Dyonic black holes in the theory of two electromagnetic potentials. II

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The results obtained in our previous paper are now extended to the case of stationary axially symmetric dyonic black boles within the theory of two electromagnetic potentials. We slightly enlarge the classical Ernst formalism by introducing, with the aid of the t and φ components of the dual potential B_μ, the magnetic potential Φ_m which, similar to the known electric potential Φ_e, also takes constant value on the black hole horizon. We analyze in detail the case of the dyonic Kerr-Newman black hole and show how the Komar mass must be evaluated correctly in this stationary dyonic model. In particular, we rigorously prove the validity of the standard Tomimatsu mass formula and point out that attempts to "improve" it made in recent years are explained by misunderstanding of the auxiliary role that singular potentials play in the description of magnetic charges. Our approach is symmetrical with respect to electric and magnetic charges and, like in the static case considered earlier, Dirac strings of all kind are excluded from the physical picture of the stationary black hole dyonic spacetimes.

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I. INTRODUCTION

In the paper [1] we have shown that the field of a magnetic charge is described correctly by the t component of the dual electromagnetic potential B_μ, so that the semi-infinite singularities accompanying the φ component of the usual potential A_μ, that must be taken into account during some mathematical calculations, cannot be considered as representing real physical characteristics of the magnetic charge. In [1] our consideration was restricted exclusively to the static spherically symmetric dyonic case that ideally suited our objective of giving simple and clear arguments in favor of our novel approach to the description of magnetic charges without Dirac strings. In the present paper we shall expand our analysis to the stationary axially symmetric dyonic black holes for which the effect of rotation introduces additional technical difficulties; however, these difficulties will be circumvented in an elegant way, clearly confirming the physical conclusions of the previous paper in a more general situation.

In the next section we shall introduce the nonzero components of the dual electromagnetic potential B_μ within the framework of the well-known Ernst formulation of the stationary axially symmetric problem [2] and define explicitly the magnetic potential Φ_m which, similar to the electric potential Φ_e introduced long ago by Carter [3], takes constant value on the black hole horizon. The advantages of the enhanced Ernst formalism are illustrated here by the example of the dyonic Kerr-Newman black hole [3,4] for which a complete set of the corresponding potentials will be constructed. The validity of the original Tomimatsu mass integral [5] will be proven in Sec. III with the aid of the symmetrical representation of the electromagnetic energy-momentum tensor. Discussion of the results obtained and conclusions can be found in Sec. IV.

II. THE ENHANCED ERNST FORMALISM

In the theory of exact solutions of the Einstein-Maxwell equations, the Ernst formalism, developed in two papers [2,6] in 1968, occupies an outstanding place as constituting the basis for various solution generating techniques and different approaches to the multipole analysis of vacuum and electrovac spacetimes. In particular, Ernst trivialized the derivation of the Kerr [7] and Kerr-Newman [8] black hole solutions that were originally obtained by means of hardly reproducible procedures.

The main idea of Ernst's formalism is to use the Papapetrou line element [9],

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \quad (1)$$

with the coordinate system $\{\rho, z, \varphi, t\}$, which describes a generic stationary axisymmetric electrovac field, its three unknown functions f, γ, and ω depending only on ρ and z, for reducing the corresponding set of the Einstein-Maxwell equations to a fundamental system of two differential equations for the complex potentials $\mathcal{E}$ and Φ of the following elegant form:

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$$
\begin{aligned}
(\mathrm{Re}\,\mathcal{E}+\Phi\bar{\Phi})\Delta\mathcal{E} &= (\boldsymbol{\nabla}\mathcal{E}+2\bar{\Phi}\boldsymbol{\nabla}\Phi)\cdot\boldsymbol{\nabla}\mathcal{E},\\
(\mathrm{Re}\,\mathcal{E}+\Phi\bar{\Phi})\Delta\Phi &= (\boldsymbol{\nabla}\mathcal{E}+2\bar{\Phi}\boldsymbol{\nabla}\Phi)\cdot\boldsymbol{\nabla}\Phi,
\end{aligned} \tag{2}
$$

where Δ and $\boldsymbol{\nabla}$ are the usual three-dimensional Laplacian and gradient operators, respectively, and a bar over a symbol means complex conjugation.

The potentials $\mathcal{E}$ and Φ are related to the metric functions f, ω, and to the φ and t components of the electromagnetic four-potential $A_\mu=(0,0,A_\varphi,A_t)$ by the equations

$$
\mathcal{E}=f-\Phi\bar{\Phi}+i\chi, \qquad \Phi=-A_t+iA'_\varphi, \tag{3}
$$

and by the systems of the first-order differential equations

$$
\begin{aligned}
\partial_\rho\omega &= -\rho f^{-2}[\partial_z\chi+2\mathrm{Im}(\bar{\Phi}\partial_z\Phi)],\\
\partial_z\omega &= \rho f^{-2}[\partial_\rho\chi+2\mathrm{Im}(\bar{\Phi}\partial_\rho\Phi)],
\end{aligned} \tag{4}
$$

and

$$
\begin{aligned}
\partial_\rho A'_\varphi &= \rho^{-1}f(\partial_z A_\varphi+\omega\partial_z A_t),\\
\partial_z A'_\varphi &= -\rho^{-1}f(\partial_\rho A_\varphi+\omega\partial_\rho A_t),
\end{aligned} \tag{5}
$$

so that the knowledge of $\mathcal{E}$ and Φ permits one to find the functions f, ω, A_t, and A_φ from (3) to (5), while for the determination of the remaining metric function γ one has to solve the system

$$
\begin{aligned}
\partial_\rho\gamma &= \frac{1}{4}\rho f^{-2}[(\partial_\rho\mathcal{E}+2\bar{\Phi}\partial_\rho\Phi)(\partial_\rho\bar{\mathcal{E}}+2\Phi\partial_\rho\bar{\Phi})\\
&\quad-(\partial_z\mathcal{E}+2\bar{\Phi}\partial_z\Phi)(\partial_z\bar{\mathcal{E}}+2\Phi\partial_z\bar{\Phi})]\\
&\quad-\rho f^{-1}(\partial_\rho\Phi\partial_\rho\bar{\Phi}-\partial_z\Phi\partial_z\bar{\Phi}),\\
\partial_z\gamma &= \frac{1}{2}\rho f^{-2}\mathrm{Re}[(\partial_\rho\mathcal{E}+2\bar{\Phi}\partial_\rho\Phi)(\partial_z\bar{\mathcal{E}}+2\Phi\partial_z\bar{\Phi})]\\
&\quad-2\rho f^{-1}\mathrm{Re}(\partial_\rho\bar{\Phi}\partial_z\Phi),
\end{aligned} \tag{6}
$$

the integrability condition of which are Eqs. (2).

Note that the potential $A'_\varphi=\mathrm{Im}\Phi$ is regarded in the Ernst formalism as an auxiliary function, the knowledge of which makes possible the calculation of the corresponding magnetic component A_φ of the four-potential A_μ. However, in our preceding paper [1] we have already shown that A_φ does not describe correctly the field of the magnetic charge, so it seems desirable to supplement the above formalism with the nonzero components of the dual electromagnetic four-potential $B_\mu=(0,0,B_\varphi,B_t)$ that are related to the components A_t and A_φ by the first-order differential equations. Indeed, using the one-form $B=B_t dt+B_\varphi d\varphi$, we obtain the desired relations by means of the formula

$$
dB=\star F, \tag{7}
$$

where the star denotes the Hodge dual, and F is the usual electromagnetic two-form. Taking into account that, on the one hand,

$$
dB=d(B_\nu dx^\nu)=\partial_a B_\nu dx^a\wedge dx^\nu, \tag{8}
$$

and, on the other hand,

$$
\begin{aligned}
\star F &= \star d(A_t dt+A_\varphi d\varphi)\\
&= (g^{t\beta}\partial_a A_t+g^{\varphi\beta}\partial_a A_\varphi)g^{ab}\sqrt{-g}\varepsilon_{b\beta\gamma\delta}dx^\gamma\wedge dx^\delta,
\end{aligned} \tag{9}
$$

($a,b\in\{\rho,z\}$), we get from (8) and (9), by first equating the coefficients at $d\rho\wedge dt$ and $dz\wedge dt$, the system of differential equations for B_t in terms of A_t and A_φ, namely,

$$
\begin{aligned}
\partial_\rho B_t &= \rho^{-1}f(\partial_z A_\varphi+\omega\partial_z A_t),\\
\partial_z B_t &= -\rho^{-1}f(\partial_\rho A_\varphi+\omega\partial_\rho A_t),
\end{aligned} \tag{10}
$$

and then, by equating the coefficients at $d\rho\wedge d\varphi$ and $dz\wedge d\varphi$, the analogous system for the determination of B_φ:

$$
\begin{aligned}
\partial_\rho B_\varphi &= \rho^{-1}f[(\rho^2 f^{-2}-\omega^2)\partial_z A_t-\omega\partial_z A_\varphi],\\
\partial_z B_\varphi &= -\rho^{-1}f[(\rho^2 f^{-2}-\omega^2)\partial_\rho A_t-\omega\partial_\rho A_\varphi].
\end{aligned} \tag{11}
$$

A simple inspection of formulas (5) and (10) shows that the t component of the dual potential B_μ coincides with the auxiliary potential A'_φ of the Ernst formalism, i.e.,

$$
B_t=A'_\varphi. \tag{12}
$$

Curiously, it is also possible to identify the component B_φ (up to a sign) as the potential B_2 introduced in the paper [10] by Kinnersley as part of various matrix potentials of his solution generating method; in particular, it arises as the imaginary part of Kinnersley's potential Φ_2.

The knowledge of the full set of the components A_t, A_φ, B_t, and B_φ of the four-potentials A_μ and B_μ allows one to analyze the electric and magnetic fields of the dyonic black hole solutions in a symmetrical way advocated long ago by Schwinger [11]. For example, as was shown by Carter [3], the electric potential Φ_e determined as the combination

$$
\Phi_e=-A_t-\omega^{-1}A_\varphi, \tag{13}
$$

assumes constant value on the black hole horizon. Having introduced explicitly the dual potential B_μ into the Ernst formalism, we can now define "symmetrically" the magnetic counterpart of Φ_e as

$$
\Phi_m=B_t+\omega^{-1}B_\varphi, \tag{14}
$$

the magnetic potential Φ_m also taking constant value on the horizon, which may be considered an important result following from our approach.

The above said can be well illustrated by the dyonic Kerr-Newman black hole solution. Following [4], we write its defining Ernst potentials $\mathcal{E}$ and Φ in the form

$$\mathcal{E} = \frac{\sigma x - m - iay}{\sigma x + m - iay}, \qquad \Phi = \frac{q + ip}{\sigma x + m - iay},$$
$$x = \frac{1}{2\sigma}(r_+ + r_-), \qquad y = \frac{1}{2\sigma}(r_+ - r_-),$$
$$r_\pm = \sqrt{\rho^2 + (z \pm \sigma)^2}, \qquad \sigma = \sqrt{m^2 - a^2 - q^2 - p^2}, \tag{15}$$

where the parameters m, a, q, and p stand, respectively, for the mass, angular momentum per unit mass, electric, and magnetic charges of the black hole (we restrict our consideration to the real-valued σ only).

The corresponding metric functions f, γ, and ω have the form

$$f = \frac{\sigma^2(x^2-1) - a^2(1-y^2)}{(\sigma x + m)^2 + a^2 y^2},$$
$$e^{2\gamma} = \frac{\sigma^2(x^2-1) - a^2(1-y^2)}{\sigma^2(x^2-y^2)},$$
$$\omega = -\frac{a(1-y^2)[2m(\sigma x + m) - q^2 - p^2]}{\sigma^2(x^2-1) - a^2(1-y^2)}, \tag{16}$$

while for the electric and magnetic components A_t and A_φ of the four-potential A_μ we have the expressions

$$A_t = -\frac{q(\sigma x + m) - apy}{(\sigma x + m)^2 + a^2 y^2},$$
$$A_\varphi = -py + \frac{a(1-y^2)[q(\sigma x + m) - apy]}{(\sigma x + m)^2 + a^2 y^2}, \tag{17}$$

where the integration constant on the right-hand side of A_φ has been chosen equal to zero, thus determining the case with two magnetic “strings.”

Turning now to the components of the dual four-potential B_μ, we see that B_t is obtainable as just the imaginary part of the Ernst potential Φ, while B_φ must be found by solving the system (11). The resulting expressions are

$$B_t = \frac{p(\sigma x + m) + aqy}{(\sigma x + m)^2 + a^2 y^2},$$
$$B_\varphi = -qy - \frac{a(1-y^2)[p(\sigma x + m) + aqy]}{(\sigma x + m)^2 + a^2 y^2}, \tag{18}$$

where the choice of the integration constant in B_φ is the same as for A_φ and defines a pair of electric “Dirac strings.”

The only plausible conclusion that can be drawn from the structure of the components (17) and (18) is that the field of the electric charge q in the dyonic Kerr-Newman solution is described by A_t, and the field of the magnetic charge p is determined by B_t, both components A_t and B_t being well behaved and asymptotically flat. In turn, the components A_φ and B_φ possessing the string singularities do not define the singularity structure of this dyonic black hole solution, playing exclusively auxiliary mathematical roles in some calculations. For instance, the components A_φ and B_φ are needed for the evaluation of the electric and magnetic potentials Φ_e and Φ_m on the horizon ($\rho = 0$, $-\sigma < z < \sigma$, or $x = 1$):

$$\Phi_e^H = -A_t - \omega^{-1} A_\varphi|_{x=1} = \frac{q(m+\sigma)}{(m+\sigma)^2 + a^2},$$
$$\Phi_m^H = B_t + \omega^{-1} B_\varphi|_{x=1} = \frac{p(m+\sigma)}{(m+\sigma)^2 + a^2}. \tag{19}$$

After introducing the angular momentum $J = ma$, and also recalling that ω takes a constant value on the horizon, so that

$$\omega^{-1}(x=1) \equiv \Omega^H = \frac{a}{(m+\sigma)^2 + a^2}, \tag{20}$$

one can see that the above formulas verify the Smarr mass relation [12]

$$m = \sigma + 2J\Omega^H + q\Phi_e^H + p\Phi_m^H. \tag{21}$$

We now turn to the discussion of the evaluation of the Komar mass [13] in the dyonic Kerr-Newman solution, the issue that also addresses the question of the distribution of that mass.

III. VALIDATING TOMIMATSU’S MASS INTEGRAL FORMULA

To calculate the Komar [13] mass M of a rotating charged black hole, Tomimatsu [5] derived a simple formula

$$M = -\frac{1}{8\pi}\int_H \omega \partial_z \chi d\varphi dz, \tag{22}$$

where the integral is taken over the horizon of the black hole. Formula (22) was widely used for years in application to nonisolated black holes in the presence of other black holes or exterior gravitational fields. In the case of the dyonic Kerr-Newman black hole, (22) assumes the form

$$M = -\frac{1}{4}\omega^H[\chi(y=1) - \chi(y=-1)], \tag{23}$$

where both ω and χ must be taken on the horizon ($x = 1$). It is not difficult to verify that the corresponding M calculated with the help of (23) coincides with the mass parameter m in (15).

However, the validity of the mass formula (22) in the presence of magnetic charge was questioned in the paper [14]. The authors of [14] used during their calculations the conventional representation of the electromagnetic energy-momentum tensor

$$T^{\mu}{}_{\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}\right), \qquad (24)$$

which led them to a specific dyonic configuration with two magnetic Dirac strings and an additional electromagnetic term in the integrand of (22), both strings carrying portions of nonzero mass, so that the mass parameter m becomes the sum of three different contributions—one coming from the surface integral evaluated on the horizon, and two others arising from the singular "massive" Dirac strings. Although the version of the dyonic Kerr-Newman black hole presented in [14] is manifestly physically inconsistent (see [15] for the discussion of unphysical features of that model), the mathematical computation of the Komar integral performed in [14] looks correct (albeit with some misprints). The explanation of such a seemingly puzzling situation is quite simple in the framework of the ideas developed in [1] and in the present paper: the pathologies of a specific representation of the electromagnetic energy-momentum tensor formally taking part in the calculation of the Komar mass integral should not be ascribed to the dyonic model itself since the singularity structure of the magnetic charge is determined by the well-behaved component B_t, and not by the function A_φ. In this respect, the desire to automatically associate the singularities of the auxiliary potentials with the intrinsic properties of the dyonic black hole would have forced the authors of [14], after using a different representation of $T^{\mu}{}_{\nu}$ involving say the dual electromagnetic tensor $\tilde{F}^{\mu\nu}$ only, to draw a new conclusion that it is the *electric* string singularities of the component B_φ that contribute to the expression of the Komar mass, with zero contribution coming from the magnetic charge.

As has already been shown in [1], the choice of the energy-momentum tensor $T^{\mu}{}_{\nu}$ in the symmetrical representation

$$T^{\mu}{}_{\nu} = \frac{1}{8\pi}(F^{\mu\alpha}F_{\nu\alpha} + \tilde{F}^{\mu\alpha}\tilde{F}_{\nu\alpha}), \qquad (25)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \tilde{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad (26)$$

permits one to avoid singular sources during the calculation of the Komar mass integral, reducing the calculational procedure exclusively to the integrals over the black hole horizon. Although the paper [1] treated the static case, the rotation of the black hole does not really change the qualitative picture of the nonrotating model, and below we shall demonstrate that the Komar mass of the dyonic black hole is obtainable straightforwardly by means of the original Tomimatsu's mass integral formula (22), without the need to consider any singular terms outside the horizon.

In his article [5], Tomimatsu started with the same standard integral for the calculation of the Komar mass that has been recently used in the papers [1,14],

$$\begin{aligned} M_K &= \frac{1}{4\pi}\int_\infty D^\nu k^\mu d\Sigma_{\mu\nu} \\ &= \frac{1}{4\pi}\int_{\partial\mathcal{M}} D^\nu k^\mu d\Sigma_{\mu\nu} + \frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu, \end{aligned} \qquad (27)$$

with the same decomposition into the surface and bulk integrals.

By choosing the horizon of the black hole as $\partial\mathcal{M}$, Tomimatsu computed the first integral on the right-hand side of (27) and obtained

$$\begin{aligned} \frac{1}{4\pi}\int_H D^\nu k^\mu d\Sigma_{\mu\nu} &= \frac{1}{8\pi}\int_H [-\omega\partial_z\chi + 2\omega \mathrm{Im}(\Phi\partial_z\bar{\Phi})]d\varphi dz \\ &= \frac{1}{8\pi}\int_H [-\omega\partial_z\chi + 2\omega(A_t\partial_z B_t \\ &\quad - B_t\partial_z A_t)]d\varphi dz, \end{aligned} \qquad (28)$$

and he also rewrote the bulk integral on the right-hand side of (27) in the form

$$\frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu = -2\int_{\mathcal{M}} T^t{}_t\sqrt{-g}d^3x, \qquad (29)$$

and the correctness of formulas (28) and (29) was not objected in [14]. The authors of [14], however, questioned Tomimatsu's result of computing the integral on the right-hand side of (29), namely,

$$-2\int_{\mathcal{M}} T^t{}_t\sqrt{-g}d^3x = -\frac{1}{4\pi}\int_H \omega\,\mathrm{Im}(\Phi\partial_z\bar{\Phi})d\varphi dz, \qquad (30)$$

which, together with (28), gives formula (22). Though they rightly pointed out that the representation (24) of the energy-momentum tensor used by Tomimatsu requires additionally taking account of two singular string sources, which modifies the horizon contribution (22) of the Komar mass, they still erroneously ascribed the *formal* mass distribution due to singularities of the auxiliary function to the genuine dyonic Kerr-Newman space. Actually, we have a strong impression that Tomimatsu obtained his formula (22) after deliberately suppressing the additional electromagnetic term discussed in [14], with the idea of getting a physically consistent expression for the Komar mass of a black hole. On the other hand, the authors of [14] have restored the additional electromagnetic term in Tomimatsu's formula (22) for mathematical consistency, but this has led them to the physically incorrect result for the mass distribution in a dyonic black hole.

Remarkably, the validity of Tomimatsu's mass integral (22) can be readily demonstrated by employing the symmetrical representation of the electromagnetic energy-momentum tensor (25) for the evaluation of the integral on the right-hand side of (29). Then, following the steps outlined in the paper [1] for that representation, the bulk integral on the left-hand side of (30) reduces to the surface integral over the horizon, yielding

$$-2\int_{\mathcal{M}} T^t{}_t\sqrt{-g}d^3x = \frac{1}{4\pi}\int_H (A_t\partial_z B_\varphi - B_t\partial_z A_\varphi)d\varphi dz, \quad (31)$$

where, at the last stage of the computation, we have used the substitutions

$$\rho^{-1}f[(\rho^2 f^{-2} - \omega^2)\partial_\rho A_t - \omega\partial_\rho A_\varphi] = -\partial_z B_\varphi \quad (32)$$

and

$$\rho^{-1}f[(\rho^2 f^{-2} - \omega^2)\partial_\rho B_t - \omega\partial_\rho B_\varphi] = \partial_z A_\varphi, \quad (33)$$

the latter relation being the corollary of the first equations of the systems (10) and (11).

Now, combining formulas (28) and (31) in one, and also taking into account that ω assumes a constant value on the horizon, we get for the Komar integral (27) the expression

$$M_K = \frac{1}{8\pi}\int_H [-\omega\partial_z\chi + 2\omega A_t\partial_z(B_t + \omega^{-1}B_\varphi) + 2\omega B_t\partial_z(-A_t - \omega^{-1}A_\varphi)]d\varphi dz, \quad (34)$$

and lastly, after noting that the second and third terms in the integrand of (34) vanish because these contain the derivatives of the potentials Φ_e and Φ_m, both potentials taking constant values on the horizon, we obtain the final expression for the Komar mass

$$M_K = -\frac{1}{8\pi}\int_H \omega\partial_z\chi d\varphi dz, \quad (35)$$

which fully coincides with Tomimatsu's formula (22).

Therefore, the use of the symmetrical representation (25) of the electromagnetic energy-momentum tensor during the calculation of the Komar mass integral leads straightforwardly to the original formula obtained by Tomimatsu in the paper [5]. We think this gives us a nice example of a brilliant physical intuition prevailing over scholastic mathematical estimates.

IV. DISCUSSION AND CONCLUSIONS

The derivation of formula (35) exclusively involving the integrals over the event horizon unequivocally suggests that the whole Komar mass evaluated in this way is located inside the horizon of the black hole. In this respect, it seems remarkable that in the generic expression (27) for the Komar mass the integration is set to be performed over a sphere of infinite radius, thus giving an opportunity to use, if necessary, singular functions during the computational process. The presence of the electromagnetic field obviously complicates the evaluation of the Komar mass, both technically and conceptually, compared with the pure vacuum case since, as we have seen in our previous paper and in the present one, the correct choice of the representation of the electromagnetic energy-momentum tensor is required to avoid the presence of artificial singularities in the dyonic black holes; consequently, in the case when an unsymmetrical representation of the energy-momentum tensor is employed, a very accurate physical interpretation of the results obtained is needed. Thus, the use of the representation (24) in the paper [14] urged the authors of that paper to evaluate the mass integral (27) with the help of the pathological φ component of the potential A_μ. So, it is not a surprise that they could only arrive, within the framework of their approach, at the mass distribution spreading along the whole symmetry axis, and this purely technical result was erroneously claimed by them to be an intrinsic property of the dyonic Kerr-Newman black hole. At the same time, what those authors really did was simply calculating in a not rational way the same value of the Komar mass (located entirely inside the black hole horizon) that otherwise follows directly from Tomimatsu's formula (22) when the symmetrical representation of the energy-momentum tensor is used. It is also clear that since a certain part of the total Komar mass m calculated in the paper [14] for the Kerr-Newman dyon comes from the string singularities, the horizon contribution there differs from the value obtainable by means of Tomimatsu's formula in the absence of Dirac strings, which explains the appearance of the additional electromagnetic term in the mass formula of the paper [14].

Summarizing the results obtained in our short series of two papers, it should be first of all pointed out that the knowledge of only the four-potential A_μ is generically not sufficient for a correct description of the electromagnetic field which also requires the knowledge of the dual four-potential B_μ. In the case of stationary axially symmetric fields, these potentials A_μ and B_μ have the nonzero t and φ components, namely A_t, A_φ, B_t, and B_φ, among which it is the t components A_t and B_t that are the basic key functions defining the physical properties of the electric and magnetic field, respectively, in particular their singularity structure, while the φ components A_φ and B_φ play an auxiliary role in the description of the electromagnetic field, and the singularities of the functions A_φ and B_φ are not characteristic of the proper electric or magnetic field.

We have shown that the use of a specific representation of the electromagnetic energy-momentum tensor is able to provoke erroneous interpretations of the physical properties of dyonic black holes: thus, the choice of the canonical representation (24) for $T^\mu{}_\nu$ in the Komar mass integral leads to the appearance of magnetic Dirac strings [16] as the sources of mass, while the representation of $T^{\mu}{}_{\nu}$ involving only the dual electromagnetic tensor $\tilde{F}^{\mu\nu}$ (see formula (8) of [1]) gives rise to massive Dirac strings generated by the electric charge. This naturally singles out the symmetrical representation (25) of $T^{\mu}{}_{\nu}$ as the most appropriate one for the dyonic solutions because no contributions due to string singularities emerge during the evaluation of the mass integral with the help of (25).

It follows directly from our analysis that Dirac strings (magnetic and electric ones) must be excluded from the physical picture of dyonic spacetimes. Nevertheless, the semi-infinite singularities that are characteristic mathematical attributes of the components A_φ and B_φ in the presence of nonzero magnetic and electric net charges still remain a legitimate part of the general mathematical toolkit and are expected to be taken into account as purely mathematical objects in some calculations involving the functions A_φ and B_φ.

Bearing in mind our basic idea that electromagnetism is necessarily a theory of two electromagnetic potentials, we have slightly enlarged the well-known Ernst formalism by explicitly introducing into it the components B_t and B_φ of the dual electromagnetic potential B_μ. This improves the formalism in two ways. First, it now permits a unified symmetrical treatment of the electric and magnetic fields, in particular the introduction for the first time of the magnetic potential Φ_m which takes a constant value on the horizon, half a century later than Carter's electric potential Φ_e [3]. Second, after our amendment, the Ernst formalism looks not only more complete but also logistically refined: the Ernst auxiliary magnetic function A'_φ, which was needed before just for computing the "genuine" component A_φ of A_μ, and which we identified as the t component of the dual potential B_μ, now plays, alongside A_t, the leading role in the description of the electromagnetic field, while A_φ plays the role of an auxiliary function. This, in our opinion, enriches the Ernst formalism conceptually, as the knowledge of the electromagnetic Ernst potential $\Phi = -A_t + iB_t$ supplies us directly with the explicit expressions of the physical components of the electromagnetic four-potentials determining the intrinsic properties of the electromagnetic field, without the need of finding A_φ. We notice in this respect that it is the component B_t, and not A_φ, that takes part for instance in the definition of the relativistic multipole moments of the electromagnetic field [17–21], which gives us another good illustration of a generic secondary role of the component A_φ in the physical analysis.

We hope that our present paper, as well as the paper [1], presenting some new ideas about the description of magnetic charges, could also be helpful in the search and experimental detection of dyonic sources. Of course, a natural expectation would be that some known elementary particles, in addition to electric charges they have, might also carry magnetic charges, such particles thus being the dyonic objects. Taking the dyonic Kerr-Newman solution considered in Sec. II as the simplest model for a stationary dyon, we observe that the corresponding magnetic dipole moment of the source is aq, while the electric dipole moment is equal to $-ap$, the latter moment arising due to the rotation of the magnetic charge. Therefore, the presence of the electric dipole moment in elementary particles might be considered in principle as an indirect indication that the particles are endowed with nonzero magnetic charges.

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