Type-*D* solutions of the Einstein-Euler-Heisenberg nonlinear electrodynamics with a cosmological constant

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(Received 2 February 2023; accepted 22 February 2023; published 13 March 2023)

We establish the Einstein-Euler-Heisenberg system of field equations with cosmological constant (EEH-Λ) for type-D metrics within the null tetrad formalism. Then we determine all type-D solutions to the EEH-Λ equations; among the derived solutions are the Einstein-Euler-Heisenberg-Λ generalizations of the Bertotti-Robinson, Reissner-Nordström, Newman-Unti-Tamburino-B (NUT-B) $(+)$, and Kerr-Newman solutions. Moreover it is shown that the (static) C–metric is not compatible with the Euler-Heisenberg electrodynamics.

DOI: 10.1103/PhysRevD.107.064026

I. INTRODUCTION

Nowadays black holes are of utmost importance to be investigated theoretically [1], since they can give insight into matters like quantum gravity, the holographic correspondence and statistical new perspectives, among others. But equally interesting is their astrophysical relevance related to observations of their shadows [2,3]. Also the dynamics of stars in the neighborhood of the galactic centers [4,5] is associated with the presence of supermassive black holes. Moreover gravitational waves observations has lead to a catalog of compact objects binaries [6]. These scenarios can be approached by means of solutions in the framework of the Einstein general relativity and last decades have witnessed the development of techniques to determine solutions to the Einstein field equations coupled to a variety of fields in spacetimes with two Killing vectors related to stationarity and axisymmetry, or to staticity and spherical symmetry, like the Petrov type-D metrics [7], a class of solutions characterized by two double principal null directions.

On the other hand, the QED effective theory after one-loop of nonperturbative quantization is the Euler-Heisenberg nonlinear electrodynamics [8]. The vacuum is treated as a specific type of medium, the polarizability properties of which are determined by clouds of virtual charges surrounding the real ones. It accounts for vacuum corrections to the Maxwell-Lorentz theory. These effects become significant when the electromagnetic field strengths approach the critical values $E_{\rm cr} \approx m_e^2 c^3/(e\hbar) \approx 10^{16}$ V/cm or $B_{\rm cr} \approx 10^9$ T. The situation of strong magnetic fields has astrophysical interest since neutron stars and magnetars can generate magnetic fields in the range of 10^6–10^9 T, then processes like photon splitting and pair conversion are expected to occur in their vicinity [9,10].

Efforts are currently in progress for measuring some nonlinear electromagnetic effects, we mention just a few of them: Light-light interactions can be studied using heavy ion collisions; the electromagnetic (EM) field strengths produced, for example by a lead (Pb) nucleus would be up to 10^{25} V m^{-1} and it has been measured light by light scattering in Pb + Pb collisions at the Large Hadron Collider [11]. Other experimental proposals include the measurement of photon splitting in strong magnetic fields [12,13], the search for vacuum polarization with laser beams crossing magnetic fields and the detection of vacuum birefringence with intense laser pulses [14,15]. There is a suggestion for the detection of QED vacuum nonlinearities using waveguides [16]. Vacuum pair production, known as the Sauter-Schwinger effect [17], was a prediction of the Euler-Heisenberg (EH) nonlinear electrodynamics. However, the necessary electric field strength, $E_{\rm cr}$, corresponding to a critical laser intensity of about

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$I_{\rm cr} = 4.3 \times 10^{29}$ W/cm^2 [18–20], is at least four orders of magnitude larger than the presently feasible laser experimental arrangements. Moreover, the phase velocity of an electromagnetic wave traveling through intense EM fields will be altered due to vacuum polarization. Light trajectories can be determined by means of the effective optical metric [21,22] and there are theoretical proposals to measure these effects with a Michelson interferometer [23].

Therefore, it is interesting to couple EH nonlinear electromagnetic fields with Einstein general relativity, particularly with type D metrics whose most general metric was derived by Plebański and Demiański [7].

In this paper we establish the Einstein-Euler-Heisenberg equations with cosmological constant (EEH- Λ) for type D metrics using the null tetrad formalism; first we focus on the static solutions, that include as particular cases the Schwarzschild, Bertotti-Robinson (BR), Reissner-Nordström (RN), and the Levi-Civita or C-metric. Regarding the stationary branch of the type D metrics there are the Carter family that includes the NUT-B$(+)$, the Kerr and the Kerr-Newman solutions that we generalize to include the EH electromagnetic field and the cosmological constant that can be positive (de Sitter) or negative (anti–de Sitter). We also show the impossibility of coupling the (static) C-metric geometry [24] with the EH electromagnetic field. Some static EEH solutions were previously addressed, for instance the solutions derived by Ruffini *et al.* [25] corresponding to electric [26] and magnetic monopoles, as well as for dyonic black holes [27]. The stationary EEH solutions were addressed only recently, and in [28] the EEH generalization of the Kerr-Newman solution has been derived.

The organization of the paper is as follows: In Sec. II generalities of the EH nonlinear electrodynamics (NLED) are revisited. In Sec. III we present the type D metric and the electromagnetic field in the null tetrad formalism, the alignment of the eigenvectors of the electromagnetic field with the real vectors of the null tetrad allows to have only two nonvanishing components of the electromagnetic field. In Sec. IV, we address the four possible cases of static solutions, i.e., the BR, RN, anti-RN, and the C-metric, we prove that the latter one is not compatible with the Euler-Heisenberg nonlinear electrodynamics. In Sec. V we focus on the stationary type D and determine the EEH generalization of the NUT-B$(+)$ and of the Kerr-Newman solutions. Conclusions are presented in Sec. VI and we include an Appendix with the explicit expression of the Ricci tensor for the type D metrics.

II. EULER-HEISENBERG THEORY

From the study of the Dirac's positron theory W. Heisenberg and H. Euler [8] proposed in 1936 a nonlinear electrodynamics theory. They derived a Lagrangian that depends in nonlinear way of the two Maxwell-Lorentz electromagnetic invariants constructed with the Faraday tensor $F_{\mu\lambda}$, i.e., $F = -F^{\mu\lambda}F_{\mu\lambda}/4 = (E^2 - B^2)/2$ and $G = -\tilde{F}^{\mu\lambda}F_{\mu\lambda}/4 = \vec{B}\cdot\vec{E}$, with the Faraday dual tensor defined by $\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and is given by

$$\mathcal{L}_{\rm EH}(F,G) = F + \frac{\mu}{2}\left(F^2 + \frac{7}{4}G^2\right), \tag{1}$$

where for shortness we use μ as the nonlinearity parameter of the EH theory that in terms of the fine structure constant, $\alpha = e^2/c\hbar$, reads

$$\mu = \frac{16\alpha^2}{45 m_e^4}; \tag{2}$$

in terms of the critical fields it is of the order $\mu = 20\alpha/225\pi E_{cr}^2$. The linear electromagnetic Maxwell-Lorentz theory is recovered if $\mu = 0$, $\mathcal{L}_{\rm Maxwell}(F) = F$.

The four dimensional action of general relativity with cosmological constant Λ coupled to EH-NLED is given by [29],

$$S = \frac{1}{4\pi}\int_{M^4} d^4x\sqrt{-g}\left[\frac{1}{4}(R - 2\Lambda) - \mathcal{L}_{\rm EH}(F,G)\right], \tag{3}$$

where g is the determinant of the metric tensor, R is the Ricci scalar, $\mathcal{L}_{\rm EH}(F,G)$ is the EH Lagrangian given in Eq. (1).

Regarding NLED there are two possible frameworks, one is the usual F-framework in terms of the Faraday electromagnetic field tensor $F_{\mu\nu}$. The other one is the P-framework in terms of the Legendre dual tensor $P_{\mu\nu}$ as the main field, defined by

$$d\mathcal{L}_{\rm EH}(F,G) = -\frac{1}{2}P^{\mu\nu}dF_{\mu\nu}, \tag{4}$$

$P_{\mu\nu}$ coincides with $F_{\mu\nu}$ for the linear Maxwell theory. In general it reads

$$P_{\mu\nu} = \mathcal{L}_F F_{\mu\nu} + \mathcal{L}_G \tilde{F}_{\mu\nu}, \tag{5}$$

where the subscript X in $\mathcal{L}$ denotes the derivative, $\mathcal{L}_X = d\mathcal{L}/dX$. In our EH case it reads

$$P_{\mu\nu} = (1 + \mu F)F_{\mu\nu} + \frac{7\mu}{4}G\tilde{F}_{\mu\nu}. \tag{6}$$

The tensor $P_{\mu\nu}$ corresponds to the electric field strength **D** and the magnetic field **H** and Eqs. (5) are the constitutive or material relations between **D**, and **H** with the electric field **E** and the magnetic field strength **B**. The Legendre transformation of $\mathcal{L}_{\rm EH}$ defines the structural function $\mathcal{H}$ as [29],

$$\mathcal{H}(s,t) = \frac{1}{2}P^{\mu\nu}F_{\mu\nu} + \mathcal{L}_{\rm EH}. \tag{7}$$

The two independent invariants s and t associated to the P-framework are defined as

$$s = \frac{1}{4} P_{\mu\nu} P^{\mu\nu}, \qquad t = -\frac{1}{4} \tilde{P}^{\mu\nu} P_{\mu\nu}. \tag{8}$$

with $\tilde{P}^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} P_{\rho\sigma}$.

Neglecting second and higher order terms in μ, the structural function for the EH theory takes the form

$$\mathcal{H}(s,t) = s + \frac{\mu}{2}\left(s^2 + \frac{7}{4}t^2\right). \tag{9}$$

The electromagnetic Faraday-Maxwell equations, in the absence of sources are given by [29],

$$\nabla_\mu P^{\mu\nu} = 0, \qquad \nabla_\mu \tilde{F}^{\mu\nu} = 0, \tag{10}$$

where ∇_μ is the covariant derivative. Working with the structural function $\mathcal{H}(s,t)$ one has the technical advantage that $P_{\mu\nu}$ satisfies (10) and $F_{\mu\nu}$ is given directly by the material equations as a function of $P_{\mu\nu}$,

$$F_{\mu\nu} = \mathcal{H}_s P_{\mu\nu} + \mathcal{H}_t \tilde{P}_{\mu\nu} = P_{\mu\nu} - \mu\left[s P_{\mu\nu} + \frac{7}{4} t^* P_{\mu\nu}\right]. \tag{11}$$

The gravitational field equations are given by

$$G_{\mu\nu} = 8\pi T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{12}$$

where the energy momentum tensor $T_{\mu\nu}$ for the EH theory in the P-framework is given by

$$4\pi T_{\mu\nu} = (P^{\beta}_{\cdot\mu} P_{\nu\beta} + g_{\mu\nu} s)(1 - \mu s) + g_{\mu\nu}\frac{\mu}{2}\left(s^2 + \frac{7}{4}t^2\right). \tag{13}$$

III. TYPE-*D* METRICS AND THEIR IMPORTANCE IN GR AND ASTROPHYSICS

The interest in Petrov type-*D* metrics relies on their physical relevance; as we already mentioned, Schwarzschild, Reissner-Nordström, Kerr, Kerr–Newman solutions belong to the type-*D* metrics family; also are included the metrics describing two accelerating sources as the C-metric [24], and the nonexpanding Kundt's class, among others [30–32].

Since in this work we aim to look for the space-time created by compact gravitating objects, we will determine all static and some stationary type-*D* metrics with cosmological constant with the Euler-Heisenberg matter as a source. We use the type-*D* metric as an ansatz and determine then if the Euler-Heisenberg nonlinear electromagnetic field can be coupled to the corresponding geometry. This is not always the case, as we shall see for the Levi-Civita or C-metric. We shall employ in the search for the type-*D* EEH-Λ solutions the null tetrad formalism. The most general type-*D* metric, which allows for aligned gravitational and electromagnetic fields, in the coordinates (x, y, τ, σ) is given by [7],

$$ds^2 = \frac{1}{\Omega^2}\left\{\frac{\Sigma}{\mathcal{P}}dx^2 + \frac{\mathcal{P}}{\Sigma}[d\tau + p(y)d\sigma]^2 + \frac{\Sigma}{\mathcal{Q}}dy^2 - \frac{\mathcal{Q}}{\Sigma}[d\tau + m(x)d\sigma]^2\right\}, \tag{14}$$

where $\Omega = \Omega(x,y)$, $\mathcal{P} = \mathcal{P}(x)$, $\mathcal{Q} = \mathcal{Q}(y)$ and $\Sigma = m(x) - p(y)$. Type–D metrics are distinguished by possessing two obvious symmetries, stationarity and axisymmetry or staticity and spherical symmetry, associated to the two Killing vectors ∂_τ and ∂_σ. But also there are hidden symmetries related to the existence of Killing tensors. Those symmetries are associated with the integrability of geodesics and the separability of Hamilton-Jacobi and Klein-Gordon equations [33].

In terms of the null tetrad e^a, the line element can be written as

$$ds^2 = 2e^1 \otimes e^2 + 2e^3 \otimes e^4, \qquad e^1 = \bar{e^2}, \qquad e^3 = \bar{e^3}, \qquad e^4 = \bar{e^4}, \tag{15}$$

where the upper bar means complex conjugate and we denote the quantities in the null tetrad formalism with latin letters, $a, b, c, \ldots,$ sub(super)scripts, while tensors in coordinate frame are denoted with greek sub(super)scripts. The null tetrad for the type-*D* metric (14) is given by

$$e^1 = \frac{1}{\Omega\sqrt{2}}\left\{\sqrt{\frac{\Sigma}{\mathcal{P}}}dx + i\sqrt{\frac{\mathcal{P}}{\Sigma}}[d\tau + p(y)d\sigma]\right\} = \bar{e^2},$$
$$e^{3,4} = \frac{1}{\Omega\sqrt{2}}\left\{\sqrt{\frac{\Sigma}{\mathcal{Q}}}dy \pm \sqrt{\frac{\mathcal{Q}}{\Sigma}}[d\tau + m(x)d\sigma]\right\}. \tag{16}$$

A. The conformal factor

Following [34] the metric functions $\Omega(x,y), m(x)$ and $p(y)$ have to be determined from

$$\begin{aligned}
\Sigma(m_{,xx} - p_{,yy}) - (m_{,x})^2 - (p_{y})^2 &= 0,\\
4\Sigma\Omega_{,xx} &= \Omega(m_{,xx} + p_{,yy}),\\
4\Sigma\Omega_{,yy} &= -\Omega(m_{,xx} + p_{,yy}),\\
2\Sigma\Omega_{,xy} - m_{,x}\Omega_{,y} + p_{,y}\Omega_{,x} &= \Omega(m_{,xx} + p_{,yy}),\\
2m_{,x}\Omega_{,x} + 2p_{,y}\Omega_{,y} &= \Omega(m_{,xx} + p_{,yy}).
\end{aligned} \tag{17}$$

The stationary solutions correspond to the three cases: (i) $m_{,x} = 0$; $p_{,y} \neq 0$, (ii) $m_{,x} \neq 0$; $p_{,y} = 0$, and (iii) $m_{,x} \neq 0$; $p_{,y} \neq 0$, while static cases are $m_{,x} = 0 = p_{,y}$, i.e., m and p are constant, that we shall assume to be $m = 1/2$ and $p = -1/2$, then $\Sigma = 1$.

B. The electromagnetic equations

For type-D solutions one can always align the directions of the real null vectors e^3 and e^4 along the double Debever-Penrose (DP) vectors or principal null directions, such that the only nonzero curvature components of $C^{(a)}$, $a = 1, \ldots, 5$, which characterizes the conformal curvature tensor, C_{abcd}, is $C^{(3)}$ [7]. It is convenient for electromagnetic solutions that the eigenvectors of the electromagnetic field tensor F_{ab} be aligned along the principal null directions as well. Hence the nonvanishing components of F_{ab} are F_{12} and F_{34}, correspondingly P_{12} and P_{34}; the dual in the null tetrad formalism is given by $\tilde{P}_{ab} = -\frac{1}{2}\varepsilon_{abcd}P^{cd}$ then $\tilde{P}_{12} = P_{34}$ and $\tilde{P}_{34} = P_{12}$.

The electromagnetic Faraday-Maxwell equations are $\nabla_a P^{ab} = 0$ and $\nabla_a \tilde{F}^{ab} = 0$; F_{ab} and $\tilde{P}_{ab}$ are curls and then can be written as the gradient of an electromagnetic potential; these equations are comprised in $d\omega = 0$, where the electromagnetic two-form ω is given by

$$\omega = \frac{1}{2}(F_{ab} + \tilde{P}_{ab})e^a \wedge e^b. \tag{18}$$

Considering the eigenvectors of the electromagnetic field aligned along the real null vectors e^3 and e^4 then gives

$$\omega = (F_{12} + P_{34})e^1 \wedge e^2 + (F_{34} + P_{12})e^3 \wedge e^4. \tag{19}$$

By aligning the eigenvectors of the electromagnetic field along the real null vectors e^3 and e^4 we can parametrize the electromagnetic components as [34]

$$F_{12} = iB, \quad F_{34} = E, \quad P_{12} = iH, \quad P_{34} = D, \tag{20}$$

where B, E, H, and D are real. Since F_{12} and F_{34}, and correspondingly P_{12} and P_{34}, are the only nonvanishing electromagnetic components, the corresponding invariants s and t read

$$s = \frac{1}{2}(H^2 - D^2), \qquad t = i\vec{H}\cdot\vec{D}. \tag{21}$$

where $P_{0i} = D_i$ and $\tilde{P}_{0i} = H_i$. The electromagnetic two-form ω can be expressed in terms of the fields E, B, D and H as

$$\omega = (iB + D)e^1 \wedge e^2 + (E + iH)e^3 \wedge e^4. \tag{22}$$

The constitutive or material relations for the EH structural function $\mathcal{H}$ in (9) are given by

$$\begin{aligned} F_{12} &= (1 + \mu s)P_{12} + \frac{7}{4}\mu t P_{34} \\ F_{34} &= (1 + \mu s)P_{34} + \frac{7}{4}\mu t P_{12}. \end{aligned} \tag{23}$$

With the parametrization (20) and the substitution of the invariants (21) the material relations read

$$\begin{aligned} E &= \left[1 - \frac{\mu}{2}\left(D^2 + \frac{5}{2}H^2\right)\right]D = \kappa_1 D, \\ B &= \left[1 + \frac{\mu}{2}\left(H^2 + \frac{5}{2}D^2\right)\right]H = \kappa_2 H. \end{aligned} \tag{24}$$

In the spirit of the Lorentz theory of electrons we can write

$$\begin{aligned} D &= \epsilon_0 E = \left[1 + \frac{\mu}{2}(H^2 + D^2) + \frac{3}{4}\mu H^2\right]E = \frac{E}{\kappa_1} \\ H &= \frac{B}{\mu_0} = \left[1 - \frac{\mu}{2}(H^2 + D^2) - \frac{3}{4}\mu D^2\right]B = \frac{B}{\kappa_2}, \end{aligned} \tag{25}$$

so that $(\kappa_1)^{-1}$ can be interpreted as an electric permittivity ϵ_0 and κ_2 as a magnetic permeability μ_0 of an effective material that is a result of the vacuum polarization. Note also that the second term in ϵ_0 and μ_0 is proportional to the energy density $\rho = (H^2 + D^2)/(8\pi)$ of the electromagnetic field,

$$\begin{aligned} \epsilon_0 &= \left(1 + 4\pi\mu\rho + \frac{3}{4}\mu H^2\right) \\ \mu_0 &= \left(1 + 4\pi\mu\rho + \frac{3}{4}\mu D^2\right), \end{aligned} \tag{26}$$

that turns out to be greater than the linear vacuum value. Besides the energy density term, the electric permitivity ϵ_o has a magnetic correction in the last term, $3\mu H^2/4$, while the magnetic permeability μ_0 has an electric correction from the term $3\mu D^2/4$. When the EH nonlinear parameter $\mu = 0$ the linear vacuum values, $\epsilon_0 = 1$ and $\mu_0 = 1$, are recovered, hence $B = H$, and $D = E$. If we substitute the wedge products $e^1 \wedge e^2$ and $e^3 \wedge e^4$ for the null tetrad (16),

$$e^1 \wedge e^2 = \frac{\imath}{\Omega^2}[d\tau + p(y)d\sigma] \wedge dx \tag{27}$$

$$e^3 \wedge e^4 = \frac{1}{\Omega^2}[d\tau + m(x)d\sigma] \wedge dy, \tag{28}$$

then the Faraday-Maxwell equations for the field strengths and excitations given by Eq. (20) in case of a type-D metric are given by

$$\begin{aligned} \partial_x\left(\frac{D\kappa_1 + iH}{\Omega^2}\right) &= i\partial_y\left(\frac{D + iH\kappa_2}{\Omega^2}\right), \\ \partial_x\left(m(x)\frac{D\kappa_1 + iH}{\Omega^2}\right) &= i\partial_y\left(p(y)\frac{D + iH\kappa_2}{\Omega^2}\right). \end{aligned} \tag{29}$$

This is a set of four equations to determine $D(x,y)$ and $H(x,y)$, for a given conformal factor Ω determined from (17). The fields E and B are given by the material relations (24).

C. Einstein-Euler-Heisenberg-Λ equations for type-*D* metrics

The energy–momentum tensor T_{ab} for the EH field is given in Eq. (13), and remembering that $8\pi T^M_{ab} = P_{ac}P_{b\cdot}{}^{c} - g_{ab}s$ is the energy-momentum tensor of the Maxwell theory, then T_{ab} can be written as

$$8\pi T_{ab} = (8\pi T^M_{ab})(1-\mu s) + \mu\left(s^2 + \frac{7}{4}t^2\right)g_{ab}, \qquad (30)$$

where s and t are the electromagnetic invariants defined in (21). In the null tetrad formalism the components different from zero of the energy momentum tensor are T_{12} and T_{34}. With the adopted parametrization, and the energy-momentum tensor of the Maxwell theory being $8\pi T^M_{12} = -(D^2+H^2)$ and $8\pi T^M_{34} = D^2 + H^2$, the nonvanishing components of the energy–momentum tensor are

$$8\pi T_{12} = -(D^2+H^2) + \frac{\mu}{4}(3D^4 - H^4 + 5H^2D^2)$$
$$8\pi T_{34} = (D^2+H^2) + \frac{\mu}{4}(3H^4 - D^4 + 5H^2D^2). \qquad (31)$$

Let us now consider the Einstein equations with cosmological constant Λ and with the EH nonlinear electromagnetic field as a source

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} + \Lambda g_{ab}. \qquad (32)$$

The expression for the scalar curvature R is

$$R = -4\Lambda - 4\mu\left(s^2 + \frac{7}{4}t^2\right) \qquad (33)$$

and with (31) and using that $-8\pi T^M_{12} = (D^2+H^2) = 8\pi T^M_{34}$, the nonvanishing components of the Ricci tensor R_{ab} are R_{12} and R_{34}. Then the field equations (32) reduce to

$$R_{12} = -(D^2+H^2) + \frac{\mu}{4}(3D^4 - H^4 + 5H^2D^2) - \Lambda$$
$$R_{34} = (D^2+H^2) + \frac{\mu}{4}(3H^4 - D^4 + 5H^2D^2) - \Lambda. \qquad (34)$$

For completeness, the expressions for R_{12} and R_{34} for the type D metric (14) are outlined in the Appendix. The equations to be solved are then the gravitational field equations (34) together with the Maxwell-Faraday equations (29).

IV. STATIC EINSTEIN-EULER-HEISENBERG-Λ SOLUTIONS

The static case of the type-D metrics (14) corresponds to $m_{,x} = 0 = p_{,y}$. That means, $m(x)$ and $p(y)$ being constant in (17), while the conformal factor is of the form $\Omega = a + bx + cy$ with integration constants a, b, and c. The principal null directions e^3 and e^4 are geodesic, shear-free and twist-free. Additionally, for $\Omega = \Omega(x)$, the principal directions are non-expanding, while for $\Omega = \Omega(y)$, the principal congruences are expanding. We assume $m = 1/2$ and $p = -1/2$ what gives $\Sigma = 1$. The line element (14) will be expressed in new coordinates, $(\tau, \sigma) \mapsto (t, \phi)$ given by $d\phi = d\tau - d\sigma/2$ and $dt = d\tau + d\sigma/2$, so that the metric can be written in diagonal form

$$ds^2 = \frac{1}{\Omega^2}\left\{\frac{dx^2}{\mathcal{P}} + \mathcal{P}d\phi^2 + \frac{dy^2}{\mathcal{Q}} - \mathcal{Q}dt^2\right\}. \qquad (35)$$

Then the electromagnetic two-form ω in (22) is

$$\omega = \frac{i}{\Omega^2}(D + i\kappa_2 H)d\phi\wedge dx + \frac{1}{\Omega^2}(D\kappa_1 + iH)dt\wedge dy, \quad (36)$$

then the Faraday-Maxwell equations (29) reduce to

$$\partial_x\left(\frac{D\kappa_1 + iH}{\Omega^2}\right) = 0, \qquad \partial_y\left(\frac{D + i\kappa_2 H}{\Omega^2}\right) = 0. \qquad (37)$$

While using the expressions for R_{12} and R_{34} (see the Appendix) the EEH-Λ equations for the static case are given by

$$\mathcal{P}_{,xx} + \frac{2\mathcal{Q}_{,y}}{y} - \frac{2\mathcal{Q}(y)}{y^2} = -2y^2\left[D^2 + H^2 + \frac{\mu}{4}(3H^4 - D^4 + 5H^2D^2) + \Lambda\right]$$
$$\mathcal{Q}_{,yy} - \frac{2\mathcal{Q}_{,y}}{y} + \frac{2\mathcal{Q}(y)}{y^2} = 2y^2\left[D^2 + H^2 - \frac{\mu}{4}(-H^4 + 3D^4 + 5H^2D^2) - \Lambda\right]. \qquad (38)$$

Now we shall integrate the four cases of the form of $\Omega(x, y)$ corresponding to $m(x)$ and $p(y)$ being constant: (A) The Bertotti-Robinson metric with $\Omega = 1$, (B) the Reissner-Nordström metric with $\Omega = y$, (C) the anti Reissner-Nordström metric with $\Omega = x$, and (D) the C-metric or Levi-Civita metric with $\Omega = x + y$.

A. EH generalization of the Bertotti-Robinson solution

If $b = 0 = c$ and $a = 1$, then $\Omega = 1$, and the Faraday-Maxwell (FM) Eqs. (37) reduce to

$$\partial_x H = 0 \qquad \partial_y(\kappa_2 H) = 0 \qquad (39)$$

$$\partial_y D = 0 \qquad \partial_x(\kappa_1 D) = 0, \tag{40}$$

whose solution is $H = C_1$ and $D = C_2$, C_1, C_2 being constants to be determined from the EEH-Λ Eqs. (38) that amount to:

$$\mathcal{P}_{,xx} = -2(D^2+H^2) - \frac{\mu}{2}(3H^4 - D^4 + 5D^2H^2) - 2\Lambda$$
$$\mathcal{Q}_{,yy} = 2(D^2+H^2) - \frac{\mu}{2}(3D^4 - H^4 + 5D^2H^2) - 2\Lambda. \tag{41}$$

Since $\mathcal{P} = \mathcal{P}(x)$ and $\mathcal{Q} = \mathcal{Q}(y)$, then the right-hand side of the equations cannot depend on both coordinates (x, y), leaving then for the electromagnetic fields that the only acceptable solutions are $D = \text{const}$ and $H = \text{const}$, in agreement with MF equations; then the metric functions $\mathcal{P}(x)$ and $\mathcal{Q}(y)$ are second degree polynomials given by

$$\mathcal{P}(x) = \alpha_1 + \beta x + \gamma_1 x^2$$
$$\mathcal{Q}(y) = \alpha_2 + \delta y + \gamma_2 y^2, \tag{42}$$

where $\alpha_1, \alpha_2, \beta$ and δ are arbitrary constants, and γ_1 and γ_2 are determined from the EEH-Λ equations, Eqs. (41), as

$$\gamma_1 = -(D^2+H^2) - \frac{\mu}{4}(3H^4 - D^4 + 5D^2H^2) - \Lambda$$
$$\gamma_2 = (D^2+H^2) - \frac{\mu}{4}(3D^4 - H^4 + 5D^2H^2) - \Lambda. \tag{43}$$

The EH generalization of the Bertotti-Robinson solution with cosmological constant Λ consists then in the line element (35) with the metric functions (42) and (43), with D and H being uniform electric and magnetic fields, respectively. In the case $\mu = 0$ we recover the linear Bertotti-Robinson solution consisting of the spacetime generated by two uniform electric and magnetic fields E and B [35], with the metric functions $\mathcal{P}(x)$ and $\mathcal{Q}(y)$ given in (42) with $\gamma_1 = -(E^2 + B^2)$ and $\gamma_2 = (E^2 + B^2)$.

B. Reissner-Nordström generalization with EH field

The Reissner-Nordström metric, that represents a charged black hole, is the type-D metric, Eq. (35) characterized by the conformal factor $\Omega = y$. It is convenient to change variables $y \mapsto 1/y$; then the Maxwell-Faraday equations reduce to

$$\partial_x[(D\kappa_1 + iH)y^2] = 0,$$
$$\partial_y[(D + i\kappa_2 H)y^2] = 0. \tag{44}$$

Considering the electromagnetic field generated by electric and magnetic charges, Q_e and Q_m, respectively, also called the dyonic solution, the D and H fields are given by

$$D = \frac{Q_e}{y^2}, \qquad H = \frac{Q_m}{y^2}, \tag{45}$$

then E and B are given through the material relations Eqs. (24) by

$$E = \frac{Q_e}{y^2}\left(1 - \frac{\mu}{2}\frac{Q_e^2}{y^4} - \frac{5}{4}\mu\frac{Q_m^2}{y^4}\right)$$
$$B = \frac{Q_m}{y^2}\left(1 + \frac{\mu}{2}\frac{Q_m^2}{y^4} + \frac{5}{4}\mu\frac{Q_e^2}{y^4}\right). \tag{46}$$

Substituting the electromagnetic fields D and H in (38) and solving we find

$$\mathcal{P}(x) = \alpha + \beta x - \epsilon x^2 \tag{47}$$

$$\mathcal{Q}(y) = \epsilon y^2 - 2My - \frac{\Lambda}{3}y^4 + Q_e^2 + Q_m^2$$
$$- \frac{\mu}{20y^4}(Q_e^4 + Q_m^4 + 5Q_m^2 Q_e^2), \tag{48}$$

where α and β are constants, $\epsilon = -1, 0, 1$, and M can be identified as the BH mass. The metric function $\mathcal{Q}(y)$ can also be written in terms of the screened electric and magnetic charges $\hat{Q}_e$ and $\hat{Q}_m$ as

$$\mathcal{Q}(y) = \epsilon y^2 - 2My - \frac{\Lambda}{3}y^4 + \hat{Q}_e^2 + \hat{Q}_m^2 \tag{49}$$

with

$$\hat{Q}_e^2(y) = Q_e^2\left(1 - \frac{\mu Q_e^2}{20y^4} - \frac{\mu Q_m^2}{8y^4}\right)$$
$$\hat{Q}_m^2(y) = Q_m^2\left(1 - \frac{\mu Q_m^2}{20y^4} - \frac{\mu Q_e^2}{8y^4}\right). \tag{50}$$

To recover the canonical form of the RN solution we have to change y, x coordinates as $y \mapsto r$ and $x \mapsto \cos\theta$; then with $\alpha = 1$, $\beta = 0$, $\epsilon = 1$, $P = (1 - x^2) \mapsto \sin^2\theta$, then the line element (35) is

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dr^2}{\mathcal{Q}(r)} - \mathcal{Q}(r)dt^2, \tag{51}$$

with $\mathcal{Q}(r)$ given by

$$\mathcal{Q}(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{\hat{Q}_e^2(r)}{r^2} + \frac{\hat{Q}_m^2(r)}{r^2}, \tag{52}$$

To determine the purely electric or magnetic case we make $Q_m = 0$ or $Q_e = 0$, respectively in the previous solution. In the case $\Lambda = 0$ this solution reduces to the dyonic solution derived from the Lagrangian formalism in [25].

C. Anti-Reissner-Nordström generalization with EH field

Following the same method as in Subsection B the solution corresponding to $\Omega = x$ (anti-Reissner-Nordström) can be derived. However it is simpler to obtain the EH anti-Reissner-Nordström from the EH Reissner-Nordström solution by a coordinate transformation: $x \mapsto y$, $y \mapsto x$, $\tau \mapsto i\tau$, and $\sigma \mapsto i\sigma$. Redefining the electromagnetic charges, such that the electric charge becomes the magnetic one and conversely, the mass parameter becomes the NUT parameter or magnetic mass, n [34]. Since this solution lacks of horizon it does not admit a BH interpretation. In order not to be redundant we omit this derivation.

$$
\begin{aligned}
ds^2 &= \frac{x^2}{\mathcal{P}}dx^2 + \frac{\mathcal{P}}{x^2}dt^2 + x^2\left[\frac{1}{\mathcal{Q}}dy^2 - \mathcal{Q}d\phi^2\right], \\
\mathcal{Q}(y) &= \alpha + \beta y - \epsilon y^2, \\
\mathcal{P}(x) &= \epsilon x^2 + 2nx - \frac{\Lambda}{3}x^4 - (Q_e^2 + Q_m^2) \\
&\quad - \frac{\mu}{20x^4}(Q_e^4 + Q_m^4 + 5Q_m^2 Q_e^2).
\end{aligned} \tag{53}
$$

and the electromagnetic fields are given by

$$
\begin{aligned}
\partial_y[(D\kappa_1 + iH)x^2] &= 0 \\
\partial_x[(D + i\kappa_2 H)x^2] &= 0.
\end{aligned} \tag{54}
$$

the D and H fields are given by

$$
D = \frac{Q_m}{x^2}, \qquad H = \frac{Q_e}{x^2}, \tag{55}
$$

while E and B are given through the material relations (24).

D. There is not EH generalization of the Levi-Civita or C-metric

The case with the conformal factor given by $\Omega = x + y$ is the Levi-Civita or C-metric and it admits the interpretation of the field produced by the motion of two accelerated charges [24]. We will show that the C-metric does not admit the EH generalization, and the only acceptable solution is the linear Maxwell one. The electromagnetic equations (37) are in this case

$$
\partial_x\left(\frac{D\kappa_1 + iH}{(x+y)^2}\right) = 0, \qquad \partial_y\left(\frac{D + i\kappa_2 H}{(x+y)^2}\right) = 0, \tag{56}
$$

with κ_1 and κ_2 given by (24). Then (56) implies

$$
\begin{aligned}
D &= (x+y)^2 h(x), & H &= (x+y)^2 g(y), \\
D\kappa_1 &= (x+y)^2 f(y), & H\kappa_2 &= (x+y)^2 j(x),
\end{aligned} \tag{57}
$$

where the functions $h(x)$, $g(y)$, $f(y)$, and $j(x)$ are arbitrary. Using the definitions of κ_1 and κ_2 and Eqs. (57) we obtain that

$$
\begin{aligned}
\kappa_1 &= 1 - \frac{\mu}{2}D^2 - \frac{5}{4}\mu H^2 = \frac{f(y)}{h(x)} \\
\kappa_2 &= 1 + \frac{\mu}{2}H^2 + \frac{5}{4}\mu D^2 = \frac{j(x)}{g(y)},
\end{aligned} \tag{58}
$$

that using $D = (x+y)^2 h(x)$ and $H = (x+y)^2 g(y)$ can be written as

$$
\begin{aligned}
f(y) &= h(x)\left(1 - \frac{\mu}{2}(x+y)^4\left[h(x)^2 + \frac{5}{2}g(y)^2\right]\right) \\
j(x) &= g(y)\left(1 + \frac{\mu}{2}(x+y)^4\left[g(y)^2 + \frac{5}{2}h(x)^2\right]\right).
\end{aligned} \tag{59}
$$

These equations can only be satisfied if $h(x)$, $g(y)$, $f(y)$, and $j(x)$ are constants and $\mu = 0$, therefore $D = C_1(x+y)^2$ and $H = C_2(x+y)^2$, being C_1 and C_2 constants and $\kappa_1 = 1$ and $\kappa_2 = 1$ that is actually the Maxwell case. For completeness we include the metric functions $\mathcal{P}(x)$ and $\mathcal{Q}(y)$ of the linear case [24], that is the line element (35) with

$$
\begin{aligned}
\mathcal{P}(x) &= \gamma - \frac{\Lambda}{6} + p_1 x + p_2 x^2 + p_3 x^3 - (Q_e^2 + Q_m^2)x^4 \\
\mathcal{Q}(y) &= -\gamma - \frac{\Lambda}{6} + p_1 y - p_2 y^2 + p_3 y^3 + (Q_e^2 + Q_m^2)y^4,
\end{aligned} \tag{60}
$$

where γ, p_1, p_2, p_3 are constants and Q_e, Q_m are the electric and magnetic charges, respectively, sources of the electromagnetic fields $E = D = Q_e(x+y)^2$ and $B = H = Q_m(x+y)^2$

Then we conclude that the EH electromagnetic field is not compatible with the C-metric geometry. The reason might be that the preferred direction given by the motion of the charged BHs breaks the spherical symmetry, being then impossible to couple the C-metric spacetime with the electromagnetic field as was done in cases (A), (B), and (C).

V. STATIONARY TYPE-D SOLUTIONS OF THE EINSTEIN-EULER-HEISENBERG EQUATIONS

It is worthwhile to stress the fact that these stationary solutions cannot be obtained from the static ones by means of complex translations, since this method does not work for nonlinear sources. This issue is addressed in detail in [36].

The class of stationary metrics arises as solutions of the EEH system with the metric Eq. (14) in the case that $m(x)$ and $p(y)$ are not simultaneously constant. This includes the Plebański class of metrics, and the most important solution in this class is the Kerr-Newman one. It is worth to mention that solutions to the Einstein-Born-Infeld equations, both static and stationary, were derived in [34]. Stationary solutions within NLE with a Lagrangian given in terms of the coordinates has been derived by García-Díaz in [37] and the EEH generalization of the Kerr-Newman solution was presented in [28].

A. The EEH generalization of the NUT-B(+) metric

The importance of this metric has been related to the magnetic monopole, however the rod singularity, associated to the NUT parameter l, has made that some people disregard its importance, however see [33,38]. It is also worthwhile to mention that the Born-Infeld nonlinear generalization of the NUT-B(+) metric was presented in [34] and analyzed in [39].

This solution arises for $p_{,y}=0$ and $m_{,x}\neq 0$, and with $m(x)=-2lx$. The metric is of the form

$$ds^2=(l^2+y^2)\left[\mathcal{P}(x)d\sigma^2+\frac{dx^2}{\mathcal{P}(x)}\right] + \frac{(l^2+y^2)}{\mathcal{Q}(y)}dy^2-\frac{\mathcal{Q}(y)}{(l^2+y^2)}(d\tau-2lxd\sigma)^2. \quad (61)$$

The electromagnetic two-form ω reads

$$\omega=i(l^2+y^2)(D+iH\kappa_2)d\sigma\wedge dx + (D\kappa_1+iH)(d\tau-2lxd\sigma)\wedge dy. \quad (62)$$

The closure of the electromagnetic two-form, $d\omega=0$ in this case amounts to the equations

$$\partial_x(D\kappa_1+iH)=0,$$
$$\partial_y[D(l^2+y^2)]+2lH=0,$$
$$\partial_y[\kappa_2 H(l^2+y^2)]-2lD\kappa_1=0. \quad (63)$$

From the first equation we know then that $D=D(y)$ and $H=H(y)$; while the second equation is satisfied by

$$D=Q_e\frac{(l^2-y^2)}{(l^2+y^2)^2},\qquad H=Q_e\frac{2ly}{(l^2+y^2)^2}. \quad (64)$$

In the stationary case even if the BH only possesses electric charge, the rotation induces a magnetic dipole moment, linked to the NUT parameter l. While the third equation is satisfied to first order in μ by the screened fields

$$\tilde{E}=D\left(1-\frac{\mu}{2}D^2+\frac{9}{4}\mu H^2\right)$$
$$\tilde{B}=H\left(1+\frac{\mu}{2}H^2-\frac{9}{4}\mu D^2\right). \quad (65)$$

On the other hand the Einstein equations, Eqs. (34), reduce for this metric to

$$-\mathcal{P}_{,xx}-\frac{2y\mathcal{Q}_{,y}}{(l^2+y^2)}+\frac{2\mathcal{Q}(y)}{(l^2+y^2)} =2(l^2+y^2)\left[(D^2+H^2)(1-\mu s)+\mu\left(s^2+\frac{7}{4}t^2\right)+\Lambda\right]$$
$$-\mathcal{Q}_{,yy}+\frac{2y\mathcal{Q}_{,y}}{(l^2+y^2)}-\frac{2\mathcal{Q}(y)}{(l^2+y^2)} =2(l^2+y^2)\left[-(D^2+H^2)(1-\mu s)+\mu\left(s^2+\frac{7}{4}t^2\right)+\Lambda\right]. \quad (66)$$

Summing and subtracting the previous equations lead to the equivalent system of equations,

$$-\mathcal{P}_{,xx}-\mathcal{Q}_{,yy}=2\Sigma\left\{2\Lambda+2\mu\left(s^2+\frac{7}{4}t^2\right)\right\}$$
$$\mathcal{P}_{,xx}+\frac{2y\mathcal{Q}_{,y}}{(l^2+y^2)}-\frac{2\mathcal{Q}(y)}{(l^2+y^2)} =2\Sigma\left\{-(D^2+H^2)(1-\mu s)-\Lambda-\mu\left(s^2+\frac{7}{4}t^2\right)\right\}, \quad (67)$$

with $\Sigma=l^2+y^2$. Since the functions involved in the previous equations depend on y, it must be that $\mathcal{P}(x)_{,xx}=-2\epsilon$, where the parameter $\epsilon=1,0,-1$ determines the geometry of the metric sector $g_2=dx^2/P+Pd\sigma^2$.

The metric function $\mathcal{Q}(y)$ is given by

$$\mathcal{Q}(y)=\epsilon(y^2-l^2)-2My-\Lambda\left(\frac{y^4}{3}+2l^2y^2-l^4\right) + Q_e^2+\mu\mathcal{F}(y), \quad (68)$$

where $\mathcal{F}(y)$ should satisfy

$$\mathcal{F}(y)''=-4\Sigma\left(s^2+\frac{7}{4}t^2\right), \quad (69)$$

after integration we obtain

$$\mathcal{F}(y)=\frac{1}{3}\left(s^2+\frac{7}{4}t^2\right)(-\Sigma^2+l^4-4l^2y^2)+\text{const}, \quad (70)$$

and adjusting the constant from the second equation, one has that

$$\mathcal{P}_{,xx}+\frac{2y\mathcal{Q}_{,y}}{(l^2+y^2)}-\frac{2\mathcal{Q}(y)}{(l^2+y^2)} =2\Sigma\left(-\frac{Q_e^2}{\Sigma^2}(1-\mu s)-\Lambda-\mu\left(s^2+\frac{7}{4}t^2\right)\right), \quad (71)$$

then the metric function $\mathcal{Q}$ turns out to be

$$\mathcal{Q}(y) = \epsilon(y^2 - l^2) - 2My - \Lambda\left(\frac{y^4}{3} + 2l^2y^2 - l^4\right) + Q_e^2(1-\mu s) + \mu\left(s^2 + \frac{7}{4}t^2\right)\left(-\frac{y^4}{3} - 2l^2y^2 + l^4\right). \tag{72}$$

In order to obtain the screening, one should remember that

$$s = \frac{1}{2}(D^2 - H^2) = \frac{Q_e^2}{2\Sigma^4}(\Sigma^2 - 8l^2y^2),$$
$$s^2 + \frac{7}{4}t^2 = \frac{1}{4}(D^4 + H^4 + 5H^2D^2) = \frac{Q_e^4}{4\Sigma^4}\left(1 + \frac{12l^2y^2}{\Sigma^4}(y^2 - l^2)^2\right), \tag{73}$$

Regarding the physical properties of the NUT-B$(+)$ solution it is worth to mention that the also called Taub–NUT solution, first derived by Taub (1951) [40] and then by Newman *et al.* (1963) [41], has been the object of interest in many aspects. Derived by a generalization of the Schwarzschild metric to include the NUT parameter, several interpretations of this parameter have arisen and still are under debate, for instance it has been related to the gravomagnetic monopole strength, or the twist of an electromagnetic universe [42].

The NUT (also called Taub–NUT) solution is stationary and axisymmetric but it is not globally asymptotically flat because it has one semi-infinite singularity on the symmetry axis at $\theta = \pi$, and Bonnor [43] interpreted this singularity as a semi-infinite massless source endowed with a nonzero angular momentum. Misner [44] addressed as well NUT spaces discovering that either closed timelike geodesics appear or there is the singularity also called the "Misner string." These aspects and geodesic incompleteness at the horizons were addressed in [45] as well as the complete set of analytical solutions of the geodesic equation in Taub-NUT space-times in terms of the Weierstrass elliptic functions were derived.

NUT spaces has also been explored in connection with the AdS/CFT conjecture [46]. Besides, NUT–Reissner–Nordstrom spaces that are asymptotically de Sitter (dS) have given counterexamples to the dS/CFT paradigm [47]. Regarding thermodynamics studies, the entropy of the Taub–NUT solution has also been relevant in the context of Euclidean solutions or instantons [48,49]. Moreover, it has been shown that in NUT spaces the entropy/area relationship does not hold [50,51]. Hawking [52] was also interested in elucidating the role of the NUT charge in the context of quantum gravity. Questions like if hairy black holes possessing a magnetic mass do exist or the new effects that arise due to the presence of a NUT charge were addressed in [53]. In [54] it was pointed out the importance of the NUT charge in the context of low energy string theory, and how some symmetries remain unnoticed if the NUT charge is not included [55].

In a proposal by Atiyah, Manton, and Schroers [56] as a generalization of the Skyrme model, they assigned a geometrical and topological interpretation to the electric charge and baryon and lepton numbers, static particles are described in terms of gravitational instantons, where the electrically charged particles correspond to noncompact asymptotically locally flat instantons; in this context the Taub–NUT instanton is a model for the electron. The geometry of the Taub–NUT skyrmion has been addressed in [57].

The case NUT-B$(-)$ corresponds to $p_{,y} \neq 0$ and $m_{,x} = 0$, with $p(y) = 2ly$. The EEH generalization of the NUT-B$(-)$ metric can be obtained following a similar method.

B. The EH generalization of the Kerr-Newman metric

The Plebański class of metrics is characterized by $p_{,y} \neq 0$ and $m_{,x} \neq 0$, that without loss of generality can be given by $p(y) = -y^2$ and $m(x) = x^2$, while $\Omega = a + bxy$. As already mentioned, the most important solution of this class is the Kerr-Newman one with $a = 1$ and $b = 0$. The importance of Kerr metric cannot be exaggerated, since it represents a rotating black hole and observations of stars trajectories near the center of our galaxy agree with the geodesics of test particles in a Kerr spacetime [5]. Its charged version is the Kerr-Newman solution and its generalization considering uniform nonlinear EH electromagnetic fields, arising from QED after one loop of nonperturbative quantization, has been recently derived in [28], that we now generalize to include the cosmological constant Λ. The metric is of the form of Eq. (14), with $p(y) = -y^2$, $m(x) = x^2$, and $\Omega(x, y) = 1$, it is given by

$$ds^2 = \frac{\Sigma}{\mathcal{P}}dx^2 + \frac{\mathcal{P}}{\Sigma}(d\tau - y^2 d\sigma)^2 + \frac{\Sigma}{\mathcal{Q}}dy^2 - \frac{\mathcal{Q}}{\Sigma}(d\tau + x^2 d\sigma)^2, \tag{74}$$

where $\Sigma = x^2 + y^2$, $\mathcal{P} = \mathcal{P}(x)$, and $\mathcal{Q} = \mathcal{Q}(y)$.

The electromagnetic field equations, Eq. (29), now read

$$\partial_x(D\kappa_1 + iH) = i\partial_y(D + iH\kappa_2),$$
$$\partial_x[x^2(D\kappa_1 + iH)] = -i\partial_y[y^2(D + iH\kappa_2)], \tag{75}$$

since we know the solutions for $\mu = 0$, i.e., the Kerr-Newman electromagnetic field, we make the ansatz,

$$D = Q_e\frac{(y^2 - x^2)}{(x^2 + y^2)^2}, \qquad H = -Q_e\frac{2xy}{(x^2 + y^2)^2}, \tag{76}$$

while the metric function satisfy the Einstein equations (34) in terms of the electromagnetic invariants

$$\begin{aligned}&\frac{\mathcal{P}(x)}{\Sigma^2}-\frac{\mathcal{Q}(y)}{\Sigma^2}-\frac{x\mathcal{P}_{,x}}{\Sigma^2}+\frac{y\mathcal{Q}_{,y}}{\Sigma^2}+\frac{\mathcal{P}_{,xx}}{2\Sigma}\\&\quad=-(D^2+H^2)+\mu s(D^2+H^2)-\mu\left(s^2+\frac{7}{4}t^2\right)-\Lambda\\&\quad-\frac{\mathcal{P}(x)}{\Sigma^2}+\frac{\mathcal{Q}(y)}{\Sigma^2}+\frac{x\mathcal{P}_{,x}}{\Sigma^2}-\frac{y\mathcal{Q}_{,y}}{\Sigma^2}+\frac{\mathcal{Q}_{,yy}}{2\Sigma}\\&\quad=(D^2+H^2)-\mu s(D^2+H^2)-\mu\left(s^2+\frac{7}{4}t^2\right)-\Lambda,\end{aligned}\tag{77}$$

or since $D^2+H^2=Q_e^2/\Sigma^2$ they reduce to

$$\begin{aligned}&\frac{\mathcal{P}(x)}{\Sigma^2}-\frac{\mathcal{Q}(y)}{\Sigma^2}-\frac{x\mathcal{P}_{,x}}{\Sigma^2}+\frac{y\mathcal{Q}_{,y}}{\Sigma^2}+\frac{\mathcal{P}_{,xx}}{2\Sigma}\\&\quad=-\frac{Q_e^2}{\Sigma^2}(1-\mu s)-\mu\left(s^2+\frac{7}{4}t^2\right)-\Lambda\\&\quad-\frac{\mathcal{P}(x)}{\Sigma^2}+\frac{\mathcal{Q}(y)}{\Sigma^2}+\frac{x\mathcal{P}_{,x}}{\Sigma^2}-\frac{y\mathcal{Q}_{,y}}{\Sigma^2}+\frac{\mathcal{Q}_{,yy}}{2\Sigma}\\&\quad=\frac{Q_e^2}{\Sigma^2}(1-\mu s)-\mu\left(s^2+\frac{7}{4}t^2\right)-\Lambda,\end{aligned}\tag{78}$$

that, as can be check, have a solution of the form

$$\begin{aligned}&\mathcal{P}(x)=a^2-\epsilon x^2-\frac{\Lambda}{3}x^4\\&\mathcal{Q}(y)=a^2+Q_e^2-2My+\epsilon y^2-\frac{\Lambda}{3}y^4+\mu\mathcal{G}(y),\end{aligned}\tag{79}$$

where $\mathcal{G}(y)$ should satisfy

$$\begin{aligned}&\mathcal{G}(y)''=-4\Sigma\left(s^2+\frac{7}{4}t^2\right)\\&\mathcal{G}(y)-y\mathcal{G}(y)'=-\Sigma^2\left(\frac{Q_e^2}{\Sigma^2}s-\left(s^2+\frac{7}{4}t^2\right)\right),\end{aligned}\tag{80}$$

The above equations are solved by

$$\mathcal{G}(y)=\left(s^2+\frac{7}{4}t^2\right)\left(-\frac{y^4}{3}-2y^2x^2+x^4\right)-Q_e^2 s,\tag{81}$$

that then gives the metric function $\mathcal{Q}(y)$ as

$$\begin{aligned}\mathcal{Q}(y)&=a^2-2My+\epsilon y^2-\frac{\Lambda}{3}y^4+Q_e^2(1-\mu s)\\&\quad+\mu\left(s^2+\frac{7}{4}t^2\right)\left(-\frac{y^4}{3}-2y^2x^2+x^4\right),\end{aligned}\tag{82}$$

To get the screening remember that

$$\begin{aligned}s&=\frac{1}{2}(D^2-H^2)=\frac{Q_e^2}{2\Sigma^4}(\Sigma^2-8x^2y^2),\\s^2+\frac{7}{4}t^2&=\frac{1}{4}(D^4+H^4+5H^2D^2)\\&=\frac{Q_e^4}{4\Sigma^4}\left\{1+\frac{12x^2y^2}{\Sigma^4}(y^2-x^2)^2\right\},\end{aligned}\tag{83}$$

The determined solution for the Kerr-like metric with a EH nonlinear electromagnetic source does coincide with the solution presented in [28], as we show in what follows. From Eq. (92) in [28] we have

$$m'(r)=\frac{Q_e^2}{2r^2}(1-\mu s)+\mu\left(s^2+\frac{7}{4}t^2\right)\left(\frac{\Sigma^2}{2r^2}\right),\tag{84}$$

and integrating we obtain

$$\begin{aligned}m(r)&=M-\frac{Q_e^2}{2r}(1-\mu s)\\&\quad-\frac{\mu}{2r}\left(s^2+\frac{7}{4}t^2\right)\left(-\frac{r^4}{3}-2r^2a^2\cos^2\theta+a^4\cos^4\theta\right),\end{aligned}\tag{85}$$

such that

$$\begin{aligned}-2rm(r)&=-2Mr+Q_e^2(1-\mu s)\\&\quad+\mu\left(s^2+\frac{7}{4}t^2\right)\left(-\frac{r^4}{3}-2r^2a^2\cos^2\theta+a^4\cos^4\theta\right),\end{aligned}\tag{86}$$

and the metric component g_{tt} is given by

$$g_{tt}(r,\cos\theta)=1-\frac{2rm(r)}{\Sigma}.\tag{87}$$

While from our Eqs. (79) in the Kerr-like metric Eq. (74) the $g_{\tau\tau}$ component is given by

$$g_{\tau\tau}(x,y)=\frac{\mathcal{P}(x)-\mathcal{Q}(y)}{\Sigma},\tag{88}$$

that substituting the explicit form of $\mathcal{P}(x)$ and $\mathcal{Q}(y)$ from Eqs. (70) and (73), respectively,

$$\begin{aligned}g_{\tau\tau}(x,y)&=-\epsilon+\frac{\Lambda}{3}(y^2-x^2)\\&\quad+\frac{1}{\Sigma}\left\{2My-Q_e^2(1-\mu s)\right.\\&\quad\left.-\mu\left(s^2+\frac{7}{4}t^2\right)\left(-\frac{y^4}{3}-2y^2x^2+x^4\right)\right\},\end{aligned}\tag{89}$$

then we notice that making $\epsilon=1$, $\Lambda=0$, $y\mapsto r$, $x\mapsto a\cos\theta$, $\tau\mapsto t$ and $\sigma\mapsto\phi$, we obtain the line element in [28], that in fact we have generalized introducing the cosmological constant Λ.

VI. CONCLUSIONS

In this paper, using the null tetrad formalism, the most important type-D spacetimes that can be coupled to the Euler-Heisenberg nonlinear electrodynamics with cosmological constant are determined. For type-D solutions one can always align the directions of the real null vectors e^3 and e^4 along the principal null directions. The most convenient setting is to align the eigenvectors of the electromagnetic field tensor P_{ab} with e^3 and e^4. Then the nonvanishing components of P_{ab} are P_{34} and P_{12}, that we parametrize as the electric field strength $\mathbf{D}$ and the magnetic field $\mathbf{H}$, respectively. Correspondingly, for F_{ab} the nonvanishing components are F_{12} and F_{34}, parametrized as the electric field $\mathbf{E}$ and the magnetic field strength $\mathbf{B}$, respectively. We work in the frame of the structural function $\mathcal{H}(s,t)$ which is a function of the electromagnetic invariants associated to P^{ab}. This scheme has the advantage that the electromagnetic fields (D,H) satisfy the equation, $\nabla_a P^{ab}=0$, and through the constitutive equations we determined the electric field and the magnetic field strength (E,B). All the static type-D solutions (commuting Killing vectors) are generalized with the EH nonlinear electromagnetic field and cosmological constant and include the Bertotti-Robinson and the Reissner-Nordström solutions with both electric and magnetic charges. The anti-Reissner-Nordström solution can be obtained from the corresponding EEH-Λ RN by means of a coordinate transformation. Moreover, it is shown that, due to the form of the conformal factor that depends on two coordinates, it is impossible to couple the C-metric to the EH electromagnetic field. We have also determined the EEH-Λ generalization (de Sitter and anti–de Sitter) of the most important stationary solutions, namely, the NUT-B$(+)$ and Kerr-Newman metrics, the former was unknown so far and the latter was already presented in [28]. These solutions have a BH interpretation and are important in view of the recent improved observations of BH and compact objects [3].

ACKNOWLEDGMENTS

This work was partially supported by DAAD fellowships AM: Ref. No.: 91699151 and NB: Ref. No.: 91834104. N. B. also acknowledges partial support by CONACyT (Mexico) Grant No. 284489. C. L. is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy-EXC-2123 "QuantumFrontiers"-390837967, the CRC 1464 "Relativistic and Quantum-based Geodesy" (TerraQ), and by the Research Training Group GRK1620 "Models of Gravity".

APPENDIX: RICCI TENSOR FOR TYPE-D METRICS

The nonvanishing components of the Ricci tensor for the type D metrics (14), that are the input into Eqs. (34) are given by

$$\begin{aligned}
-2R_{12} &= \frac{\Omega^2}{2\Sigma}\left\{-2P_{,xx}+2P_{,x}\left(4\frac{\Omega_{,x}}{\Omega}+\frac{m_{,x}}{\Sigma}\right)+P\left[6\frac{\Omega_{,xx}}{\Omega}-12\left(\frac{\Omega_{,x}}{\Omega}\right)^2-4\frac{m_{,x}}{\Sigma}\frac{\Omega_{,x}}{\Omega}+\frac{m_{,xx}}{\Sigma}-\frac{3}{2\Sigma^2}[(m_{,x})^2+(p_{,y})^2]\right]\right.\\
&\quad\left.-Q\left[-2\frac{\Omega_{,yy}}{\Omega}+12\left(\frac{\Omega_{,y}}{\Omega}\right)^2+4\frac{p_{,y}}{\Sigma}\frac{\Omega_{,y}}{\Omega}-\frac{p_{,yy}}{\Sigma}-\frac{3}{2\Sigma^2}[(m_{,x})^2+(p_{,y})^2]\right]+2Q_{,y}\left(2\frac{\Omega_{,y}}{\Omega}+\frac{p_{,y}}{\Sigma}\right)\right\},\\
-2R_{34} &= \frac{\Omega^2}{2\Sigma}\left\{-2Q_{,yy}+2Q_{,y}\left(4\frac{\Omega_{,y}}{\Omega}-\frac{p_{,y}}{\Sigma}\right)-P\left[-2\frac{\Omega_{,xx}}{\Omega}+12\left(\frac{\Omega_{,x}}{\Omega}\right)^2-4\frac{m_{,x}}{\Sigma}\frac{\Omega_{,x}}{\Omega}+\frac{m_{,xx}}{\Sigma}-\frac{3}{2\Sigma^2}[(m_{,x})^2+(p_{,y})^2]\right]\right.\\
&\quad\left.+Q\left[6\frac{\Omega_{,yy}}{\Omega}-12\left(\frac{\Omega_{,y}}{\Omega}\right)^2+4\frac{p_{,y}}{\Sigma}\frac{\Omega_{,y}}{\Omega}-\frac{p_{,yy}}{\Sigma}-\frac{3}{2\Sigma^2}[(m_{,x})^2+(p_{,y})^2]\right]+2P_{,x}\left(2\frac{\Omega_{,x}}{\Omega}-\frac{m_{,x}}{\Sigma}\right)\right\},
\end{aligned} \tag{A1}$$

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