Topology of black hole thermodynamics in Lovelock gravity

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In this work, we present a convenient method to perform the topological analysis of black hole thermodynamics. Utilizing the spinodal curve, thermodynamic critical points of a black hole are endowed with a topological quantity, Brouwer degree, which reflects intrinsic properties of the system under smooth deformations. Particularly, in our setup it can be easily calculated without an exact solution of critical points. This enables us to conveniently investigate the topological transition between different thermodynamic systems, and give a topological classification for them. In this framework, topology of Lovelock AdS black holes with spherical horizon geometry is explored. Results show that charged black holes in arbitrary dimensions can be classified into the same topology class, whereas the $d = 7$ and $d \geq 8$ uncharged black holes are in different topology classes. Moreover, we revisit the relation between different phase structures of these black holes from the viewpoint of topology. Some general topological properties of critical points are also discussed.

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I. INTRODUCTION

The study of black hole thermodynamics continues to be one of the most exciting areas in gravitational theory. Of especial interest are the phase transitions in asymptotically anti–de Sitter (AdS) black holes, motivated by the straightforward definition of thermodynamic equilibrium and the possible interpretation in the context of gauge/gravity duality [1–3]. Early examples include the Hawking-Page phase transition occurred between thermal radiation and large AdS black holes [4], which can be interpreted as the confinement/deconfinement phase transition of gauge field [5], and the van der Waals (VdW) like phase transition found between charged small and large AdS black holes [6–9].

A topic of active research in recent years is the interpretation of the cosmological constant as the thermodynamic pressure [10–12]. This perspective, known as the extended phase space, has shed new insights into the thermodynamics and phase transitions of AdS black holes, including understanding the Hawking-Page phase transition as a solid/liquid transition [13], and strengthening the analogy between van der Waals fluids and charged AdS black holes [14–17]. Plenty of novel phase behaviors are also discovered in this framework, such as the reentrant large/small/large black hole phase transition [18,19], triple points where small/large/intermediate black holes can coexist [20,21], superfluid black holes [22] admitting a λ-line phase transition reminiscent of the superfluid transition in liquid ^{4}He, etc.

In spite of phase transitions differing in their forms, critical points always emerge and record crucial thermodynamic information. Locally, a second-order phase transition occurs at the critical point. Critical exponents can be derived from the behavior of physical quantities near this point, which are believed to be universal and related to general features of the physical system [23]. For black hole systems, in most cases, they were found to have the standard set of critical exponents expected from mean field theory [24]. A special case is the isolated critical point from Lovelock gravity, which was shown to possess nonmean-field exponents [25]. Globally, the presence of multiple critical points[1] generally implies intriguing phase behaviors, such as the reentrant phase transition (typically two critical points [18,19]), triple point (typically three critical points [20,21]), and in particular the superfluid black hole which admits infinite critical points [22]. Therefore, discovering the nature of critical points is quite valuable, which can provide additional insight into black hole thermodynamics and may also reveal more features about quantum gravity.

To this aim, topology has been introduced to the critical points of black hole thermodynamics [26]. This approach

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[1]Here we take the unphysical critical points (with negative pressure or temperature) into account.

begins by constructing a two-dimensional vector field with zero points designed to be critical points of the thermodynamic system. Following Duan's ϕ-mapping topological current theory [27,28], one can assign a *topological charge* for each critical point to reflect its local property. Concretely, the first-order phase transition can extend from the critical point with negative topological charge, but not from the one with positive topological charge. Moreover, topological charges at critical points can be added together to construct a *total topological charge* for the thermodynamic system. This allows us to determine the global property of the thermodynamic system, and classify various thermodynamic systems into a few classes. Thermodynamics of several AdS black holes has been revisited and classified utilizing such an approach [26,29,30]. It is also interesting to see that the isolated critical point can be interpreted as a topological phase transition of a "vortex/antivortex pair" [31].

In this paper, we present an alternative method to perform the topological analysis of thermodynamics and phase transitions of black holes. This method relies on a topological quantity *Brouwer degree* [32], which is invariant under smooth deformations of the system and reflects system's intrinsic properties. Utilizing features of *spinodal curve* [17], including its continuous differentiability and relation to critical points, we construct such a topological quantity for the thermodynamic system. In particular, as we will see, this quantity can be directly calculated by using a mathematical formula without an exact solution of critical points. This enables us to conveniently probe the topological transition between different thermodynamic systems, and give them a topological classification. We shall employ this approach to explore the topological properties of AdS black holes in Lovelock gravity [33,34]. On the one hand, these black holes possess rich phase behaviors [35–37], providing an excellent arena for discovering topological features of black hole thermodynamics. On the other hand, the topology of black hole thermodynamics in Gauss-Bonnet gravity, namely the 2nd-order Lovelock gravity, has been investigated [29]. It would be interesting to examine whether the higher curvature gravity, such as the 3rd-order Lovelock gravity, would affect the topological properties of black hole thermodynamics.

The structure of the paper is as follows. In Sec. II, we give a brief introduction to the Brouwer degree. Then, by use of the spinodal curve, we relate the Brouwer degree to balck hole thermodynamics. In Sec. III, the topology of Lovelock black holes' thermodynamics is investigated. The charged case and uncharged case are discussed separately. In Sec. IV, further discussions on the relation between topological degree (charge) and critical point are given. Finally, we summarize and discuss our results in Sec. V.

II. BROUWER DEGREE AND SPINODAL CURVE

Consider an open and bounded set $X \subset \mathbb{R}^n$ with (at least) a C^1-smooth map $f: X \to \mathbb{R}^n$. Let $y \in f \backslash f(\partial X)$ be a regular value of f, then the set $f^{-1}(y) = \{x_1, x_2, \cdots\}$ with $x_n \in X$ has a finite number of points, such that $f(x_n) = y$. Suppose the Jacobian $J(x_n) = \det(\partial f / \partial x_n) \neq 0$, one can define a topological quantity, called the Brouwer degree of the map [32]

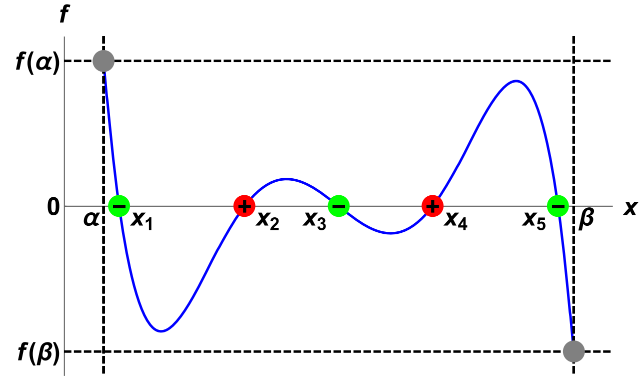


FIG. 1. A continuously differentiable function $f(x)$ with $x \in [\alpha, \beta]$, and $f(\alpha) \neq 0$, $f(\beta) \neq 0$.

$$\deg(f, X, y) = \sum_{x_n \in f^{-1}(y)} \mathrm{sgn} J(x_n), \tag{1}$$

where sgn denotes the sign function

$$\mathrm{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}. \tag{2}$$

This quantity is a topological characteristic of the map itself, which does not depend on the choice of the regular value y and remains constant under continuous deformations of the map.

Now we focus on the one-dimensional case. Let $X = [\alpha, \beta] \subset \mathbb{R}$ and $f: X \to \mathbb{R}$ be a continuously differentiable function with $f(\alpha) \neq 0$ and $f(\beta) \neq 0$, and we choose $y = 0$ such that $\{x_n\} = f^{-1}(y)$ be the set of zeroes of f, as shown in Fig. 1. The nonzero Jacobian $J(x_n)$ now restricts $f'(x_n) \neq 0$, i.e., 0 is a regular value of f. Then the Brouwer degree (1) can be expressed as

$$\deg(f, X, 0) = \sum_{x_n \in f^{-1}(0)} \mathrm{sgn} f'(x_n). \tag{3}$$

Following Refs. [38,39], we can associate a topological charge $Q_n \equiv \mathrm{sgn} f'(x_n)$ for each zero point, and then sum over all contributions to construct the total topological charge (degree)

$$Q_{\mathrm{total}} = \sum_n Q_n. \tag{4}$$

Note that the topological charge Q_n can have two values, -1 or $+1$, according to the slope at the zero point x_n, as shown in Fig. 1. The green and red points represent the zero points with negative and positive topological charges respectively. In the case of Fig. 1, the total topological charge can be easily read as $Q_{\text{total}} = -1$.

Moreover, we notice that there is another simple way to calculate the total topological charge, which is accomplished by using the following formula [40]:

$$\frac{1}{2}[\text{sgn}f(\beta) - \text{sgn}f(\alpha)] = \sum_{x_n \in f^{-1}(0)} \text{sgn}f'(x_n). \tag{5}$$

From this equation, we can directly obtain the total topological charge by studying the asymptotic behavior of f, even if the zero points are undetermined. For the case of Fig. 1, we have $Q_{\text{total}} = (-1-1)/2 = -1$, which is consistent with the result obtained before. Note that Eq. (5) must be used with caution, which can only be applicable for a continuously differentiable function with a nonzero boundary, see Ref. [40] for more discussions.

To apply this tool to the topological study of thermodynamics, one has to endow the function f and its zero points with specific physical significance. Recall that for a general state equation $T = T(S, P, z^i)$, the critical point can be identified with

$$\left(\frac{\partial T}{\partial S}\right)_{P,z^i} = 0, \qquad \left(\frac{\partial^2 T}{\partial S^2}\right)_{P,z^i} = 0. \tag{6}$$

Using the first equation, one can eliminate the parameter P, and then get the spinodal curve $T_{\text{sp}} = T(S, z^i)$ [17]. Now the condition (6) turns into

$$(\partial_S T_{\text{sp}})_{z^i} = 0. \tag{7}$$

Hence, we can let $f \equiv (\partial_S T_{\text{sp}})_{z^i}$, and thus zero points of f exactly become critical points of the thermodynamic system. In this context, we can endow a topological charge for each critical point, and a total topological charge for the system to investigate the global properties.

To make this claim more clear, we first take a simple example—the charged AdS black hole system. Identifying the cosmological constant as the thermodynamic pressure P, the state equation in d-dimensional spacetime reads [41]

$$T = \frac{16\pi r_{\text{h}}^2\left(P - \frac{2\pi q^2 r_{\text{h}}^{4-2d}}{\omega_{d-2}^2}\right) + d(d-5) + 6}{4\pi(d-2)r_{\text{h}}}, \tag{8}$$

where r_{h} is the horizon radius, q the electric charge, and $\omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ the volume of a unit d-sphere. After simple calculation, the spinodal curve can be obtained as

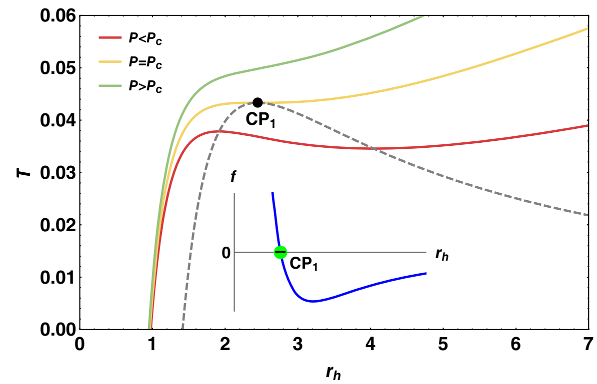


FIG. 2. Isobaric curves and spinodal curve (gray dashed line) for the charged AdS black hole. Inset: $f \equiv (\partial_{r_{\text{h}}} T_{\text{sp}})_q$ vs r_{h} diagram. We have set $d = 4$ and $q = 1$.

$$T_{\text{sp}} = \frac{1}{2\pi r_{\text{h}}}\left(-\frac{32\pi^2 q^2 r_{\text{h}}^{6-2d}}{\omega_{d-2}^2} + d - 3\right). \tag{9}$$

Taking $d = 4$, $q = 1$ for example, we show the isobaric curves and the spinodal curve in Fig. 2. It is clear that the critical point CP_1 is exactly the extreme point of spinodal curve. Note that the condition for critical point $(\partial_S T_{\text{sp}})_q = 0$ is equal to $(\partial_{r_{\text{h}}} T_{\text{sp}})_q = 0$.

Now we construct the function

$$f \equiv (\partial_{r_{\text{h}}} T_{\text{sp}})_q = \frac{16\pi(2d-5)q^2 r_{\text{h}}^{4-2d}}{\omega_{d-2}^2} - \frac{d-3}{2\pi r_{\text{h}}^2}, \tag{10}$$

and it is obvious that this function is continuously differentiable. It is also easy to see that for any $d \geq 4$ and $q > 0$,

$$f(r_{\text{h}} \to 0^+) \sim \frac{(2d-5)q^2 r_{\text{h}}^{4-2d}}{\omega_{d-2}^2} \to +\infty, \tag{11}$$

$$f(r_{\text{h}} \to +\infty) \sim -\frac{d-3}{r_{\text{h}}^2} \to 0^-, \tag{12}$$

and thus f admits a nonzero boundary. For these features, we can directly use Eq. (5) to calculate the total topological charge:

$$\begin{aligned} Q_{\text{total}} &= \frac{1}{2}[\text{sgn}f(r_{\text{h}} \to +\infty) - \text{sgn}f(r_{\text{h}} \to 0^+)] \\ &= \frac{1}{2}(-1-1) = -1. \end{aligned} \tag{13}$$

Therefore we recover the $d = 4$, $q = 1$ result obtained in Ref. [26] using a different method, and further show that for any $d \geq 4$ and $q > 0$, the charged AdS black holes share the same topological charge -1. This also implies that there is *at least* one critical point presented in the system, otherwise, a zero topological charge will be obtained. Note that we capture this information without actually solving Eq. (7).

To obtain the topological charge for each critical point, one can study the behavior of function f at zero points. In the inset of Fig. 2, we display the $d=4$, $q=1$ case. Obviously, there is a zero point with negative slope, indicating a critical point CP_1 with the topological charge

$$Q_{\mathrm{CP}_1} = \mathrm{sgn} f'(\mathrm{CP}_1) = -1. \tag{14}$$

In fact, using the condition (6), one can show that charged AdS black holes only have one critical point, and thus $Q_{\mathrm{CP}_1} \equiv Q_{\mathrm{total}} = -1$. Moreover, since both $f = (\partial_{r_h} T_{\mathrm{sp}})_q = 0$ and $f' = (\partial_{r_h, r_h} T_{\mathrm{sp}})_q < 0$ are satisfied, we conclude that a critical point with negative topological charge denotes a maximum point of spinodal curve. As shown in Fig. 2, near such a point, the stable black hole branches ($C_P = T(\partial_S T)^{-1}_{P,z^i} > 0$) are on both sides, while the unstable black hole branch is in the middle, and one can draw a first-order phase transition line in $T-S$ plane between these stable black hole branches using Maxwell's equal area law. However, whether such a phase transition can actually occur should be carefully examined by the free energy. This will be discussed further in Sec. IV.

On the contrary, a critical point with positive topological charge denotes a minimum point of spinodal curve. Near this point, the unstable black hole branches are on both sides, whereas the stable black hole branch is in the middle, and thus one cannot draw a first-order phase transition line.

It is worth noting that in the above discussion, we suppose $f'(x_n) \neq 0$. In this case, it is obvious that the adjacent critical points must have opposite topological charges, as shown in Fig. 1. Under the continuous deformations of f, if $f'(x_j) = 0$ is satisfied for some zero point x_j, there will be a critical point with zero slope. One can treat this point as the result of the *annihilation* of adjacent critical points, and endow it with a zero topological charge. Since the total topological charge does not change in this process, Eq. (5) is still applicable. Moreover, similar topological discussion can also be done in the $P-V$ criticality and $q-\Phi$ criticality, see Appendixes A and B for more details.

So far, we have constructed a convenient method to investigate the topology of black hole thermodynamics. The physical significance of the topological charge has also been clearly described in the context of charged AdS black holes. In the following sections, we shall focus on a more complex system—Lovelock black holes, in which more interesting topological properties will be disclosed.

III. THERMODYNAMIC TOPOLOGY OF LOVELOCK BLACK HOLES

The action of the Lovelock gravity in d-dimensional spacetime with a Maxwell field is given by [33]

$$I = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(\sum_{k=0}^{k_{\max}} \hat{\alpha}_{(k)} \mathcal{L}^{(k)} - 4\pi G_N F_{ab} F^{ab} \right), \tag{15}$$

where $\hat{\alpha}_{(k)}$ are the Lovelock coupling constants and $\mathcal{L}^{(k)}$ are the $2k$-dimensional Euler densities, given by the contraction of k powers of the Riemann tensor

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{(k)} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_k b_k}{}^{c_k d_k}, \tag{16}$$

where $\delta^{(k)} = \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k}$ is totally antisymmetric in both sets of indices. The $\mathcal{L}^{(0)}$, $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ correspond to the cosmological constant term, Einstein–Hilbert term and quadratic Gauss–Bonnet term, respectively. The integer $k_{\max} = [\frac{d-1}{2}]$ restricts that only for $d > 2k$, $\mathcal{L}^{(k)}$ contributes to the equations of motion.

Considering the charged AdS black holes in Lovelock gravity with static spherically symmetric, the ansatz is given by

$$ds^2 = -g(r) dt^2 + g(r)^{-1} dr^2 + r^2 d\Omega^2_{(\kappa)d-2}, \tag{17}$$

$$F = \frac{Q}{r^{d-2}} dt \wedge dr, \tag{18}$$

where Q denotes the electric charge, $d\Omega^2_{(\kappa)d-2}$ is the line element of a $(d-2)$-dimensional compact space with volume denoted by $\Sigma^{(\kappa)}_{d-2}$ and constant curvature $(d-2)(d-3)\kappa$, in which $\kappa = 0, +1, -1$ correspond to flat, spherical, hyperbolic black hole horizon geometries respectively.

For this ansatz, the field equations derived from Eq. (15) can be reduced to [42–44]

$$\sum_{k=0}^{k_{\max}} \alpha_k \left(\frac{\kappa - g}{r^2} \right)^k = \frac{16\pi G_N M}{(d-2)\Sigma^{(\kappa)}_{d-2} r^{d-1}} - \frac{8\pi G_N Q^2}{(d-2)(d-3)} \frac{1}{r^{2d-4}}. \tag{19}$$

where

$$\alpha_0 = \frac{\hat{\alpha}_{(0)}}{(d-1)(d-2)}, \qquad \alpha_1 = \hat{\alpha}_{(1)},$$
$$\alpha_k = \hat{\alpha}_{(k)} \prod_{n=3}^{2k} (d-n) \quad \text{for } k \geq 2, \tag{20}$$

and M is the ADM mass of the black hole. To avoid the possible solutions with naked singularities [45] and conform to the string tension explanation in heterotic string theory, we focus on the $\alpha_k > 0$ case in the followings. Specially, to recover general relativity in the small curvature limit, α_1 is set to be 1.

By using the Hamiltonian formalism, the thermodynamic quantities of the black hole, including the black hole mass M, the temperature T, the entropy S and the electric potential Φ, are calculated as [44]

$$M = \frac{\Sigma_{d-2}^{(\kappa)}(d-2)}{16\pi G_N}\sum_{k=0}^{k_{\max}}\alpha_k\kappa^k r_+^{d-1-2k} + \frac{\Sigma_{d-2}^{(\kappa)}}{2(d-3)}\frac{Q^2}{r_+^{d-3}}, \tag{21}$$

$$T = \frac{1}{4\pi r_+ D(r_+)}\left[\sum_k \kappa\alpha_k(d-2k-1)\left(\frac{\kappa}{r_+^2}\right)^{k-1} - \frac{8\pi G_N Q^2}{(d-2)r_+^{2(d-3)}}\right], \tag{22}$$

$$S = \frac{\Sigma_{d-2}^{(\kappa)}(d-2)}{4G_N}\sum_{k=0}^{k_{\max}}\frac{k\kappa^{k-1}\alpha_k r_+^{d-2k}}{d-2k}, \tag{23}$$

$$\Phi = \frac{\Sigma_{d-2}^{(\kappa)}Q}{(d-3)r_+^{d-3}}, \tag{24}$$

where r_+ is the horizon radius determined as the largest root of $f(r_+) = 0$, and

$$D(r_+) = \sum_{k=1}^{k_{\max}} k\alpha_k(\kappa r_+^{-2})^{k-1}. \tag{25}$$

Identifying the cosmological constant $\Lambda = -\hat{\alpha}_0/2$ as the thermodynamic pressure,

$$P = -\frac{\Lambda}{8\pi G_N} = \frac{\hat{\alpha}_0}{16\pi G_N}, \tag{26}$$

the extended first law of black hole thermodynamics reads [36,46]

$$\delta M = T\delta S + V\delta P + \sum_{k=1}^{K}\Psi^{(k)}\delta\alpha_k, \tag{27}$$

where $\Psi^{(k)}$ represents the quantity conjugate to α_k,

$$\Psi^{(k)} = \frac{\Sigma_{d-2}^{(\kappa)}(d-2)}{16\pi G_N}\kappa^{k-1}r_+^{d-2k}\left[\frac{\kappa}{r_+} - \frac{4\pi k T}{d-2k}\right], \tag{28}$$

and V is the thermodynamic volume conjugate to P,

$$V = \frac{16\pi G_N \Psi^{(0)}}{(d-1)(d-2)}. \tag{29}$$

Working in an ensemble that fixes α_k for $k \geq 1$, in terms of the following dimensionless quantities [36]:

$$Q = \frac{q}{\sqrt{2}}\alpha_3^{\frac{d-3}{4}}, \qquad \alpha = \frac{\alpha_2}{\sqrt{\alpha_3}}, \qquad m = \frac{16\pi M}{(d-2)\Sigma_{d-2}^{(\kappa)}\alpha_3^{\frac{d-3}{4}}},$$

$$r_+ = v\alpha_3^{\frac{1}{4}}, \qquad T = \frac{t\alpha_3^{-\frac{1}{4}}}{d-2}, \qquad p = 4\sqrt{\alpha_3}P, \tag{30}$$

one can reinterpret (22) as the state equation for 3rd-order Lovelock $U(1)$ charged black holes,

$$t = \frac{1}{4\pi v(2\alpha\kappa v^2 + v^4 + 3)}\Big[-4\pi q^2 v^{10-2d} + (d-7)(d-2)\kappa + \alpha(d-5)(d-2)v^2 + (d-3)(d-2)\kappa v^4 + 4p\pi v^6\Big], \tag{31}$$

and the condition (6) for critical points now becomes

$$\left(\frac{\partial t}{\partial v}\right)_{p,q,\alpha} = 0, \qquad \left(\frac{\partial^2 t}{\partial v^2}\right)_{p,q,\alpha} = 0. \tag{32}$$

The possible phase transitions can be investigated based on the behavior of the Gibbs free energy in the “canonical ensemble,” given by [46,47]

$$G = M - TS = G(P, T, Q, \alpha_1, \ldots, \alpha_{k_{\max}}). \tag{33}$$

with the dimensionless counterpart [36]

$$\begin{aligned} g(t,p,q,\alpha) = \frac{1}{\Sigma_{d-2}^{(\kappa)}}\alpha_3^{\frac{3-d}{4}}G = &-\frac{1}{16\pi(3+2\alpha\kappa v^2+v^4)}\Bigg[\frac{4\pi p v^{d+3}}{(d-1)(d-2)} - \kappa v^{d+1} + \frac{24\pi\kappa p\alpha v^{d+1}}{(d-1)(d-4)} - \frac{\alpha v^{d-1}(d-8)}{d-4} \\ &+ \frac{60\pi p v^{d-1}}{(d-1)(d-6)} - \frac{2\kappa\alpha^2 v^{d-3}(d-2)}{d-4} + \frac{4\kappa v^{d-3}(d+3)}{d-6} - \frac{3\alpha v^{d-5}(d-2)(d-8)}{(d-4)(d-6)} - \frac{3\kappa v^{d-7}(d-2)}{d-6}\Bigg] \\ &+ \frac{q^2}{4(3+2\alpha\kappa v^2+v^4)(d-3)v^{d-3}}\left[\frac{v^4(2d-5)}{d-2} + \frac{2\alpha\kappa(2d-7)v^2}{d-4} + \frac{3(2d-9)}{d-6}\right]. \end{aligned} \tag{34}$$

The stable state corresponds to the global minimum of this quantity for its fixed parameters t, p, q, and α.

For $\kappa = 0$, one can find there is no critical point, which means that any planar black holes of higher-order Lovelock gravity in arbitrary dimensions do not exhibit critical behavior. Thus there is no urge to study their thermodynamic topology. On the other hand, the thermodynamic topology for the $\kappa = -1$ case has been investigated and an intriguing result has been given; the isolated critical point—a peculiar thermodynamic critical point that occurs in the phase diagram of hyperbolic black holes, can be interpreted as a topological phase transition of a vortex/antivortex pair [31]. In what follows, we shall concentrate on the thermodynamic topology of $\kappa = +1$ case, in which rich phase behaviors such as triple points and reentrant phase transitions exist, and thus interesting topological properties would be expected.

A. Charged case

We first study the topology of thermodynamics in the charged case, i.e., $q > 0$. From Eq. (31) and the first equation of (32), the spinodal curve is given by

$$t_{\rm sp} = \frac{d-2}{2\pi v^{2d+1}(6\alpha v^2 + v^4 + 15)}\Big[-4\pi q^2 v^{10} + 3(d-7)v^{2d} + 2(d-5)\alpha v^{2d+2} + (d-3)v^{2d+4}\Big], \tag{35}$$

and then we can define

$$f \equiv (\partial_v t_{\rm sp})_{q,\alpha} = -\frac{d-2}{2\pi v^{2d+2}(6\alpha v^2 + v^4 + 15)^2}\Big[60\pi(9-2d)q^2 v^{10} + 24\pi\alpha(7-2d)q^2 v^{12} + 4\pi(5-2d)q^2 v^{14} + 45(d-7)v^{2d} + 4\alpha(6d-57)v^{2d+2} + 6(-10\alpha^2 + 2\alpha^2 d - 5d + 5)v^{2d+4} - 12\alpha v^{2d+6} + (d-3)v^{2d+8}\Big]. \tag{36}$$

Obviously, this function is continuously differentiable. In addition, for any $d \geq 7$ and $q > 0$, we have

$$f(v\to 0^+) \sim \frac{(d-2)(2d-9)q^2}{v^{2d-8}(6\alpha v^2 + v^4 + 15)^2} \to +\infty,$$
$$f(v\to +\infty) \sim -\frac{(d-2)(d-3)v^6}{(6\alpha v^2 + v^4 + 15)^2} \to 0^-, \tag{37}$$

hence f admits an nonzero boundary. By use of Eq. (5), the total topological charge can be directly calculated as

$$Q_{\rm total} = \frac{1}{2}[{\rm sgn}f(v\to +\infty) - {\rm sgn}f(v\to 0^+)] = \frac{1}{2}(-1-1) = -1. \tag{38}$$

Since this result does not depend on the values of parameters (q, α) and dimension d, we conclude that charged Lovelock AdS black holes with spherical horizon geometry share the same topological charge, indicating they can be classified into the same topology class. More interestingly, they belong to the same topology class as the charged AdS black holes, which may imply some similarities between their thermodynamics.

Actually, in the $d = 7$ case, one can find that the equation of state only admits one critical point, and the system exhibits the typical small/large black hole phase structure. While in the $d = 8$ case, the state equation displays one or three critical points (including the unphysical ones) in appropriate parameter ranges, and a triple point may arise when a small charge q is added to the black hole, at which the small, large and intermediate black holes can coexist together. Taking $\alpha = 2.8$ and $q = 0.0175$ for example, we display such a phase structure in Fig. 3. The invariant total topological charge shown above thus suggests that the small/intermediate/large black hole phase structure can be interpreted as the *topological transformation* of small/large black hole phase structure. According to the behavior of f shown in the inset of Fig. 3, it is easy to verify that the total topological charge indeed takes -1:

$$Q_{\rm total} = Q_{{\rm CP}_1} + Q_{{\rm CP}_2} + Q_{{\rm CP}_3} = -1 + 1 - 1 = -1. \tag{39}$$

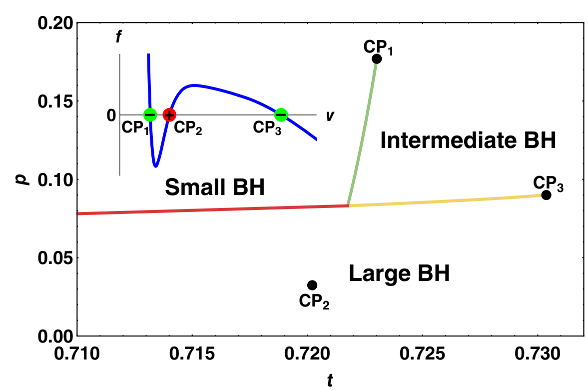


FIG. 3. Small/intermediate/large black phase structure for the charged AdS Lovelock black hole with spherical horizon geometry. Inset: $f \equiv (\partial_v T_{\rm sp})_{q,\alpha}$ vs v diagram. We have set $(d, \alpha, q) = (8, 2.8, 0.0175)$.

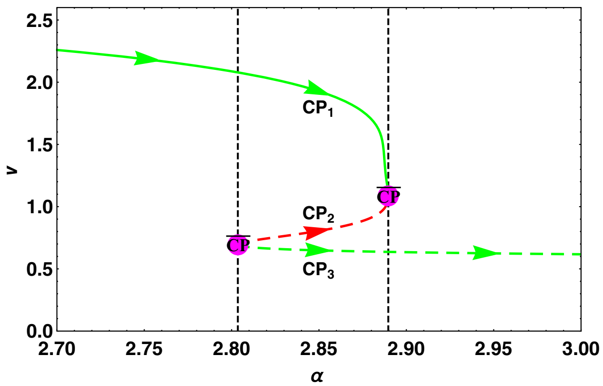


FIG. 4. Pair creation and annihilation of critical points for the charged AdS Lovelcock black hole. The green and red curves represent the branches with negative and positive topological charges, respectively. The black dashed line on the left denotes $\alpha_1 \approx 2.804$, while the right one denotes $\alpha_2 \approx 2.890$. The arrows refer to the direction of increasing α. We have set $d = 8$ and $q = 0.022$.

Meanwhile, the invariant total topological charge also indicates that critical points must *emerge*, *or annihilate in pairs*, between the ones with *opposite* topological charges. Similar phenomena can also be seen in Refs. [38,39,48–52]. To actually observe such a behavior, we take $d = 8$ and consider a process of varying α and fixing $q = 0.022$. In this process, α can be treated as a "time evolution factor" of the system. The critical points are numerically solved using condition (32), and the corresponding topological charge for each critical point is obtained according to the slope of f. Results are summarized in Fig. 4. When $\alpha < \alpha_1 \approx 2.804$, there is only one critical point CP_1 with topological charge -1. When α goes exactly to α_1, besides the original CP_1, we observe a critical point $\overline{\mathrm{CP}}$ with zero slope of f, i.e., zero topological charge. Further increasing α, CP_1 still exists, while $\overline{\mathrm{CP}}$ generates two new critical points:

$$\overline{\mathrm{CP}} \to \mathrm{CP}_2 + \mathrm{CP}_3, \tag{40}$$

in which CP_2 has topological charge $+1$, whereas CP_3 has topological charge -1. With the increasing of α, these two critical points move away from each other, while CP_1 and CP_2 get closer. When α goes to $\alpha_2 \approx 2.890$, we observe a reverse process—two critical points merge into one critical point:

$$\mathrm{CP}_1 + \mathrm{CP}_2 \to \overline{\mathrm{CP}}, \tag{41}$$

where again $\overline{\mathrm{CP}}$ has zero topological charge. Beyond α_2, $\overline{\mathrm{CP}}$ also disappears, and only CP_3 is left. Interestingly, creation and annihilation do not need to occur between the same pair of critical points. The most important feature is that they must occur between critical points with opposite topological charges, to leave the total topological charge unchanged.

B. Uncharged case

Now we turn to study the uncharged case. Setting $q = 0$ in Eq. (35), the spinodal curve reduces to

$$t_{\mathrm{sp}} = \frac{d-2}{2\pi v(6\alpha v^2 + v^4 + 15)}\Big[3(d-7) + 2(d-5)\alpha v^2 + (d-3)v^4\Big], \tag{42}$$

and we define

$$f \equiv (\partial_v t_{\mathrm{sp}})_\alpha = -\frac{d-2}{2\pi v^2(6\alpha v^2 + v^4 + 15)^2}\Big[45(d-7) + 4\alpha(6d-57)v^2 + 6(-10\alpha^2 + 2\alpha^2 d - 5d + 5)v^4 - 12\alpha v^6 + (d-3)v^8\Big]. \tag{43}$$

Analogous to the charged case, this function is continuously differentiable. However, unlike the charged case, we find that this function has different asymptotic behaviors in different dimensions. Specially, when $d = 7$, we have

$$f(v \to 0^+)_{d=7} \sim \alpha, \quad f(v \to +\infty)_{d=7} \sim -\frac{v^6}{(6\alpha v^2 + v^4 + 15)^2} \to 0^-. \tag{44}$$

While for $d > 7$, it becomes

$$f(v \to 0^+)_{d>7} \sim -\frac{(d-2)(d-7)}{v^2(6\alpha v^2 + v^4 + 15)^2} \to -\infty, \quad f(v \to +\infty)_{d>7} \sim -\frac{v^6}{(6\alpha v^2 + v^4 + 15)^2} \to 0^-. \tag{45}$$

For positive α_k, $\alpha > 0$. From Eq. (5), we know that the total topological charge in different dimensions should be given as

$$Q_{\mathrm{total}} = \frac{1}{2}[\mathrm{sgn} f(v \to +\infty) - \mathrm{sgn} f(v \to 0^+)] = \begin{cases} -1 & \text{for } d = 7 \\ 0 & \text{for } d > 7. \end{cases} \tag{46}$$

Therefore, the total topological charge of $d = 7$ uncharged Lovelock black holes is the same as the ones of charged AdS black holes and charged Lovelock AdS black holes, indicating they can be classified into the same topology class. A detailed analysis shows that the equation of state admits only one critical point and the system demonstrates

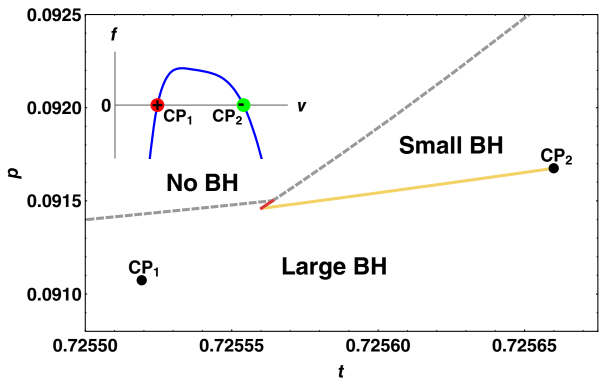


FIG. 5. Reentrant phase structure for the $d = 8$ uncharged Lovelock black hole. Inset: $f \equiv (\partial_v t_{\mathrm{sp}})_\alpha$ vs v diagram.

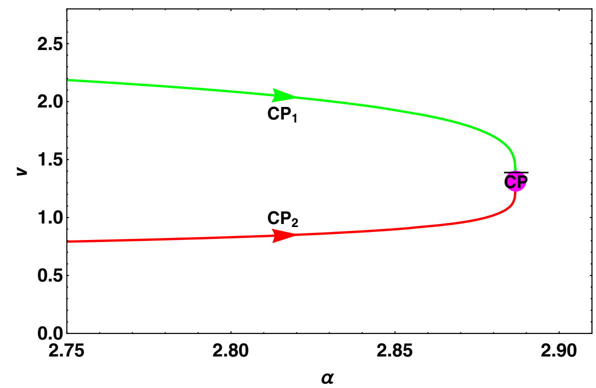


FIG. 6. Pair annihilation of critical points for the uncharged AdS Lovelock black hole. The green and red curves represent the branches with negative and positive topological charges, respectively. The arrows refer to the direction of increasing α. Two branches intersect at $\alpha_1 \approx 2.886$. We have set $d = 8$.

the small/large black hole phase structure, which may be expected for a same topology class. While the total topological charge of the $d > 7$ black holes is different from the $d = 7$ ones, indicating they are in different topology classes. Such a difference in topology may imply some distinct differences in their thermodynamics. Indeed, a new phase transition—the reentrant phase transition—was found in the $d > 7$ case [35,36].

Taking $d = 8$ and $\alpha = 2.884$ for example, we show the corresponding phase diagram in Fig. 5. Different from the small/large black hole phase transition, the reentrant phase transition consists of a VdW-like first-order phase transition (yellow curve) and a zeroth-order phase transition (red curve). Moreover, two critical points exist in the phase diagram. From the inset displayed in Fig. 5, one can see that these critical points possess opposite topological charges, such that

$$Q_{\mathrm{total}} = Q_{\mathrm{CP}_1} + Q_{\mathrm{CP}_2} = 0. \tag{47}$$

On the other hand, due to the invariant total topological charge for $d > 7$ black holes, one can expect that there exists pair creation or annihilation of critical points. To observe this behavior, we numerically solve the critical points for $d = 8$ black holes with different values of α, and then obtain the topological charge for each critical point by reading the corresponding slope of f. Results are shown in Fig. 6. It is clear that when $\alpha < \alpha_1 \approx 2.886$, there are two branches of critical points, CP_1 and CP_2, possessing opposite topological charges. With the increasing of α, they get closer to each other. When α goes exactly to α_1, two critical points merge into one:

$$\mathrm{CP}_1 + \mathrm{CP}_2 \to \overline{\mathrm{CP}}, \tag{48}$$

where $\overline{\mathrm{CP}}$ has zero topological charge. Further increasing α, $\overline{\mathrm{CP}}$ disappears, and there is no critical point anymore.

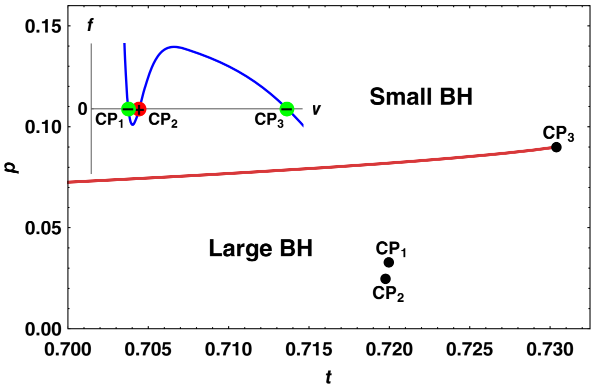
IV. TOPOLOGICAL CHARGE AND REAL CRITICAL POINT

In this section, we would like to discuss the relation between the topological charge and the critical point in more detail. In Sec. II, we have pointed out that a critical point with negative topological charge denotes a maximum point of spinodal curve. Near such a point, using Maxwell's equal law, one can draw a first-order phase transition line in the $T - VT - S$ plane. While for a critical point with positive topological charge, it denotes a minimum point of spinodal curve, and one cannot draw a first-order phase transition line near such a point. In Ref. [26], the authors conclude that the first-order phase transition can extend from the conventional critical point (topological charge -1), while the presence of the novel critical point (topological charge $+1$) cannot serve as an indicator of the presence of the first-order phase transition near it. In other words, the topological charge can be used to distinguish the *real* critical point (second-order phase transition point) and the *pseudo* one.

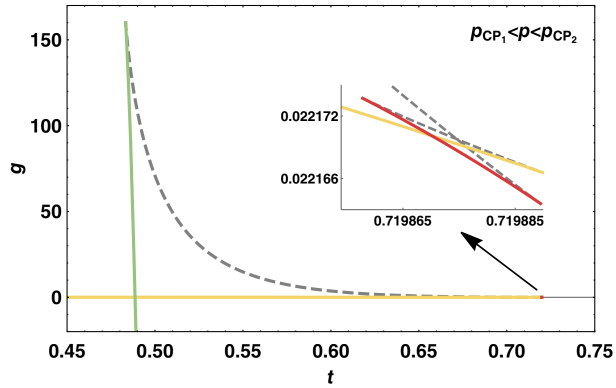
For the black holes we studied, this conclusion holds for the vast majority of cases. It is clear that for the phase structures of most interest shown in Figs. 3 and 5, the conventional critical points indeed connect with the first-order phase transition curve, but the novel ones do not. Nevertheless, we observe a special case in which this conclusion is not applicable, as shown in Fig. 7(a). Such a case occurs in the $d = 8$ charged Lovelock AdS black hole, with $q = 0.021$ and $\alpha = 2.8$. Similar to the small/intermediate/large black hole phase structure, three critical points exist in the system, with topological charges

$$Q_{\mathrm{CP}_1} = -1, \qquad Q_{\mathrm{CP}_2} = +1, \qquad Q_{\mathrm{CP}_3} = -1. \tag{49}$$

However, there are only two stable phases, the large black hole and the small black hole, together with a first-order



(a)



(b)

FIG. 7. (a) Small/large black hole phase structure for the $d = 8$ charged Lovelock black hole. (b) $g - t$ diagram. We have set $q = 0.021$ and $\alpha = 2.8$.

phase transition curve (red line) between them. We can find that only CP_3 connects with the phase transition curve, whereas CP_1 and CP_2 are below it. Hence, in this case, the topological charge is failing to identify whether a critical point is real or pseudo. In fact, near the critical pressure and temperature of CP_1, one can also observe the swallowtail behavior in the Gibbs free energy curve, as shown in the inset of Fig. 7(b). But the black hole phases here do not correspond to a minimal energy, and thus CP_1 is also a pseudocritical point. It is worth noting that a similar case was also observed in the $d = 6$ charged Gauss-Bonnet AdS black holes [29].

Although the negative topological charge may not indicate a real critical point, it still can be a *necessary condition*; the real critical points *can only* emerge from the critical points with negative topological charge.

V. CONCLUSIONS AND DISCUSSIONS

Starting from the Brouwer degree, we have constructed an approach to probe the topological properties of black hole thermodynamics. The spinodal curve was shown to be a powerful tool to derive the topological information of thermodynamic system. By defining the derivative of spinodal curve as a new function f, we can associate a topological charge (degree) for each zero point, i.e., the critical point, and sum over all contributions to construct a total topological charge for the thermodynamic system. Utilizing a mathematical formula, Eq. (5), the analytic calculation of total topological charge is now straightforward; one just needs to examine the asymptotic behavior of spinodal curve's derivative. This enables us to investigate the topological transition between different thermodynamic systems, and to give a topological classification for them conveniently.

As an example, we first investigated the topology of charged AdS black holes in arbitrary dimensions. We showed that for any $d \geq 4$ and $q > 0$, the charged AdS black holes have the same topological charge -1, which generalizes the $d = 4$, $q = 1$ result obtained in Ref. [26].

Then we turned to study the topology of a more complex system—Lovelock AdS black holes. In particular, we focused on the $\kappa = +1$ case, i.e., the black holes with spherical horizon geometry. For the charged case, it was demonstrated that $d \geq 7$ black holes with arbitrary parameters have the same topological charge -1. This indicates that spherical charged Lovelock AdS black holes should be classified into the same topology class with charged AdS black holes, as well as the charged Gauss-Bonnet AdS black holes [29]. Thus, it seems that the higher curvature corrections do not change the topology class of black hole thermodynamics in $U(1)$ charged black holes. On the other hand, it is known that these black holes hold two different phase structures—the small/large black hole phase structure and small/intermediate/large black hole phase structure. While from the viewpoint of topology, they are equivalent; the small/intermediate/large black hole phase structure can be interpreted as the topological transformation of the small/large one.

For the uncharged case, we found that the $d = 7$ (topological charge -1) and $d \geq 8$ (topological charge 0) black holes are in different topology classes. Such a topological change is found to be accompanied by a change in phase structures—from the small/large black hole phase structure to the reentrant phase structure. Combing the relation between phase structures in the charged case, we conclude that the topological change can be a prognostic indicator of the change in phase structures, but *not vice versa*.

Some general topological properties of critical points were also discussed: (i) For a black hole system with nonzero topological charge, there is at least one critical point. (ii) For a black hole system with invariant topological charge, critical points must emerge, or annihilate in pairs, between the ones with opposite topological charges. (iii) Real critical points can only emerge from the critical points with negative topological charge. Especially, if a system has a total topological charge -1, from (i), we know that there is at least one critical point. Combing property (ii), this explains why there is always an odd number of critical points in the system, instead of an even number. Analogously, for a system with topological charge 0, there may be no critical points or an even number of critical points. These results confirm the parity conjecture of critical points proposed in Ref. [29], which says that, "for odd (even) number of critical points, the total topological charge is an odd (even) number." Note that the critical points we are talking about here include the ones with negative critical pressure and temperature, and exclude the very special ones with topological charge 0.

It is worth emphasizing that, to obtain the total topological charge, one does not need to get an exact solution for critical points. Some useful information can be directly obtained from this topological quantity, such as the existence and number (odd or even) of critical points, as well as the possible transition in phase structures. This would be quite helpful for the investigation on black hole thermodynamics.

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APPENDIX A: TOPOLOGY IN $P-V$ CRITICALITY

In the $P-V$ criticality, the state equation can be generally expressed as $P = P(V, T, z^i)$, with the critical points identified by

$$\left(\frac{\partial P}{\partial V}\right)_{T,z^i} = 0, \qquad \left(\frac{\partial^2 P}{\partial V^2}\right)_{T,z^i} = 0. \tag{A1}$$

Using the first equation, one can eliminate the parameter T and obtain the spinodal curve $P_{\rm sp} = P(V, z^i)$. Then a function $f \equiv (\partial_V P_{\rm sp})_{z^i}$ can be constructed to investigate the topological properties of critical points in the $P-V$ criticality. Taking the charged AdS black hole system for example, the state equation reads

$$P = \frac{\frac{32\pi^2 q^2 r_{\rm h}^{6-2d}}{\omega_{d-2}^2} + 4\pi(d-2)r_{\rm h}T - d(d-5) - 6}{16\pi r_{\rm h}^2}, \tag{A2}$$

which is obtained by rearranging Eq. (8). The spinodal curve is calculated as

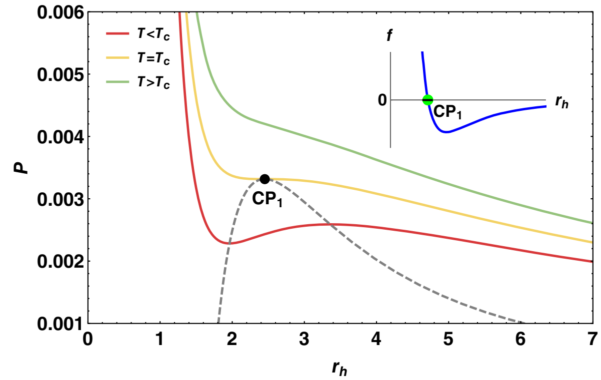

FIG. 8. Isothermal curves and spinodal curve (gray dashed line) for the charged AdS black hole in the $P-r_h$ plane. Inset: $f \equiv (\partial_{r_{\rm h}} P_{\rm sp})_q$ vs $r_{\rm h}$ diagram. We have set $d = 4$ and $q = 1$.

$$P_{\rm sp} = \frac{d(d-5)+6}{16\pi r_{\rm h}^2} - \frac{2\pi(2d-5)q^2}{\omega_{d-2}^2 r_{\rm h}^{2d-4}}. \tag{A3}$$

As displayed in Fig. 8, the extreme point of spinodal curve exactly corresponds to the thermodynamic critical point. The function f can be constructed as

$$f \equiv (\partial_{r_{\rm h}} P_{\rm sp})_q = -\frac{d(d-5)+6}{8\pi r_{\rm h}^3} + \frac{4\pi^2(d-2)(2d-5)q^2}{\pi\omega_{d-2}^2 r_{\rm h}^{2d-3}}, \tag{A4}$$

which is continuously differentiable, and for any $d \geq 4$ and $q > 0$,

$$f(r_{\rm h} \to 0^+) \sim \frac{4\pi^2(d-2)(2d-5)q^2}{\pi\omega_{d-2}^2 r_{\rm h}^{2d-3}} \to +\infty, \tag{A5}$$

$$f(r_{\rm h} \to +\infty) \sim -\frac{d(d-5)+6}{8\pi r_{\rm h}^3} \to 0^-, \tag{A6}$$

and thus f also admits a nonzero boundary. With Eq. (5), the total topological charge reads

$$\begin{aligned} Q_{\rm total} &= \frac{1}{2}[{\rm sgn} f(r_{\rm h} \to +\infty) - {\rm sgn} f(r_{\rm h} \to 0^+)] \\ &= \frac{1}{2}(-1-1) = -1. \end{aligned} \tag{A7}$$

This result is the same as the one we obtained in Sec. II via $T-S$ criticality, which may be expected due to the equivalence of critical point conditions (6) and (A1). The nonzero value suggests that at least one critical point presents in the system. Moreover, as displayed in Fig. 8, near the critical point with negative topological charge, the stable black hole branches are on both sides, while the unstable black hole branch is in the middle, and one can draw a first-order phase transition line in P–V plane between these stable black hole branches using Maxwell's equal area law. But note that whether such a phase transition can actually occur should be carefully examined by the free energy, as we discussed in Sec. IV.

A slightly complex example is the charged spherical Lovelock AdS black hole, whose state equation in terms of the dimensionless quantities Eq. (30) is given by

$$p = \frac{t}{v} - \frac{(d-3)(d-2)}{4\pi v^2} + \frac{2\alpha t}{v^3} - \frac{\alpha(d-5)(d-2)}{4\pi v^4} + \frac{3t}{v^5} - \frac{(d-7)(d-2)}{4\pi v^6} + \frac{q^2}{v^{2(d-2)}}, \tag{A8}$$

with the spinodal curve

$$\begin{aligned} p_{\rm sp} = & \frac{1}{4\pi v^{2(d+3)}(6\alpha v^2 + v^4 + 15)} \Big[-12\pi(2d-9)q^2 v^{10} \\ & - 8\pi\alpha(2d-7)q^2 v^{12} - 4\pi(2d-5)q^2 v^{14} \\ & + 3(d-7)(d-2)v^{2d} + 3\alpha(d-9)(d-2)v^{2d+2} \\ & + 2(d-2)(-5\alpha^2 + \alpha^2 d - 2d - 4)v^{2d+4} \\ & + \alpha(d-9)(d-2)v^{2d+6} + (d-3)(d-2)v^{2d+8} \Big]. \end{aligned} \tag{A9}$$

The function $f \equiv (\partial_v p_{\rm sp})_{q,\alpha}$ can be calculated as

$$\begin{aligned} f = & \frac{(d-2)(2\alpha v^2 + v^4 + 3)}{2\pi v^{2d+7}(6\alpha v^2 + v^4 + 15)^2} \Big[60\pi(2d-9)q^2 v^{10} \\ & + 24\pi\alpha(2d-7)q^2 v^{12} + 4\pi(2d-5)q^2 v^{14} \\ & - 45(d-7)v^{2d} - 12\alpha(2d-19)v^{2d+2} \\ & - 6(-10\alpha^2 + 2\alpha^2 d - 5d + 5)v^{2d+4} \\ & + 12\alpha v^{2d+6} - (d-3)v^{2d+8} \Big]. \end{aligned} \tag{A10}$$

This function is continuously differentiable. In addition, for any $d \geq 7$ and $q > 0$, we have

$$f(v \to 0^+) \sim \frac{(d-2)(2d-9)(3 + 2\alpha v^2 + v^4)q^2}{v^{2d-3}(6\alpha v^2 + v^4 + 15)^2} \to +\infty, \tag{A11}$$

$$f(v \to +\infty) \sim -\frac{(d-2)(d-3)(3v + 2\alpha v^3 + v^5)}{2\pi(6\alpha v^2 + v^4 + 15)^2} \to 0^-, \tag{A12}$$

hence f admits a nonzero boundary. The total topological charge can be then obtained as

$$\begin{aligned} Q_{\rm total} &= \frac{1}{2}[{\rm sgn} f(v \to +\infty) - {\rm sgn} f(v \to 0^+)] \\ &= \frac{1}{2}(-1 - 1) = -1. \end{aligned} \tag{A13}$$

Again, this result is consistent with the one we obtained in Sec. III.

APPENDIX B: TOPOLOGY IN $q-\Phi$ CRITICALITY

One can also observe the VdW-like critical behaviors in the $q-\Phi$ plane, by identifying the electric charge q with the fluid pressure, the electric potential Φ with the fluid volume, and the black hole inverse temperature $\beta = 1/T$ with the fluid temperature [7–9,14]. It would be interesting to examine the applicability of our topological discussion in such case. The state equation now can be generally expressed as $q = q(\Phi, \beta, z^i)$, with the critical points identified by

$$\left(\frac{\partial q}{\partial \Phi}\right)_{\beta, z^i} = 0, \qquad \left(\frac{\partial^2 q}{\partial \Phi^2}\right)_{\beta, z^i} = 0. \tag{B1}$$

Using the first equation, one can eliminate the parameter β and get the spinodal curve $q_{\rm sp} = q(\Phi, z^i)$. Then a function $f \equiv (\partial_\Phi q_{\rm sp})_{z^i}$ can be constructed to investigate the topological properties of critical points in the $q-\Phi$ criticality. Similarly, taking the charged AdS black hole system for example, the state equation reads

$$\begin{aligned} & 2(d-3)^2 q \Phi^{\frac{2d-5}{d-3}} - (d-2)(d-3) q \Phi^{\frac{1}{d-3}} \\ & + \frac{(d-2)(4\pi)^{\frac{d-2}{d-3}} q^{\frac{d-2}{d-3}}}{(d-3)^{\frac{1}{d-3}} \omega_{d-2}^{\frac{1}{d-3}} \beta} \\ & - \frac{(d-1)(d-2)(4\pi)^{\frac{2}{d-3}} q^{\frac{d-1}{d-3}}}{(d-3)^{\frac{2}{d-3}} \omega_{d-2}^{\frac{2}{d-3}} l^2 \Phi^{\frac{1}{d-3}}} = 0, \end{aligned} \tag{B2}$$

which is obtained by inserting

$$\Phi = \frac{4\pi q}{(d-3)\omega_{d-2} r_{\rm h}^{d-3}} \tag{B3}$$

into Eq. (8). The spinodal curve can be then calculated as

$$\begin{aligned} & q_{\rm sp}^{\frac{2}{d-3}} - \frac{(d-3)^{\frac{d-1}{d-3}} \omega_{d-2}^{\frac{2}{d-3}} l^2 \Phi^{\frac{2}{d-3}}}{(4\pi)^{\frac{2}{d-3}} (d-1)(d-2)} \\ & \times [(d-2) - 2(d-3)(2d-5)\Phi^2] = 0. \end{aligned} \tag{B4}$$

i.e.,

$$q_{\rm sp} = \frac{(d-3)^{\frac{d-1}{2}}\omega_{d-2}l^{d-3}\Phi}{4\pi(d-1)^{\frac{d-3}{2}}(d-2)^{\frac{d-3}{2}}} \times [(d-2)-2(d-3)(2d-5)\Phi^2]^{\frac{d-3}{2}}, \quad \text{(B5)}$$

with $\Phi \in (0, a)$ where $a = \sqrt{\frac{d-2}{2(d-3)(2d-5)}}$. Taking $d = 4$, $l = 1$ for example, we show the isothermal curves and the spinodal curve in Fig. 9. It is clear that the critical point CP_1 is exactly the extreme point of spinodal curve. Moreover, the black hole branches upon the spinodal curve with positive heat capacity $C_q = T(\partial_S T)^{-1}_{q,z^i}$ are stable, while the branches below the spinodal curve with negative C_q are unstable [6–9].

The function f can be calculated as

$$f \equiv (\partial_\Phi q_{\rm sp})_l = \frac{(d-3)^{\frac{d-1}{2}}\omega_{d-2}l^{d-3}}{4\pi(d-1)^{\frac{d-3}{2}}(d-2)^{\frac{d-3}{2}}} \times [1-2(d-3)(2d-5)\Phi^2] \times [(d-2)-2(d-3)(2d-5)\Phi^2]^{\frac{d-5}{2}}. \quad \text{(B6)}$$

It is continuously differentiable. Especially, for $d = 4$, $\Phi \in (0, 1/\sqrt{3})$, and

$$f(\Phi \to 0^+) \sim l/\sqrt{3}, \quad \text{(B7)}$$

$$f\left(\Phi \to \frac{1}{\sqrt{3}}\right) \sim -\frac{l}{\sqrt{\sqrt{3}-3\Phi}} \to -\infty, \quad \text{(B8)}$$

as shown in the inset of Fig. 9.

For $d = 5$, $\Phi \in (0, \frac{1}{2}\sqrt{\frac{3}{5}})$, and

$$f(\Phi \to 0^+) \sim \frac{\pi l^2}{2}, \qquad f\left(\Phi \to \frac{1}{2}\sqrt{\frac{3}{5}}\right) \sim -\pi l^2. \quad \text{(B9)}$$

For $d > 5$, $\Phi \in (0, a)$, and

$$f(\Phi \to 0^+) \sim \frac{(d-3)^{\frac{d-1}{2}}\omega_{d-2}l^{d-3}}{4\pi(d-1)^{\frac{d-3}{2}}}, \qquad f(\Phi \to a) \to 0^-. \quad \text{(B10)}$$

Thus f admits a nonzero boundary.

With Eq. (5), the total topological charge reads

$$Q_{\rm total} = \frac{1}{2}[{\rm sgn}f(\Phi \to a) - {\rm sgn}f(\Phi \to 0^+)] \quad \text{(B11)}$$

$$= \frac{1}{2}(-1-1) = -1, \quad \text{(B12)}$$

which is independent of $d \geq 4$ and $l > 0$. This implies that there is at least one critical point presented in the system.

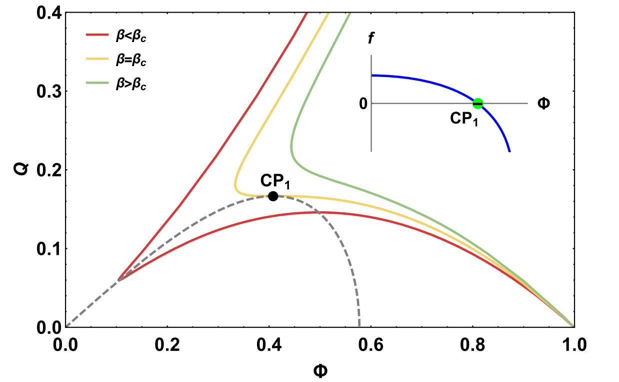

FIG. 9. Isothermal curves and spinodal curve (gray dashed line) for the charged AdS black hole in the $q-\Phi$ plane. Inset: $f \equiv (\partial_\Phi q_{\rm sp})_l$ vs Φ diagram. We have set $d = 4$ and $l = 1$.

An exact solution for Eq. (B1) shows that there is only one critical point with

$$\Phi_c = \frac{1}{\sqrt{2(d-3)(2d-5)}}, \quad \text{(B13)}$$

$$\beta_c = \frac{(2d-5)\pi l}{(d-3)\sqrt{(d-1)(d-2)}}, \quad \text{(B14)}$$

$$q_c = \frac{(d-3)^{\frac{2d-5}{2}}\omega_{d-2}l^{d-3}}{4\pi(d-1)^{\frac{d-3}{2}}(d-2)^{\frac{d-3}{2}}\sqrt{4d-10}}. \quad \text{(B15)}$$

Moreover, near the critical point with negative topological charge, one can draw a first-order phase transition line in the $q-\Phi$ plane between the stable black hole branches using Maxwell's equal area law. Again, whether such a phase transition can actually occur should be carefully examined by the free energy.

After similar calculations, the same topological charge is obtained for the $d \geq 4$ charged AdS black holes in the $\beta - r_{\rm h}$ criticality [6]. We have also tried to calculate the topological charges for charged Lovelock AdS black holes in the $q-\Phi$ and $\beta - v$ criticalities. Here $\beta = 1/t$, t is the reduced Hawking temperature and v denotes the reduced horizon radius, as shown in Eq. (30). In the $q-\Phi$ criticality, we find that it is difficult to obtain an analytic expression for the spinodal curve $q_{\rm sp}$ due to the complexity of the state equation given by Eqs. (24) and (31). In the $\beta - v$ criticality, although an analytic expression for $f(v) \equiv (\partial_v \beta_{\rm sp})_{l,\alpha}$ can be obtained, the specific boundary for v (required by a positive q^2 in the first condition of critical points) is difficult to calculate analytically. These hinder us from performing topological analysis using the current method. Such issues may be addressed by employing the method provided in our recent work [53]. We hope that through our efforts, these issues can be completely solved in the future.

[1] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).

[3] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).

[4] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti-de Sitter space, Commun. Math. Phys. **87**, 577 (1983).

[5] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. **2**, 505 (1998).

[6] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Charged AdS black holes and catastrophic holography, Phys. Rev. D **60**, 064018 (1999).

[7] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Holography, thermodynamics and fluctuations of charged AdS black holes, Phys. Rev. D **60**, 104026 (1999).

[8] X. N. Wu, Multicritical phenomena of Reissner-Nordstrom anti-de Sitter black holes, Phys. Rev. D **62**, 124023 (2000).

[9] C. Niu, Y. Tian, and X. N. Wu, Critical phenomena and thermodynamic geometry of RN-AdS black holes, Phys. Rev. D **85**, 024017 (2012).

[10] M. M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Classical Quantum Gravity **17**, 399 (2000).

[11] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the mechanics of AdS black holes, Classical Quantum Gravity **26**, 195011 (2009).

[12] B. P. Dolan, The cosmological constant and the black hole equation of state, Classical Quantum Gravity **28**, 125020 (2011).

[13] D. Kubiznak and R. B. Mann, Black hole chemistry, Can. J. Phys. **93**, 999 (2015).

[14] D. Kubiznak and R. B. Mann, P-V criticality of charged AdS black holes, J. High Energy Phys. 07 (2012) 033.

[15] S. W. Wei and Y. X. Liu, Insight into the Microscopic Structure of an AdS Black Hole from a Thermodynamical Phase Transition, Phys. Rev. Lett. **115**, 111302 (2015); **116**, 169903(E) (2016).

[16] S. W. Wei, Y. X. Liu, and R. B. Mann, Repulsive Interactions and Universal Properties of Charged Anti–de Sitter Black Hole Microstructures, Phys. Rev. Lett. **123**, 071103 (2019).

[17] S. W. Wei, Y. X. Liu, and R. B. Mann, Ruppeiner geometry, phase transitions, and the microstructure of charged AdS black holes, Phys. Rev. D **100**, 124033 (2019).

[18] N. Altamirano, D. Kubiznak, and R. B. Mann, Reentrant phase transitions in rotating anti–de Sitter black holes, Phys. Rev. D **88**, 101502 (2013).

[19] A. Dehyadegari and A. Sheykhi, Reentrant phase transition of Born-Infeld-AdS black holes, Phys. Rev. D **98**, 024011 (2018).

[20] N. Altamirano, D. Kubizňák, R. B. Mann, and Z. Sherkatghanad, Kerr-AdS analogue of triple point and solid/liquid/gas phase transition, Classical Quantum Gravity **31**, 042001 (2014).

[21] S. W. Wei and Y. X. Liu, Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space, Phys. Rev. D **90**, 044057 (2014).

[22] R. A. Hennigar, R. B. Mann, and E. Tjoa, Superfluid Black Holes, Phys. Rev. Lett. **118**, 021301 (2017).

[23] L. E. Reichl, *A Modern Course in Statistical Physics* (Wiley, New York, 1998).

[24] D. Kubiznak, R. B. Mann, and M. Teo, Black hole chemistry: Thermodynamics with Lambda, Classical Quantum Gravity **34**, 063001 (2017).

[25] B. P. Dolan, A. Kostouki, D. Kubiznak, and R. B. Mann, Isolated critical point from Lovelock gravity, Classical Quantum Gravity **31**, 242001 (2014).

[26] S. W. Wei and Y. X. Liu, Topology of black hole thermodynamics, Phys. Rev. D **105**, 104003 (2022).

[27] Y. S. Duan and M. L. Ge, SU(2) gauge theory and electrodynamics with N magnetic monopoles, Sci. Sin. **9**, 1072 (1979).

[28] Y. S. Duan, The structure of the topological current, Stanford Linear Accelerator Center (SLAC) Report No. SLAC-PUB-3301, 1984.

[29] P. K. Yerra and C. Bhamidipati, Topology of black hole thermodynamics in Gauss-Bonnet gravity, Phys. Rev. D **105**, 104053 (2022).

[30] P. K. Yerra and C. Bhamidipati, Topology of Born-Infeld AdS black holes in 4D novel Einstein-Gauss-Bonnet gravity, Phys. Lett. B **835**, 137591 (2022).

[31] M. B. Ahmed, D. Kubiznak, and R. B. Mann, Vortex/anti-vortex pair creation in black hole thermodynamics, arXiv:2207.02147.

[32] D. George and M. Jean, *Brouwer Degree* (Springer, New York, 2021).

[33] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys. (N.Y.) **12**, 498 (1971).

[34] P. Wang, H. Wu, H. Yang, and S. Ying, Derive Lovelock gravity from string theory in cosmological background, J. High Energy Phys. 05 (2021) 218.

[35] H. Xu, W. Xu, and L. Zhao, Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions, Eur. Phys. J. C **74**, 3074 (2014).

[36] A. M. Frassino, D. Kubiznak, R. B. Mann, and F. Simovic, Multiple reentrant phase transitions and triple points in Lovelock thermodynamics, J. High Energy Phys. 09 (2014) 080.

[37] S. H. Hendi, S. Panahiyan, and B. Eslam Panah, Extended phase space of black holes in Lovelock gravity with nonlinear electrodynamics, Prog. Theor. Exp. Phys. **2015**, 103E01 (2015).

[38] L. B. Fu, Y. S. Duan, and H. Zhang, Evolution of the Chern-Simons vortices, Phys. Rev. D **61**, 045004 (2000).

[39] P. V. P. Cunha, E. Berti, and C. A. R. Herdeiro, Light-Ring Stability for Ultracompact Objects, Phys. Rev. Lett. **119**, 251102 (2017).

[40] T. Liddell, The topological degree and applications, Scientific report, Australia: Australian Mathematical Sciences Institute Vacation Research Scholarships Report, 2015.

[41] S. Gunasekaran, R. B. Mann, and D. Kubiznak, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, J. High Energy Phys. 11 (2012) 110.

[42] D. G. Boulware and S. Deser, String Generated Gravity Models, Phys. Rev. Lett. **55**, 2656 (1985).

[43] J. T. Wheeler, Symmetric solutions to the maximally Gauss-Bonnet extended Einstein equations, Nucl. Phys. **B273**, 732 (1986).
[44] R. G. Cai, A note on thermodynamics of black holes in Lovelock gravity, Phys. Lett. B **582**, 237 (2004).
[45] R. C. Myers and J. Z. Simon, Black hole thermodynamics in Lovelock gravity, Phys. Rev. D **38**, 2434 (1988).
[46] D. Kastor, S. Ray, and J. Traschen, Smarr formula and an extended first law for Lovelock gravity, Classical Quantum Gravity **27**, 235014 (2010).
[47] D. Kastor, S. Ray, and J. Traschen, Mass and free energy of Lovelock black holes, Classical Quantum Gravity **28**, 195022 (2011).
[48] Y. S. Duan and X. Liu, Knotlike cosmic strings in the early universe, J. High Energy Phys. 02 (2004) 028.
[49] Z. B. Cao, Y. X. Liu, and Y. S. Duan, Some topological properties of anyons, Commun. Theor. Phys. **50**, 1155 (2008).
[50] S. W. Wei and Y. X. Liu, Topology of equatorial timelike circular orbits around stationary black holes, arXiv:2207.08397.
[51] S. W. Wei, Y. X. Liu, and R. B. Mann, Black Hole Solutions as Topological Thermodynamic Defects, Phys. Rev. Lett. **129**, 191101 (2022).
[52] P. K. Yerra, C. Bhamidipati, and S. Mukherji, Topology of critical points and Hawking-Page transition, Phys. Rev. D **106**, 064059 (2022).
[53] N. C. Bai, L. Song, and J. Tao, Reentrant phase transition in holographic thermodynamics of Born-Infeld AdS black hole, arXiv:2212.04341.