Dyonic black holes in the theory of two electromagnetic potentials. I

C. J. Ramírez-Valdez,[1,2] H. García-Compeán,[1] and V. S. Manko[1]

[1]*Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, A.P. 14-740, 07000 Ciudad de México, Mexico*

[2]*Theoretical Particle Physics and Cosmology Group, Department of Physics, King's College London, University of London, Strand, London, WC2R 2LS, United Kingdom*

(Received 23 January 2023; accepted 9 February 2023; published 8 March 2023)

In the present paper we argue that it is advantageous to study the dyonic black hole spacetimes within the theory of two electromagnetic potentials, and we use the dyonic Reissner-Nordström solution to demonstrate that the field of the monopole magnetic charge is correctly described by the t component of the dual electromagnetic potential. As a result, the Dirac string associated with the φ component of the usual electromagnetic four-potential becomes just a mathematical object, without any physical content, that arises in some calculations when one employs unsymmetrical representations of the electromagnetic field. We use three different, though equivalent, forms of the electromagnetic energy-momentum tensor to calculate the Komar mass of the Reissner-Nordström black hole, and in one case the Dirac string is linked to the magnetic charge, in another to the electric charge, while the third, symmetrical case, is string free.

DOI: 10.1103/PhysRevD.107.064016

I. INTRODUCTION

The idea that magnetic charges are located "at the end of an unobservable string, which is the line along which the electromagnetic potentials are singular" belongs to Dirac [1], and one may think that it introduces certain physical asymmetry between electricity and magnetism. This unsymmetrical approach to electromagnetism was criticized by Schwinger [2] who advocated the symmetrical viewpoint embodying invariance under charge rotation which leads to the integer quantization condition, as opposed to Dirac's "half-integer" condition. An important ingredient of Schwinger's symmetrical approach was the introduction of a second electromagnetic vector potential defined nonlocally in terms of the field strengths; it looks like Schwinger's remarkable intuition was telling him that the magnetic charge cannot be properly described exclusively by means of the ordinary potential A_μ.

The dyonic black hole solutions within the framework of general relativity were first considered by Carter [3] who introduced the magnetic charge parameter into the Reissner-Nordström (RN) and Kerr-Newman (KN) [4] spacetimes on physical grounds. However, his analysis of the thermodynamic properties of black holes was restricted to the case of zero magnetic charge only, most probably to avoid the problem of singular electromagnetic sources. The dyonic solutions also widely arise in other field theories (see, e.g., [5,6]), which shows generic interest in the magnetic monopoles in modern theoretical physics, thus motivating and justifying efforts aimed at their correct description.

A few years ago, a discussion of the Dirac strings in dyons sprang up in relation to the problem of the mass distribution in the dyonic KN black hole, when in the paper [7] such distribution was assumed to be the same as in the usual electrically charged KN black hole, while in the paper [8] a mathematical evaluation of the mass integral gave rise to a model with two additional semi-infinite massive sources due to Dirac strings. Although the latter model was already criticized for its unphysical features [9], we believe a convincing analytical demonstration of the incorrectness of the entire Dirac-string concept is still needed to clarify and broaden our knowledge about the dyonic spacetimes in general and magnetic charges in particular. In the present paper, the first of a short series of two papers, we consider the static RN dyonic black hole solution which, in our opinion, is the best example of a spacetime for the presentation and illustration of both the basic ideas on the description of magnetic monopoles and the related mathematical calculations, while in the second paper [10] we shall extend our approach to the stationary spacetimes and the dyonic KN black hole. It is precisely the spherical symmetry of the dyonic RN solution that helped us actually realize that the φ component of the potential A_μ is nothing more than an auxiliary mathematical function whose singularity structure should not be ascribed to the RN dyon itself, whereas the field of the magnetic charge is correctly described by the t component of the dual electromagnetic potential B_μ that does share the spherical symmetry of the RN spacetime. The reader will see that the presence or absence of the string term in the mass integrals essentially depends on the choice of the specific representation of the energy-momentum tensor, and one representation even gives rise to a "Dirac string" associated with the electric charge.

Our paper is organized as follows. In the next section we consider the Maxwell equations in the symmetrical form and give three different, though equivalent, representations of the energy-momentum tensor of the electromagnetic field in terms of the usual and dual electromagnetic tensors. Here we also present the dyonic RN solution and calculate two nonzero components of the corresponding dual four-potential B_μ. In Sec. III the Komar mass [11] of the dyon RN solution is calculated in three different ways, clearly demonstrating the auxiliary mathematical character of the components endowed with singular Dirac strings. The results obtained are discussed in Sec. IV.

II. TWO-POTENTIAL FORMULATION OF MAXWELL'S EQUATIONS AND THE DYONIC RN SOLUTION

Motivated by Schwinger's symmetrical approach to the description of dyons [2], we write the vacuum Maxwell equations in the absence of currents in the form

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0, \qquad \partial_\nu(\sqrt{-g}\tilde{F}^{\mu\nu}) = 0, \tag{1}$$

where

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{2}$$

is the dual electromagnetic tensor.

Equations (1) imply the existence of the potentials A_μ and B_μ, such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \tilde{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{3}$$

which in the language of differential forms rewrites as

$$F = dA, \qquad \star F = dB, \tag{4}$$

the star symbol denoting Hodge dual.

The energy-momentum tensor of the electromagnetic field is normally taken in the form

$$T^\mu{}_\nu = \frac{1}{4\pi}\left(F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}\right), \tag{5}$$

and, as will be shown in the next section, it is precisely this representation of $T^\mu{}_\nu$ that leads to the appearance of singular terms due to the magnetic field in the mass integrals. Apart from (5), it is advantageous to have two other equivalent representations of $T^\mu{}_\nu$ which involve the dual electromagnetic tensor $\tilde{F}^{\mu\nu}$. For this purpose we use the identity [12]

$$A^{\mu\alpha}B_{\nu\alpha} - \tilde{A}^{\mu\alpha}\tilde{B}_{\nu\alpha} = \frac{1}{2}\delta^\mu_\nu A^{\alpha\beta}B_{\alpha\beta}, \tag{6}$$

which is valid for any two antisymmetric tensors $A^{\mu\nu}$ and $B_{\mu\nu}$, and their duals $\tilde{A}^{\mu\nu}$ and $\tilde{B}_{\mu\nu}$. Then the second representation of $T^\mu{}_\nu$ takes the symmetrical form

$$T^\mu{}_\nu = \frac{1}{8\pi}(F^{\mu\alpha}F_{\nu\alpha} + \tilde{F}^{\mu\alpha}\tilde{F}_{\nu\alpha}), \tag{7}$$

while for the third representation in terms of the dual tensor only we get

$$T^\mu{}_\nu = \frac{1}{4\pi}\left(\tilde{F}^{\mu\alpha}\tilde{F}_{\nu\alpha} - \frac{1}{4}\delta^\mu_\nu\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}\right). \tag{8}$$

It is our purpose to demonstrate that the field of the magnetic charge is better described by the dual potential B_μ than by A_μ. So, we can take a dyonic RN black hole as the simplest model for our analysis, described by the metric [3]

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$f = 1 - \frac{2m}{r} + \frac{q^2+p^2}{r^2}, \tag{9}$$

with the corresponding electromagnetic field defined by the one-form

$$A = A_t dt + A_\varphi d\varphi = -\frac{q}{r}dt - p\cos\theta d\varphi, \tag{10}$$

where m, q, and p are the parameters of mass, electric charge, and magnetic charge, respectively.

The RN metric (9) represents a static spherically symmetric spacetime of point charges for which we now should calculate the components of the dual potential B_μ. These can be found by solving the following differential equations:

$$\partial_r B_t = -\frac{1}{r^2\sin\theta}\partial_\theta A_\varphi, \qquad \partial_\theta B_t = \frac{f}{\sin\theta}\partial_r A_\varphi,$$
$$\partial_r B_\varphi = -f^{-1}\sin\theta\partial_\theta A_t, \qquad \partial_\theta B_\varphi = r^2\sin\theta\partial_r A_t, \tag{11}$$

which are obtainable from the second equation in (4) by taking the dual of F and by noting that $dB = d(B_\nu dx^\nu)$. From (10) and (11) we readily get

$$B = B_t dt + B_\varphi d\varphi = \frac{p}{r}dt - q\cos\theta d\varphi, \tag{12}$$

where the integration constants have been assigned zero values.

By comparing the expressions (10) and (12), we can see that both A and B have a well-behaved t component, as well as a string φ component. Taking into account the spherical symmetry of the dyonic RN spacetime, it would be plausible to draw a conclusion that the electric field is determined by the t component $A_t = -q/r$ of A, whereas the magnetic field is defined by the t component $B_t = p/r$

of B, both A_t and B_t sharing spherical symmetry of the RN solution. In this respect, having two electromagnetic potentials at hand, the affirmation that the magnetic monopole charge p is described by the string component $A_\varphi = -p\cos\theta$ would be equivalent to affirming that the electric field of a pointlike charge q is defined by $B_\varphi = -q\cos\theta$, with an "electric Dirac string" consisting of two semi-infinite singularities at $\theta = 0, \pi$. Therefore, in view of the auxiliary mathematical role of the components A_φ and B_φ it would be obviously wrong to ascribe the string singularity of the former to the field of the magnetic charge p, and the string singularity of the latter to the field of the electric charge q, on equal grounds.

We shall now illustrate a purely mathematical character of the components A_φ and B_φ by calculating the Komar mass of the dyonic RN solution in three different ways.

III. CALCULATION OF THE KOMAR MASS INTEGRAL

The Komar mass is defined by the surface integral

$$M_K = -\frac{1}{8\pi}\int_\infty \star dk, \tag{13}$$

where $k = g_{tt}dt$ is the covector associated to the timelike Killing vector ∂_t.

Let us first see how (13) can be evaluated straightforwardly just using the metric (9), for which purpose we calculate (13) for some sphere of constant radius r and then take the limit $r \to \infty$. By noting that in our case

$$\star dk = \partial_r g_{tt} r^2 \sin\theta d\theta \wedge d\varphi, \tag{14}$$

we have

$$\begin{aligned} M_r &= -\frac{1}{8\pi}\int_{r=\mathrm{const}} (-\partial_r f) r^2 \sin\theta d\theta d\varphi = \frac{1}{2} r^2 \partial_r f \\ &= m - \frac{q^2+p^2}{r}, \end{aligned} \tag{15}$$

so that

$$M_K = \lim_{r\to\infty} M_r = m. \tag{16}$$

To analyze the contribution of the electromagnetic field into the mass integral (13) in more detail, it is advantageous to rewrite (13) in the form

$$\begin{aligned} M_K = \frac{1}{4\pi}\int_\infty D^\nu k^\mu d\Sigma_{\mu\nu} &= \frac{1}{4\pi}\int_{\partial\mathcal{M}} D^\nu k^\mu d\Sigma_{\mu\nu} \\ &+ \frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu \end{aligned} \tag{17}$$

by means of Ostrogradsky's formula, where $k^\mu = \delta^\mu_t$ (here and below we have adopted some of the notations and conventions of the paper [8] for the reader's convenience).

If $\partial\mathcal{M}$ is chosen as a sphere of constant radius r, then the first integral on the right-hand side of (17) is just M_r in (15), and in particular if $r = r_+ = m + \sqrt{m^2 - q^2 - p^2}$, r_+ being the radius of the event horizon (the case that is of interest to us), then

$$\frac{1}{4\pi}\int_H D^\nu k^\mu d\Sigma_{\mu\nu} = m - \frac{q^2+p^2}{r_+}. \tag{18}$$

Following [8], we now introduce the electromagnetic field explicitly into the "geometrical" formula for M_K by writing the bulk integral from (17) in the form

$$\frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu = -2\int_{\mathcal{M}} T^\mu{}_\nu k^\nu dS_\mu \tag{19}$$

with the aid of the well-known relations

$$D_\nu D^\nu k^\mu = -R^\mu{}_\nu k^\nu = -8\pi T^\mu{}_\nu k^\nu. \tag{20}$$

Below we will calculate the integral on the right-hand side of (19) for three different (but equivalent) representations, (5), (7), and (8), of the energy-momentum tensor $T^\mu{}_\nu$. Of course, in all three cases we must get the same result $(q^2 + p^2)/r_+$, as the integrals (16) and (18) are known.

A. The canonical representation

Note that in this representation, given by formula (5), the bulk integral (19) will contain the function A_φ explicitly after the Ostrogradsky theorem is applied for converting (19) into the surface integral, and hence the contribution of the "magnetic Dirac string" must be taken into account. Bearing this in mind, we get

$$\begin{aligned} \frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu &= -2\int_{\mathcal{M}} T^t{}_t\sqrt{-g}d^3x \\ &= -\frac{1}{4\pi}\int_{\mathcal{M}} (F^{ta}F_{ta} - F^{\varphi a}F_{\varphi a})\sqrt{-g}d^3x \\ &= \frac{1}{4\pi}\int_{\mathcal{M}} \partial_a[\sqrt{-g}(F^{ta}A_t - F^{\varphi a}A_\varphi)]d^3x \\ &= \frac{1}{4\pi}\int_{\Sigma_a} (F^{ta}A_t - F^{\varphi a}A_\varphi)d\Sigma_a \\ &= \frac{1}{4\pi}\int_H F^{tr}A_t d\Sigma_r - \frac{1}{4\pi}\int_S F^{\varphi\theta}A_\varphi d\Sigma_\theta, \end{aligned} \tag{21}$$

($a \in \{r, \theta\}$) where "H" refers to the horizon and "S" refers to the string. In the last step we have taken into account that

$$\int_S F^{t\theta}A_t d\Sigma_\theta = 0, \qquad \int_H F^{\varphi r}A_\varphi d\Sigma_r = 0, \tag{22}$$

because $F^{t\theta} = 0$ and $F^{\varphi r} = 0$. Finally, we readily obtain

$$\int_H F^{tr} A_t d\Sigma_r = \int_H F_{tr} A_t r^2 \sin\theta d\theta d\varphi = 4\pi q^2/r_+,$$
$$\int_S F^{\varphi\theta} A_\varphi d\Sigma_\theta = 2\lim_{\theta\to\pi} \int_{r_+}^{\infty} \int_0^{2\pi} F_{\varphi\theta} A_\varphi \frac{1}{r^2 \sin\theta} dr d\varphi$$
$$= -4\pi p^2/r_+, \quad (23)$$

which leads to $(q^2 + p^2)/r_+$ for (21).

Note that in this representation of $T^\mu{}_\nu$ the contribution of the electric charge into the bulk integral (21) comes from the horizon, and the contribution of the magnetic charge comes from the string.

B. The dual representation

This case, defined by formula (8), is fully analogous to the previous one, with the roles of the electric and magnetic fields interchanged:

$$\begin{aligned} \frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu &= -\frac{1}{4\pi}\int_{\mathcal{M}} (\tilde{F}^{ta}\tilde{F}_{ta} - \tilde{F}^{\varphi a}\tilde{F}_{\varphi a})\sqrt{-g}d^3x \\ &= \frac{1}{4\pi}\int_{\mathcal{M}} (\tilde{F}^{ta}\partial_a B_t - \tilde{F}^{\varphi a}\partial_a B_\varphi)\sqrt{-g}d^3x \\ &= \frac{1}{4\pi}\int_{\mathcal{M}} \partial_a[\sqrt{-g}(\tilde{F}^{ta}B_t - \tilde{F}^{\varphi a}B_\varphi)]d^3x \\ &= \frac{1}{4\pi}\int_{\Sigma_a} (\tilde{F}^{ta}B_t - \tilde{F}^{\varphi a}B_\varphi)d\Sigma_a \\ &= \frac{1}{4\pi}\int_H \tilde{F}^{tr}B_t d\Sigma_r - \frac{1}{4\pi}\int_S \tilde{F}^{\varphi\theta}B_\varphi d\Sigma_\theta, \end{aligned} \quad (24)$$

where we have taken into account that

$$\int_S \tilde{F}^{t\theta} B_t d\Sigma_\theta = 0 \quad \text{and} \quad \int_H \tilde{F}^{\varphi r} B_\varphi d\Sigma_r = 0. \quad (25)$$

The evaluation of the last two integrals in (24) yields

$$\int_H \tilde{F}^{tr} B_t d\Sigma_r = 4\pi p^2/r_+, \qquad \int_S \tilde{F}^{\varphi\theta} B_\varphi d\Sigma_\theta = -4\pi q^2/r_+, \quad (26)$$

and in this representation of the energy-momentum tensor it is the electric charge that develops an "electric Dirac string," so that this time the electrostatic contribution into the bulk integral (24) comes from the string, while the contribution of the magnetic charge comes from the horizon!.

C. The symmetrical representation

In the representation (7) only the well-behaved components of the electromagnetic potentials are involved in the calculations of the bulk integral (19), so that no any auxiliary string contribution arises during the application of Ostrogradsky's theorem converting the bulk integral into the surface integral:

$$\begin{aligned} \frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu &= -\frac{1}{4\pi}\int_{\mathcal{M}} (F^{ta}F_{ta} + \tilde{F}^{ta}\tilde{F}_{ta})\sqrt{-g}d^3x \\ &= \frac{1}{4\pi}\int_{\mathcal{M}} (F^{ta}\partial_a A_t + \tilde{F}^{ta}\partial_a B_t)\sqrt{-g}d^3x \\ &= \frac{1}{4\pi}\int_{\mathcal{M}} \partial_a[\sqrt{-g}(F^{ta}A_t + \tilde{F}^{ta}B_t)]d^3x \\ &= \frac{1}{4\pi}\int_{\Sigma_a} (F^{ta}A_t + \tilde{F}^{ta}B_t)d\Sigma_a \\ &= \frac{1}{4\pi}\int_H F^{tr}A_t d\Sigma_r + \frac{1}{4\pi}\int_H \tilde{F}^{tr}B_t d\Sigma_r, \end{aligned} \quad (27)$$

and evaluation of the last two integrals readily gives

$$\int_H F^{tr} A_t d\Sigma_r = 4\pi q^2/r_+, \qquad \int_H \tilde{F}^{tr} B_t d\Sigma_r = 4\pi p^2/r_+. \quad (28)$$

Therefore, in the symmetrical representation of $T^\mu{}_\nu$, the calculation of the Komar mass of the dyonic RN source reduces to evaluation of the surface integrals over the event horizon only. As we have shown, the choice of the particular representation does not alter the final result when the singularity structure of the functions involved in the concrete calculational scheme is carefully taken into account.

IV. DISCUSSION AND CONCLUSIONS

The analysis carried out in the previous two sections clearly shows that the problem of the Dirac string associated in the literature with the magnetic charge is actually an artificial mathematical issue arising as a result of a wrong identification of the potential describing the field of the magnetic monopole. Thus we have seen that the same contributions into the mass integral can be made by the horizon or string terms, and these are interrelated as follows:

$$\begin{aligned} \int_H F^{tr} A_t d\Sigma_r &= -\int_S \tilde{F}^{\varphi\theta} B_\varphi d\Sigma_\theta = 4\pi q \Phi_e, \\ \int_H \tilde{F}^{tr} B_t d\Sigma_r &= -\int_S F^{\varphi\theta} A_\varphi d\Sigma_\theta = 4\pi p \Phi_m, \end{aligned} \quad (29)$$

where we have introduced the horizon values of the electric and magnetic potentials Φ_e and Φ_m by the well-known formulas

$$\Phi_e = q/r_+, \qquad \Phi_m = p/r_+, \tag{30}$$

and now it is manifest that Φ_m is just the dual component B_t evaluated on the horizon.

It should also be stressed that the distribution of the Komar mass along the horizon and the magnetic (or electric) string singularity of the component A_φ (or B_φ) appearing during the computation of the mass integral (13) is just a mathematical abstraction that should not be interpreted as reflecting the real physical distribution of mass in the dyonic RN black hole, which is of course spherically symmetric. In this respect it would probably be worth drawing analogy with the static vacuum Weyl gravitational fields which all satisfy the Laplace equation $\Delta\psi = 0$ for an auxiliary function ψ, but the real physical field is $f = \exp\psi$ which apparently has a different singularity structure than ψ.

A curious feature of the bulk integral (19) additionally pointing at its auxiliary technical character is that it does not seem to be actually involved in the Smarr mass formula [13], the latter important relation following directly from the surface integral (18) evaluated on the horizon. Indeed, after rewriting (18), on the one hand, in terms of the potentials Φ_e and Φ_m as

$$\frac{1}{4\pi}\int_H D^\nu k^\mu d\Sigma_{\mu\nu} = m - q\Phi_e - p\Phi_m, \tag{31}$$

and recalling, on the other hand, that, as was shown by Carter [3],

$$\frac{1}{4\pi}\int_H D^\nu k^\mu d\Sigma_{\mu\nu} = \frac{\kappa}{4\pi}\mathcal{A}, \tag{32}$$

where κ is the surface gravity and $\mathcal{A}$ the area of the event horizon, we immediately arrive at the Smarr relation verified by the dyonic RN black hole

$$m = \frac{\kappa}{4\pi}\mathcal{A} + q\Phi_e + p\Phi_m. \tag{33}$$

As a final remark, it would probably be worth mentioning that our results suggesting the nonexistence of magnetic and electric Dirac strings are particularly important in application to the systems of many dyonic black holes, for which a correct calculation of individual Komar masses would be practically impossible in the presence of numerous string singularities. Our symmetrical approach in which the individual masses are evaluated on the horizons, and hence are entirely located inside the horizons, does not have this kind of problem, confirming for instance the definition of the Komar mass in a binary system of magnetically charged Reissner-Nordström black holes [14].

ACKNOWLEDGMENTS

We would like to thank the referee for valuable suggestions. This work was partially supported by CONACyT of Mexico and by “Secretaría de Educación, Ciencia, Tecnología e Innovación de la Ciudad de México (SECTEI)” of Mexico City.

[1] P. A. M. Dirac, The theory of magnetic poles, Phys. Rev. **74**, 817 (1948).
[2] J. Schwinger, Magnetic charge and the charge quantization condition, Phys. Rev. D **12**, 3105 (1975).
[3] B. Carter, Black hole equilibrium states, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach Science Publishers, New York, 1973), p. 56.
[4] E. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating charged mass, J. Math. Phys. (N.Y.) **6**, 918 (1965).
[5] M. Cvetič and A. A. Tseytlin, Solitonic strings and BPS saturated dyonic black holes, Phys. Rev. D **53**, 5619 (1996).
[6] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, Killing horizons: Negative temperatures and entropy super-additivity, Phys. Rev. D **98**, 106015 (2018).
[7] V. S. Manko and H. García-Compeán, Smarr formula for black holes endowed with both electric and magnetic charges, Classical Quantum Gravity **35**, 064001 (2018). [arXiv:1506.03870 [gr-qc]].
[8] G. Clément and D. Gal'tsov, On the Smarr formula for rotating dyonic black holes, Phys. Lett. B **773**, 290 (2017).
[9] H. García-Compeán, V. S. Manko, and E. Ruiz, Comments on two papers of Clément and Gal'tsov, arXiv:2006.00793.
[10] H. García-Compeán, V. S. Manko, and C. J. Ramírez-Valdez, following paper, Dyonic black holes in the theory of two electromagnetic potentials. II, Phys. Rev. D **107**, 064017 (2023).
[11] A. Komar, Covariant conservation laws in general relativity, Phys. Rev. **113**, 934 (1959).
[12] J. A. Wheeler, *Geometrodynamics* (Academic Press Inc., New York, 1962), p. 239.
[13] L. Smarr, Mass Formula for Kerr Black Holes, Phys. Rev. Lett. **30**, 71 (1973).
[14] V. S. Manko, E. Ruiz, and J. Sánchez-Mondragón, Analogs of the double-Reissner-Nordström solution in magnetostatics and dilaton gravity: Mathematical description and basic physical properties, Phys. Rev. D **79**, 084024 (2009).