravity whose field equations are of second order in the metric functions.

Research of EBHs has been of recent interest particularly in the thermodynamics of asymptotically anti–de Sitter (AdS) black holes. The geometry of the event horizon was shown to greatly affect the thermodynamic phenomena [20–23]. While investigation of black holes in AdS space has been of great interest since the discovery the AdS/conformal field theory correspondence [24,25], interest in de Sitter black holes is motivated primarily by cosmological considerations [26], though some motivation comes from considerations of holographic duality [27].

We begin with a brief review of Lovelock gravity and EBHs before moving on to Gauss-Bonnet solutions. We provide general polynomial equations which determine the horizon locations as well as the Kretschmann scalar, which will function as a diagnostic for finding spacetime singularities. We then discuss the positive mass case and how

*b2hull@uwaterloo.ca
†rbmann@uwaterloo.ca

horizon geometry can dictate whether or not a black hole may exist. We then provide an analysis of the negative mass case, where we obtain a bound on the minimum allowed mass for a black hole. We also present a closed form expression that dictates the bounds on the horizon geometry for which a negative mass black hole is admitted as a vacuum solution. We show that it is possible to obtain massless de Sitter black holes with the proper set of parameters, and discuss this in the penultimate section before summarizing our work.

II. LOVELOCK GRAVITY AND EXOTIC BLACK HOLES

For a Lovelock theory of gravity the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{16\pi G_N}\sum_{k=0}^{K}\hat{\alpha}_k\mathcal{L}^{(k)}, \tag{2.1}$$

where $\hat{\alpha}_k$ are the Lovelock coupling constants and $\mathcal{L}^{(k)}$ are the Euler densities with dimension $2k$, which are

$$\mathcal{L}^{(k)} = \frac{1}{2^k}\delta^{a_1b_1\ldots a_kb_k}_{c_1d_1\ldots c_kd_k}R^{c_1d_1}_{a_1b_1}\ldots R^{c_kd_k}_{a_kb_k} \tag{2.2}$$

with δ representing the generalized fully antisymmetric Kronecker delta. The first few terms are

$$\mathcal{L}^{(0)} = 1, \quad \mathcal{L}^{(1)} = R, \quad \mathcal{L}^{(2)} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$$

with R the Ricci scalar and $\mathcal{L}^{(2)}$ the Gauss-Bonnet term. From the Lagrangian density (2.1) we may construct an action

$$S = \int d^dx\sqrt{-g}\left(\frac{1}{16\pi G_N}\sum_{k=0}^{K}\hat{\alpha}_k\mathcal{L}^{(k)} + \mathcal{L}_m\right) \tag{2.3}$$

including a matter term $\mathcal{L}_m$. Variation with respect to the metric g^{ab} yields the field equations

$$\sum_{k=0}^{K}\hat{\alpha}_{(k)}\mathcal{G}^{(k)}_{ab} = 8\pi G_N T_{ab}, \tag{2.4}$$

where T_{ab} is the stress energy tensor of the matter field and $\mathcal{G}^{(k)}_{ab}$ are the Lovelock tensors

$$\mathcal{G}^{(k)a}_{b} = -\frac{1}{2^{(k+1)}}\delta^{ac_1d_1\ldots c_kd_k}_{be_1f_1\ldots e_kf_k}R^{e_1f_1}_{c_1d_1}\ldots R^{e_kf_k}_{c_kd_k}. \tag{2.5}$$

We will consider only vacuum solutions with $\mathcal{L}_m = 0$ henceforth, so the field equations will be (2.4) with the stress energy being set to zero. The parameter $K \leq \frac{d-1}{2}$, where d is the dimension of spacetime, and sets the maximal degree of nonlinearity in the curvature. Values of K larger than this make no contribution to the field equations and are topological invariants in the action.

The metric ansatz we will make use of is

$$\begin{aligned} ds^2 &= \mathsf{g}_{ij}dy^idy^j + \gamma_{\alpha\beta}dx^\alpha dx^\beta \\ &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Sigma^2_{d-2}, \end{aligned} \tag{2.6}$$

where $y^i = (t, r)$ and $\mathsf{g}_{ij} = \mathrm{diag}(-f(r), \frac{1}{f(r)})$. The line element $d\Sigma^2_{d-2}$ of the base manifold (or transverse space) has metric $\gamma_{\alpha\beta}$ and coordinates x^α. We shall assume it to be compact, with volume Σ_{d-2}, the analog of the volume of a unit sphere. If this space is not compact then quantities such as M/Σ_{d-2} will be regarded as finite.

The only other condition on the transverse space is that it is an Einstein space, $R_{\alpha\beta} \propto \gamma_{\alpha\beta}$. The most commonly studied special case is when the base space has constant curvature, with curvature parameter $\kappa = (-1, 0, 1)$, which corresponds to negative, flat, and positive curvature respectively. Solutions in which this latter condition is relaxed, so that the transverse space is only an Einstein space [11], shall be referred to as exotic black holes [10,28].

Introducing this metric into the field equations yields a resulting polynomial equation for $f(r)$ [10]

$$\sum_{n=0}^{K}\frac{b_n}{r^{2n}}\left(\sum_{k=n}^{K}\alpha_k\binom{k}{n}\left(\frac{-f(r)}{r^2}\right)^{k-n}\right) = \frac{16\pi G_N M}{(d-2)\Sigma_{d-2}r^{d-1}}, \tag{2.7}$$

where M represents the mass of the black hole and Σ_{d-2} the volume of the base space. The quantities α_k are rescaled Lovelock couplings defined as

$$\begin{aligned} \alpha_0 &= \frac{\hat{\alpha}_0}{(d-1)(d-2)}, \qquad \alpha_1 = \hat{\alpha}_1, \\ \alpha_k &= \hat{\alpha}_k\prod_{n=1}^{2k}(d-n) \quad \text{for } k \geq 2. \end{aligned} \tag{2.8}$$

The cosmological constant is defined through the zeroth Lovelock coupling constant $\Lambda = -\frac{\hat{\alpha}_0}{2}$.

Imposing only the condition that the transverse space is an Einstein space admits the following possibilities:

$$\begin{aligned} \hat{\mathcal{G}}^{(n)\alpha}_{\beta} &= -\frac{(d-3)!b_n}{2(d-2n-3)!}\delta^\alpha_\beta, \\ \hat{\mathcal{L}}^{(n)} &= \frac{(d-2)!b_n}{(d-2n-2)!} \end{aligned} \tag{2.9}$$

for its Lovelock tensor and associated intrinsic Euler density respectively. The constants b_n we refer to as topological terms, and can take any value in $\mathbb{R}$. They define the topology of the base manifold and the geometry of the event horizon from the condition $f(r_+) = 0$ that defines the horizon location r_+. Without loss of generality we set $b_0 = 1$; to recover Einstein gravity in the proper limit of vanishing $\alpha_{k\geq 2}$, we will set $\alpha_1 = 1$. The mass M is defined through the

Hamiltonian formalism, and is the conserved charge of the timelike Killing vector of the background spacetime. When the dimension is $d = 2K + 1$ the transverse space is restricted to be of constant curvature [29,30].

The Kretschmann scalar for a metric of the form (2.6) is

$$R^{abcd}R_{abcd} = \left(\frac{d^2 f(r)}{dr^2}\right)^2 + 2\frac{(d-2)}{r^2}\left(\frac{df(r)}{dr}\right)^2 + 2\frac{(d-2)(d-3)f(r)^2}{r^4} - 4\frac{R[\gamma]f(r)}{r^4} + \frac{\mathcal{K}[\gamma]}{r^4}, \tag{2.10}$$

where $R[\gamma]$ and $\mathcal{K}[\gamma]$ are the Ricci and Kretschmann scalars of the transverse space respectively.

We pause to make a brief comment before considering solutions to (2.7). The transverse space given in (2.6) is not of constant curvature. Hence the solutions we obtain are, strictly speaking, not asymptotically de Sitter. However (as we will see) they do resemble asymptotically de Sitter solutions insofar as their cosmological horizon properties are concerned. With this in mind, we shall continue to refer to our solutions as de Sitter black holes.

III. GAUSS-BONNET SOLUTIONS

Setting $K = 2$ in (2.7) yields

$$\frac{\alpha_2 f^2}{r^4} + \left(-\frac{1}{r^2} - \frac{2b_1\alpha_2}{r^4}\right)f + \alpha_0 + \frac{b_1}{r^2} + \frac{b_2\alpha_2}{r^4} = \frac{16\pi M}{(d-2)\Sigma_{d-2}r^{d-1}} \tag{3.1}$$

with the solutions

$$f = f_\pm(\mathrm{m}) \equiv \frac{r^2 + 2b_1\alpha_2 \pm \sqrt{(b_1^2 - b_2)4\alpha_2^2 + r^4(1 - 4\alpha_2\alpha_0) + \frac{8\mathrm{m}\alpha_2}{r^{d-5}}}}{2\alpha_2}, \tag{3.2}$$

where we have written

$$\mathrm{m} \equiv \frac{8\pi M}{(d-2)\Sigma_{d-2}}. \tag{3.3}$$

The two solutions are distinguished by their behavior in the $\alpha_2 \to 0$ limit. The f_- branch is referred to as the Einstein branch as it has a smooth $\alpha_2 \to 0$ limit, giving

$$\lim_{\alpha_2 \to 0} f_-(\mathrm{m}) = \alpha_0 r^2 + b_1 - \frac{2\mathrm{m}}{r^{d-3}}, \tag{3.4}$$

whereas the f_+ solution does not have a smooth limit as $\alpha_2 \to 0$; it is referred to as the Gauss-Bonnet branch.

For the remainder of this paper we will only consider the Einstein branch for analysis; we will also choose $b_1 = 1$, so that as $\alpha_2 \to 0$ we recover the standard Schwarzschild (anti–) de Sitter solution. We will only examine solutions with $d \geq 6$, since the $d = 5$ case only admits constant curvature solutions in the transverse space. The cosmological constant is

$$\Lambda = -\frac{\hat{\alpha}_0}{2} = -\frac{\alpha_0(d-1)(d-2)}{2} \tag{3.5}$$

and we shall set $\alpha_0 < 0$ since we are interested in asymptotically de Sitter solutions. This allows us to write

$$f_- = \frac{r^2 + 2\alpha_2 - \sqrt{(1-b_2)4\alpha_2^2 + r^4\left(1 + 4\alpha_2\frac{2\Lambda}{(d-1)(d-2)}\right) + \frac{8\mathrm{m}\alpha_2}{r^{d-5}}}}{2\alpha_2}, \tag{3.6}$$

where we note $\Lambda > 0$. In the large r limit, the function (3.6) has the same features as the metric function of an asymptotically de Sitter spacetime.

It is convenient at this point to introduce a set of dimensionless variables

$$r = x\sqrt{\alpha_2}, \qquad \Lambda = \frac{(d-1)(d-2)z}{2\alpha_2},$$
$$\mathrm{m} = m\alpha_2^{\frac{d-3}{2}}, \qquad z > 0 \tag{3.7}$$

so that

$$f = 1 + \frac{x^2}{2} - \frac{\sqrt{(1+4z)x^4 + 4(1-b_2) + \frac{8m}{x^{d-5}}}}{2}. \tag{3.8}$$

The horizon(s) are located at the roots of f, which can be found from solving the polynomial

$$x^{d-1}z - x^{d-3} - b_2 x^{d-5} + 2m = 0 \text{ with } d \geq 6, \quad z > 0. \tag{3.9}$$

For a black hole solution we must have $f(x) > 0$ for $x > x_+$ for some range of x, where x_+ locates the outermost event horizon of the black hole.

Superficially it appears that there will be two or zero positive roots to this polynomial from Descartes's rule of signs. However if we allow both b_2 and m to be either positive or negative the number of possible positive roots are then the following:

$$\begin{aligned}
&\text{If } m > 0, && \text{then two roots} \to (x_c, x_+) \text{ or zero roots for any } b_2,\\
&\text{if } m < 0 \text{ and } b_2 > 0, && \text{then one root} \to x_c,\\
&\text{if } m < 0 \text{ and } b_2 < 0, && \text{then three roots} \to (x_-, x_+, x_c) \text{ or one root} \to x_c,
\end{aligned} \tag{3.10}$$

where x_c is the cosmological horizon, x_- is the inner horizon, and x_+ is the outer horizon of the black hole.

A. Positive mass

Consider first positive mass. We plot the behavior of f in Fig. 1 for sample values: $b_2 = -2.5$, $z = 0.05$, and $d = 6$, while varying m. For small nonzero m we have an asymptotically de Sitter black hole (left diagram in Fig. 1) whereas for sufficiently large m there is no static black hole solution and we have an expanding spacetime with r and t interchanging roles (middle diagram). As m increases from small to large values, there is one value of m where f has a double root and the horizons merge, illustrated in the right diagram in Fig. 1. This is a Gauss-Bonnet generalization of the Nariai solution. This structure is the same in all dimensions $d \geq 6$. Only quantitative differences are seen as we increase the dimension: The spread between x_+ and x_c increases and the maximum value of f slightly increases. Consequently we will illustrate our results in six dimensions unless otherwise stated.

We may also examine what happens when we hold all the parameters fixed except for b_2, which is seen in Fig. 2. For a fixed mass m and cosmological parameter z increasing the value of b_2 will shift f upward. In this manner the geometry of the horizon plays an important role regarding the existence of a black hole solution, namely that a minimum allowed value of b_2 is required. This minimum value occurs when f has a double root, corresponding to a Nariai type of black hole solution. In de Sitter space this is referred to as the maximum allowed mass a black hole can have, other parameters being fixed. To determine the value of $x_+ = x_c$ for the Nariai-type solution, the equations

$$f = 0, \qquad \frac{\partial f}{\partial x} = 0 \tag{3.11}$$

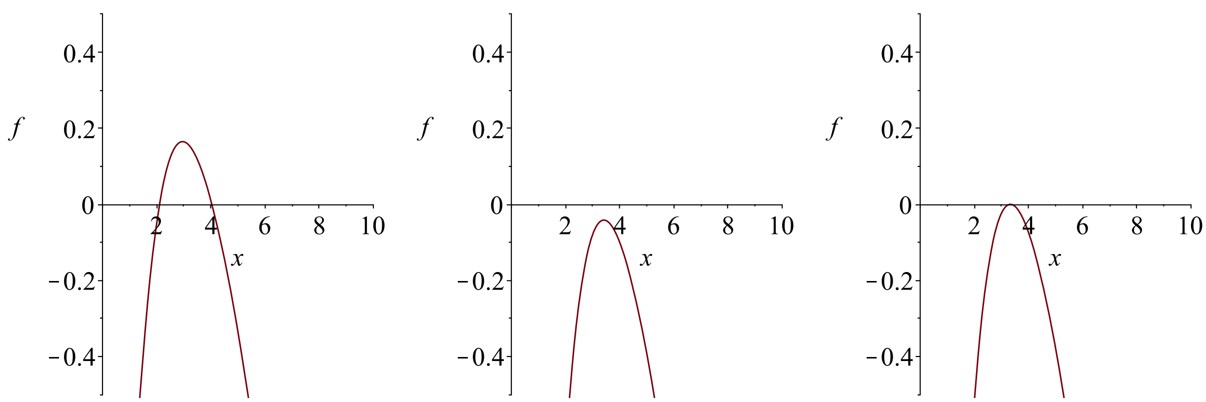

FIG. 1. Positive mass black hole $b_2 = -2.5$, $z = 0.5$, $d = 6$. Left: black hole with mass $m = 1$ showing two distinct real roots. Center: mass $m = 5$, showing no roots, and a naked singularity. Right: mass $m = 4.0638$ showing a double root.

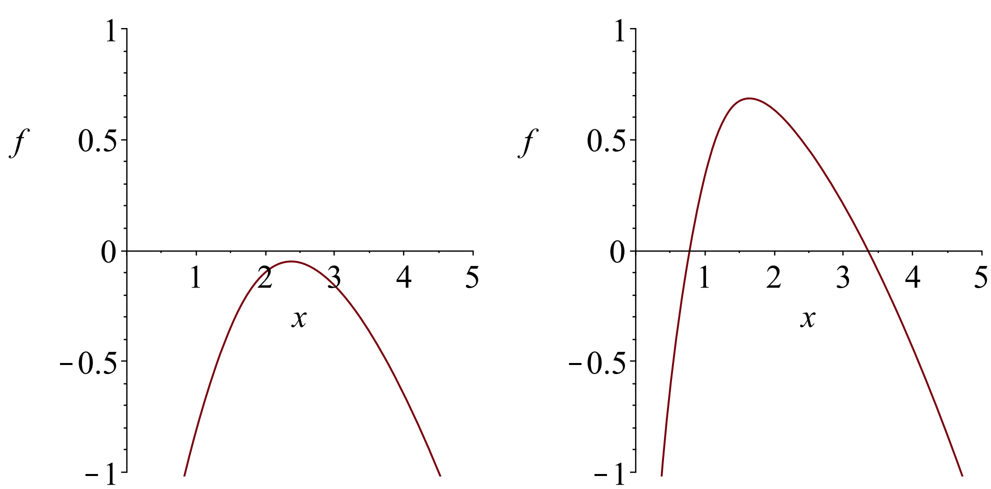

FIG. 2. Positive mass $m = 1$, $z = 0.1$, $d = 6$. Left: naked singularity with $b_2 = -2$ showing no real roots. Right: de Sitter black hole with $b_2 = 2$ showing a cosmological horizon and event horizon $x_c > x_+$.

must be simultaneously solved. In six dimensions there is no closed form solution since the latter equation is a quintic polynomial in x. For the parameters in Fig. 2 we obtain numerically $b_{2_{\min}} \approx -1.62$. For b_2 less than this the spacetime will have a naked singularity at the origin.

We can also make another observation, shown in Fig. 3. That is, for a certain set of parameters f can become discontinuous for some range of x. This discontinuity is due to the argument inside the square root becoming negative. Whenever this occurs the Kretschmann scalar (2.10) diverges, yielding a spacetime singularity. Solutions to the following polynomial

$$(4-4b_2)x^{d-5}+(4z+1)x^{d-1}+8m=0 \tag{3.12}$$

yield the singularity location(s). From Descartes's rule of signs this will only have a root when $b_2 > 1$. This is a necessary but not sufficient condition; as illustrated in the right image of Fig. 3, increasing the value of z allows the two discontinuous branches to join, yielding a smooth continuous metric function f. If we hold all parameters fixed, increasing z shifts f down vertically and reduces the spread between the black hole and cosmological horizons, allowing the possibility of a smooth metric function between them.

B. Negative mass

Negative mass solutions have more interesting behavior, since if $b_2 < 0$ there can now be three roots to (3.9). We illustrate this in Fig. 4. The left image depicts three horizons, while the center and right images show the two possible cases where horizons can merge. In the center image $x_c = x_+$ (a Nariai-type solution) and in the right $x_- = x_+$ (an extremal black hole with a cosmological horizon). These latter two extremes constrain the range of allowed values of $b_2 < 0$ that yield black hole solutions of negative mass. For the case illustrated in Fig. 4, we approximately have $-1.3 \geq b_2 > -3$.

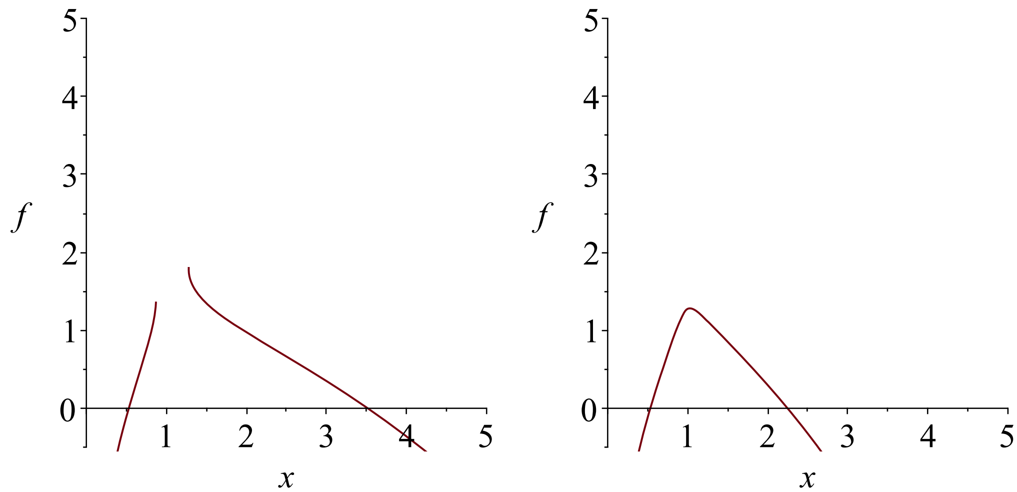

FIG. 3. Positive mass $m = 1$, $d = 6$, $b_2 = 3.5$. Left: $z = 0.1$ two discontinuous branches with two spacetime singularities. Right: $z = 0.3$ de Sitter black hole.

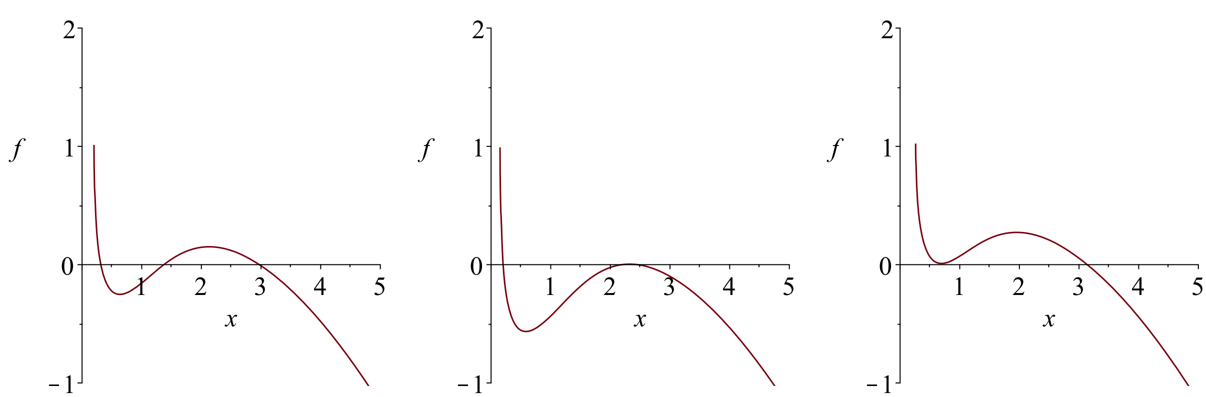

FIG. 4. Negative mass: $m = -0.3$, $z = 0.09$, $d = 6$. Left: $b_2 = -2$ showing three distinct horizons. Center: $b_2 = -3$ showing one double root depicting a Nariai-like solution. Right: $b_2 = -1.3$ showing an extremal black hole with a distinct cosmological horizon.

We also find that as $x \to 0$, f does not have a smooth limit, and instead becomes complex at some finite positive value of x. For $m < 0$ and $b_2 < 0$ there is only one sign change in the coefficients of the terms in (3.12) and hence this equation has only one positive root, at which point the Kretschmann scalar (2.10) diverges. For the admissible range of b_2, this singularity is always behind a horizon as can be seen in Fig. 4. For the more generic 3-horizon case shown in the left diagram in Fig. 4, the solution to (3.12) is $x = 0.1999637662$.

1. Allowed values of b_2 for valid negative mass black hole solutions

To determine the range of b_2 that allows for a valid black hole solution of negative mass in any dimension, we must solve the following set of equations:

$$f = 0, \quad \frac{\partial f}{\partial x} = 0 \ \text{ for } b_2, x \ \text{ with } \{x, b_2 | \in \mathbb{R} > 0\}. \tag{3.13}$$

The first equation is given by (3.9) and yields

$$b_2 = x_+^4 z - x_+^2 + 2x_+^{5-d} m, \tag{3.14}$$

where x_+ solves

$$2x_+^{d-1} z - x_+^{d-3} - m(d-5) = 0 \tag{3.15}$$

from the former equation. There is no analytic solution to this set of equations except in certain dimensions.

For $d = 6$ Eq. (3.15) is a quintic, and there is no analytic solution for arbitrary m, z. For the parameters used in Fig. 4, we obtain numerically

$$\{x_{1+} = 0.6897087623, b_{2_{\max}} = -1.325264590\},$$
$$\{x_{2+} = 2.328863088, b_{2_{\min}} = -3.033847194\} \tag{3.16}$$

refining the values given above.

In seven dimensions however, we can find a closed form expression. The equations become

$$b_2 = \frac{x_+^6 z - x_+^4 + 2m}{x_+^2}, \tag{3.17}$$

$$2x_+^6 z - x_+^4 - 2m = 0, \tag{3.18}$$

which admits analytic solutions for arbitrary m, z. The latter equation is a cubic in x_+^2; writing $m = -|\mathbf{m}|$ we get

$$2y^3 z - y^2 + 2|\mathbf{m}| = 0 \quad \text{with} \quad y = x_+^2, z > 0. \tag{3.19}$$

Now through Descartes's rule of signs, the possible number of distinct positive roots is either two or zero, while the number of negative roots will always be one; this latter case is inadmissible since we must have $y > 0$.

To have two distinct positive real roots for y (and thus the same number of distinct positive values for x_+), we require the discriminant

$$\Delta_3 = 8|\mathbf{m}|(1 - 54z^2|\mathbf{m}|) > 0 \tag{3.20}$$

for (3.19), implying

$$-\frac{1}{54z^2} < m < 0. \tag{3.21}$$

This condition must be satisfied to have three distinct roots. Writing (3.19) as a depressed cubic, the solutions (for x) are easily written as

$$x_k = \sqrt{\frac{1}{3z}\cos\left(\frac{1}{3}\arccos\left(1 - 108|\mathbf{m}|z^2\right) - \frac{2\pi k}{3}\right) - \frac{1}{6z}}$$
$$k = 0, 1, 2. \tag{3.22}$$

For all values of m satisfying (3.20), only $k = 0, 1$ yield real solutions, where $x_0 > x_1$. The value x_0 corresponds to the location of the Nariai-like black hole solution whereas x_1 locates the extremal black hole of negative mass. From (3.17), the respective minimal and maximal values of b_2 are

$$b_{2_{\min}} = \frac{8\cos(A)^3 - 432|\mathbf{m}|z^2 - 36\cos(A)^2 + 30\cos(A) - 7}{72z^2\cos(A) - 36z},$$
$$A = \frac{1}{3}\arccos(1 - 108|\mathbf{m}|z^2),$$
$$b_{2_{\max}} = \frac{8\cos(B)^3 - 432|\mathbf{m}|z^2 - 36\cos(B)^2 + 30\cos(B) - 7}{72z^2\cos(B) - 36z},$$
$$B = \frac{1}{3}\arccos(1 - 108|\mathbf{m}|z^2) - \frac{2\pi}{3}, \tag{3.23}$$

and provided $b_{2_{\max}} > b_2 > b_{2_{\min}}$, there will be a valid negative mass black hole with two horizons and a de Sitter cosmological horizon.

We can make a further interesting observation about (3.20). If we saturate the inequality (3.21) from above, setting set $m = 0$, a range of interesting phenomena can occur. It is possible to have a black hole, a Nariai-like black hole, pure de Sitter space, or a naked singularity. These scenarios are analyzed in the next section.

IV. MASSLESS BLACK HOLES

The saturation of (3.21) from above allows for the possibility of zero mass exotic black holes. An intriguing feature of these possible black holes is that they are independent of the dimension of the spacetime. Taking (3.8) and setting the mass to zero gives us

$$f = 1 + \frac{x^2}{2} - \frac{\sqrt{(1 + 4z)x^4 + 4 - 4b_2}}{2}. \tag{4.1}$$

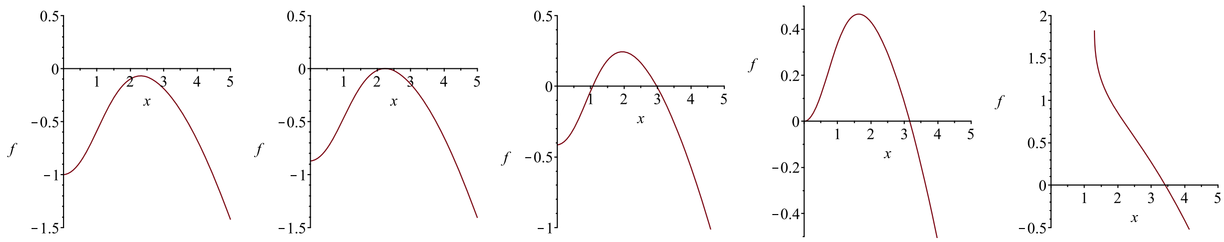

FIG. 5. Massless solutions $z = 01$. From left to right the b_2 values are −3, −2.5, −1, 0, 2. It depicts a naked singularity at the origin, a Nariai-like black hole, a de Sitter black hole, pure de Sitter space, and then a nontrivial naked singularity outside of the origin, respectively.

The root equation (3.15) is now

$$x^4 z - x^2 - b_2 = 0 \tag{4.2}$$

which is a quadratic equation in x^2 so it can be written as

$$y^2 z - y - b_2 = 0. \tag{4.3}$$

We may again analyze the roots of this polynomial by Descartes's rule of signs. Assuming $z > 0$, then

$$\begin{aligned} &\text{for } b_2 < 0, && \text{two or zero roots,} \\ &\text{for } b_2 \geq 0, && \text{one root.} \end{aligned} \tag{4.4}$$

As a quadratic the discriminant is immediately written down as $\Delta_2 \equiv 4zb_2 + 1$. For a valid black hole solution, we must have $\Delta_2 \geq 0$. The saturation of this inequality yields the Nariai solution. The unsaturated inequality gives us, similar to the negative mass case, a condition on b_2 as a function of z that must be satisfied for a black hole to exist: $-\frac{1}{4z} < b_2 < 0$. If b_2 is not within these bounds there will be a naked singularity located at the origin.

For $b_2 > 0$ the space is pure de Sitter, provided (3.12) has no solutions so as to avoid singularities. These equations are easily solved for any d, yielding

$$x = \frac{\sqrt{2}(b_2 - 1)^{1/4}}{(1 + 4z)^{1/4}}, \tag{4.5}$$

and so we conclude that we must have $0 \leq b_2 < 1$ to obtain a pure de Sitter space without any naked singularities. The naked singularity that would occur for $b_2 \geq 1$ is at a finite value of x given by (4.5). The different scenarios are outlined in Fig. 5. The three leftmost images display negative values of b_2 respectively showing a naked singularity, Nariai black hole, and de Sitter black hole. The two rightmost images are for zero and positive values of b_2 respectively, with the former being pure de Sitter space and the latter a naked singularity outside of the origin. For increasing values of $b_2 > 0$ the termination point of f at $x = 0$ moves up the vertical axes until it reaches a maximum point at $f(x, b_2, z) \to f(0, 1, 0.1) = 1$.

V. CONCLUSION

We have shown that metrics of the form (2.6), with $\gamma_{\alpha\beta}$ being the metric of an arbitrary Einstein space, can possess both negative mass and massless asymptotically de Sitter black hole solutions in Gauss-Bonnet gravity. These $M \leq 0$ black holes are all exotic black holes, and their existence depends on the relative values of the parameters b_2, z, and m. For the massless case, there is a direct relationship between b_2 and z with b_2 being bounded from below by z. For $M < 0$ black holes there are more restrictions on the parameters. The range of m is bounded by z, as given by (3.21). There is also a minimum and maximum allowed value of b_2 admitting such solutions.

We close by noting that the existence of such horizon geometries is still an open question. This is due to the fact that field equations for the transverse space are an overconstrained partial differential equation system, where certain topological parameters b_n may not provide a solution. However we are not aware of any theorem forbidding the existence of nontrivial solutions to (2.9).

With the discovery of negative mass black holes in de Sitter space interesting questions arise. These include the behavior of geodesics, formation from gravitational collapse, and possible pair production in the early Universe of such objects. Their thermodynamic behavior is also likely to yield interesting surprises in comparison to their anti–de Sitter counterparts [20,21].

ACKNOWLEDGMENTS

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

[1] K. Schwarzschild, Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, Deutsch. Akad. Wiss. Berlin, Kl. Math. Phys. Tech. **1916**, 189 (1916).

[2] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. (N.Y.) **6**, 918 (1965).

[3] S. Aminneborg, I. Bengtsson, S. Holst, and P. Peldan, Making anti–de Sitter black holes, Classical Quantum Gravity **13**, 2707 (1996).

[4] R. B. Mann, Pair production of topological anti–de Sitter black holes, Classical Quantum Gravity **14**, L109 (1997).

[5] R. B. Mann, Topological black holes: Outside looking in, Ann. Isr. Phys. Soc. **13**, 311 (1997).

[6] R. Mann, Black holes of negative mass, Classical Quantum Gravity **14**, 2927 (1997).

[7] W. L. Smith and R. B. Mann, Formation of topological black holes from gravitational collapse, Phys. Rev. D **56**, 4942 (1997).

[8] M. C. Johnson, M. B. Paranjape, A. Savard, and N. Tapia-Arellano, Stable, thin wall, negative mass bubbles in de Sitter space-time, Gen. Relativ. Gravit. **52**, 80 (2020).

[9] S. Mbarek and M. B. Paranjape, Negative mass bubbles in de Sitter spacetime, Phys. Rev. D **90**, 101502 (2014).

[10] S. Ray, Birkhoff's theorem in Lovelock gravity for general base manifolds, Classical Quantum Gravity **32**, 195022 (2015).

[11] G. Dotti and R. J. Gleiser, Obstructions on the horizon geometry from string theory corrections to Einstein gravity, Phys. Lett. B **627**, 174 (2005).

[12] G. Dotti, J. Oliva, and R. Troncoso, Exact solutions for the Einstein-Gauss-Bonnet theory in five dimensions: Black holes, wormholes, and spacetime horns, Phys. Rev. D **76**, 064038 (2007).

[13] G. Dotti, J. Oliva, and R. Troncoso, Vacuum solutions with nontrivial boundaries for the Einstein-Gauss-Bonnet theory, Int. J. Mod. Phys. A **24**, 1690 (2009).

[14] G. Dotti, J. Oliva, and R. Troncoso, Static solutions with nontrivial boundaries for the Einstein-Gauss-Bonnet theory in vacuum, Phys. Rev. D **82** (2010).

[15] J. Oliva, All the solutions of the form $M_2 \times w\Sigma_{d-2}$ for Lovelock gravity in vacuum in the Chern-Simons case, J. Math. Phys. (N.Y.) **54**, 042501 (2013).

[16] S. Ohashi and M. Nozawa, Lovelock black holes with a nonconstant curvature horizon, Phys. Rev. D **92**, 064020 (2015).

[17] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1982).

[18] K. S. Stelle, Renormalization of higher-derivative quantum gravity, Phys. Rev. D **16**, 953 (1977).

[19] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys. (N.Y.) **12**, 498 (1971).

[20] B. R. Hull and R. B. Mann, Thermodynamics of exotic black holes in Lovelock gravity, Phys. Rev. D **104**, 084032 (2021).

[21] B. Hull and F. Simovic, Exotic black hole thermodynamics in third-order Lovelock gravity (to be published).

[22] N. Farhangkhah and M. H. Dehghani, Lovelock black holes with nonmaximally symmetric horizons, Phys. Rev. D **90**, 044014 (2014).

[23] N. Farhangkhah and Z. Dayyani, Extended phase space thermodynamics for third-order Lovelock black holes with nonmaximally symmetric horizons, Phys. Rev. D **104**, 024068 (2021).

[24] E. Witten, Anti–de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. **2**, 505 (1998).

[25] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).

[26] Planck Collaboration, Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. **641**, A6 (2020).

[27] A. Strominger, The ds/CFT correspondence, J. High Energy Phys. 01 (2001) 034.

[28] S. Ray, *Exotic black holes in Lovelock gravity* (2015).

[29] G. Dotti, J. Oliva, and R. Troncoso, Static solutions with nontrivial boundaries for the Einstein-Gauss-Bonnet theory in vacuum, Phys. Rev. D **82**, 024002 (2010).

[30] G. Dotti, J. Oliva, and R. Troncoso, Exact solutions for the Einstein-Gauss-Bonnet theory in five dimensions: Black holes, wormholes and spacetime horns, Phys. Rev. D **76**, 064038 (2007).